Half-Space Proximal: A New Local Test for Extracting a Bounded Dilation Spanner of a Unit Disk Graph

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Abstract. We give a new local test, called a *Half-Space Proximal* or *HSP test*, for extracting a sparse directed or undirected subgraph of a given unit disk graph. The HSP neighbors of each vertex are unique, given a fixed underlying unit disk graph. The HSP test is a fully distributed, computationally simple algorithm that is applied independently to each vertex of a unit disk graph. The directed spanner obtained by this test is shown to be strongly connected, has out-degree at most six, its dilation is at most $2\pi + 1$, contains the minimum weight spanning tree as its subgraph and, unlike the Yao graph, it is rotation invariant. Since no coordinate assumption is needed to determine the HSP nodes, the test can be applied in any metric space.

1 Introduction

An *ad-hoc network* is a network consisting of transmitters, often called *hosts*, that is established as needed, typically without any assistance from a fixed infrastructure. It is assumed that each host can communicate with all the hosts within its transmission range with a single transmission, called a hop. Typically, not all hosts are within the transmission range of each other and the transmission ranges of all hosts are identical. We will additionally assume that each host knows its location, its coordinates in the plane, obtained by a low energy GPS device or by other means.

Such an ad-hoc network can be represented by a *unit disk graph* (UDG) in which the vertices are points in the Euclidean plane at coordinates corresponding to the geographical location of the hosts. Two nodes are connected by an edge if their Euclidean distance is less than a given unit, where the unit represents the common transmission range of the hosts. Due to the use of unit disk graphs for ad-hoc network representations, computations in the UDG are of interest in computer science.

A subgraph of the UDG is called a *geometric graph*. The length of an edge $[u, v]$ between adjacent vertices u and v of a geometric graph is defined to be the Euclidean distance between u and v. Given a path p in a geometric graph G the length of the path is the sum of the Euclidean lengths of the edges of p. Thus, for any pair of vertices u, v of a geometric graph G we define the distance $d_G(u, v)$ to be the length of the shortest path between u and v in G. Let G be a geometric graph and G' be a spanning subgraph of G. If two vertices of G are connected by an edge e in G and the distance of these vertices in G' is equal to k then we say that the dilation of e is equal to k. We call G' a $t-$*spanner* of G if the dilation of any pair of adjacent vertices of G is at most t. A geometric graph is planar if no two of its edges represented by the straight line segments intersect each other.

When a UDG contains regions with many vertices, the graph may contain a large number of edges, or in an extreme case it may contain a complete subgraph (all the nodes are reachable). For many applications, like routing, energy efficient broadcast, power optimizations, etc., it is often preferable to extract from a given UDG a subgraph having some specific properties, e.g., being planar, or close in weight to a minimal spanning tree, or a $t-$spanner [7, 17, 4, 13]. In an ad-hoc network the topology of the whole network is typically not available in the nodes of the network due to the lack of a central infrastructure, the reduced amount of memory available and the possible mobility of the hosts. Thus, in these situations, the extraction of a suitable geometric subnetwork must be done in a distributed manner in the network using local information. Ideally, there should be a simple algorithm that is executed by each node of the network using only information on nodes reachable within a fixed number of hops, called a fixed-hop neighborhood. This algorithm would determine which edges of the UDG incident with the node are retained for the suitable geometric subgraph. Such algorithms are called *tests* and the geometric graphs which are obtained in this manner are usually called *local proximity graphs* [9].

For extracting a planar subgraph of a given UDG one can use the *Relative Neighborhood test* [17], the *Gabriel test* [7], or the *Morelia test* [3]. Given a UDG G, the spanner $RNG(G)$ is obtained by applying the RNG test to every edge of the UDG: edge $[u, v]$ is retained in $RNG(G)$ if there is no vertex z such that $\max\{d_G(u, z), d_G(v, z)\} < d_G(u, v)$. The Gabriel and the Morelia tests have a different condition to retain edges for the spanner and the graphs produced by applying the tests are denoted $G(G)$ and $M(G)$, respectively. Given a UDG G we have $RNG(G) \subseteq G(G) \subseteq M(G)$, but none of the graphs is a $t-$spanner for any fixed number t.

