Hadronic decays of the tau lepton :
$\tau^- \to (\pi\pi\pi)^- \nu_\tau$ within Resonance Chiral Theory [1]

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Abstract. τ decays into hadrons foresee the study of the hadronization of vector and axial-vector QCD currents, yielding relevant information on the dynamics of the resonances entering into the processes. We analyse $\tau \to \pi\pi\pi\nu_\tau$ decays within the framework of the Resonance Chiral Theory, comparing this theoretical scheme with the experimental data, namely ALEPH spectral function and branching ratio. Hence we get values for the mass and on-shell width of the $a_1(1260)$ resonance, and provide the structure functions that have been measured by OPAL and CLEO-II.

INTRODUCTION

The implementation of Quantum Chromodynamics (QCD) in the energy region populated by light-flavoured resonances ($M_\rho \lesssim E \lesssim 2$ GeV, being M_ρ the mass of the $\rho(770)$) is a demanding task that involves poorly known aspects such as bound and resonant states, duality and hadronization mechanisms. Though *ad hoc* Breit-Wigner parameterisations have been widely employed in the literature [1, 2] they are not necessarily consistent with the underlying theory, as they seem to violate the chiral symmetry of massless QCD [3, 4].

τ decays into hadrons allow to study the hadronization properties of vector and axial-vector QCD currents and, accordingly, to determine intrinsic properties of the dynamics generated by resonances [5]. At very low energies, typically $E \ll M_\rho$, Chiral Perturbation Theory (χPT) [6] is the Effective Field Theory of QCD. Still the $\tau \to \pi\pi\pi\nu_\tau$ decays, through their full energy spectrum, are driven by the $\rho(770)$ and $a_1(1260)$ resonances mainly, in an energy region where the invariant hadron momentum approaches the masses of the resonances. Hence χPT is no longer applicable to the study of the whole spectrum but only to the very low energy domain [7]. The standard procedure that has been followed [1] to deal with these decays has been to modulate the amplitudes with a Breit-Wigner parameterization, fixing the normalization in order to match the leading $\mathcal{O}(p^2)$ χPT. Nevertheless, its deviation of the chiral behaviour at higher orders [3, 4] could spoil any outcome provided by the analysis of data.

Lately several experiments have collected good quality data on $\tau \to \pi\pi\pi\nu_\tau$, such as

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branching ratios and spectra [8] or structure functions [9]. Their analysis within a model-independent framework is highly desirable if one wishes to collect information on the hadronization of the relevant QCD currents.

THE RESONANCE CHIRAL THEORY OF QCD

At energies $E \sim M_\rho$ the resonance mesons are active degrees of freedom that have to be properly included into the pertinent Lagrangian. The procedure, put forward in Refs. [10, 11] and known as Resonance Chiral Theory (RχT), is ruled by the approximate chiral symmetry of QCD, that drives the interaction of Goldstone bosons (the lightest octet of pseudoscalar mesons), and the $SU(3)_V$ assignments of the resonance multiplets. This construction is embedded within a comprehensive framework guided by the large number of colours (N_C) limit of QCD [12]. The $1/N_C$ expansion tells us that, at leading order, we should only consider the tree level diagrams given by a local Lagrangian with infinite zero-width states in the spectrum. This is precisely the role of RχT. However in most processes, like hadron tau decays, we need to include finite widths that only appear at next-to-leading order in the large-N_C expansion and, moreover, we will only include one multiplet of vector and axial-vector resonances in our theory. Thus, in practice, we have to model this large-N_C expansion to some extent.

The final hadron system in the $\tau \to \pi\pi\pi\nu_\tau$ decays spans a wide energy region, namely $3m_\pi \lesssim E \lesssim M_\tau$ that is populated by many resonances. RχT is the appropriate framework to work with and we consider the Lagrangian [4, 10] :

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}\chi\mathrm{T}} &= \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+ \rangle + \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + i\frac{G_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle \\
&\quad + \mathscr{L}_{\mathrm{kin}}^{\mathrm{V}} + \mathscr{L}_{\mathrm{kin}}^{\mathrm{A}} + \sum_{i=1}^{5}\lambda_i\,\mathscr{O}_{\mathrm{VAP}}^i ,
\end{aligned}
\tag{1}
$$

where all the couplings are real, being F the decay constant of the pion in the chiral limit, and the operators $\mathscr{O}_{VAP}^i$ are given by :

