G. KOMATSU
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HADAMARD'S VARIATIONAL FORMULA FOR THE SZEGÖ KERNEL

BY GEN KOMATSU*

§1. **A variational formula.** The present note is concerned with the Hadamard's (first) variational formula for the Szegö kernel associated with a strictly pseudo-convex domain in $\boldsymbol{C}^n$ with $n \geqq 3$. A similar formula for the Bergman kernel has been given in [7].

Let $\Omega^0 \subset \boldsymbol{C}^n$ with $n \geqq 1$ be a bounded domain with smooth boundary $\partial\Omega^0$. Every smoothly perturbed domain of Ω^0 can be parametrized by a small function $\rho \in C^\infty(\partial\Omega^0; \boldsymbol{R})$ in such a way that the boundary of that domain Ω^ρ is given by

$$\partial\Omega^\rho = \{\zeta + \rho(\zeta)\boldsymbol{\nu}(\zeta) ; \zeta \in \partial\Omega^0\}, \tag{1}$$

where $\boldsymbol{\nu}(\zeta) = \partial/\partial\nu_\zeta$ denotes the unit exterior normal vector to Ω^0 at $\zeta \in \partial\Omega^0$ identified with an element of $\boldsymbol{C}^n$.

Let $S^\rho(z, w)$ for $z, w \in \Omega^\rho$ denote the Szegö kernel associated with Ω^ρ, which is the reproducing kernel associated with the space $L_b^2 H(\Omega^\rho)$ of holomorphic functions in Ω^ρ with L^2 boundary values equipped with the $L^2(\partial\Omega^\rho)$ scalar product. With $\delta\rho \in C^\infty(\partial\Omega^0; \boldsymbol{R})$ and $z, w \in \Omega^\rho$ fixed arbitrarily, we set

$$\delta S^\rho(z, w) = \frac{d}{d\varepsilon} S^{\rho+\varepsilon\delta\rho}(z, w)|_{\varepsilon=0}, \tag{2}$$

which is the Hadamard's first variation of $S^\rho(z, w)$ at ρ in the direction $\delta\rho$. Our purpose is to show that, for a certain class of domains Ω^0, the variation (2) at $\rho = 0$ exists and is given by

$$-\delta S^0(z, w) = \int_{\partial\Omega^0} \frac{\partial}{\partial\nu_\zeta}[S^0(z, \zeta)S^0(\zeta, w)] \cdot \delta\rho(\zeta) d\sigma^0(\zeta) + \int_{\partial\Omega^0} S^0(z, \zeta)S^0(\zeta, w)H^0(\zeta)\delta\rho(\zeta)d\sigma^0(\zeta), \tag{3}$$

where $d\sigma^0(\zeta)$ denotes the induced surface measure of $\partial\Omega^0 \subset \boldsymbol{C}^n$ at ζ, and $H^0(\zeta)$ stands for the mean curvature of $\partial\Omega^0$ at ζ multiplied by $2n-1$. A concrete statement of our result is given as follows:

THEOREM. *If $\Omega^0 \subset \boldsymbol{C}^n$ is strictly pseudo-convex with $n \geqq 3$, then the variation (2) at $\rho = 0$ exists and is given by* (3).

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Note that the right hand side of (3) makes sense, for if $\Omega^0 \subset C^n$ is strictly pseudo-convex then $S^0(\cdot, \cdot)$ extends smoothly to $(\overline{\Omega^0}\times\overline{\Omega^0})\backslash\Delta(\partial\Omega^0)$, where $\Delta(\partial\Omega^0)$ denotes the diagonal of $\partial\Omega^0\times\partial\Omega^0$ (see Boutet de Monvel and Sjöstrand [1], see also Kerzman and Stein [5]).

Remark 1. As will be seen in Section 3, the variational formula (3) is valid whenever the Szegö kernel associated with Ω^ρ depends smoothly on ρ in the sense of (6) in Section 2.