For extracting a spanner of a given UDG having a bounded dilation of edges, one can use a *Yao test* [18] that is defined as follows. Let k be an integer greater or equal to 6. From each vertex v of a unit disk graph G draw rays separated by $2\pi/k$ angles, starting with a ray in the horizontal line. A cone is defined as the space between two rays and including one of the rays so that the plane is partitioned into k cones. Yao test retains in each cone the shortest edge $[u, v]$ of G, if any exists. The collection of these oriented edges form the *directed Yao graph* $\overrightarrow{Y_k}(G)$. The *undirected Yao graph* $Y_k(G)$ is obtained by omitting the direction of edges. It has been shown [10] that the Yao graph is a $\frac{1}{1-2\sin\pi/k}$-spanner and, clearly, its out-degree is at most k. Unlike the spanners obtained by the RNG, Gabriel, or Morelia test, the graph $Y_k(G)$ depends on the exact position of the cones. Thus if G' is obtained by a rotation of a unit disk graph G then, in general, $Y_k(G)$ is not a rotation of $Y_k(G')$.

In this paper we propose a new local test for constructing a t-spanner of a UDG, called the Half-Space Proximal test, or HSP test for short. In Section 2, we give a definition of the HSP test and show that, similarly to the Yao test, the spanner obtained by the HSP test has a bounded dilation, out-degree at most 6, and is strongly connected, and it contains the minimum weight spanning tree as its subgraph. However, unlike the Yao test, the HSP test applied to a rotation of the UDG G yields a rotation of the HSP spanner of G. Thus, the graph properties of the HSP spanner are independent of the orientation of the unit disk graph in the plane. Section 3 contains experimental results involving HSP and Yao spanners of randomly generated unit disk graphs of different densities.

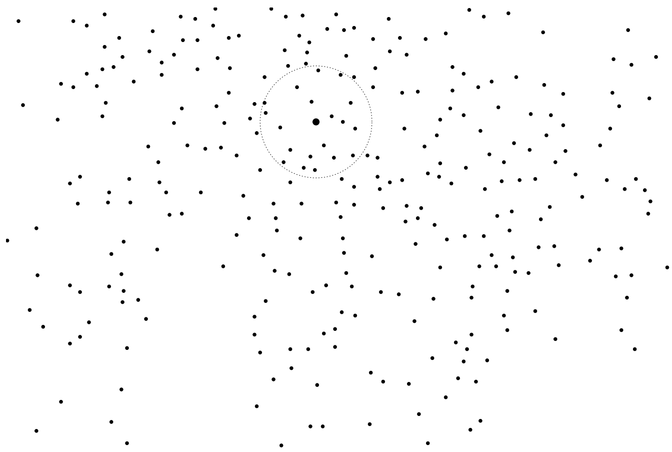
2 Half-Space Proximal Spanner and its properties

We assume that graph $G = (V, E)$ is a unit geometric graph where each node v has the coordinates v_x, v_y in the Euclidean plane and each vertex is assigned a unique integer label.

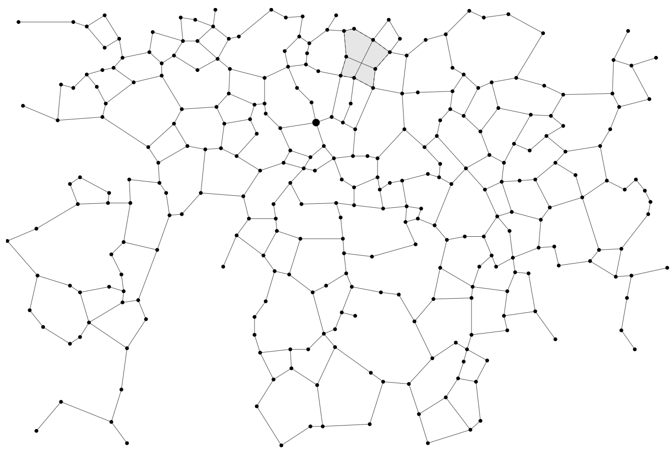
2.1 HSP test

Input: a vertex u of a geometric graph and a list L_1 of edges incident with v.
Output: A list of directed edges L_2 which are retained for the $\overrightarrow{HSP}(G)$ graph.

