Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 764 – 777

doi:10.3906/mat-1209-27

Half-lightlike submanifolds with planar normal sections in R_2^4

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Received: 14.09.2012 • **Accepted:** 11.05.2013 • **Published Online:** 25.04.2014 • **Printed:** 23.05.2014

Abstract: We investigate half-lightlike submanifolds with planar normal sections of 4-dimensional pseudo-Euclidean space. We obtain necessary and sufficient conditions for a half-lightlike submanifold of R_2^4 such that it has degenerate or nondegenerate planar normal sections.

Key words: Half-lightlike submanifold, planar normal sections

1. Introduction

Surfaces with planar normal sections in Euclidean spaces were first studied by Chen [1]. In [6], Y.H. Kim studied Surfaces with planar normal sections in semi-Riemann setting. As far as we know, however, this topic has not been studied in lightlike geometry. Therefore, as a first step, in this paper we study half-lightlike submanifolds with planar normal sections in R_2^4.

Let M be a hypersurface in R_2^4. For a point p in M and a lightlike vector ξ tangent to M at p that spans radical distribution, the vector ξ and transversal space $tr(TM)$ to M at p determine a 2-dimensional subspace $E(p,\xi)$ in R_2^4 through p. The intersection of M and $E(p,\xi)$ gives a lightlike curve γ in a neighborhood of p, which is called the normal section of M at the point p in the direction of ξ. Let v be a spacelike vector tangent to M at p $(v \in S(TM))$. The vector v and transversal space $tr(TM)$ to M at p then determine a 2-dimensional subspace $E(p,v)$ in R_2^4 through p. In this case, the intersection of M and $E(p,v)$ gives a spacelike curve γ in a neighborhood of p which is called the normal section of M at p in the direction of v. According to both situations given above, M is said to have degenerate pointwise and nondegenerate pointwise planar normal sections, respectively, if each normal section γ at p satisfies $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ [1,7,5,4].

2. Preliminaries

The codimension 2 lightlike submanifold (M,g) is called a half-lightlike submanifold if rank$(radTM) = 1$. In this case, there exist 2 complementary nondegenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $RadTM$ in TM and $TM^{\perp}$ respectively, called the screen and coscreen distribution on M. Then we have the following 2 orthogonal decompositions:

$$TM = RadTM \oplus_{orth} S(TM), TM^{\perp} = RadTM \oplus_{orth} S(TM^{\perp}),$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum.

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2010 *AMS Mathematics Subject Classification:* 53C42, 53C50.

We know from [2] that, for any smooth null section ξ of $RadTM$ on a coordinate neighborhood $U \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltrTM)$ satisfying

$$\bar{g}(N,\xi) = 1, \quad \bar{g}(N,N) = \bar{g}(N,X) = 0, \forall X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$, and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the lightlike transversal vector field, lightlike transversal bundle, and transversal vector bundle of M with respect to the screen $S(TM)$, respectively. Since $RadTM$ is a 1-dimensional vector subbundle of $TM^{\perp}$ we may consider a supplementary distribution D to $RadTM$ such that it is locally represented by u.

We call D a screen transversal bundle of M. Thus, we say that the vector bundle $tr(TM)$ is defined over M by

$$tr(TM) = D \oplus_{orth} Itr(TM).$$

Therefore:

$$\begin{aligned} T\bar{M} &= S(TM) \perp (RadTM \oplus tr(TM)) \\ &= S(TM) \perp D \perp (RadTM \oplus Itr(TM)). \end{aligned} \tag{2.1}$$

Denote by P the projection of TM on $S(TM)$ with respect to the decomposition (2.1). Then we write

$$X = PX + \eta(X)\xi, \quad \forall X \in \Gamma(TM),$$

where η is a local differential 1-form on M defined by $\eta(X) = g(X,N)$. Suppose $\bar{\nabla}$ is the metric connection on $\bar{M}$. Since $\{\xi, N\}$ is locally a pair of lightlike sections on $U \subset M$, we define symmetric $F(M)$-bilinear forms D_1 and D_2 and 1-forms $\rho_1, \rho_2, \varepsilon_1$, and ε_2 on U. Using (2.1), we put

$$\bar{\nabla}_X Y = \nabla_X Y + D_1(X,Y) N + D_2(X,Y) u \tag{2.2}$$

$$\bar{\nabla}_X N = -A_N X + \rho_1(X) N + \rho_2(X) u \tag{2.3}$$

$$\bar{\nabla}_X u = -A_u X + \varepsilon_1(X) N + \varepsilon_2(X) u \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y$, $A_N X$, and $A_u X$ belong to $\Gamma(TM)$. We call D_1 and D_2 the lightlike second fundamental form and screen second fundamental form of M with respect to $tr(TM)$, respectively. Both A_N and A_u are linear operators on $\Gamma(TM)$. The first one is $\Gamma(S(TM))$-valued, called the shape operator of M. Since u is a unit vector field, (2.4) implies $\varepsilon_2(X) = 0$. In a similar way, since ξ and N are lightlike vector fields, from (2.2)–(2.4) we obtain

$$D_1(X,\xi) = 0, \ \bar{g}(A_N X, N) = 0, \bar{g}(A_u X, Y) = \epsilon D_2(X,Y) + \varepsilon_1(X)\eta(Y), \tag{2.5}$$

$$\varepsilon_1(X) = -\epsilon D_2(X,\xi), \quad \forall X \in \Gamma(TM). \tag{2.6}$$