$$
\begin{aligned}
\mathscr{O}_{\mathrm{VAP}}^1 &= \langle [V^{\mu\nu}, A_{\mu\nu}]\chi_- \rangle , \\
\mathscr{O}_{\mathrm{VAP}}^2 &= i\langle [V^{\mu\nu}, A_{\nu\alpha}]h_\mu^\alpha \rangle , \\
\mathscr{O}_{\mathrm{VAP}}^3 &= i\langle [\nabla^\mu V_{\mu\nu}, A^{\nu\alpha}]u_\alpha \rangle , \\
\mathscr{O}_{\mathrm{VAP}}^4 &= i\langle [\nabla^\alpha V_{\mu\nu}, A_\alpha^\nu]u^\mu \rangle , \\
\mathscr{O}_{\mathrm{VAP}}^5 &= i\langle [\nabla^\alpha V_{\mu\nu}, A^{\mu\nu}]u_\alpha \rangle .
\end{aligned}
\tag{2}
$$

The notation is that of Ref. [10]. Notice that we are using the antisymmetric tensor formulation to describe the spin 1 resonances and, consequently, we do not consider the $\mathscr{O}(p^4)$ χPT Lagrangian of Goldstone bosons [11].

Our Lagrangian theory has eight a priori unknown coupling constants, namely F_V, F_A, G_V and λ_i, $i = 1,...5$. Phenomenology could provide direct information on them; for instance F_V could be extracted from the measured $\Gamma(\rho^0 \to e^+e^-)$, G_V from $\Gamma(\rho^0 \to$

$\pi^+\pi^-$), F_A from $\Gamma(a_1 \to \pi\gamma)$ and the λ_i appear in $\Gamma(a_1 \to \rho\pi)$ or the $\tau \to \pi\pi\pi\nu_\tau$ processes themselves. It is conspicuous, though, that $\mathscr{L}_{R\chi T}$ is not QCD for arbitrary values of the couplings. Hence if we want to comprehend more about QCD in this non-perturbative regime we should try to learn about the determination of the couplings from the underlying theory [13]. On this account we will implement several known features of the strong interaction theory in the following.

The QCD ruled short–distance behaviour of the vector and axial-vector form factors in the large–N_C limit (approximated with only one octet of vector resonances) constrains the couplings of $\mathscr{L}_{R\chi T}$ in Eq. (1), which must satisfy [11] :

$$
\begin{aligned}
1 - \frac{F_V\,G_V}{F^2} &= 0 \;, \\
2F_V G_V - F_V^2 &= 0 \;.
\end{aligned}
\tag{3}
$$

In addition, the first Weinberg sum rule, in the limit where only the lowest narrow resonances contribute to the vector and axial–vector spectral functions, leads to

$$
F_V^2 - F_A^2 = F^2 \;.
\tag{4}
$$

In this way all three couplings F_V, G_V and F_A can be written in terms of the pion decay constant : $F_V = \sqrt{2}F$, $G_V = F/\sqrt{2}$ and $F_A = F$. These results are well satisfied phenomenologically and we have adopted them. In the next section we will comment on an analogous study of the λ_i couplings.

THE AXIAL-VECTOR FORM FACTORS IN $\tau^- \to (\pi\pi\pi)^- \nu_\tau$

The decay amplitudes for the $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ and $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$ processes can be written as

$$
\mathscr{M}_\pm = -\frac{G_F}{\sqrt{2}} V_{ud}\,\bar{u}_{\nu_\tau}\gamma^\mu (1-\gamma_5)u_\tau\, T_{\pm\mu} \;,
\tag{5}
$$

$$
T_{\pm\mu}(p_1,p_2,p_3) = \langle \pi_1(p_1)\pi_2(p_2)\pi^\pm(p_3)|\mathscr{A}_\mu|0\rangle \;,
\tag{6}
$$

as in the isospin limit there is no contribution of the vector current to these processes. In $T_{\pm\mu}(p_1,p_2,p_3)$ the π^+ is the one in $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ and π^- that in $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$. The hadronic tensor can be written in terms of three form factors, F_1, F_2 and F_P, as [14] :

$$
T^\mu = V_1^\mu F_1 + V_2^\mu F_2 + Q^\mu F_P \;,
\tag{7}
$$

where

$$
\begin{aligned}
V_1^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right)(p_1 - p_3)_\nu \;, \\
V_2^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right)(p_2 - p_3)_\nu \;, \\
Q^\mu &= p_1^\mu + p_2^\mu + p_3^\mu \;.
\end{aligned}
\tag{8}
$$

FIGURE 1. Diagrams contributing to the hadronic amplitude $T_{\pm\mu}$ at $\mathcal{O}(p^2)$ χPT.