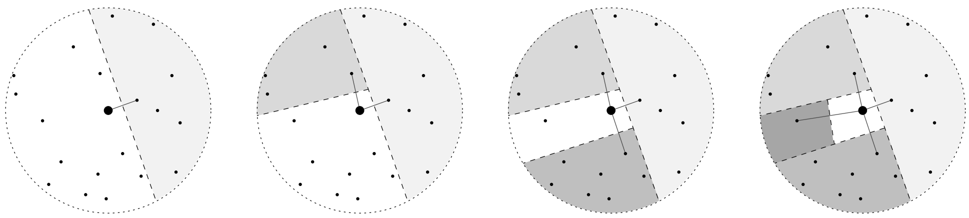
1. Set the forbidden area $F(u)$ to be $\oslash$.
2. Repeat the following while L_1 is not empty.
   (a) Remove from L_1 the shortest edge, say $[u, v]$, (any tie is broken by smaller end-vertex label) and insert in L_2 directed edge (u, v) with u being the initial vertex.
   (b) Add to $F(u)$ the open half-plane determined by the line perpendicular to the edge $[u, v]$ in the middle of the edge and containing the vertex v. (Notice that the points of the line do not belong to the forbidden area)
   (c) Scan the list L_1 and remove from it any edge whose end-vertex is in $F(u)$.



(a) Original set of points (note the zooming, below)



(b) The HSP spanner



(c) 1st neighbor (d) 2nd neighbor (e) 3rd neighbor (f) 4th neighbor

Fig. 1. Applying the HSP test to an UDG. The original set of points (a) and the resulting indirected graph (b). A zooming around the vicinity of a selected node (c) $\cdots$ (e). Notice the crossing (shaded area) in (b)

An illustration of the HSP test applied to an UDG is given in Figure 1, zooming is applied to a selected node and the forbidden area is shaded.

We stated the test using the forbidden half space due to its easy visualization. Computationally, the elimination of edge $[u, z]$ by an edge $[u.v]$ is done when the Euclidean distance from z to v is less than the Euclidean distance from z to u. Furthermore notice that the proximity test can be done without any explicit use of the coordinates, the test can be accomplished in any metric space. The HSP test is a local test since all we need to know in each vertex is the set of edges incident with it.

Definition 1. *Let G be a UDG with vertex set V. The oriented graph $\overrightarrow{HSP}$ (G) is defined to be the graph with vertex set V whose edges are obtained by applying the HSP test to each vertex in V. The undirected graph $HSP(G)$ is obtained from $\overrightarrow{HSP}(G)$ by omitting the directions of edges.*

Theorem 1. *If G is a connected UDG then the digraph $\overrightarrow{HSP}$ (G) has out-degree at most 6 and is strongly connected.*

Proof. Let u be a vertex of G and $[u, v]$ be an edge that is selected for $\overrightarrow{HSP}$ (G) by the above algorithm. The forbidden area generated by $[u, v]$ is a half-plane determined by the line perpendicular to the edge $[u, v]$ at the middle of the edge $[u, v]$. Furthermore, the next edge selected from u for $\overrightarrow{HSP}(G)$ cannot be shorter than $[u, v]$. This implies that the end-vertex v' of the next edge selected from u for $\overrightarrow{HSP}(G)$ is in the area outside the circle around u of radius equal to the length of $[u.v]$ and inside the half-plane containing u. This means that angle vuv' is at least $\pi/3$. Since the angle between any two edges selected from u for $\overrightarrow{HSP}(G)$ is at least $\pi/3$, the out-degree of u is at most 6. Notice that the degree 6 would be possible only if the selected edges form a regular hexagon.