Next, consider the decomposition (2.1), and then we have

$$\nabla_X PY = \nabla^*_X PY + E_1(X,PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -A^*_\xi X + u_1(X)\xi, \tag{2.8}$$

where $\nabla^*_X PY$ and A^*_ξ belong to $\Gamma(S(TM))$. A^*_ξ is a linear operator on $\Gamma(TM)$ and ∇^* is a metric connection on $S(TM)$. We call E_1 the local second fundamental form of $S(TM)$ with respect to $Rad(TM)$ and A^*_ξ the shape operator of the screen distribution. The geometric object from Gauss and Weingarten equations (2.2)–(2.4) on one side and (2.7) and (2.8) on the other side are related by

$$\begin{aligned} E_1(X,PY) &= g(A_N X,PY), D_1(X,PY) = g\left(A_\xi^* X,PY\right), \\ u_1(X) &= -\rho_1(X), A_\xi^*\xi = 0, \end{aligned} \tag{2.9}$$

for any $X,Y \in \Gamma(TM)$. A half-lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a normal vector field $\acute{Z} \in \Gamma(tr(TM))$ on M, called an affine normal curvature vector field of M, such that

$$h(X,Y) = D_1(X,Y)N + D_2(X,Y)u = \acute{Z}\bar{g}(X,Y), \quad \forall X,Y \in \Gamma(TM).$$

In particular, (M,g) is said to be totally geodesic if its second fundamental form $h(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$. By direct calculation it is easy to see that M is totally geodesic if and only if both the lightlike and the screen second fundamental tensors D_1 and D_2 respectively vanish on M. Moreover, from (2.3), (2.5), (2.6), and (2.9) we obtain

$$A_\xi = A_u = \varepsilon_1 = \rho_2 = 0.$$

The notion of screen locally conformal half-lightlike submanifolds was introduced by Duggal and Sahin [3] as follows.

A half-lightlike submanifold M, of a semi-Riemannian manifold, is called screen locally conformal if on any coordinate neighborhood U there exists a nonzero smooth function φ such that for any null vector field $\xi \in \Gamma\left(TM^\perp\right)$ the relation

$$A_N X = \varphi A_\xi^* X, \quad \forall X \in \Gamma\left(TM_{|U}\right) \tag{2.10}$$

holds between the shape operators A_N and A_ξ^* of M and $S(TM)$, respectively [3].

On the other hand, the notion of minimal half-lightlike submanifolds has been defined by Bejancu and Duggal as follows.

Definition 2.1 *Let M be a half-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. We then say that M is a minimal half-lightlike submanifold if $\left(tr\,|_{S(TM)}\, h = 0\right)$ and $\varepsilon_1(X) = 0$ [3].*

Definition 2.2 *A half-lightlike submanifold M is said to be irrotational if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$, where $\xi \in \Gamma(RadTM)$ [3].*

For a half-lightlike M, since $D_1(X,\xi) = 0$, the above definition is equivalent to $D_2(X,\xi) = 0 = \varepsilon_1(X)$, $\forall X \in \Gamma(TM)$.

3. Planar normal sections of half-lightlike hypersurfaces in R_2^4

In this section we consider half-lightlike submanifolds having planar normal section. First, we consider degenerate planar normal sections.

3.1. Degenerate planar normal sections in half-lightlike submanifolds

Let M be a half-lightlike submanifold in R_2^4. Now we investigate the conditions for a half-lightlike submanifold of R_2^4 to have degenerate planar normal sections.

Theorem 3.1 *Let M be a half-lightlike submanifold in R_2^4. Then M has degenerate planar normal sections if and only if*

$$D_2(\xi,\xi)u \wedge \bar{\nabla}_\xi D_2(\xi,\xi)u = 0, \tag{3.1.1}$$

where D_2 is the screen second fundamental form of M.

Proof If γ is a null curve, for a point p in M, we have

$$\begin{aligned} \gamma'(s) &= \xi, && (3.1.2)\\ \gamma''(s) &= \nabla_\xi \xi + D_2(\xi,\xi)u, && (3.1.3)\\ \gamma'''(s) &= \nabla_\xi\nabla_\xi\xi + D_2(\nabla_\xi\xi,\xi)u && (3.1.4)\\ &\quad +\xi(D_2(\xi,\xi))u + D_2(\xi,\xi)(-A_u\xi + \varepsilon_1(\xi)N). \end{aligned}$$

From the definition of planar normal section and using $Rad(TM) = Sp\{\xi\}$, we get

$$\nabla_\xi\xi \wedge \xi = 0 \text{ and } \nabla_\xi\nabla_\xi\xi \wedge \xi = 0. \tag{3.1.5}$$

