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# Hadamard's variation of the Green kernels of heat equations and their traces I

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**§1. Introduction.** Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\gamma$. Let $\rho(x)$ be a $C^\infty$ real valued function on $\gamma$ and $\nu_x$ be the exterior unit normal vector at $x\in\gamma$. For any sufficiently small $\varepsilon\geqq 0$, let $\Omega_\varepsilon$ be the bounded domain whose boundary $\gamma_\varepsilon$ is defined by

$$\gamma_\varepsilon=\{x+\varepsilon\rho(x)\nu_x\,;\,x\in\gamma\}.$$

Let $G_\varepsilon(x, y)$ be the Green's function of the Dirichlet boundary value problem for the Laplacian, that is, $G_\varepsilon(x, y)$ has the following properties:

$$\begin{cases} -\Delta_x G_\varepsilon(x, y)=\delta(x-y) & x,\ y\in\Omega_\varepsilon \\ G_\varepsilon(x, y)=0 & x\in\gamma_\varepsilon,\ y\in\Omega_\varepsilon\,. \end{cases}$$

We abbreviate $G_0(x, y)$ as $G(x, y)$. For any $x,\ y\in\Omega$ satisfying $x\neq y$, we put

$$\delta G(x, y)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(G_\varepsilon(x, y)-G(x, y))\,.$$

Then the celebrated Hadamard variational formula is the following:

$$\delta G(x, y)=\int_\gamma \frac{\partial G(x, z)}{\partial\nu_z}\,\frac{\partial G(y, z)}{\partial\nu_z}\,\rho(z)d\sigma_z\,, \tag{1.1}$$

where $\partial/\partial\nu_z$ denotes the exterior normal derivative with respect to $z$ and $d\sigma_z$ denotes the surface element of $\gamma$ at $z$.

In [7], Hadamard proved the formula (1.1) in the case that $\rho(z)$ did not change its sign. And he also proved it when $\gamma$ was of class $C^\omega$.

Proof of the formula (1.1) for general $\rho(z)\in C^\infty(\gamma)$ can be found, for example, in Garabedian [5], Garabedian-Schiffer [6]. Based on (1.1), many authors derived interesting facts about the Green's function and the results about the theory of functions of one complex variable. See Bergmann-Schiffer [2] and Schiffer-Spencer [12]. Recently, new applications of the formula (1.1) have appeared.

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See Aomoto [1], Fujiwara-Tanikawa-Yukita [4]. In Fujiwara-Ozawa [3], we generalized the formula (1.1) to the variational formulas of the Green kernel of elliptic normal boundary value problems. See also Peetre [11].

In this note, in § 3, we give the Hadamard variational formulas of the Green kernels of the heat equations with the Dirichlet and the third boundary condition. In § 4, § 5 we give the Hadamard variational formulas of the traces of the Green kernels of the heat equations with the Dirichlet boundary condition.

We now state the results in this note.

Let $k$ be a fixed number satisfying $0 \leqq k \leqq 1$. Let $U_\varepsilon(x, y, t)$ denote the fundamental solution of the heat equation with the boundary condition (1.3), that is, $U_\varepsilon(x, y, t)$ has the following properties (1.2), (1.3) and (1.4):

$$(\partial_t - \Delta_x) U_\varepsilon(x, y, t) = 0 \qquad x, y \in \Omega_\varepsilon,\ t > 0 \tag{1.2}$$

$$\Big(k \frac{\partial}{\partial \nu_x^\varepsilon} + (1-k)\Big) U_\varepsilon(x, y, t) = 0 \qquad x \in \gamma_\varepsilon,\ y \in \Omega_\varepsilon,\ t > 0 \tag{1.3}$$

$$\lim_{t \to +0} U_\varepsilon(x, y, t) = \delta(x-y) \qquad x, y \in \Omega_\varepsilon, \tag{1.4}$$

where $\partial/\partial \nu_x^\varepsilon$ denotes the exterior normal derivative with respect to $x$ at the boundary $\gamma_\varepsilon$. We abbreviate $U_0(x, y, t)$ as $U(x, y, t)$.

Put

$$\delta U(x, y, t) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(U_\varepsilon(x, y, t) - U(x, y, t))$$

for any $x, y \in \Omega$ and $t > 0$ if the right hand side exists. Then we have the following

THEOREM 1. *If $k=0$, then $\delta U(x, y, t)$ exists and it is given by*

$$\delta U(x, y, t) = \int_0^t d\tau \int_\gamma \frac{\partial U(x, z, t-\tau)}{\partial \nu_z} \frac{\partial U(y, z, \tau)}{\partial \nu_z} \rho(z) d\sigma_z . \tag{1.5}$$

To state the next result, we shall use the following notation. We fix $z \in \gamma$ and take an orthonormal basis $(z_1, \cdots, z_{n-1})$ on the tangent hyperplane at $z$. Then we put

$$\langle \nabla_\gamma a(z), \nabla_\gamma b(z) \rangle = \sum_{j=1}^n \frac{\partial a}{\partial z_j}(z) \frac{\partial b}{\partial z_j}(z) \tag{1.6}$$

for any $a(z), b(z) \in C^\infty(\gamma)$. The left hand side of (1.6) is independent of the choice of orthonormal basis on the tangent hyperplane.

We have the following

THEOREM 2. *If $0 < k \leqq 1$, then*

$$\delta U(x, y, t) = -\int_0^t d\tau \int_\gamma \langle \nabla_\gamma U(x, z, t-\tau), \nabla_\gamma U(y, z, \tau) \rangle \rho(z) d\sigma_z \tag{1.7}$$

$$-\frac{\partial}{\partial t}\int_0^t d\tau\int_\gamma U(x, z, t-\tau)U(y, z, \tau)\rho(z)d\sigma_z$$

$$+\int_0^t d\tau\int_\gamma U(x, z, t-\tau)U(y, z, \tau)(p^2-(n-1)pH_1(z))\rho(z)d\sigma_z .$$

*Here* $p=k^{-1}(1-k)$ *and* $H_1(z)$ *denotes the first mean curvature of* $\gamma$ *at* $z$ *with respect to the inner normal vector.*

Let $T_\gamma(t; \varepsilon)$ denote the trace of $U_\varepsilon(x, y, t)$ on $\Omega_\varepsilon$, which is defined by

$$T_\gamma(t; \varepsilon)=\int_{\Omega_\varepsilon} U_\varepsilon(x, x, t)dx . \tag{1.8}$$

For any $t>0$, we put

$$\delta T_\gamma(t)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(T_\gamma(t; \varepsilon)-T_\gamma(t; 0))$$

if the right hand side exists. Then we have the following

THEOREM 3. *When* $k=0$, *then* $\delta T_\gamma(t)$ *exists for* $t>0$ *and*

$$\delta T_\gamma(t)=\int_\Omega \delta U(x, x, t)dx . \tag{1.9}$$

*And the right hand side of* (1.9) *can be written explicitly as*

$$t\int_\gamma \frac{\partial^2}{\partial\nu_y\partial\nu_w}U(y, w, t)\Big|_{y=w=z}\rho(z)d\sigma_z . \tag{1.10}$$

Let $0\geqq\lambda_1\geqq\lambda_2\geqq\cdots$ be the eigenvalues of the Laplacian with the Dirichlet boundary condition on $\gamma$. We arrange them repeatedly according to their multiplicities. Let $\{\varphi_j(x)\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of the Laplacian. We assume that $\varphi_j(x)$ belongs to the eigenspace associated with $\lambda_j$. It is well known that

$$U(x, y, t)=\sum_{j=1}^\infty e^{\lambda_j t}\varphi_j(x)\varphi_j(y) . \tag{1.11}$$

Therefore by Theorem 1 and (1.9), we have the following

THEOREM 4. *If* $k=0$, *then*

$$\delta T_\gamma(t)=t\sum_{j=1}^\infty e^{\lambda_j t}\int_\gamma\Big(\frac{\partial\varphi_j}{\partial\nu_z}(z)\Big)^2\rho(z)d\sigma_z . \tag{1.12}$$

Let $\lambda_j(\varepsilon)$ be the $j$-th eigenvalue of the Laplacian with the Dirichlet boundary condition on $\gamma_\varepsilon$. In [7], Hadamard gave the following variational formula when $\lambda_j(0)\equiv\lambda_j$ is a simple eigenvalue

$$\delta\lambda_j=\int_\gamma\Big(\frac{\partial\varphi_j}{\partial\nu_z}(z)\Big)^2\rho(z)d\sigma_z , \tag{1.13}$$

where we put

$$\delta\lambda_j=\lim_{\varepsilon\to 0}\varepsilon^{-1}(\lambda_j(\varepsilon)-\lambda_j(0)).$$

The precise proof of (1.13) was given in Garabedian-Schiffer [**6**] when $\lambda_j(0)$ is simple.