We show the strong connectivity of $\overrightarrow{HSP}$ (G) by showing that if $[u, v]$ is an edge of G than there is a directed path from u to v in $\overrightarrow{HSP}$ (G). Assume that there exist edges in G such that there is no directed path between the end-vertices of the edges in $\overrightarrow{HSP}(G)$. Let $[u, v]$ be the shortest edge of G such that there is no directed path from u to v in $\overrightarrow{HSP}(G)$. According to the construction, one possibility for $[u, v]$ not being in $\overrightarrow{HSP}(G)$ is that there exists an edge $[u, z]$ in $\overrightarrow{HSP}(G)$ such that $[u, z]$ is of length shorter or equal to the length of $[u, v]$ and v is in the forbidden area generated by the edge $[u, z]$. This implies that the vertices u, v, z form a triangle with the angle vuz being at most $\pi/3$. Since G is a UDG and the distance between z and v is strictly less than the distance between u and v, edge $[z, v]$ is in G, and furthermore, there exists a directed path from z to v in $\overrightarrow{HSP}(G)$. Since $[u, z]$ is an edge in $\overrightarrow{HSP}(G)$, there is a directed path from u to v in $\overrightarrow{HSP}(G)$. □

One can lower the highest out-degree of any $HSP-$spanner to 5. As mentioned in the proof, the list L_2 contains 6 edges after the execution of the $HSP-$test on vertex v when v is the center of a regular hexagon. In this case we may always remove from the list the directed edge closest to, say the vertical line drawn through vertex v in the clockwise direction. It is easy to check that this results in a strongly connected spanner of degree at most 5. Since there is a sense of orientation for this edge deletion, this out-degree at most 5 spanner depends on the rotation of the graph. In cases when the degree reduction is more important, one can use the degree reduction to 5 using the above test.

One can ask whether or not the degree of the spanning subgraph could be further improved. The answer is negative, by considering a star graph of degree 5 in which all edges are of length 1 and the angles between two consecutive edges is $2\pi/5$. This is a UDG and the only spanning subgraph is equal to G and thus the degree is necessarily 5 in some cases.

Theorem 2. *Let G be a geometric UDG and $\overrightarrow{HSP}(G)$ be the digraph constructed from G by the above algorithm. Then the stretch factor of $\overrightarrow{HSP}(G)$ is at most $2\pi + 1$.*

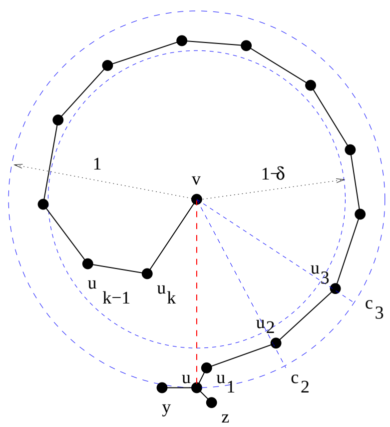


Fig. 2. Upper bound on the dilation of an edge.

Proof. Let u be a vertex of G and $[u, v]$ be an edge of G of length $r \leq 1$ such that the edge is not selected by u for $\overrightarrow{HSP}(G)$. Then there exist an edge $[u, u_1]$ in G which is selected by u for $\overrightarrow{HS}(G)$ such that $[u, u_1]$ is shorter than $[u, v]$ and the angle u_1uv is less than $\pi/2$. Thus the edge $[u, u_1]$ makes the vertex v to be in the forbidden area (see Figure 1). If the edge $[u_1, v]$ is in $\overrightarrow{HSP}(G)$ than the stretch factor is less than 3, else we can argue inductively that there exists a sequence of vertices $u_0 = u, u_1, u_2, u_3, \ldots, u_{k+1} = v$ such that (see Figure 2):

1. for every i, $0 \leq i \leq k$, there is an edge $[u_i, u_{i+1}]$ in $\overrightarrow{HS}(G)$,
2. for every i, $0 \leq i \leq k-1$, the length of $[u_i, u_{i+1}]$ is less than the Euclidean distance between u_i and v,
3. for every i, $0 \leq i \leq k-1$, the angle $u_{i+1}u_iv$ is less than $\pi/2$,
4. for every i, $0 \leq i \leq k-1$, the vertices $u_0, u_1, u_2, \ldots, u_k$ are in either clockwise or anticlockwise order around v,
5. for every i, $0 \leq i \leq k-1$, the Euclidean distance between u_{i+1} and v is smaller than the Euclidean distance between u_i and v,
6. the sum of the angles $\sum_{i=0}^{k} u_ivu_{i+1} < 2\pi$