Assume that M has planar degenerate normal sections. Then

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0. \tag{3.1.6}$$

Thus, by using (3.1.2)–(3.1.5) in (3.1.6) one can see that $D_2(\xi,\xi)u$ and $D_2(\nabla_\xi\xi,\xi)u + \xi(D_2(\xi,\xi))u - D_2(\xi,\xi)A_u\xi + D_2(\xi,\xi)\varepsilon_1(\xi)N$ are linearly dependent. Taking the derivative of $D_2(\xi,\xi)u$, we obtain

$$\bar{\nabla}_\xi D_2(\xi,\xi)u = \xi(D_2(\xi,\xi))u - D_2(\xi,\xi)A_u\xi + D_2(\xi,\xi)\varepsilon_1(\xi)N,$$

where γ is assumed to be parameterized by a distinguished parameter. Hence, we get

$$D_2(\xi,\xi)u \wedge \bar{\nabla}_\xi D_2(\xi,\xi)u = 0.$$

Conversely, assume that $D_2(\xi,\xi)u \wedge \bar{\nabla}_\xi D_2(\xi,\xi)u = 0$ for the degenerate tangent vector ξ of M at p. In this case, either $D_2(\xi,\xi)u = 0$ or $\bar{\nabla}_\xi D_2(\xi,\xi)u = 0$. If $D_2(\xi,\xi)u = 0$, then M is totally geodesic in $\bar{M}$ and M is totally umbilical. Thus, we obtain

$$\begin{aligned} \gamma'(s) &= \xi\,, && (3.1.7)\\ \gamma''(s) &= u_1(\xi)\xi, && (3.1.8)\\ \gamma'''(s) &= \xi(u_1(\xi))\xi + u_1^2(\xi)\xi. && (3.1.9) \end{aligned}$$

which give that M has degenerate planar normal sections. On the other hand, if $\bar{\nabla}_\xi D_2(\xi,\xi)u = 0$, then M is screen conformal. Hence, we have

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = \xi \wedge (\nabla_\xi\xi + D_2(\xi,\xi)u) \wedge (\nabla_\xi\nabla_\xi\xi + D_2(\xi,\nabla_\xi\xi)u + \bar{\nabla}_\xi D_2(\xi,\xi)u) = 0.$$

Hence, we complete the proof. □

Now we define a function

$$\begin{aligned} L_p : RadT_pM &\rightarrow R, \\ \xi &\rightarrow L_p(\xi) = D_2^2(\xi,\xi)\,\epsilon, \end{aligned}$$

where $p \in M$ and $\gamma(0) = p$. If $L_p(\xi) = D_2^2(\xi,\xi)\,\epsilon = 0$, then we obtain $D_2(\xi,\xi) = 0$ and $\varepsilon_1(\xi) = 0$. From (3.1.7), (3.1.8), and (3.1.9) we find $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$. Hence, M has degenerate planar normal sections.

We say that the curve γ has a vertex at the point p if the curvature κ of γ satisfies $\frac{d\kappa^2(p)}{ds} = 0$ and $\kappa^2 = \langle \gamma''(s), \gamma''(s) \rangle$. Now let M have degenerate planar normal sections. Then $L_p = 0$, and so $D_2(\xi,\xi) = 0$. Hence, we get

$$h(\xi,\xi) = D_2(\xi,\xi)\,u = 0, \left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0,$$

which gives $\bar{\nabla} h = 0$. Moreover, we have

$$\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s) \rangle = 0$$

for any $p \in M$.

Consequently, we have the following result.

Corollary 3.2 *Let M be a half-lightlike submanifold in R_2^4 with degenerate planar normal sections such that*

$$\begin{aligned} L_p : RadT_pM &\rightarrow R, \\ \xi &\rightarrow L_p(\xi) = D_2^2(\xi,\xi)\,\epsilon, \end{aligned}$$

where $p \in M$. Then the following statements are equivalent:

1. $D_2(\xi,\xi) = 0$,
2. $\left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0$,
3. $\bar{\nabla} h = 0$,
4. *For any $p \in M$, $\kappa = 0$.*

Now, let us assume that a half-lightlike submanifold M of R_2^4 has degenerate planar normal sections. Then for null vector $\xi \in RadTM$, we have

$$\nabla_\xi \xi \neq 0, \tag{3.1.10}$$

where $\xi = \gamma'(s)$, namely, the normal section γ is not a geodesic arc on a sufficiently small neighborhood of p. Then from (3.1.2)–(3.1.4) we write

$$\gamma'''(s) = a(s)\,\gamma''(s) + b(s)\,\gamma'(s),$$

where, a and b are differentiable functions for all $p \in M$. Hence, we get $D_2(\xi,\xi) = \varepsilon_1(\xi) = 0$.

Consequently, we have the following:

Corollary 3.3 *Let a half-lightlike submanifold M in R_2^4 have degenerate planar normal sections. If the normal section γ for any p is not a geodesic arc on a sufficiently small neighborhood of p, then $D_2 = 0$ at $RadTM$.*

Next, assume that γ is parameterized by a distinguished parameter, namely, γ is a geodesic arc on a small neighborhood of $p = \gamma(0)$, i.e. $\nabla_\xi \xi = 0$. Since $u_1(\xi) = \rho_1(\xi) = 0$, we obtain

$$\begin{aligned} \gamma'(0) &= \xi, \\ \gamma''(0) &= D_2(\xi,\xi)u\ , \qquad (3.1.11) \\ \gamma'''(0) &= \bar{\nabla}_\xi D_2(\xi,\xi)u = \xi(D_2(\xi,\xi))u - D_2(\xi,\xi)A_u\xi - \epsilon D_2^2(\xi,\xi)N. \qquad (3.1.12) \end{aligned}$$