If $\lambda_j$ are all simple eigenvalue, we have by heuristic consideration

$$\text{(1.14)}\qquad \delta T_\gamma(t)=\delta\Big(\sum_{j=1}^{\infty} e^{\lambda_j t}\Big)$$

$$=t\sum_{j=1}^{\infty}\delta\lambda_j e^{\lambda_j t}.$$

Put (1.13) into (1.14), then we get the formula (1.12).

In this note, we derive (1.12) without assumption on the multiplicities of eigenvalues, since we use the fundamental solution of the heat equation to derive (1.12).

Recently the author was informed by Professor Jiro Watanabe that he obtained the variational formulas of the eigenvalues of the Laplacian with the Dirichlet and the third condition with no assumption on the multiplicities of the eigenvalues. He did not use the fundamental solution of the heat equation for his studies.

We make some remarks on this paper. For any $\lambda\in\boldsymbol{C}\backslash(0,\infty)$, let $G_\varepsilon(x,y,\lambda)$ be the Green's function of the Dirichlet boundary value problem for $\Delta-\lambda$, that is, $G_\varepsilon(x,y,\lambda)$ satisfies the following properties:

$$\begin{cases}(-\Delta_x+\lambda)G_\varepsilon(x,y,\lambda)=\delta(x-y) & x,y\in\Omega_\varepsilon\\ G_\varepsilon(x,y,\lambda)=0 & x\in\gamma_\varepsilon,\ y\in\Omega_\varepsilon.\end{cases}$$

For any $x, y\in\Omega$ satisfying $x\neq y$, put

$$\delta G(x,y,\lambda)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(G_\varepsilon(x,y,\lambda)-G_0(x,y,\lambda)).$$

Then, the Hadamard variational formula states that

$$\delta G(x,y,\lambda)=\int_\gamma\frac{\partial G(x,z,\lambda)}{\partial\nu_z}\frac{\partial G(y,z,\lambda)}{\partial\nu_z}\rho(z)d\sigma_z.$$

On the other hand, it is well known that $U_\varepsilon$ is given by the Laplace transformation of $G_\varepsilon$, that is,

$$\text{(1.15)}\qquad U_\varepsilon(x,y,t)=\frac{1}{2\pi i}\int_A e^{t\lambda}G_\varepsilon(x,y,\lambda)d\lambda$$

for some path $A$ in the complex plane. From Theorem 1, we see that $\delta U(x,y,t)$

is the Laplace transformation of $\delta G(x, y, \lambda)$. Therefore Theorem 1 assures the commutativity of the Laplace transformation $\mathcal{L}$ and $\delta$, that is, $\delta(\mathcal{L}G)=\mathcal{L}(\delta G)$. Then in a sense Theorem 1 is thought as the Laplace transformation of the Hadamard formula. We can give another proof of $\delta(\mathcal{L}G)=\mathcal{L}(\delta G)$ by using (1.15). For this purpose, we need a kind of uniform estimate for $\varepsilon^{-1}(G_{\varepsilon}(x, y, \lambda) -G_{0}(x, y, \lambda))-\delta G(x, y, \lambda)$ with respect to $\lambda$ on $A$. Such a uniform estimate is not too difficult to give, of course not trivial, but here the author gives a direct proof of Theorem 1 using calculus concerning the heat equation. Theorem 1 enables us to prove Theorem 4 which is the main aim of this note.

It is well known that we can derive information on the asymptotic behaviour of the eigenvalues $\lambda_j$ when $j\to\infty$ from the properties of $T_{\gamma}(t)$ when $t$ tends to zero. From Theorem 4 the following problem occurs. Give an asymptotic properties of

$$\sum_{-\lambda_j<\lambda}\left(\frac{\partial\varphi_j}{\partial\nu}(x)\right)^2\Big|_{x\in\gamma}$$

when $\lambda\to\infty$. In [8] (See also Erratum in [9]) we gave an answer to this problem. The author thinks Theorem 4 as a crossline of the Hadamard formula and the asymptotic properties of eigenfunctions at the boundary.

The main line of the proof of the theorems will start in § 3. § 2 is devoted to a preparatory lemma. We give proofs of theorems 1 and 2 in § 3, a proof of Theorem 3 in § 4. In § 5, we give a proof of Theorem 4.

The results in this paper were announced in [8], [9]. The reader may also refer to [10] in which we gave an expository statement on Hadamard's variational formula of the eigenvalues of the Laplacian.

In a subsequent paper, we give the Hadamard variational formula of the trace of the Green kernels of the heat equation with the third boundary condition.

## § 2. Examination of the $\varepsilon$-dependence of $U_{\varepsilon}$.

The main aim of this section is to prove Lemma 2.3 which will be used in a later section.

Firstly we give some notations. Let $T$ be a positive number. Let $J(T)=\Omega\times(0, T)$ be the cylindrical domain which will be abbreviated as $J$. Put

$$d((x, t), (y, s))=(|x-y|^2+|t-s|)^{1/2}$$

for any $(x, t), (y, s)\in J$, where $|x-y|$ is the Euclidean distance in $R^n$. Let $C^0(J)$ denote the space of continuous function in $J$. Put

$$|u|_{\alpha}^{J}=\text{supremum}\frac{|u(x, t)-u(y, s)|}{d((x, t), (y, s))^{\alpha}}$$

for any $u\in C^0(J)$. Here $0<\alpha<1$. Put

$$\|u\|_\alpha^J=\sup_{(x,t)\in J}|u(x,t)|+|u|_\alpha^J$$

and

$$C^\alpha(J)=\{u\,;\,\|u\|_\alpha^J<\infty\}.$$

If $\partial^{\beta+\gamma}u/\partial x^\beta\partial t^\gamma$ is a continuous function for any multi-indices $\beta$, $\gamma$ satisfying $2|\gamma|+|\beta|\leqq m$, then we define

$$\|u\|_{m+\alpha}^J=\sum_{2|\gamma|+|\beta|\leq m}\left\|\frac{\partial^{\beta+\gamma}u}{\partial x^\beta\partial t^\gamma}\right\|_\alpha^J$$

for each integer $m$. We define the weighted Hölder space $C^{m+\alpha}(J)$ as follows:

$$C^{m+\alpha}(J)=\left\{u\,;\,\frac{\partial^{\beta+\gamma}u}{\partial x^\beta\partial t^\gamma}\text{ is continuous for any }\beta,\gamma\right.$$
$$\left.\text{such that }2|\gamma|+|\beta|\leqq m\text{ and }\|u\|_{m+\alpha}^J<\infty\right\}.$$

We consider the following problem:

$$(2.1)\qquad\begin{cases}\left(\dfrac{\partial}{\partial t}-\Delta_x\right)u(x,t)=f(x,t) & (x,t)\in J\\ u(x,0)=0 & x\in\Omega\\ \left(k\dfrac{\partial}{\partial\nu_x}+(1-k)\right)u(x,t)=b(x,t) & (x,t)\in\gamma\times[0,T),\end{cases}$$

where $k$ is a fixed constant in $[0,1]$ and where $f(x,t)$, $b(x,t)\in C^\infty(\bar\Omega\times[0,T))$. For the sake of simplicity we assume that

$$\left(\frac{\partial^j}{\partial t^j}f\right)(x,0)=0\quad\text{and}\quad\left(\frac{\partial^j}{\partial t^j}b\right)(x,0)=0$$

for any $j=0,1,\cdots$. Then the problem (2.1) satisfies the compatibility condition of order $\infty$ in the sense of p. 98 of Solonnikov [13]. We abbreviate $(\Omega\times\{0\})\cup\{\gamma\times(0,t)\}$ as $\mathfrak{B}$. A function $g(x,t)$ in $\mathfrak{B}$ is said to be of $C^{m+\alpha}(\mathfrak{B})$ class if there exists a function $G(x,t)\in C^{m+\alpha}(J)$ such that $G=g$ on $\mathfrak{B}$. Put

$$\|g\|_{m+\alpha}^{\mathfrak{B}}=\underset{\substack{G\in C^{m+\alpha}(J)\\ G=g\text{ on }\mathfrak{B}}}{\text{infimum}}\|G\|_{m+\alpha}^J.$$

The following two lemmas are well known.