The form factors F_1 and F_2 have a transverse structure in the total hadron momenta Q_μ and drive a $J^P = 1^+$ transition. Bose symmetry under interchange of the two identical pions in the final state demands that $F_1(Q^2, s, t) = F_2(Q^2, t, s)$ where $s = (p_1 + p_3)^2$ and $t = (p_2 + p_3)^2$. Meanwhile F_P accounts for a $J^P = 0^-$ transition that carries pseudoscalar degrees of freedom and vanishes with the square of the pion mass. Its contribution to the spectral function of $\tau \to \pi\pi\pi\nu_\tau$ goes like m_π^4/Q^4 and, accordingly, it is very much suppressed with respect to that coming from F_1 and F_2. We will not consider it in the following.

In the low Q^2 region, the matrix element in Eq. (6) can be calculated using χPT. At $\mathcal{O}(p^2)$ one has two contributions, arising from the diagrams in Fig. 1. The sum of both graphs yields

$$T_{\pm\mu}^\chi = \mp \frac{2\sqrt{2}}{3F} \left\{ V_{1\mu} + V_{2\mu} \right\} . \tag{9}$$

We now include the resonance–mediated contributions to the amplitude, to be evaluated through the interacting terms in $\mathscr{L}_{R\chi T}$ Eq. (1). The relevant diagrams to be taken into account are those shown in Fig. 2. We get

$$
\begin{aligned}
T_{\pm\mu}^R &= \mp \frac{\sqrt{2} F_V G_V}{3F^3} \left[\alpha(Q^2, s, t) V_{1\mu} + \alpha(Q^2, t, s) V_{2\mu} \right] \\
&\pm \frac{4 F_A G_V}{3F^3} \frac{Q^2}{Q^2 - M_A^2} \left[\beta(Q^2, s, t) V_{1\mu} + \beta(Q^2, t, s) V_{2\mu} \right] .
\end{aligned} \tag{10}
$$

The functions $\alpha(Q^2, s, t)$ and $\beta(Q^2, s, t)$ are :

$$\alpha(Q^2, s, t) = -3 \frac{s}{s - M_V^2} + \left(\frac{2G_V}{F_V} - 1 \right) \left\{ \frac{2Q^2 - 2s - u}{s - M_V^2} + \frac{u - s}{t - M_V^2} \right\} , \tag{11}$$

$$\beta(Q^2, s, t) = -3 (\lambda' + \lambda'') \frac{s}{s - M_V^2} + F(Q^2, s) \frac{2Q^2 + s - u}{s - M_V^2} + F(Q^2, t) \frac{u - s}{t - M_V^2} , \tag{12}$$

with

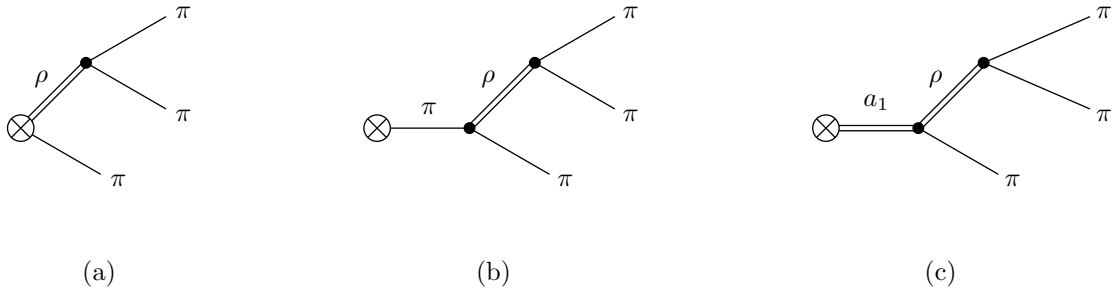$$F(Q^2, s) = -\lambda_0 \frac{m_\pi^2}{Q^2} + \lambda' \frac{s}{Q^2} + \lambda'' , \tag{13}$$

FIGURE 2. Resonance–mediated diagrams contributing to $T_{\pm\mu}$.

and depend on three combinations of the λ_i couplings in $\mathscr{L}_{R\chi T}$, that we call λ_0, λ' and λ''. Following the ideas outlined above we can get information on these combinations by implementing known aspects of asymptotic QCD. In particular we expect that the form factor of the axial-vector current into three pions should vanish at infinite transfer of momentum ($Q^2 \rightarrow \infty$). This is a consequence of the fact that its contribution to the spectral function of the axial-vector current correlator, being positive, has to add to other infinite hadronic positive contributions to reach the constant value evaluated within QCD [16]. Accordingly the proper behaviour of the $T_{\pm\mu}$ form factor imposes the constraints :

$$
\begin{aligned}
2\lambda' - 1 &= 0 \ , \\
\lambda'' &= 0 \ .
\end{aligned}
\tag{14}
$$

Hence there is only one combination of couplings left unknown, namely λ_0.