It is plausible that (3) holds if $n=1$. In fact, if $n=1$, then the Szegö kernel is expressed in terms of the Bergman kernel and the harmonic measures (see Garabedian [3]). The smooth dependence of the Bergman kernel on ρ has been established (cf. [7], Remark 2), while the harmonic measures are expressed in terms of the Poisson kernel and thus depend smoothly on ρ, cf. Section 2.

The assumption $n\geqq 3$ in Theorem above is imposed in order to use an expression of the Szegö kernel in terms of the $\bar\partial_b$-Neumann operator, see (9) in Section 2. It is likely that Theorem above is valid also for strictly pseudo-convex domains in C^2.

Remark 2. In case $n=1$, Schiffer [9] has obtained another expression for the variation (2) in terms of the Szegö kernel and the so-called adjoint kernel. It is not difficult to see that his formula follows from (3).

§ 2. Existence of the variation (2).

Setting

$$\mathcal{V}^0(\varepsilon_1)=\{\rho\in C^\infty(\partial\Omega^0\,;\,\boldsymbol{R})\,;\ |\rho(\zeta)|<\varepsilon_1 \quad \text{for } \zeta\in\partial\Omega^0\}$$

with $\varepsilon_1>0$ small, we begin with constructing a family of diffeomorphisms $e_\rho: C^n\to C^n$ for $\rho\in\mathcal{V}^0(\varepsilon_1)$ such that

$$(4)\qquad \begin{cases} e_\rho(\zeta)=\zeta+\rho(\zeta)\boldsymbol{\nu}(\zeta) \qquad \text{for}\quad \zeta\in\partial\Omega^0 \qquad (\text{cf. (1)}),\\ \mathcal{V}^0(\varepsilon_1)\ni\rho\mapsto e_\rho\in C^\infty(C^n\,;\,C^n) \quad \text{is continuous},\\ e_\rho-e_0 \text{ depends linearly on } \rho \text{ and } e_0=\text{identity}. \end{cases}$$

In particular, (4) will imply that e_ρ depends smoothly on ρ and that $e_\rho(\partial\Omega^0)=\partial\Omega^\rho$ so that $e_\rho(\Omega^0)=\Omega^\rho$. Several ways of constructing such a family $\{e_\rho\,;\,\rho\in\mathcal{V}^0(\varepsilon_1)\}$ are possible. We shall employ the one as in [7], which will be convenient for our purpose.

Given a small constant $\varepsilon_0>0$, we consider a tubular neighborhood $N(\varepsilon_0)=\{z\in C^n\,;\,|r^0(z)|<\varepsilon_0\}$ of $\partial\Omega^0$ in C^n, where $r^0\in C^\infty(C^n\,;\,\boldsymbol{R})$ is a defining function of Ω^0 such that

$$\Omega^0=\{z\in C^n\,;\,r^0(z)<0\},\quad |dr^0(z)|=1 \quad \text{for}\quad z\in N(\varepsilon_0)\,.$$

Then, every point $z\in N(\varepsilon_0)$ is uniquely expressed as $z=\zeta_z+r^0(z)\boldsymbol{\nu}(\zeta_z)$, where $\zeta_z\in\partial\Omega^0$ is the nearest point to z. Fixing a constant ε_1 with $0<\varepsilon_1<\varepsilon_0/4$, we choose $\chi_0 \in C_0^\infty(\boldsymbol{R};\boldsymbol{R})$ satisfying

$$\begin{array}{lll} \chi_0(r)=1 & \text{for} \quad |r| \leqq \varepsilon_1, & \\ & & \text{and} \quad \left|\frac{d}{dr}\chi_0(r)\right| \leqq \frac{3}{4\varepsilon_1} \quad \text{for} \quad r \in \boldsymbol{R}. \\ \chi_0(r)=0 & \text{for} \quad |r| \geqq 3\varepsilon_1, & \end{array}$$

For $\rho \in \mathcal{V}^0(\varepsilon_1)$, we define a mapping $e_\rho : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by setting

$$(5) \qquad \begin{array}{ll} e_\rho(z)=z+\chi_0(r^0(z))\rho(\zeta_z)\boldsymbol{\nu}(\zeta_z) & \text{for} \quad z \in N(\varepsilon_0), \\ e_\rho(z)=z & \text{otherwise.} \end{array}$$

Then, $\{e_\rho;\, \rho \in \mathcal{V}^0(\varepsilon_1)\}$ is a family of diffeomorphisms satisfying (4).