LEMMA 2.1 (Solonnikov [13] p. 121). *For any non negative integer $m$, there exists a positive constant $C_m$ such that*

$$(2.2)\qquad\|u\|_{m+2+\alpha}^J\leqq C_m(\|g\|_{m+1+\alpha}+\|f\|_{m+\alpha}^J)$$

*holds for the solution of the problem* (2.1). *Here* $g(x, t)$ *is equal to* $b(x, t)$ *for* $(x, t)\in\gamma\times[0, T)$ *and* $g(x, 0)\equiv 0$.

LEMMA 2.2. *There exists a positive constant* $C$ *which is independent of* $\varepsilon$ *such that*

$$|U_\varepsilon(x, y, t)|\leqq C\, t^{-n/2}\exp(-C^{-1}t^{-1}|x-y|^2) \tag{2.3}$$

*and*

$$\max_{1\leq i\leq n}\left|\frac{\partial}{\partial x_i}U_\varepsilon(x, y, t)\right|\leqq C\, t^{-(n+1)/2}\exp(-C^{-1}t^{-1}|x-y|^2) \tag{2.4}$$

*holds for any* $x, y\in\Omega_\varepsilon$, $t>0$.

If $\varepsilon$ is sufficiently small and $y\in\Omega$ is fixed, we can construct a family of diffeomorphisms $\Psi_\varepsilon:\bar{\Omega}\to\bar{\Omega}_\varepsilon$ with the following properties (2.5), (2.6) and (2.7):

$$\Psi_\varepsilon(x)=x+\varepsilon\rho(x)\nu_x \qquad \text{for any } x\in\gamma\,. \tag{2.5}$$

(2.6) We consider the diffeomorphism $\Psi_\varepsilon$ as the $R^n$-valued smooth functions on $\bar{\Omega}$, that is, $\Psi_\varepsilon\in C^\infty(\bar{\Omega}, R^n)$ equipped with the usual Fréchet space topology. We assume that the image of the mapping

$$[0, \varepsilon_0)\ni\varepsilon\longmapsto\left(\frac{\partial^m}{\partial\varepsilon^m}\right)\Psi_\varepsilon$$

is contained in a bounded subset of $C^\infty(\bar{\Omega}, R^n)$ for any fixed $m$.

(2.7) There is a relatively compact subset $K$ containing $y$ such that

$$\Psi_\varepsilon(x)=x$$

on some neighbourhood $K'$ of $K$ which satisfies $\bar{K}'\Subset\Omega$.

We define the morphism $\Psi_\varepsilon^*: C^{m+\alpha}(\Omega_\varepsilon)\ni f\mapsto\Psi_\varepsilon^*f\in C^{m+\alpha}(\Omega)$ by $(\Psi_\varepsilon^*f)(x)=f(\Psi_\varepsilon(x))$ for $x\in\Omega$. Here $C^{m+\alpha}(\Omega_\varepsilon)$ denotes the usual Hölder space. This gives the topological isomorphism between $C^{m+\alpha}(\Omega_\varepsilon)$ and $C^{m+\alpha}(\Omega)$ and it also induces the topological isomorphism between $C^{m+\alpha}(\gamma_\varepsilon)$ and $C^{m+\alpha}(\gamma)$. Also $\Psi_\varepsilon^*$ induces the isomorphism

$$\begin{array}{ccc}\Psi_\varepsilon^*\times I: C^{m+\alpha}(\Omega_\varepsilon\times[0, T)) & \xrightarrow{\sim} & C^{m+\alpha}(\Omega\times[0, T)) \\ \cup\!\!\!\!\vert & & \cup\!\!\!\!\vert \\ g & \longmapsto & ((\Psi_\varepsilon^*\times I)g)\end{array}$$

if we set $((\Psi_\varepsilon^*\times I)g)(x, t)=g(\Psi_\varepsilon(x), t)$. Here the function space $C^{m+\alpha}(\Omega_\varepsilon\times[0, T))$ is the weighted Hölder space which we introduced in the first part of this section. We abbreviate $\Psi_\varepsilon^*\times I$ as $\Psi_\varepsilon^*$.

Now we state Lemma 2.3 which will be useful in a later section.

LEMMA 2.3.

$$\lim_{\varepsilon\to 0}\|\Psi_\varepsilon^*U_\varepsilon-U\|_{m+2+\alpha}^J=0\,. \tag{2.8}$$

*Here we give a norm to* $\Psi_\varepsilon^* U_\varepsilon - U$ *considering it as a function of* $(x, t)$.

REMARK. Since $\lim_{t\to+0} U(x, y, t)=\delta(x-y)$ (resp. $\lim_{t\to+0} U_\varepsilon(\Psi_\varepsilon(x), y, t)=\delta(x-y)$), then the function space $C^{m+\alpha}(J)$ does not fit to the study of $U(x, y, t)$ (resp. $U_\varepsilon(\Psi_\varepsilon(x), y, t)$) for each fixed $y$. However $C^{m+\alpha}(J)$ fits to the study of $U(x, y, t) -(\Psi_\varepsilon^* U_\varepsilon)(x, y, t)$, since we know from (2.7) that

$$U(x, y, t)-(\Psi_\varepsilon^* U_\varepsilon)(x, y, t)\in C^\infty(\Omega\times[0, T)).$$

PROOF OF LEMMA 2.3. Let $W(x, y, t)$ be the fundamental solution of the heat equation in $R^n$. We know

$$W(x, y, t)=(4\pi t)^{-n/2}e^{-|x-y|^2/4t}.$$

Put

$$L_\varepsilon(x, y, t)=U_\varepsilon(\Psi_\varepsilon(x), y, t)-W(\Psi_\varepsilon(x), y, t).$$

Then we have

$$\begin{aligned}&\Psi_\varepsilon^* U_\varepsilon(x, y, t)-U(x, y, t)\\ &\quad=L_\varepsilon(x, y, t)-L_0(x, y, t)+W(\Psi_\varepsilon(x), y, t)-W(x, y, t).\end{aligned}$$

The function $L_\varepsilon$ satisfies the following equations

$$(2.9)\qquad \begin{cases} \left(\dfrac{\partial}{\partial t}-\Psi_\varepsilon^*\Delta_x\Psi_\varepsilon^{*-1}\right)L_\varepsilon(x, y, t)=0 \qquad x\in\Omega, \qquad t>0 \\ \left(k\dfrac{\partial}{\partial \nu_x}+(1-k)\right)L_\varepsilon(x, y, t) \\ \qquad =\varepsilon(h_\varepsilon(x, D)U_\varepsilon)(\Psi_\varepsilon(x), y, t) \\ \qquad\quad -\left(k\dfrac{\partial}{\partial \nu_x}+(1-k)\right)W(\Psi_\varepsilon(x), y, t) \qquad x\in\gamma \\ \lim_{t\to0} L_\varepsilon(x, y, t)=0 \qquad x\in\Omega, \end{cases}$$

where $h_\varepsilon(x, D)$ is the differential operator of the first order which has the form

$$h_\varepsilon(x, D)=\sum_{i=1}^n h_\varepsilon^i(x)\frac{\partial}{\partial x^i}.$$

We know from (2.6) that the image of the mapping $[0, \varepsilon_0)\ni\varepsilon\mapsto h_\varepsilon^i(x)$ is contained in a bounded subset of $C^\infty(R^n)$ for each $i$. Since $y$ is fixed, $L_\varepsilon(x, y, t)$ is a smooth function of $(x, t)$ in $\bar\Omega\times[0, T)$.