Now, let us suppose that M has degenerate planar normal sections at $\gamma(0) = p$. Therefore, from $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$, we have $\xi \wedge h(\xi,\xi) \wedge \bar{\nabla}_\xi h(\xi,\xi) = 0$. From (3.1.11) and (3.1.12), ξ, $h(\xi,\xi)$, and $\bar{\nabla}_\xi h(\xi,\xi)$ are not linearly dependent. In this case, either $h(\xi,\xi) = 0$ or $\bar{\nabla}_\xi h(\xi,\xi) = 0$. If $\bar{\nabla}_\xi h(\xi,\xi) = 0$, then we calculate

$$\begin{aligned} \langle h(\xi,\xi), h(\xi,w)\rangle &= -\left\langle \bar{\nabla}_\xi h(\xi,\xi), w\right\rangle \\ &= 0 \qquad (3.1.13) \end{aligned}$$

and

$$\begin{aligned} \langle h(\xi,\xi), h(\xi,w)\rangle &= \left\langle h(\xi,\xi), \bar{\nabla}_w \xi\right\rangle - \langle h(\xi,\xi), \nabla_w \xi\rangle \\ &= \epsilon D_2(\xi,\xi) D_2(w,\xi). \qquad (3.1.14) \end{aligned}$$

From the symmetry of bilinear forms D_1 and D_2 at $\Gamma(TM)$, hence from (3.1.13) and (3.1.14), we get $D_2 = 0$ at $\Gamma(TM)$. Furthermore, from $\bar{\nabla}_w \xi \in \Gamma(TM)$, ($\xi \in RadTM$, and $w \in \Gamma(TM)$), we see that M is irrotational. Then we have the following result.

Corollary 3.4 *Let M be a half-lightlike submanifold of R_2^4 with degenerate planar normal sections. If the normal section γ for any p is a geodesic arc on a sufficiently small neighborhood of p, then M is irrotational.*

Let M be a half-lightlike submanifold in R_2^4 with degenerate planar normal sections. Since γ is a planar curve, we write

$$\gamma'''(s) = a(s)\gamma''(s) + b(s)\gamma'(s),$$

where a and b are differentiable functions for all $p \in M$. Then (3.1.8) gives

$$\begin{aligned} a(s) &= u_1(\xi) + \xi(\ln(D_2(\xi,\xi))), \\ b(s) &= \xi(u_1(\xi)) - D_2(\xi,\xi)\rho_2(\xi)\epsilon - u_1(\xi)\xi(\ln(D_2(\xi,\xi))). \end{aligned}$$

Moreover, we have $\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s)\rangle = 0$ for any $p \in M$, which gives $D_2(\xi,\xi) = \varepsilon_1(\xi) = 0$. Thus, we obtain

$$\begin{aligned} \gamma'''(s) &= u_1^2(\xi)\xi + u_1(\xi)D_2(\xi,\xi)u \\ &\quad + \xi(\ln(D_2(\xi,\xi)))D_2(\xi,\xi)u \qquad (3.1.15) \\ &\quad + \xi(u_1(\xi))\xi - \epsilon D_2(\xi,\xi)\rho_2(\xi)\xi \end{aligned}$$

and

$$A_u\xi = \epsilon\rho_2(\xi)\xi. \tag{3.1.16}$$

Namely:

Corollary 3.5 *Let M be a half-lightlike submanifold of R_2^4 with degenerate planar normal sections, then $A_u\xi$ is $RadTM$-valued.*

Now, from (3.1.15) and (3.1.16), we obtain

$$\begin{aligned}\left(\bar{\nabla}_\xi h\right)(\xi,\xi) &= \xi\left(\ln\left(D_2(\xi,\xi)\right)\right)D_2(\xi,\xi)u \\ &\quad -\epsilon D_2(\xi,\xi)\rho_2(\xi)\xi - 2u_1(\xi)D_2(\xi,\xi)u. \end{aligned}\tag{3.1.17}$$

Let M be a half-lightlike submanifold of R_2^4 with degenerate planar normal sections. If the normal section γ for any p is not a geodesic arc on a sufficiently small neighborhood of p, then we obtain

$$D_2(\xi,\xi)u \wedge \left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0. \tag{3.1.18}$$

Conversely, we assume that (3.1.18) is satisfied for any degenerate tangent vector ξ of M. Then either $D_2(\xi,\xi)u = 0$ or $\left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0$. If $D_2(\xi,\xi)u = 0$, then from Theorem 3.1, we see that M has degenerate planar normal sections. On the other hand, if $\left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0$, then, by considering (3.1.5), we obtain

$$\gamma'''(s)\wedge\gamma''(s)\wedge\gamma'(s) = \xi\wedge D_2(\xi,\xi)u\wedge\left(\bar{\nabla}_\xi h\right)(\xi,\xi) = 0.$$

Consequently, we have the following:

Corollary 3.6 *Let M be half-lightlike submanifold of R_2^4 such that the normal section $\gamma(s)$ for any p is not a geodesic arc on a sufficiently small neighborhood of p. Then half-lightlike submanifold M has planar normal sections if and only if (3.1.18) is satisfied.*