By means of e_ρ, one can pull back in general a function f^ρ in Ω^ρ or on $\partial\Omega^\rho$ and a linear operator L^ρ acting on f^ρ as follows:

$$f_\rho = e_\rho^* f^\rho = f^\rho \circ e_\rho, \quad L_\rho f_\rho = (e_\rho^* L^\rho e_\rho^{-1*}) f_\rho = (L^\rho(f_\rho \circ e_\rho^{-1})) \circ e_\rho .$$

Let $S^\rho : L^2(\partial\Omega^\rho) \to L^2 H_b(\partial\Omega^\rho) \subset L^2(\partial\Omega^\rho)$ denote the Szegö projector associated with Ω^ρ, which is the orthogonal projector onto $L^2 H_b(\partial\Omega^\rho) = L_b^2 H(\Omega^\rho)|_{\partial\Omega^\rho}$ and is related to $S^\rho(z, w)$ by

$$S^\rho f^\rho(z) = \int_{\partial\Omega^\rho} S^\rho(z, \zeta) f^\rho(\zeta) d\sigma^\rho(\zeta) \qquad \text{for} \quad f^\rho \in L^2(\partial\Omega^\rho),$$

where $d\sigma^\rho(\zeta)$ stands for the induced surface measure of $\partial\Omega^\rho \subset \boldsymbol{C}^n$ at ζ. Then, $S_\rho = e_\rho^* S^\rho e_\rho^{-1*}$ satisfies

$$S_\rho f_\rho(z) = \int_{\partial\Omega^0} S_\rho(z, \zeta) f_\rho(\zeta) d\sigma^\rho(e_\rho(\zeta)) \qquad \text{for} \quad f \in L^2(\partial\Omega^0),$$

where we have set

$$S_\rho(z, w) = S^\rho(e_\rho(z), e_\rho(w)) \qquad \text{for} \quad (z, w) \in \Omega^0 \times \overline{\Omega^0}.$$

Observe by (5) that $S_\rho(z, w) = S^\rho(z, w)$ for $z, w \in \Omega^0 \backslash N(\varepsilon_0)$. Therefore, the variation (2) exists for $z, w \in \Omega^0 \backslash N(\varepsilon_0)$ fixed, provided that $S_\rho(z, w)$ depends smoothly on ρ as far as ρ is small with respect to the $C^\infty(\partial\Omega^0)$-topology. For the later use, we shall show that

$$(6) \qquad \mathcal{V}^2 \ni \rho \mapsto S_\rho(\cdot, w) \in C^\infty(\overline{\Omega^0}) \quad \text{is smooth}$$

with $w \in \Omega^0 \backslash N(\varepsilon_0)$ fixed, where $\mathcal{V}^2$ is a neighborhood of $0 \in C^\infty(\partial\Omega^0; \boldsymbol{R})$.

In order to prove (6), we first recall that

$$\begin{aligned} S^\rho(z, w) &= \int_{\partial\Omega^\rho} S^\rho(z, \zeta) P^\rho(w, \zeta) d\sigma^\rho(\zeta) \\ &= [S^\rho P^\rho(w, \cdot)](z) \qquad \text{for} \quad (z, w) \in \overline{\Omega^\rho} \times \Omega^\rho, \end{aligned}$$

where $P^\rho(w, \zeta)$ denotes the Poisson kernel associated with Ω^ρ, see Kerzman and Stein [5]. Then,

$$S_\rho(\cdot, w)=S_\rho P_\rho(w, \cdot) \quad \text{on} \quad \overline{\Omega^0} \quad \text{for} \quad w\in\Omega^0\backslash N(\varepsilon_0). \tag{7}$$