Simple calculations give

$$(2.10)\qquad \Psi_\varepsilon^*\Delta_x\Psi_\varepsilon^{*-1}-\Delta_x=\varepsilon\alpha_\varepsilon(x, D),$$

where $\alpha_\varepsilon(x, D)$ is a differential operator of the second order which has the form

$$(2.11)\qquad \sum_{i,j=1}^n \alpha_\varepsilon^{ji}(x)\frac{\partial^2}{\partial x^i\partial x^j}+\sum_{i=1}^n \beta_\varepsilon^i(x)\frac{\partial}{\partial x^i}+\mu_\varepsilon(x),$$

where $\alpha_\varepsilon^{ij}(x)=\alpha_\varepsilon^{ji}(x)$. Moreover there exists $\varepsilon_0>0$ such that the image of the mapping

$$[0, \varepsilon_0)\ni\varepsilon\longmapsto\alpha_\varepsilon^{ij}(x),\ \beta_\varepsilon^i(x),\ \mu_\varepsilon(x)\in C^\infty(R^n)$$

is contained in a bounded subset of $C^\infty(R^n)$.

Considering $L_\varepsilon$ as a function of $(x, t)$, we have by Lemma 2.1 and 2.2, (2.10), (2.11)

$$\sup_{0<\varepsilon<\varepsilon_0}\|L_\varepsilon\|_{m+2+\alpha}^J<\infty \tag{2.12}$$

for some $\varepsilon_0>0$.

In the next place, we consider

$$M_\varepsilon(x, y, t)=L_\varepsilon(x, y, t)-L_0(x, y, t).$$

This satisfies:

$$\left\{\begin{array}{ll}\left(\dfrac{\partial}{\partial t}-\Delta_x\right)M_\varepsilon(x, y, t)=(\Psi_\varepsilon^*\Delta_x\Psi_\varepsilon^{*-1}-\Delta_x)L_\varepsilon(x, y, t) & x\in\Omega,\ t>0\\ \left(k\dfrac{\partial}{\partial\nu_x}+(1-k)\right)M_\varepsilon(x, y, t) & \\ \quad=\varepsilon(h_\varepsilon(x, D)U_\varepsilon)(\Psi_\varepsilon(x), y, t) & \\ \quad-\left(k\dfrac{\partial}{\partial\nu_x}+(1-k)\right)(W(\Psi_\varepsilon(x), y, t)-W(x, y, t)) & x\in\gamma,\ t>0\\ \lim\limits_{t\to0}M_\varepsilon(x, y, t)=0 \quad x\in\Omega. & \end{array}\right. \tag{2.13}$$

It follows from (2.10), (2.11), (2.12) and Lemma 2.1 that

$$\sup_{0<\varepsilon<\varepsilon_0}\|\varepsilon^{-1}(L_\varepsilon-L_0)\|_{m+2+\alpha}^J<\infty \tag{2.14}$$

for some $\varepsilon_0>0$. Therefore $L_\varepsilon$ converges to $L_0$ in $C^{m+2+\alpha}(J)$ topology. By the definition of $L_\varepsilon$, we have

$$\|\Psi_\varepsilon^*U_\varepsilon-U\|_{m+2+\alpha}^J\leqq\|L_\varepsilon-L_0\|_{m+2+\alpha}^J+\|\Psi_\varepsilon^*W-W\|_{m+2+\alpha}^J.$$

It is easy to see that $\Psi_\varepsilon^*W-W$ converges to 0 in $C^{m+2+\alpha}(J)$ topology, since we have (2.6), (2.7). Summing up these facts, we get Lemma 2.3.

## § 3. Proofs of Theorem 1 and Theorem 2.

Before the proof of Theorem 1 we shall give a lemma concerning the Whitney extension of the fundamental solution of the heat equation $U$. The idea using the Whitney extension of smooth function to prove our Theorem 1 is the same as in Fujiwara-Ozawa [3].

Let $\tilde{\Omega}$ be an open bounded neighbourhood of the closure $\bar{\Omega}$ of $\Omega$. Let $\tilde{U}(z, y, \tau)$ be a smooth extension of $U(z, y, \tau)$ to $\tilde{\Omega}\times\Omega\times(0, \infty)$ with respect to

$(z, y, \tau)$ satisfying the following conditions:

(3.1) For any fixed compact subset $K$ in $\Omega$, then

$$\lim_{\tau\to+0} \sup_{\substack{y\in K\\ z\in\tilde{\Omega}\setminus\bar{\Omega}}} |\tilde{U}(z, y, \tau)|=0 .$$

$$\tilde{U}(z, y, \tau)=U(z, y, \tau) \tag{3.2}$$

for any $z\in\bar{\Omega}$, $y\in\Omega$, $\tau\in(0, \infty)$.

We have the following Lemma 3.1 which is crucial for our study.

LEMMA 3.1. *There exists a real valued function $C(y)$ such that the inequality*

$$\left|\left(\frac{\partial}{\partial t}-\Delta_z\right)\tilde{U}(z, y, t)\right|\leqq C(y)(\mathrm{dist}(z, \gamma))^2 \tag{3.3}$$

*holds for any $z\in\tilde{\Omega}\setminus\Omega$, $t\in(0, \infty)$ and for any $y\in\Omega$. Moreover we can take $C(y)$ such that*

$$\sup_{y\in K} C(y)<\infty$$

*for any compact subset $K$ of $\Omega$.*

PROOF OF LEMMA 3.1. We see that $(\partial/\partial t-\Delta_z)U(z, y, t)$ is a smooth function of $(z, y, t)\in\tilde{\Omega}\times\Omega\times(0, \infty)$ and it is equal to zero for any $(z, y, t)\in\bar{\Omega}\times\Omega\times(0, \infty)$. Therefore there exists $C(y, t)$ which depends on $y$ and $t$ such that

$$\left|\left(\frac{\partial}{\partial t}-\Delta_z\right)\tilde{U}(z, y, t)\right|\leqq C(y, t)(\mathrm{dist}(z, \gamma))^2$$

for any $z\in\tilde{\Omega}\setminus\Omega$. By Lemma 2.2 we can take $C(y, t)$ as

$$\operatorname*{supremum}_{y\in K,\, t\in(0,\infty)} C(y, t)<\infty$$

for any fixed compact subset $K$ of $\Omega$.

PROOF OF THEOREM 1. Note that in this case $k=0$. By (3.1) we have

$$U_\varepsilon(x, y, t)-U(x, y, t)=-\int_0^t d\tau\frac{\partial}{\partial\tau}\left(\int_{\Omega_\varepsilon} U_\varepsilon(x, z, t-\tau)\tilde{U}(z, y, \tau)dz\right) \tag{3.4}$$

for any fixed $x$, $y\in\Omega$ and $t>0$. In the next place, we have

$$\begin{aligned} U_\varepsilon(x, y, t)&-U(x, y, t)\\ &=\int_0^t d\tau\int_{\Omega_\varepsilon} \Delta_z U_\varepsilon(x, z, t-\tau)\tilde{U}(z, y, \tau)dz\\ &\quad-\int_0^t d\tau\int_{\Omega_\varepsilon} U_\varepsilon(x, z, t-\tau)\Delta_z\tilde{U}(z, y, \tau)dz+O(\varepsilon^2)\end{aligned} \tag{3.5}$$

by using Lemma 2.2 and Lemma 3.1.