Now, let the normal section γ be a geodesic arc on a sufficiently small neighborhood of p, namely, $\nabla_\xi\xi = 0 = u_1(\xi)$. Since M has degenerate planar normal sections, we obtain

$$\gamma'''(s)\wedge\gamma''(s)\wedge\gamma'(s) = (\xi\wedge D_2(\xi,\xi)u\wedge D_2(\xi,\xi)A_u\xi) + (\xi\wedge D_2(\xi,\xi)u\wedge D_2(\xi,\xi)\varepsilon_1(\xi)N).$$

From Corollary 3.5, we have $D_2(\xi,\xi) = 0$ and $\varepsilon_1(\xi) = 0$. Thus, we have the following result:

Corollary 3.7 *Let M be a half-lightlike submanifold with degenerate planar normal section of R_2^4. The normal section γ for any p is a geodesic arc on a sufficiently small neighborhood of p. Then $D_2(\xi,\xi) = 0$ or $\varepsilon_1(\xi) = 0$.*

Let M be a screen conformal half-lightlike submanifold of $R_2^4(c)$ with degenerate planar normal sections. We denote the Riemann curvature tensors of $\bar{M}$ and M by $\bar{R}$ and R, and hence we have

$$\begin{aligned}\bar{g}\left(\bar{R}(X,Y)Z,PW\right) &= \varphi\left[D_1(X,Z)D_1(Y,PW) - D_1(Y,Z)D_1(X,PW)\right] \\ &\quad +\epsilon\left[D_2(X,Z)D_2(Y,PW) - D_2(Y,Z)D_2(X,PW)\right]. \end{aligned}\tag{3.1.19}$$

Let $p \in M$ and ξ be a null vector of T_pM. A plane H of T_pM is called a null plane directed by ξ if it contains ξ, $\bar{g}(\xi, W) = 0$ for any $W \in H$ and there exists $W_0 \in H$ such that $\bar{g}(W_0, W_0) \neq 0$. Then the null sectional curvature of H with respect to ξ and $\bar{\nabla}$ is defined by

$$K_{\xi}(H) = \frac{R_p(W, \xi, \xi, W)}{g_p(W, W)}. \tag{3.1.20}$$

Since $v \in \Gamma(S(TM))$ and $\xi \in \Gamma(RadTM)$, we have

$$\begin{aligned} K_{\xi}(H) &= \varphi\left[D_1(v,\xi)D_1(\xi,v) - D_1(\xi,\xi)D_1(v,v)\right] \\ &\quad +\epsilon\left[D_2(v,\xi)D_2(\xi,v) - D_2(\xi,\xi)D_2(v,v)\right]. \end{aligned}$$

By using $D_1(v,\xi) = 0$ in the last equation, we obtain

$$K_{\xi}(H) = \epsilon\left[D_2(v,\xi)D_2(\xi,v) - D_2(\xi,\xi)D_2(v,v)\right]. \tag{3.1.21}$$

Consequently, we have the following:

Corollary 3.8 *Let M be a screen conformal half-lightlike submanifold of $R_2^4(c)$ with degenerate planar normal sections. If M is minimal, then $K_{\xi}(H) = 0$.*

Example 3.9 *Consider a surface M in R_2^4 given by the equation*

$$x^3 = \frac{1}{\sqrt{2}}\left(x^1 + x^2\right); \quad x^4 = \frac{1}{2}\log\left(1 + \left(x^1 - x^2\right)^2\right).$$

It is easy to see that M is a totally umbilical half-lightlike submanifold of R_2^4. Then by straightforward calculations we obtain

$$D_2(\xi,\xi) = 0.$$

Therefore, the intersection of M and $E(p,\xi)$ gives a lightlike curve γ in a neighborhood of p, which is called the normal section of M at point p in the direction of ξ, namely

$$\begin{aligned} \gamma'(s) &= \xi, \\ \gamma''(s) &= \bar{\nabla}_{\xi}\xi = 0. \end{aligned}$$

Hence, we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0.$$

3.2. Nondegenerate planar normal sections in half-lightlike submanifolds

In this subsection we investigate the conditions for a screen conformal half-lightlike submanifold M of R_2^4 to have nondegenerate planar normal sections.

Theorem 3.10 *Let M be a screen conformal half-lightlike submanifold in R_2^4. M has spacelike planar normal sections if and only if*

$$T(v,v) \wedge \bar{\nabla}_v T(v,v) = 0, \tag{3.2.1}$$

where $v \in \Gamma(S(TM))$ and $T(v,v) = E_1(v,v)\xi + D_1(v,v)N + D_2(v,v)u$.