Finally an additional comment on the result for $T^R_{\pm\mu}$ is required. The form factors in Eq. (10) include zero–width $\rho(770)$ and $a_1(1260)$ propagator poles, leading to divergent phase–space integrals in the calculation of the $\tau \rightarrow \pi\pi\pi\nu_\tau$ decay width as the kinematical variables go along the full energy spectrum. The result can be regularized through the inclusion of resonance widths, which means to go beyond the leading order in the $1/N_C$ expansion, and implies the introduction of some additional theoretical inputs. This issue has been analysed in detail within the resonance chiral effective theory in Ref. [15] and, accordingly, we include off–shell widths for both resonances [4].

THEORY VERSUS EXPERIMENT

To analyse the experimental data we will only consider the dominating $J^P = 1^+$ driven axial–vector form factors, that satisfy $T_{+\mu} = -T_{-\mu}$ hence providing the same predictions for both $\tau^- \rightarrow \pi^+\pi^-\pi^-\nu_\tau$ and $\tau^- \rightarrow \pi^-\pi^0\pi^0\nu_\tau$ processes in the isospin limit.

We have fitted the experimental values for the $\tau^- \rightarrow \pi^+\pi^-\pi^-\nu_\tau$ branching ratio and normalized spectral function obtained by ALEPH [8] and we get a reasonable $\chi^2/d.o.f. = 64.5/52$. This is shown in Fig. 3. Hence we get the axial–vector $a_1(1260)$ parameters $M_A = (1.203 \pm 0.003)\,\text{GeV}$ and $\Gamma_{a_1}(M_A^2) = (0.48 \pm 0.02)\,\text{GeV}$, where the errors are only statistical. We also obtain a value for the still unknown combination of λ_i couplings : $\lambda_0 = 11.9 \pm 0.4$. However, as pointed out in Ref. [17], this value seems too large when
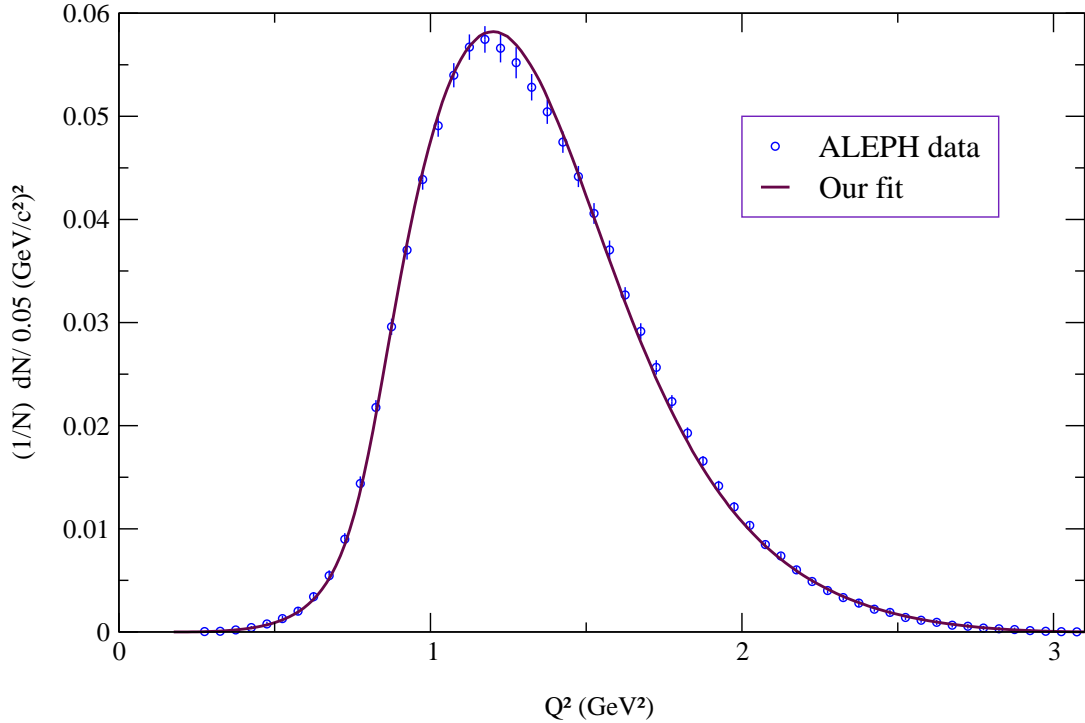
FIGURE 3. Fit to the ALEPH data [8] for the normalized $\tau^- \to \pi^+\pi^-\pi^- \nu_\tau$.

additional QCD constraints are imposed. The origin of the discrepancy could be the small sensitivity of the tau decay amplitude to this parameter, as it only appears multiplied by the mass of the pion (13), together with an improvable implementation of the off-shell width of the $a_1(1260)$ resonance.