By Green's identity and by the Dirichlet boundary condition for $U_\varepsilon$ at $\gamma_\varepsilon$, we have

$$(3.6)\qquad U_\varepsilon(x, y, t)-U(x, y, t)$$
$$=-\int_0^t d\tau\int_{\gamma_\varepsilon}\Big(\frac{\partial U_\varepsilon}{\partial \nu_z^\varepsilon}\Big)(x, z, t-\tau)\tilde{U}(z, y, \tau)d\sigma_z^\varepsilon+O(\varepsilon^2)\,.$$

To simplify the right hand side of (3.6), we require the following well known Lemma 3.2.

LEMMA 3.2. *Let $g(\varepsilon, z)\equiv f(\varepsilon, z+\varepsilon\rho(z)\nu_z)$ be a function of $(\varepsilon, z)\in(-\varepsilon_0, \varepsilon_0)\times\gamma$, then*

$$(3.7)\qquad \frac{\partial}{\partial\varepsilon}\Big(\int_{\gamma_\varepsilon} f(\varepsilon, q)d\sigma_q^\varepsilon\Big)\Big|_{\varepsilon=0}$$
$$=\int_\gamma f(0, z)(n-1)H_1(z)\rho(z)d\sigma_z+\int_\gamma\Big(\frac{\partial g}{\partial\varepsilon}\Big)(\varepsilon, z)\Big|_{\varepsilon=0}d\sigma_z\,,$$

*where $\varepsilon_0$ is a sufficiently small positive number and $H_1(z)$ denotes the first mean curvature of $\gamma$ at $z$.*

Fix $t>0$. We apply Lemma 3.2 to

$$g(\varepsilon, z)=\int_0^t\Big(\frac{\partial U}{\partial\nu_z^\varepsilon}\Big)(x, z+\varepsilon\rho(z)\nu_z, t\quad\tau)\tilde{U}(z+\varepsilon\rho(z)\nu_z, y, \tau)d\tau$$

for $z\in\gamma$. Then we get

$$f(0, z)=\int_0^t\frac{\partial U}{\partial\nu_z}(x, z, t-\tau)U(z, y, \tau)d\tau$$
$$=0$$

by the boundary condition for $U$ at $\gamma$. Therefore we have only to calculate the term $(\partial g/\partial\varepsilon)(\varepsilon, z)|_{\varepsilon=0}$. By the mean value theorem and by the boundary condition for $U$ at $\gamma$, we have

$$\tilde{U}(z+\varepsilon\rho(z)\nu_z, y, \tau)=\varepsilon\rho(z)\Big(\frac{\partial\tilde{U}}{\partial\nu_z}\Big)(z, y, \tau)\Big|_{z=z_\theta}\,,$$

where $z_\theta=z+\varepsilon\rho(z)\theta(z)\nu_z$ for some $\theta(z)$ satisfying $0<\theta(z)<1$. Then by Lemma 2.3 we get

$$\Big(\frac{\partial g}{\partial\varepsilon}\Big)(\varepsilon, z)\Big|_{\varepsilon=0}=\int_0^t\frac{\partial U}{\partial\nu_z}(x, z, t-\tau)\frac{\partial U}{\partial\nu_z}(z, y, \tau)\rho(z)d\tau\,.$$

The proof of Theorem 1 is complete.

PROOF OF THEOREM 2. Also in the case $0<k\leqq 1$, we have (3.3) and (3.4). By Green's identity and by the boundary condition (1.3), we get

(3.8) $U_\varepsilon(x, y, t)-U(x, y, t)$

$$=-\int_0^t d\tau\int_{\gamma_\varepsilon} U_\varepsilon(x, z, t-\tau)\Big(\Big(\frac{\partial\tilde{U}}{\partial\nu_z^\varepsilon}\Big)(z, y, \tau)+p\tilde{U}(z, y, \tau)\Big)d\sigma_z^\varepsilon+O(\varepsilon^2),$$

where $p=k^{-1}(1-k)$. In order to simplify the right hand side of (3.8), we shall give a geometrical observation. We fix $z\in\gamma$ and take an orthonormal basis $(z_1, \cdots, z_{n-1})$ on the tangent hyperplane at $z$ and we consider $(z_1, \cdots, z_n)$-coordinates as global coordinates in $R^n$. Then we have

$$\frac{\partial}{\partial\nu_z^\varepsilon}=\frac{\partial z_1}{\partial\nu_z^\varepsilon}\frac{\partial}{\partial z_1}+\cdots+\frac{\partial z_{n-1}}{\partial\nu_z^\varepsilon}\frac{\partial}{\partial z_{n-1}}+\frac{\partial z_n}{\partial\nu_z^\varepsilon}\frac{\partial}{\partial z_n}.$$

By simple calculus, we obtain

$$\frac{\partial z_j}{\partial\nu_z^\varepsilon}=-\varepsilon\frac{\partial\rho}{\partial z_j}\mu_\varepsilon+O(\varepsilon^2)$$

for $j=1, \cdots, n-1$ and

$$\frac{\partial z_n}{\partial\nu_z^\varepsilon}=\mu_\varepsilon+O(\varepsilon^2),$$

where

$$\mu_\varepsilon=\Big(1+\varepsilon^2\sum_{j=1}^{n-1}\frac{\partial\rho}{\partial z_j}(z)^2\Big)^{-1/2}.$$

We have the following

LEMMA 3.3. *For an arbitrary fixed* $v(z)\in C^\infty(R^n)$,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\Big\{\Big(\frac{\partial v}{\partial\nu_z^\varepsilon}\Big)(z(\varepsilon))-\frac{\partial v}{\partial\nu_z}(z)\Big\}=-\langle\nabla_\gamma\rho(z), \nabla_\gamma v(z)\rangle+\rho(z)\frac{\partial^2 v}{\partial z_n^2}(z). \tag{3.9}$$

*Here* $z(\varepsilon)=z+\varepsilon\rho(z)\nu_z$ *and* $\langle\,,\rangle$ *denotes the inner product which is defined by* (1.6).

PROOF OF LEMMA 3.3. We have

(3.10) $\Big(\dfrac{\partial v}{\partial\nu_z^\varepsilon}\Big)(z(\varepsilon))$

$$=\mu_\varepsilon\Big\{-\varepsilon\Big(\frac{\partial\rho}{\partial z_1}\frac{\partial v}{\partial z_1}+\cdots+\frac{\partial\rho}{\partial z_{n-1}}\frac{\partial v}{\partial z_{n-1}}\Big)(z(\varepsilon))+\Big(\frac{\partial v}{\partial z_n}\Big)(z(\varepsilon))\Big\}+O(\varepsilon^2),$$

$$\Big(\frac{\partial v}{\partial z_n}\Big)(z(\varepsilon))=\frac{\partial v}{\partial z_n}(z)+\varepsilon\rho(z)\frac{\partial^2 v}{\partial z_n^2}(z)+O(\varepsilon^2) \tag{3.11}$$

and

$$\Big(\frac{\partial v}{\partial z_j}\Big)(z(\varepsilon))=\frac{\partial v}{\partial z_j}(z)+O(\varepsilon) \tag{3.12}$$

for $j=1, \cdots, n-1$. Summing up (3.10), (3.11) and (3.12), we have the desired result.