Proof Let M be a screen conformal half-lightlike submanifold and γ a spacelike curve on M. Then we have

$$\gamma'(s) = v, \tag{3.2.2}$$
$$\gamma''(s) = \bar{\nabla}_v v = \nabla^*_v v + E_1(v,v)\,\xi + D_1(v,v)\,N + D_2(v,v)\,u, \tag{3.2.3}$$

$$\begin{aligned}\gamma'''(s) = {} & \nabla^*_v\nabla^*_v v + E_1(v,\nabla^*_v v)\,\xi + D_1(v,\nabla^*_v v)\,N + D_2(v,\nabla^*_v v)\,u \\ & + v(E_1(v,v))\,\xi + v(D_1(v,v))\,N + v(D_2(v,v))\,u \\ & - E_1(v,v)\,A^*_\xi v + E_1(v,v)\,u_1(v)\,\xi + E_1(v,v)\,D_2(v,\xi)\,u \\ & - D_1(v,v)\,A_N v + D_1(v,v)\,\rho_1(v)\,N + D_1(v,v)\,\rho_2(v)\,u \\ & - D_2(v,v)\,A_u v + D_2(v,v)\,\varepsilon_1(v)\,N,\end{aligned} \tag{3.2.4}$$

where ∇^* is the induced connection of M' and $\gamma'(s) = v$, $\gamma'(0) = v$. From the definition of a planar normal section and $S(TM) = Sp\{v\}$ we have

$$v \wedge \nabla^*_v v = 0 \text{ and } v \wedge \nabla^*_v\nabla^*_v v = 0. \tag{3.2.5}$$

Assume that M has planar nondegenerate normal sections. Then we have

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0.$$

Thus, from (3.2.5),

$$T(v,v) = E_1(v,v)\,\xi + D_1(v,v)\,N + D_2(v,v)\,u$$

and

$$\begin{aligned}\bar{\nabla}_v T(v,v) = {} & E_1(v,\nabla^*_v v)\,\xi + D_1(v,\nabla^*_v v)\,N + D_2(v,\nabla^*_v v)\,u \\ & + v(E_1(v,v))\,\xi + v(D_1(v,v))\,N + v(D_2(v,v))\,u \\ & - E_1(v,v)\,A^*_\xi v + E_1(v,v)\,u_1(v)\,\xi + E_1(v,v)\,D_2(v,\xi)\,u \\ & - D_1(v,v)\,A_N v + D_1(v,v)\,\rho_1(v)\,N + D_1(v,v)\,\rho_2(v)\,u \\ & - D_2(v,v)\,A_u v + D_2(v,v)\,\varepsilon_1(v)\,N\end{aligned}$$

are linearly dependent, where γ is assumed to be parameterized by arc length. Thus, we obtain

$$T(v,v) \wedge \bar{\nabla}_v T(v,v) = 0.$$

Conversely, we assume that $T(v,v) \wedge \bar{\nabla}_v T(v,v) = 0$ for a spacelike tangent vector v of M at p. Then either $T(v,v) = 0$ or $\bar{\nabla}_v T(v,v) = 0$. If $T(v,v) = 0$, then from (3.2.2), (3.2.3), (3.2.4), and (3.2.5), M has degenerate planar normal sections. If $\bar{\nabla}_v T(v,v) = 0$, from (3.2.5), we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = v \wedge T(v,v) \wedge \bar{\nabla}_v T(v,v) = 0.$$

□

Example 3.11 *Let M be a half-lightlike submanifold of the 4-dimensional semi-Riemann space $\left(R_2^4, \bar{g}\right)$ of index 2, as given in Example 3.9. Now, for a point p in M and a spacelike vector U_2 tangent to M at p $(U_2 \in S(TM))$, the vector U_2 and transversal space $tr(TM)$ to M at p determine a 2-dimensional subspace $E(p, U_2)$ in R_2^4 through p. The intersection of M and $E(p, U_2)$ gives a spacelike curve γ in a neighborhood of p. Now we research half-lightlike submanifolds of the R_2^4 semi-Riemannian manifold to have the condition of nondegenerate planar normal sections. Hence, we obtain*

$$\begin{aligned}
U_1 &= \sqrt{2}\left(1+\left(x^1-x^2\right)^2\right)\partial_1+\left(1+\left(x^1-x^2\right)^2\right)\partial_3+\sqrt{2}\left(x^1-x^2\right)\partial_4,\\
U_2 &= \sqrt{2}\left(1+\left(x^1-x^2\right)^2\right)\partial_1+\left(1+\left(x^1-x^2\right)^2\right)\partial_3-\sqrt{2}\left(x^1-x^2\right)\partial_4,\\
\xi &= \partial_1+\partial_2+\sqrt{2}\partial_3,\\
u &= 2\left(x^2-x^1\right)\partial_2+\sqrt{2}\left(x^2-x^1\right)\partial_3+\left(1+\left(x^1-x^2\right)\right)\partial_4.
\end{aligned}$$

$$N=-\frac{1}{2}\partial_1+\frac{1}{2}\partial_2+\frac{1}{\sqrt{2}}\partial_3$$

and

$$\begin{aligned}
\gamma' &= U_2=\sqrt{2}\left(1+\left(x^1-x^2\right)^2\right)\partial_1+\left(1+\left(x^1-x^2\right)^2\right)\partial_3-\sqrt{2}\left(x^1-x^2\right)\partial_4,\\
\gamma'' &= 2\left(1+\left(x^1-x^2\right)^2\right).\left\{2\left(x^2-x^1\right)\partial_2+\sqrt{2}\left(x^2-x^1\right)\partial_3+\partial_4\right\},\\
\gamma''' &= \sqrt{2}\left(1+\left(x^1-x^2\right)^2\right)\left[\begin{array}{c} 4\left(1+3\left(\left(x^1-x^2\right)^2\right)\right)\partial_2 \\ +2\sqrt{2}\left(1+3\left(\left(x^1-x^2\right)^2\right)\right)\partial_3-4\left(x^1-x^2\right)\partial_4 \end{array}\right]\\
&\quad +\frac{4\sqrt{2}\left(x^2-x^1\right)^3}{\left(1+\left(x^1-x^2\right)^4\right)}\left(1+\left(x^1-x^2\right)^2\right)\left[\begin{array}{c} 2\left(x^2-x^1\right)\partial_2 \\ +\sqrt{2}\left(x^2-x^1\right)\partial_3+\left(1+x^1-x^2\right)\partial_4 \end{array}\right].
\end{aligned}$$