Ultimately we predict the integrated structure functions w_A, w_C, w_D and w_E [14], that we compare with the experimental results for $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$ in Fig. 4. In spite of the large errors the predictions follow notably the depicted trend. As analysed in Ref. [4] a variance between the ALEPH data on one side and the OPAL and CLEO-II on the other are at the origin of the seemingly inconsistent result for w_A in the high Q^2 region.

In conclusion it can be inferred that, within the present experimental errors and for the studied observables, there is no evidence of relevant contributions in $\tau \to \pi\pi\pi\nu_\tau$ decays beyond those of the $\rho(770)$ and $a_1(1260)$ resonances.
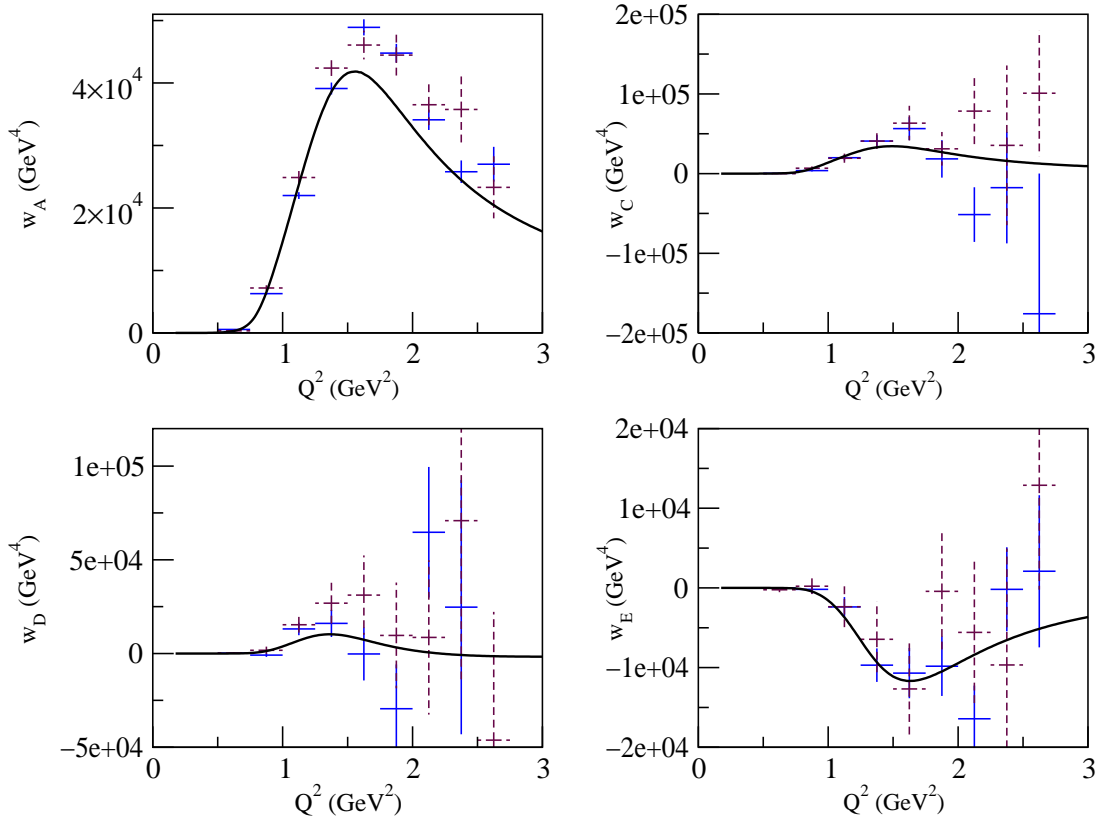
ACKNOWLEDGMENTS

FIGURE 4. Theoretical values for the w_A, w_C, w_D and w_E integrated structure functions in comparison with the experimental data from CLEO-II (solid) and OPAL (dashed) [9].

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