Now we continue the proof of Theorem 2. By Lemma 2.3 and Lemma 3.3, we obtain

$$\text{(3.13)}\quad \lim_{\varepsilon\to 0}\varepsilon^{-1}\Big\{\Big(\frac{\partial\tilde{U}}{\partial\nu_z^{\varepsilon}}\Big)(z(\varepsilon),\ y,\ \tau)+p\tilde{U}(z(\varepsilon),\ y,\ \tau)-\Big(\frac{\partial U}{\partial\nu_z}(z,\ y,\ \tau)+pU(z,\ y,\ \tau)\Big)\Big\}$$
$$=\Big(\frac{\partial^2 U}{\partial\nu_z^2}(z,\ y,\ \tau)+p\frac{\partial U}{\partial\nu_z}(z,\ y,\ \tau)\Big)\rho(z)-\langle\nabla_\gamma\rho(z),\ \nabla_\gamma U(z,\ y,\ \tau)\rangle\,,$$

where $p=k^{-1}(1-k)$. Combining (3.13) with Lemma 3.2, we have

$$\text{(3.14)}\quad \delta U(x,\ y,\ t)=-\int_0^t d\tau\int_\gamma\frac{\partial^2 U}{\partial\nu_z^2}(z,\ y,\ \tau)U(x,\ z,\ t-\tau)\rho(z)d\sigma_z$$
$$+k^{-2}(1-k)^2\int_0^t d\tau\int_\gamma U(x,\ z,\ t-\tau)U(z,\ y,\ \tau)\rho(z)d\sigma_z$$
$$+\int_0^t d\tau\int_\gamma\langle\nabla_\gamma\rho(z),\ \nabla_\gamma U(z,\ y,\ \tau)\rangle U(x,\ z,\ t-\tau)d\sigma_z\,.$$

Now it is well known that the Laplacian $\Delta$ is represented as

$$\frac{\partial^2}{\partial\nu_z^2}+\nabla_\gamma^2+(n-1)H_1(z)$$

on the boundary $\gamma$. Here $\nabla_\gamma^2$ denotes the Laplacian on the submanifold $\gamma$ of $R^n$ with the metric induced from that of $R^n$. Therefore we have

$$\text{(3.15)}\quad \delta U(x,\ y,\ t)=-\int_0^t d\tau\int_\gamma\langle\nabla_\gamma U(x,\ z,\ t-\tau),\ \nabla_\gamma U(z,\ y,\ \tau)\rangle\rho(z)d\sigma_z$$
$$-\int_0^t d\tau\int_\gamma U(x,\ z,\ t-\tau)\frac{\partial}{\partial\tau}U(z,\ y,\ \tau)\rho(z)d\sigma_z$$
$$+\int_0^t d\tau\int_\gamma p(p-(n-1)H_1(z))U(x,\ z,\ t-\tau)U(z,\ y,\ \tau)\rho(z)d\sigma_z\,,$$

where $p=k^{-1}(1-k)$. On the other hand, we have

$$\text{(3.16)}\quad \int_0^t d\tau\Big(\frac{\partial}{\partial\tau}\int_\gamma U(x,\ z,\ t-\tau)U(z,\ y,\ \tau)\rho(z)d\sigma_z\Big)=0\,,$$

for any $x,\ y\in\Omega$ and $t>0$. Hence we get the equality

$$\text{(3.17)}\quad \int_0^t d\tau\int_\gamma U(x,\ z,\ t-\tau)\frac{\partial}{\partial\tau}U(z,\ y,\ \tau)\rho(z)d\sigma_z$$
$$=\frac{\partial}{\partial t}\int_0^t d\tau\int_\gamma U(x,\ z,\ t-\tau)U(z,\ y,\ \tau)\rho(z)d\sigma_z\,.$$

By (3.15) and (3.17) we obtain the desired result.

## §4. Proof of Theorem 3.

In this section we give a proof of Theorem 3.

We prepare some notations. We put $|\rho|=\max_{x\in\gamma}|\rho(x)|$. Let $\Gamma$ be a fixed tubular neighbourhood of $\gamma$. And let $\varepsilon_0>0$ be a small positive number such that

$$\Gamma\supset\{x\in R^n\,;\ \mathrm{dist}(x,\gamma)<4\varepsilon_0|\rho|\}.$$

Then $\gamma_{4\varepsilon_0}\subset\Gamma$. And we have $\gamma_\varepsilon\subset\Gamma$ for any $\varepsilon\in(0,\varepsilon_0)$. Let $\{I_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be a family of subdomains of $\Omega$ such that $I_\varepsilon\subset I_{\varepsilon'}$ when $\varepsilon'<\varepsilon$ and $I_\varepsilon\subset\Omega\cap\Omega_\varepsilon$ for any $\varepsilon\in(0,\varepsilon_0)$. We assume that the boundary $\tilde\gamma_\varepsilon$ of $I_\varepsilon$ is smooth and it satisfies the following properties:

(4.1) The inequality

$$(3/2)|\rho|\varepsilon\leqq\mathrm{dist}(x,\partial(\Omega\cap\Omega_\varepsilon))\leqq2|\rho|\varepsilon$$

holds for any $x\in\tilde\gamma_\varepsilon$.

(4.2) For $z\in\Gamma$, let $\alpha(z)$ be the unique point of $\gamma$ which satisfies $\mathrm{dist}(z,\alpha(z))=\mathrm{dist}(z,\gamma)$. Assume that the mapping

$$\alpha|_{\tilde\gamma_\varepsilon}:\tilde\gamma_\varepsilon\ni z\longmapsto\alpha(z)\in\gamma$$

is a smooth diffeomorphism between $\tilde\gamma_\varepsilon$ and $\gamma$ whose Jacobian satisfies $2^{-1}\leqq|\det\mathrm{Jac}(\alpha|_{\tilde\gamma_\varepsilon})|\leqq2$ for any $\varepsilon\in(0,\varepsilon_0)$.

By (4.1) we see that $\mathrm{dist}(x,\gamma)\leqq\mathrm{dist}(x,\partial(\Omega\cap\Omega_\varepsilon))+\varepsilon|\rho|\leqq3|\rho|\varepsilon$ for any $x\in\tilde\gamma_\varepsilon$ if $0<\varepsilon<\varepsilon_0$. Therefore $\tilde\gamma_\varepsilon\in\Gamma$ for any $\varepsilon\in(0,\varepsilon_0)$. So the mapping $\alpha|_{\tilde\gamma_\varepsilon}$ is well defined because $\tilde\gamma_\varepsilon\subset\Gamma$.

We begin the proof of Theorem 3. We divide the term $\varepsilon^{-1}(T_\gamma(t\,;\varepsilon)-T_\gamma(t\,;0))$ into three parts. Put

$$J_1(\varepsilon,t)=\varepsilon^{-1}\Big(\int_{\Omega_\varepsilon\setminus\Omega}U_\varepsilon(x,x,t)dx-\int_{\Omega\setminus\Omega_\varepsilon}U(x,x,t)dx\Big)\tag{4.3}$$

$$J_2(\varepsilon,t)=\varepsilon^{-1}\int_{(\Omega\cap\Omega_\varepsilon)\setminus I_\varepsilon}(U_\varepsilon(x,x,t)-U(x,x,t))dx$$

and

$$J_3(\varepsilon,t)=\varepsilon^{-1}\int_{I_\varepsilon}(U_\varepsilon(x,x,t)-U(x,x,t))dx\,.\tag{4.4}$$

Then

$$\varepsilon^{-1}(T_\gamma(t\,;\varepsilon)-T_\gamma(t\,;0))=\sum_{n=0}^{3}J_n(\varepsilon,t)\,.$$

By Lemma 2.3 we get

$$\lim_{\varepsilon\to 0} J_1(\varepsilon, t)=\int_\gamma U(x, x, t)\rho(x)d\sigma_x \qquad (4.5)$$
$$=0$$

and

$$\lim_{\varepsilon\to 0} J_2(\varepsilon, t)=0\,. \qquad (4.6)$$

Therefore we have only to prove the following claim to prove Theorem 3.

CLAIM. $\lim_{\varepsilon\to 0} J_3(\varepsilon, t)=\int_\Omega \delta U(x, x, t)dx$.