Then, by direct calculations we find

$$E_1\left(U_2, U_2\right) = 0, \tag{3.2.6}$$

$$E_1\left(U_2, \nabla^*_{U_2} U_2\right) = 0. \tag{3.2.7}$$

Thus, from (3.2.6) and (3.2.7), $T\left(U_2, U_2\right)$ and $\bar{\nabla}_{U_2} T\left(U_2, U_2\right)$ are linearly dependent. Hence we have $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s)=0$.

Proposition 3.12 *Let M be a half-lightlike submanifold in R_2^4. If M has planar normal sections, then*

$$\nabla^*_v v=0, \tag{3.2.8}$$

where γ is a normal section in the direction $v=\gamma'(s)$ for $v \in \Gamma(S(TM))$.

Proof From $v \in S(TM)$ we have

$$\langle v, v\rangle=1 \Rightarrow \langle v, \nabla^*_v v\rangle=0. \tag{3.2.9}$$

Using the definition of a normal section and (3.2.9), we complete the proof. □

Now we define a function L by

$$L(p,v) = L_p(v) = \langle T(v,v), T(v,v)\rangle$$

on $\bigcup_p M$, where $\bigcup_p M = \left\{ v \in \Gamma(TM) \mid \langle v,v\rangle^{\frac{1}{2}} = 1 \right\}$. If $L \neq 0$, then M has nondegenerate pointwise normal sections. By a vertex of curve γ we mean a point p on γ such that its curvature κ satisfies $\frac{d\kappa^2(0)}{ds} = 0$. Let M have planar normal sections. From Proposition 3.12 we obtain

$$\begin{aligned} \epsilon\kappa^2(s) &= 2E_1(v,v)\,D_1(v,v) + D_2^2(v,v)\,\epsilon, \\ \frac{1}{2}\frac{d\kappa^2(0)}{ds} &= v(E_1(v,v)\,D_1(v,v)) + v(D_2(v,v))D_2(v,v)\,\epsilon. \end{aligned}$$

If M is totally geodesic, then $D_1 = D_2 = 0$. Thus γ has a vertex. Consequently, we have the following:

Corollary 3.13 *Let M be a half-lightlike submanifold of R_2^4. If M has nondegenerate planar normal sections the submanifold is totally geodesic screen conformal at $p \in M$, if and only if normal section curve γ has a vertex at $p \in M$.*

Corollary 3.14 *Let M be a half-lightlike submanifold of R_2^4. with planar normal sections. Then normal section curve γ has a vertex and the submanifold is totally geodesic if and only if M is minimal.*

Proof If M is totally geodesic, then from $(tr\,|_{S(TM)}\, h = 0)$ and $\varepsilon_1(\xi) = 0$, we conclude.

From Corollary 8 and Corollary 9, we give: □

Corollary 3.15 *Let M be a half-lightlike submanifold in $R_2^4(c)$ with planar normal sections. Then $K_\xi(H) = 0$ if and only if normal section curve γ has a vertex at $p \in M$ where $\xi \in \Gamma(RadTM)$.*

Corollary 3.16 *Let M be a half-lightlike submanifold of R_2^4 and the normal section γ at for any p be a geodesic arc on a sufficiently small neighborhood of p. Then M has nondegenerate planar normal sections if and only if*

$$h(v,v) \wedge (\bar{\nabla}_v h)(v,v) = 0,$$

where is $h(v,v) = D_1(v,v)\,N + D_2(v,v)\,u$.

Proof If normal section γ at for any p is a geodesic arc on a sufficiently small neighborhood of p, we have

$$\begin{aligned} \gamma'(s) &= v \\ \gamma''(s) &= D_1(v,v)\,N + D_2(v,v)\,u \\ \gamma'''(s) &= v(D_1(v,v))N + v(D_2(v,v))u \\ &\quad -D_1(v,v)\,A_N v + D_1(v,v)\,\rho_1(v)\,N \\ &\quad +D_1(v,v)\,\rho_2(v)\,u - D_2(v,v)\,A_u v \\ &\quad +D_2(v,v)\,\varepsilon_1(v)\,N. \end{aligned}$$

Therefore, by taking the covariant derivative of

$$h(v,v) = D_1(v,v)\,N + D_2(v,v)\,u,$$

we obtain

$$(\bar{\nabla}_v h)(v,v) = \bar{\nabla}_v h(v,v) = \gamma'''(s),$$

which gives

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = v \wedge h(v,v) \wedge (\bar{\nabla}_v h)(v,v) = 0.$$