The items 1, 2 and 3 are due to the fact that the edge $[u_i, v]$ is not chosen for $\overrightarrow{HSP}(G)$ by u_i. If the vertices $u_0, u_1, u_2, \ldots, u_k$ are not all in clockwise or anticlockwise order then let i be the index of a vertex such that both, u_1 and u_{i+1} are both say anticlockwise from the edge $[u_1, v]$. If the distance between u_{i-1}, u_{i+1} is not more than the distance between u_i, u_{i+1} then the edge $[u_{i-1}, u_{i+1}]$ exists in G since G is a UDG and we can argue that there is a path from u to v in $\overrightarrow{HSP}(G)$ that is even shorter than the sequence $u_0 = u, u_1, u_2, u_3, \ldots, u_{k+1} = v$. If the distance between u_{i-1}, u_{i+1} is greater than the distance between u_i, u_{i+1} then by considering the angles between $[u_i, u_{i+1}]$ and $[u_i, v]$ we can argue that there is a configuration of vertices that follows a clockwise path from u to v in $\overrightarrow{HSP}(G)$ that is even shorter than the sequence $u_0 = u, u_1, u_2, u_3, \ldots, u_{k+1} = v$. The item 5 follows directly from item 3.

If $\sum_{i=0}^{k} u_ivu_{i+1} \geq 2\pi$ then there exist integers i and j, $0 \leq i < j-1$ such that the vertex u_j is in the triangle u_i, v, u_{i+1}. Since G is a UDG, either there is a path from u to v in $\overrightarrow{HSP}(G)$ that omits some of the edges between u_{i+1} and u_j and hence is shorter than the sequence $u_0 = u, u_1, u_2, u_3, \ldots, u_{k+1} = v$, or the vertices of the path must be inside a circle of radius $[u_[i+1], u_j]$, also leading to a shorter path.

Consider the circle with center v of diameter r and denote by c_i the point of intersection of the line segment from v through vertex u_i (see Figure 2). By the triangular inequality, the Euclidean length of the edge $[u_i, u_{i+1}]$ is bounded from above by $s_i + r_i - r_{i+1}$ where s_i denotes the length of the circle segment from c_i to c_{i+1} and r_i denotes the Euclidean distances between u_i and v. Thus the Euclidean length of the path specified by the sequence $u_0 = u, u_1, u_2, u_3, \ldots, u_{k+1} = v$ is at most $\sum_{i=0}^{k} s_i + r_i - r_{i+1} \leq 2\pi r + r$. Thus the stretch factor is at most $2\pi + 1$. □

The dilation given in the theorem is an upper bound on the maximal dilation of an edge in an HSP spanner. A lower bound on the maximal dilation can be obtained from graph G in Figure 3. Consider edge $[u, v]$ in this graph. Due to either edge $[u, u_1]$ or the distance from u, there is no edge from u to $v, u_2, u_3, \ldots, u_k$ in the HSP spanner of G. Thus the path length from u to v in the HSP spanner of the graph is at least $5\pi/3(1-\delta) + (1-\delta) = 5\pi/3 + 1 - \delta(5\pi/3 + 1)$. Since δ can be an arbitrarily small positive number, the dilation can be arbitrarily close to $5\pi/3 + 1$. Thus $5\pi/3 + 1$ gives a lower bound on the maximum dilation of an edge in an HSP spanner. We conjecture that the maximum dilation of an HSP spanner is close to this lower bound.

For a comparison, the upper bound of $\frac{1}{1-2\sin\pi/k}$ of Yao spanners is valid for $k \geq 7$ and this bound is larger than $2\pi + 1$ when $k = 7$. It is clear that the configuration in Figure 3 is unlikely to occur in a UDG that represents an ad-hoc network and thus the dilation of the HSP spanner of a UDG graph corresponding to an ad-hoc network should be substantially smaller.