We next recall the assumption that Ω^0 is strictly pseudo-convex, so that

$$\sum_{j,k=1}^{n}\frac{\partial^2 r^0(z)}{\partial z_j\partial\bar{z}_k}\xi_j\bar{\xi}_k\geqq C\sum_{j=1}^{n}|\xi_j|^2 \quad \text{whenever} \quad \sum_{j=1}^{n}\frac{\partial r^0(z)}{\partial z_j}\xi_j=0 \tag{8}$$

holds for each $z\in\partial\Omega^0$, where $C>0$ is a constant independent of z. Hence, if $\rho\in\mathcal{V}^0(\varepsilon_1)$ is small with respect to the $C^2(\partial\Omega^0)$-topology, say,

$$\rho\in\mathcal{V}^2=\{\rho\in\mathcal{V}^0(\varepsilon_1)\,;\,|\rho|_{C^2(\partial\Omega^0)}<\varepsilon_2\} \quad \text{with } \varepsilon_2>0 \text{ small,}$$

then Ω^ρ is strictly pseudo-convex uniformly in $\rho\in\mathcal{V}^2$ in the sense that (8) holds for each $z\in\partial\Omega^\rho$ with $r^\rho=r^0\circ e_\rho^{-1}$ in place of r^0, where $C>0$ is independent of $\rho\in\mathcal{V}^2$. If moreover $n\geqq3$ then the following formula holds:

$$S^\rho=1-\vartheta_b^\rho N_b^\rho\bar{\partial}_b^\rho, \quad \text{thus} \quad S_\rho=1-(\vartheta_b)_\rho(N_b)_\rho(\bar{\partial}_b)_\rho\,, \tag{9}$$

where $\bar{\partial}_b^\rho$ and ϑ_b^ρ denote the tangential Cauchy-Riemann operator acting on $C^\infty(\partial\Omega^\rho)$ and its $L^2(\partial\Omega^\rho)$ adjoint, respectively, and N_b^ρ stands for the $\bar{\partial}_b$-Neumann operator acting on the space $C^\infty_{(0,1)}(\partial\Omega^\rho)$ of tangential (0, 1)-forms on $\partial\Omega^\rho$ (see Kohn [6], or Folland and Kohn [2]). The definitions of $(\vartheta_b)_\rho$, $(N_b)_\rho$ and $(\bar{\partial}_b)_\rho$ will be clear except for the fact that the space $e_\rho^*C^\infty_{(0,1)}(\partial\Omega^\rho)$ may vary with ρ. However, one may modify it to be independent of ρ by considering the projection: $e_\rho^*C^\infty_{(0,1)}(\partial\Omega^\rho)\to C^\infty_{(0,1)}(\partial\Omega^0)$ (see Kuranishi [8]). The smooth dependence of this modification of the pull-back of $(N_b)_\rho$ on ρ small is involved in Kuranishi [8]. Therefore, S_ρ depends smoothly on ρ in the sense that

$$\mathcal{V}^2\times C^\infty(\partial\Omega^0)\ni(\rho, f)\mapsto S_\rho f\in C^\infty(\partial\Omega^0) \quad \text{is smooth.}$$

Since the Poisson kernel $P^\rho(w, \cdot)$ can be expressed in terms of the Green kernel, the smooth dependence of $P_\rho(w, \cdot)$ on ρ can be proved as in Hamilton [4] (the easier case). Hence, by virtue of (7), we have proved (6). In particular, the variation (2) makes sense.