From the last equation above, we have

$$h(v,v) \wedge (\bar{\nabla}_v h)(v,v) = 0.$$

Conversely, we assume that $h(v,v) \wedge (\bar{\nabla}_v h)(v,v) = 0$. In this case, we have either $h(v,v) = 0$ or $(\bar{\nabla}_v h)(v,v) = 0$. If $h(v,v) = 0$, we have $D_1(v,v) = 0$ and $D_2(v,v) = 0$. In this way, we get

$$\varepsilon_1(\xi) = 0,$$

which shows that M is minimal and has planar normal sections. On the other hand, if $(\bar{\nabla}_v h)(v,v) = 0$, from $(\bar{\nabla}_v h)(v,v) = \bar{\nabla}_v h(v,v) = \gamma'''(s) = 0$, we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0;$$

that is, M has nondegenerate planar normal sections. □

We also have the following result:

Corollary 3.17 *Let M be a half-lightlike submanifold in R_2^4 and the normal section γ for any p be a geodesic arc on a sufficiently small neighborhood of p. Then the following statements are equivalent:*

1. $(\bar{\nabla}_v h)(v,v) = 0,$
2. $\bar{\nabla} h = 0,$
3. *M has nondegenerate planar normal sections of $p \in M$ and γ has a vertex point at $p \in M$,*
4. *$D_2 = 0$ in $S(TM)$.*

Proof For curvature κ at p point of γ, we have

$$\begin{aligned} \epsilon\kappa^2(s) &= D_2^2(v,v)\,\epsilon, \\ \frac{1}{2}\epsilon\frac{d\kappa^2(s)}{ds} &= v(D_2(v,v))\,D_2(v,v)\,\epsilon, \end{aligned} \tag{3.2.10}$$

and from $\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s)\rangle$,

$$\begin{aligned} \frac{1}{2}\epsilon\frac{d\kappa^2(s)}{ds} &= \left\langle (\bar{\nabla}_v h)(v,v), h(v,v)\right\rangle \\ &= 0. \end{aligned} \tag{3.2.11}$$

Hence, from (3.2.10) and (3.2.11), we obtain $D_2(v,v) = 0$. From here, we complete the proof. □

Example 3.18 *Consider a surface M in R_1^4 given by the equation*

$$x_1 = x_3, x_2 = (1 - x_4)^{\frac{1}{2}}.$$

Then we obtain:

$$\begin{aligned} TM &= Sp\{\xi = \partial x_1 + \partial x_3, v = -x_4\partial x_2 + x_2\partial x_4\}, \\ TM^{\perp} &= Sp\{\xi = \partial x_1 + \partial x_3, u = x_2\partial x_2 + x_4\partial x_2\}. \end{aligned}$$

Therefore, we have $RadTM = Sp\{\xi\}$, $S(TM) = Sp\{v\}$, $S(TM^{\perp}) = Sp\{u\}$, and $ltr(TM) = Sp\{N = \frac{1}{2}(\partial x_1 + \partial x_3)\}$, which show that M is a half-lightlike submanifold of R_1^4. Then using the Gauss and Weingarten formulas and on account of $\nabla_v \xi = 0$, we have

$$\bar{g}\left(A_\xi^* v, v\right) = 0 \Rightarrow A_\xi^* v = 0.$$

Moreover, from (2.3) and by straightforward calculations we obtain

$$A_N v = 0$$

or $A_N v \in RadTM$. Using (2.2), we have

$$D_1(v,v) = \bar{g}\left(A_\xi^* v, v\right) = 0,$$

and since $D_1(v,\xi) = 0$, we have $D_1 = 0$. Using (2.2), we obtain

$$D_2(v,v)\,\epsilon = \bar{g}(v, A_u v).$$

Since

$$\bar{g}(\bar{\nabla}_v \bar{\nabla}_v v, N) = \bar{g}(A_u v, N) = 0,$$

$A_u v \in S(TM)$ and by straightforward calculations we obtain

$$\begin{aligned} \bar{g}\left(\bar{\nabla}_v \bar{\nabla}_v v, u\right) &= 2x_2 x_4 \\ \bar{g}\left(\bar{\nabla}_v \bar{\nabla}_v v, \xi\right) &= -\varepsilon_1(u) = 0 \\ \rho_1(v) &= -\bar{g}(A_N v, \xi) = 0, \rho_2(v) = -\epsilon\bar{g}(A_N v, u) = 0 \\ \varepsilon_1(u) &= 0 \Rightarrow D_2(v,\xi) = 0, \epsilon D_2(v,v) = -1. \end{aligned}$$

Thus, M is a screen conformal totally umbilical half-lightlike submanifold. Let $v \in S(TM)$ and $p \in M$. We denote subspace

$$E(p,v) = \{v\} \cup tr(TM)$$

and we have

$$E(p,v) \cap M = \gamma,$$

where γ is the normal section of M at p in the direction of v. Then we have

$$\begin{aligned} \gamma'(s) &= v = -x_4\partial x_2 + x_2\partial x_4 \\ \gamma''(s) &= \bar{\nabla}_v v = -2x_2\partial x_2 - 2x_4\partial x_4 \\ \gamma'''(s) &= \bar{\nabla}_v \bar{\nabla}_v v = 2x_4\partial x_2 - 2x_2\partial x_4. \end{aligned}$$

Hence,

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0;$$

that is, M has nondegenerate planar normal sections.

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