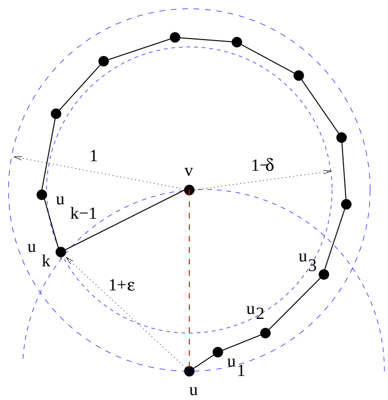


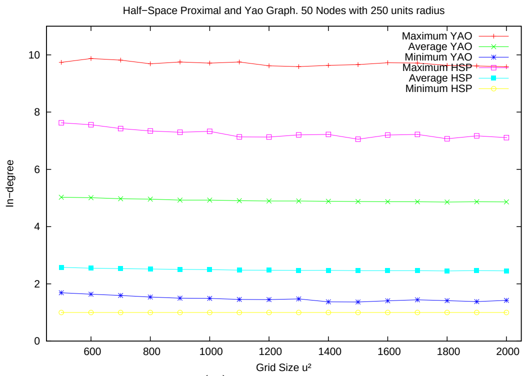
Fig. 3. HSP-spanner with edge dilation $5\pi/3 + 1$.

Theorem 3. *If G is a connected unit disk graph then a geometric minimum spanning tree of G is a subgraph of $HSP(G)$.*

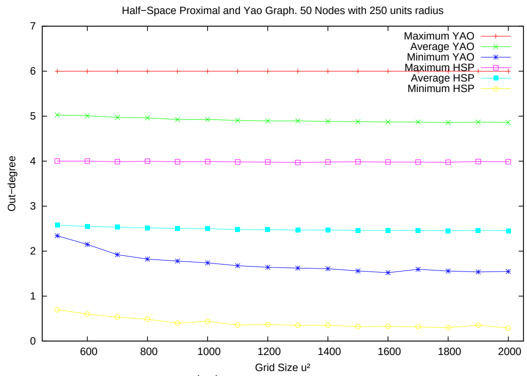
Proof. Let G be a geometric unit disk graph and T be the geometric minimum spanning tree of G that contains the maximum number of edges of $HSP(G)$. Assume that there is an edge $[u, v] \in T$ which is not in $HSP(G)$. Since the edge $[u, v]$ is not in $HSP(G)$, there exist an edge $[u, w]$ in $HSP(G)$ and either $[u, w]$ and $[w, v]$ are shorter than $[u, v]$ or $d_G[u, v] = d_G[u, w]$ and $[w, v]$ is shorter than $[u, v]$. Clearly, for one of u and v there is a path p to w in T that does not contain edge $[u.v]$. If such a path exists from v then removing $[u, v]$ from T and adding $[u, w]$ we obtain a spanning tree of the same or lower cost containing one more edge of $HSP(G)$, a contradiction. If such a path p exists from u then removing $[u, v]$ from T and adding edge $[w, v]$ instead we obtain a spanning tree of lower cost, a contradiction. □

It should be noted that, like in a Yao graph, the in-degree of $\overrightarrow{HSP}(G)$ is not bounded by any constant, and $\overrightarrow{HSP}(G)$ is not necessarily a planar graph (see Figure 1(f), shaded area). If a low in-degree is needed, one can apply to HSP spanners the technique from [1] that has been used to lower the in-degree of Yao graphs.