To estimate the term $J_3(\varepsilon, t)$ we study $\varepsilon$-dependence of $U_\varepsilon(x, x, t)$. A pair $(\varepsilon, x, t)$ satisfying $x\in I_\varepsilon$, $t>0$ is called a nice pair. If $(\varepsilon, x, t)$ is a nice pair and $\varepsilon\in(0, \varepsilon_0)$, then

$$x\in I_\varepsilon\subset \bigcap_{0<\mu<\varepsilon}\Omega_\mu$$

and $U_\varepsilon(x, x, t)$ and $U(x, x, t)$ are well defined. Now we show an explicit representation of $U_\varepsilon(x, x, t)-U(x, x, t)$ by using the mean value theorem. It is easy to see that there exists $\varepsilon_1(x)\in(0, \varepsilon_0)$ such that $x\in \bigcap_{0<\mu<\varepsilon_1(x)}\Omega_\mu$ holds for any fixed $x\in\Omega$. By Theorem 1 and by Lemma 2.3 we see that the mapping $(0, \varepsilon_1(x))\ni\varepsilon\mapsto U_\varepsilon(x, x, t)$ is of $C^1$-class with respect to $\varepsilon$ for any fixed $x\in\Omega$, $t>0$. Therefore, there exists $\theta(\varepsilon, x, t)$ such that $0<\theta(\varepsilon, x, t)<1$ and the equality

$$U_\varepsilon(x, x, t)-U(x, x, t)=\varepsilon\delta U_{\varepsilon\theta(\varepsilon,x,t)}(x, x, t) \qquad (4.7)$$

holds for any pair $(\varepsilon, x, t)$ satisfying $x\in\Omega\cap\Omega_\varepsilon$, $t>0$. Of course (4.7) holds for any nice pair. We abbreviate $\theta(\varepsilon, x, t)$ as $\theta$ if there is no fear of confusion.

In the following we shall represent the term $\delta U_{\varepsilon\theta}(x, x, t)$ explicitly. We fix $t$ and $\varepsilon\theta$. For any sufficiently small $\tilde\varepsilon$ satisfying $0<\tilde\varepsilon<(1-\theta)\varepsilon$, there is the unique function

$$\rho_{\varepsilon\theta}(y\,;\tilde\varepsilon)\in C^\infty(\gamma_{\varepsilon\theta}\times(0, (1-\theta)\varepsilon))$$

such that the set $\gamma_{\varepsilon\theta+\tilde\varepsilon}$ can be written as

$$\gamma_{\varepsilon\theta+\tilde\varepsilon}=\{y+\tilde\varepsilon\rho_{\varepsilon\theta}(y\,;\tilde\varepsilon)\nu_y^{\varepsilon\theta}\,;\ y\in\gamma_{\varepsilon\theta}\}$$

and the mapping

$$\gamma_{\varepsilon\theta+\tilde\varepsilon}\ni y\longmapsto y+\tilde\varepsilon\rho_{\varepsilon\theta}(y\,;\tilde\varepsilon)\nu_y^{\varepsilon\theta}$$

is bijective. Here $\nu_y^{\varepsilon\theta}$ denotes the exterior unit normal vector of $\gamma_{\varepsilon\theta}$ at $y$. Then by Theorem 2 we can represent $\delta U_{\varepsilon\theta}(x, x, t)$ explicitly as follows:
For any nice pair $(\varepsilon, x, t)$ we have

$$\delta U_{\varepsilon\theta}(x, x, t)=\int_0^t d\tau\int_{\gamma_{\varepsilon\theta}}\frac{\partial U_{\varepsilon\theta}(x, z, t-\tau)}{\partial\nu_z^{\varepsilon\theta}}\,\frac{\partial U_{\varepsilon\theta}(x, z, \tau)}{\partial\nu_z^{\varepsilon\theta}}\rho_{\varepsilon\theta}(z)d\sigma_z^{\varepsilon\theta}\,, \qquad (4.8)$$

where $d\sigma_z^{\varepsilon\theta}$ denotes the surface element of $\gamma_{\varepsilon\theta}$.

Now we divide $J_3(\varepsilon, t)$ into two parts. For any fixed $t>0$, $\varepsilon>0$, let $\hat{U}_\varepsilon(x, x, t)$ be a smooth extension of $U_\varepsilon(x, x, t)$ to the whole space $R^n$ with respect to $x$. Then $\hat{U}_\varepsilon(x, x, t)=U_\varepsilon(x, x, t)$ for any $x\in\Omega_\varepsilon$. For any fixed $\varepsilon\in(0, \varepsilon_0/2)$, we have $J_3(\varepsilon, t)=J_4(\varepsilon, t)+J_5(\varepsilon, t)$. Here

$$J_4(\varepsilon, t)=\int_{\Omega\setminus I_{\varepsilon_0/2}} \varepsilon^{-1}\chi_\varepsilon(x)(\hat{U}_\varepsilon(x, x, t)-U(x, x, t))dx \tag{4.9}$$

and

$$J_5(\varepsilon, t)=\int_{I_{\varepsilon_0/2}} \varepsilon^{-1}\chi_\varepsilon(x)(\hat{U}_\varepsilon(x, x, t)-U(x, x, t))dx \tag{4.10}$$

$$=\int_{I_{\varepsilon_0/2}} \varepsilon^{-1}(U_\varepsilon(x, x, t)-U(x, x, t))dx\,.$$

It should be remarked that $\varepsilon^{-1}(\hat{U}_\varepsilon(x, x, t)-U(x, x, t))\chi_\varepsilon(x)$ is a measurable function of $x$ in $\Omega$ when we fix $\varepsilon$ and $t$.

We give two lemmas which are crucial to our study of $J_3(\varepsilon, t)$.

Lemma 4.1. *We put*

$$R(x, t)=\sup_{0<\varepsilon<\varepsilon_0/2}|\varepsilon^{-1}\chi_\varepsilon(x)(\hat{U}_\varepsilon(x, x, t)-U(x, x, t))|$$

*for any* $x\in\Omega\setminus I_{\varepsilon_0/2}$, $t>0$. *Then* $R(x, t)<\infty$ *for any* $x\in\Omega\setminus I_{\varepsilon_0/2}$, $t>0$, *and*

$$\int_{\Omega\setminus I_{\varepsilon_0/2}} R(x, t)dx<\infty$$

*holds.*

Lemma 4.2. *There exists a positive continuous function* $C(t)$ *of* $t$ *such that*

$$\sup_{0<\varepsilon<\varepsilon_0/2}|\varepsilon^{-1}(U_\varepsilon(x, x, t)-U(x, x, t))|\leqq C(t)$$

*holds for any* $x\in I_{\varepsilon_0/2}$, $t>0$.

Assume that Lemmas 4.1 and 4.2 are proved, then we can apply the Lebesgue dominated convergence theorem to the integral in (4.9), (4.10) when $\varepsilon$ tends to zero and we finish our proof of Theorem 3. Therefore we have only to prove Lemmas 4.1 and 4.2 to get Theorem 3.

We use the following Lemma 4.3 to prove Lemma 4.1.

Lemma 4.3. *There exists a positive constant* $\tilde{C}$ *independent of* $\varepsilon\in(0, \varepsilon_0)$ *such that* $|x-z|\geqq\tilde{C}|x-\alpha(z)|$ *holds for any* $x\in I_\varepsilon$ *and for any* $z\in\gamma_{\varepsilon\theta}$, *where* $\alpha$ *is the mapping introduced in* (4.2).

Proof. We fix $\varepsilon\in(0, \varepsilon_0)$. We take $x\in I_\varepsilon$ and $z\in\gamma_{\varepsilon\theta}$. By (4.2) we have

$$|x-z|\geqq\mathrm{dist}(x, \partial(\Omega\cap\Omega_\varepsilon))$$
$$\geqq(3/2)|\rho|\varepsilon\,.$$

Therefore we have

$$\mu\equiv\varepsilon^{-1}|x-z|-(3/2)|\rho|$$
$$\geqq 0.$$

On the other hand $|z-\alpha(z)|\leqq\varepsilon|\rho|$, then we have

$$|x-\alpha(z)|\leqq|x-z|+|z-\alpha(z)|$$
$$=\varepsilon(\mu+(5/2)|\rho|).$$

Therefore we obtain

$$|x-\alpha(z)|^{-1}|x-z|\geqq(\mu+(5/2)|\rho|)^{-1}(\mu+(3/2)|\rho|)$$
$$\geqq 3/5.$$

We finish our proof.