§3. **Proof of the variational formula** (3). The proof is similar to that in [7]. Pick $z, w\in\Omega^0$ arbitrarily and choose $\varepsilon_0>0$ so small that $z, w\in\Omega^0\backslash N(\varepsilon_0)$. Then,

$$S_\rho(z, w)=S^\rho(z, w) \qquad \text{for} \quad \rho\in\mathcal{V}^2.$$

By the reproducing property for the Szegö kernel, we have

$$S_\rho(z, w)=S^\rho(z, w)=\int_{\partial\Omega^\rho}S^\rho(z, \zeta)S^\rho(\zeta, w)d\sigma^\rho(\zeta)$$

$$=\int_{\partial\Omega^0}S_\rho(z, \zeta)S_\rho(\zeta, w)J_b[e_\rho](\zeta)d\sigma^0(\zeta),$$

where $J_b[e_\rho]$ stands for the Jacobian determinant of the mapping e_ρ restricted to

$\partial\Omega^0$. Recalling (6), we take the variation at $\rho=0$ in the direction $\delta\rho\in C^\infty(\partial\Omega^0; \boldsymbol{R})$. Then,

$$\delta S^0(z, w)=\delta S_0(z, w)=\frac{d}{d\varepsilon}S_{\varepsilon\delta\rho}(z, w)|_{\varepsilon=0}$$

$$=\int_{\partial\Omega^0}\{(I_1)+(I_2)+(I_3)\}\,d\sigma^0(\zeta),$$

where

$$(I_1)=\delta S_0(z, \zeta)S^0(\zeta, w), \quad (I_2)=S^0(z, \zeta)\delta S_0(\zeta, w),$$
$$(I_3)=S^0(z, \zeta)S^0(\zeta, w)\delta J_b[e_0](\zeta),$$

and $\delta J_b[e_0]=\frac{d}{d\varepsilon}J_b[e_{\varepsilon\delta\rho}]|_{\varepsilon=0}$. Setting $\delta X_0=\frac{d}{d\varepsilon}e_{\varepsilon\delta\rho}|_{\varepsilon=0}$, we get

$$\delta X_0(\zeta)=\delta\rho(\zeta)\frac{\partial}{\partial\nu_\zeta}, \quad \delta J_b[e_0](\zeta)=\operatorname{div}\delta X_0(\zeta)=\delta\rho(\zeta)H^0(\zeta)$$

for $\zeta\in\partial\Omega^0$, and

$$\delta S_0(z, \zeta)=\delta S^0(z, \zeta)+\delta X_0(\zeta)S^0(z, \zeta),$$
$$\delta S_0(\zeta, w)=\delta S^0(\zeta, w)+\delta X_0(\zeta)S^0(\zeta, w) \tag{10}$$

for $\zeta\in\Omega^0$, where the vector field $\delta X_0(\zeta)$ in (10) is acting as a differential operator. Note that $\delta S^0(z, \cdot)$ and $\delta S^0(\cdot, w)$ extend smoothly to $\overline{\Omega^0}$, and that the relations in (10) remain valid for $\zeta\in\overline{\Omega^0}$. Moreover, $\delta S^0(\cdot, w)$ and $\delta S^0(z, \cdot)$ are holomorphic and conjugate holomorphic in Ω^0, respectively. Since $S^0(\cdot, \cdot)$ is hermitian symmetric with the reproducing property, we have

$$\int_{\partial\Omega^0}(I_1)d\sigma^0(\zeta)=\delta S^0(z, w)+\int_{\partial\Omega^0}\delta X_0(\zeta)S^0(z, \zeta)\cdot S^0(\zeta, w)d\sigma^0(\zeta),$$

$$\int_{\partial\Omega^0}(I_2)d\sigma^0(\zeta)=\delta S^0(z, w)+\int_{\partial\Omega^0}S^0(z, \zeta)\cdot\delta X_0(\zeta)S^0(\zeta, w)d\sigma^0(\zeta),$$

while

$$\int_{\partial\Omega^0}(I_3)d\sigma^0(\zeta)=\int_{\partial\Omega^0}S^0(z, \zeta)S^0(\zeta, w)H^0(\zeta)\delta\rho(\zeta)d\sigma^0(\zeta).$$

Summing them up, we obtain the desired variational formula (3).

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DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560
JAPAN