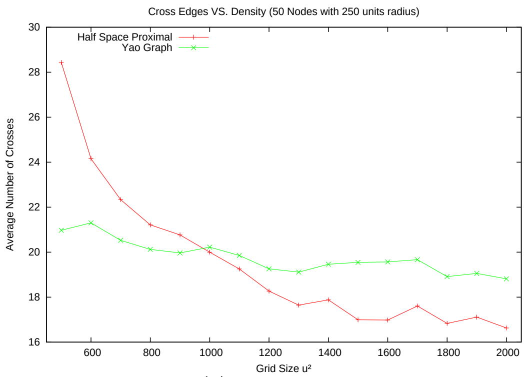
3 Experimental Results



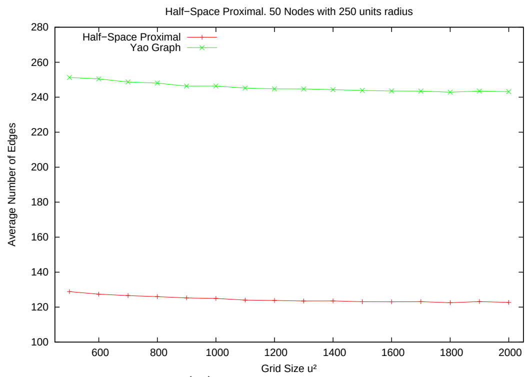
(a) In-degree



(b) Out-degree



(c) Crossings



(d) Total weight

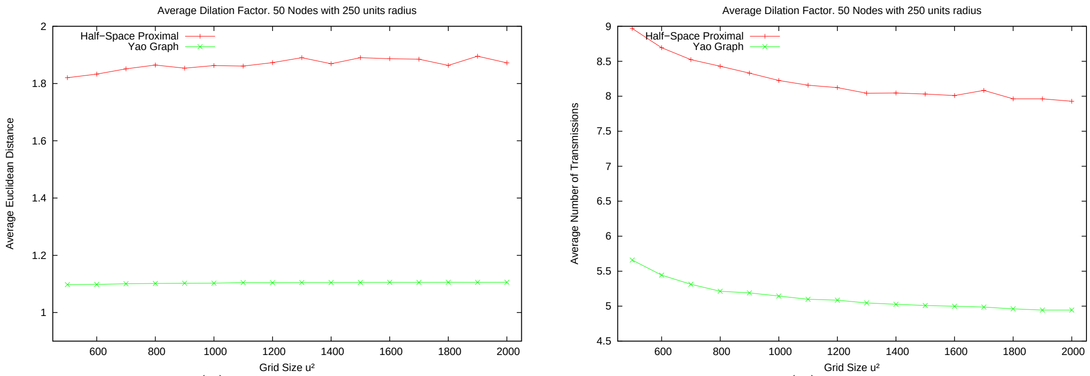
Fig. 4. We performed some experiments to compare the Yao graph, in several parameters, against the HSP graph. The in-degree and out-degree of the HSP graph is smaller for the HSP for minimum, average and maximum, (a) and (b). The number of crossings for HSP spanner is smaller than Yao spanner for more sparse graphs, and for denser graphs is exactly the reverse. For dense graphs the HSP spanner have more crosses (c). The total weigth of the HSP spanner is about half the total weigth of the Yao spanner, consistently through the density (d).

In our experiments we used a UDG with 50 nodes randomly placed in a grid area of size varying between 500 and 2000 units. The transmission radius is 250 units in all of these graphs. Thus, as the grid size becomes larger the UDG density gets smaller. For each unit disk graph HSP spanner and YAO spanner (with 6 cones) were generated both directed and undirected versions.

We measured the dependence of the following parameters of the HSP and Yao spanners on the density of UDG:

1. Minimum, maximum and average in-degree,
2. Minimum, maximum ans average out-degree,
3. The number of edges that cross each other,
4. The total weight of spanners,
5. Average Euclidean distance in the spanner,
6. The average number of hops.

See the results of experiments in Figures 3,5(a) and 5(b). The in-degree and out-degree of HSP spanners is lower that those of Yao spanners and so is the total weight of the spanners. As far as the number of crossing edges is concerned, it is higher for HSP spanners when the density is higher, but is lower for smaller densities. Due to the significantly lower in and out-degrees of HSP spanners, the average distances in HSP spanners are higher.



(a) Euclidean (b) Hops

Fig. 5. The average dilation as a function of the density

4 Conclusion

The HSP test proposed in this paper is a distributed test that gives a $(2\pi + 1)-$spanner of a UDG. The computation of the test is simple, it obviously generalizes to any metric space, and the spanner obtained by the test is independent of the exact placement of the graph in the plane. The experiments on random unit disk graphs show that the average in-degree of the spanner is very low and the number of edges that cross each other is very low for small densities. The total weight is small and a high in-degree in unlikely to occur.

Thus, HSP spanner could be very convenient in network applications where the use of spanners having these properties is needed. At present, we are investigating a generalization of the HSP test to ad-hoc networks with irregular transmission ranges [2].

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