PROOF OF LEMMA 4.1. We put $\varepsilon_2(x)=\sup_{0<\varepsilon}\{\varepsilon\,;\,x\in I_\varepsilon\}$. Then for any $x\in\Omega\backslash I_{\varepsilon_0/2}$, $\varepsilon_2(x)\leqq\varepsilon_0/2$. It is easy to see that

$$(4.11)\qquad \sup_{0<\varepsilon}|\varepsilon^{-1}\chi_\varepsilon(x)(\hat{U}_\varepsilon(x,\,x,\,t)-U(x,\,x,\,t))|$$
$$\leqq\sup_{0<\varepsilon\leqq\varepsilon_2(x)}|(U_\varepsilon(x,\,x,\,t)-U(x,\,x,\,t))|$$
$$\leqq\sup_{0<\varepsilon\leqq\varepsilon_2(x)}|\delta U_{\varepsilon\theta}(x,\,x,\,t)|$$

holds for any $x\in\Omega\backslash I_{\varepsilon_0/2}$. We put $|||\rho|||=\sup_{0<\varepsilon<\varepsilon_0}(\sup_{z\in\gamma_\varepsilon}|\rho_\varepsilon(z)|)$. Then by Lemma 2.2 and (4.8) we get

$$(4.12)\qquad |\delta U_{\varepsilon\theta}(x,\,x,\,t)|\leqq C_2C^2|||\rho|||\int_0^t d\tau\int_{\gamma_{\varepsilon\theta}}((t-\tau)\tau)^{-(n+1)/2}$$
$$\cdot\exp(-t(C\tau(t-\tau))^{-1}|x-z|^2)d\sigma_z^{\varepsilon\theta}$$

for any pair $(x,\varepsilon)$ satisfying $(x,\varepsilon)\in(\Omega\backslash I_{\varepsilon_0/2})\times(0,\,\varepsilon_2(x))$. Here $C_2$ is a positive constant independent of $\varepsilon$. From Lemma 4.3 we get

$$(4.13)\qquad \exp(-t(C\tau(t-\tau))^{-1}|x-z|^2)$$
$$\leqq\exp(-t(C\tau(t-\tau))^{-1}\tilde{C}^2|x-\alpha(z)|^2)$$

for any pair $(x,\varepsilon,z)$ satisfying $(x,\varepsilon,z)\in(\Omega\backslash I_{\varepsilon_0/2},\,(0,\,\varepsilon_2(x)),\,\gamma_{\varepsilon\theta})$. We see that the determinant of the Jacobian of the diffeomorphism

$$\alpha|_{\gamma_{\varepsilon\theta}}:\gamma_{\varepsilon\theta}\ni z\longmapsto\alpha(z)\in\gamma$$

between $\gamma_{\varepsilon\theta}$ and $\gamma$ satisfies

$$(4.14)\qquad 2^{-1}\leqq|\det \mathrm{Jac}(\alpha|_{\gamma_{\varepsilon\theta}})|\leqq 2$$

for any $\varepsilon\in(0,\,\varepsilon_0)$. By the definition of $R(x,t)$, (4.11), (4.12), (4.13) and (4.14), we obtain

$$R(x, t) \leqq C_3 \int_0^t d\tau \int_\gamma ((t-\tau)\tau)^{-(n+1)/2} \cdot \exp(-t(C\tau(t-\tau))^{-1}\tilde{C}^2|x-z|^2)d\sigma_z, \tag{4.15}$$

where $C_3 = 2C_2C^2|||\rho|||$. Therefore we obtain

$$\int_{\Omega\setminus I_{\varepsilon_0/2}} R(x, t)dx \leqq C_3 \int_\gamma d\sigma_z \int_0^t d\tau \int_{R^n} ((t-\tau)\tau)^{-(n+1)/2} \cdot \exp(-t(C\tau(t-\tau))^{-1}\tilde{C}^2|x-z|^2)dx < \infty .$$

We complete our proof of Lemma 4.1.

PROOF OF LEMMA 4.2. Since $\gamma_{\varepsilon\theta} \cap (\Omega \cap \Omega_\varepsilon) = \emptyset$ for any $\varepsilon \in (0, \varepsilon_0)$, then there is a positive constant $\varpi$ such that $\mathrm{dist}(x, \gamma_{\varepsilon\theta}) > \varpi$ for any $\varepsilon \in (0, \varepsilon_0/2)$ and $x \in I_{\varepsilon_0/2}$. The simple calculation leads to the existence of $C(t)$ in Lemma 4.2. The proof is complete.

## § 5. Proof of Theorem 4.

In this final section we give a proof of Theorem 4. Since the right hand side of (1.9) is Lebesgue integrable, we can apply Fubini's theorem. By the properties of the fundamental solution of the heat equation, we have (1.10). Put

$$Y_N(z, t) = \sum_{j=1}^N e^{\lambda_j t} \left( \frac{\partial \varphi_j}{\partial \nu_z}(z) \right)^2, \tag{5.1}$$

where $\varphi_j(z)$ is the eigenfunction of the Laplacian. When we fix $t$, then we have the uniform convergence in (5.2).

$$Y_N(z, t) \longrightarrow Y_\infty(z, t) = \frac{\partial^2 U(y, w, t)}{\partial \nu_y \partial \nu_w} \bigg|_{y=w=z}. \tag{5.2}$$

This follows from Lemma 5.1 which we prove later. By (1.9), (5.1) and (5.2), we complete the proof of Theorem 4.

LEMMA 5.1. *There exists a positive integer $r$ and a constant $\hat{C}$ such that*

$$\sup_{z\in\gamma} \left( \frac{\partial \varphi_j}{\partial \nu_z}(z) \right)^2 \leqq \hat{C} \lambda_j^r \tag{5.3}$$

*holds for any $j = 1, 2, \cdots$.*

PROOF. By the well known a priori estimates, we have

$$\|\varphi_j\|_{H^{2+m}(\Omega)} \leqq D_m(\|\Delta\varphi_j\|_{H^m(\Omega)} + \|\varphi_{j|\gamma}\|_{H^{(3/2)+m}(\gamma)}) \tag{5.4}$$

for some constant $D_m$ which depends on $m$. Here we use the $L^2$-Sobolev norm. By using (5.4) repeatedly we obtain

$$\|\varphi_j\|_{H^{2m}(\Omega)} \leqq C_5 \lambda_j^m .$$

By the Sobolev imbedding theorem, we get the desired result.

## References

[1] K. Aomoto, Formule variationnelle d'Hadamard et modèle euclidien des variétés différentiables plongées, J. Functional Analysis, **34** (1979), 493-523.
[2] S. Bergmann and M. Schiffer, Kernel functions and elliptic differential equations in mathematical physics, Academic Press, New York, 1953.
[3] D. Fujiwara and S. Ozawa, Hadamard's variational formula for the Green functions of some normal elliptic boundary value problems, Proc. Japan Acad., **54A** (1978), 215-220.
[4] D. Fujiwara, M. Tanikawa and S. Yukita, The spectrum of the Laplacian and boundary perturbation I, Proc. Japan Acad., **54A** (1978), 87-91.
[5] P.R. Garabedian, Partial differential equations, John Wiley and Sons, Inc., New York, 1964.
[6] P.R. Garabedian and M. Schiffer, Convexity of domain functionals, J. Anal. Math., **2** (1952-53), 281-368.
[7] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées, Œuvres, C.N.R.S., tom. **2** (1968), 515-631.
[8] S. Ozawa, Perturbation of domains and Green kernels of heat equations, Proc. Japan Acad., **54A** (1978), 322-325.
[9] S. Ozawa, Perturbation of domains and Green kernels of heat equations II, Ibid., **55A** (1979), 172-175.
[10] S. Ozawa, Deformation theory of domains and eigenvalues, Sûgaku no Ayumi, **18** (1979), 135-164 (in Japanese).
[11] J. Peetre, On Hadamard's variational formula, 1978, preprint.
[12] M. Schiffer and D.C. Spencer, Functionals of finite Riemann surfaces, Princeton Univ. Press, 1954.
[13] V.A. Solonnikov, On the boundary value problems for linear parabolic system of differential equations of general form, Trudy Mat. Inst. Steklov, **83** (1965), 1-184.

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