Halfspace depth and floating body

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Abstract: Little known relations of the renown concept of the halfspace depth for multivariate data with notions from convex and affine geometry are discussed. Maximum halfspace depth may be regarded as a measure of symmetry for random vectors. As such, the maximum depth stands as a generalization of a measure of symmetry for convex sets, well studied in geometry. Under a mild assumption, the upper level sets of the halfspace depth coincide with the convex floating bodies of measures used in the definition of the affine surface area for convex bodies in Euclidean spaces. These connections enable us to partially resolve some persistent open problems regarding theoretical properties of the depth.

Contents

1. Introduction

Halfspace depth and floating body are the same concept. The first is extensively studied in nonparametric statistics, the second is of great importance in convex

geometry. Until recently, the convex geometry community has not been aware of the work on data depth, and not too many researchers from the statistics community were familiar with the work on floating bodies. Of course, the motivation and the goals in both fields are different, and even their philosophies are not the same. Nonetheless, there is an abundance of results common to both directions of research. We want to explore what is common to both fields, what is known and what is not known — to build a bridge between the two disciplines of mathematics, and link these two separate worlds.

In nonparametric statistics, data depth is a generalization of order statistics and ranks to multivariate random variables. One of its aims is, for a multivariate probability distribution, to devise a distribution-specific ranking of points in the sample space. In particular, depth allows to distinguish points that fit the overall pattern of the distribution, from measurement errors and other outliers. More generally, depth is a tool that enables efficient description, visualization, and nonparametric statistical inference for multivariate data.

In convex geometry, the concept of floating body was used, among other things, to introduce the affine surface area to all convex bodies. The associated affine isoperimetric inequality is much stronger than the classical isoperimetric inequality. It provides solutions to many problems where ellipsoids are extrema.

2. Motivation and background

In classical statistics of univariate data it is well known that the ordering of data points and the corresponding rank statistics constitute powerful statistical tools, valid under very broad sets of assumptions. The median, for instance, is a rather efficient, robust, affine equivariant location estimator. Quantiles are invaluable in both visualization and inference. Rank tests provide versatile analogues to the traditional testing procedures, and unlike many standard parametric statistical tests, work under minimal assumptions imposed on the data.

In multivariate spaces, however, no natural ordering exists. For $d > 1$, we are not able to rank points x and y in $\mathbb{R}^d$ according to their magnitude, or tell whether x lies "to the left" of y. Though, for a given dataset, one can still ask how well a point fits into the overall pattern of the observations. If the data concentrate around a focal point, and follow a simple scatter structure, we can say that a point x inside the data cloud is "deeper" inside the mass of the data, than a point y that lies on the outskirts, or outside the data cloud. This notion of multivariate center-outwards ranking is formalized by the idea of data depth. In general, a depth $D(x; P)$ is a function that, given a probability distribution P on $\mathbb{R}^d$ (or an empirical measure that corresponds to a random sample from this distribution), quantifies the centrality (the depth) of a point x with respect to (w.r.t.) the geometry of P. The more x is inside the main bulk of the mass of P, the higher the depth $D(x; P)$. As such, the depth enables us to rank the points of $\mathbb{R}^d$ according to their centrality w.r.t. P, and devise the corresponding depth-rank statistics, or depth-based quantile regions.

Let us illustrate our point by giving two simple examples where the depth plays an instrumental role. In the first one, our search for sensible data ordering
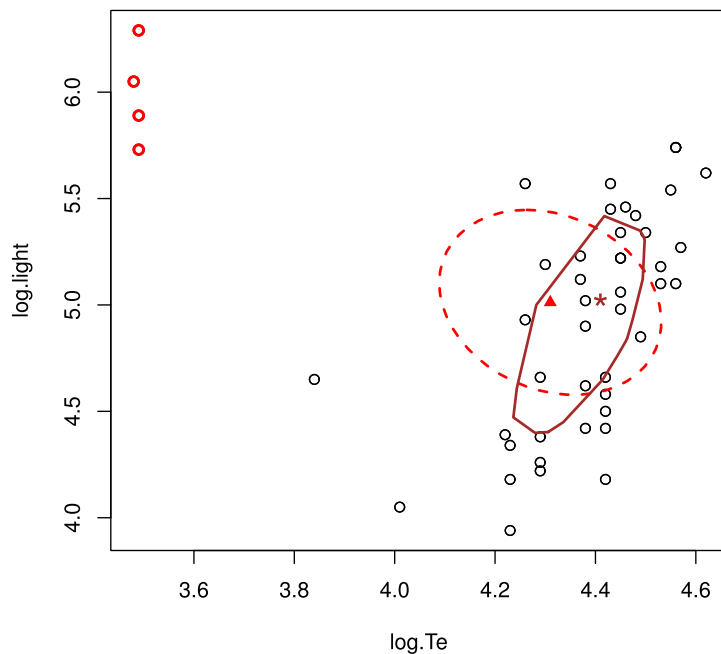
FIG 1. *The Hertzsprung-Russell diagram of the Star Cluster CYG OB1. The four giant stars (red points) attract both the sample mean (red triangle), and the sample covariance (represented by the red ellipse). The halfspace depth-based median (brown star), and the depth-based central region containing 25 % of the observations (brown polygon), provide a more appropriate representation of the location and the variability of the main data cloud.*

is motivated by the problem to define a multivariate analogue of the median. In the classification task presented afterwards, we stress the importance of general global ranking procedures in data science.

Consider the dataset of 47 bivariate observations taken from [159], displayed in Figure 1. The data correspond to the Hertzsprung-Russell diagram of the stars in the Star Cluster CYG OB1 in the Cygnus constellation. In the scatterplot, the logarithm of the effective temperature at the surface of the star (`log.Te`) is plotted, against the logarithm of its light intensity (`log.light`). The majority of the observations follows a common pattern — their data points concentrate in the south-east part of the plot, and appear to be scattered rather regularly. Four stars clearly do not follow that pattern, and could be considered as outliers (the red points in Figure 1). Those are known to be stars of different characteristics (so-called giant stars). Let us determine the location of the random sample. The sample mean (red triangle) is attracted towards the outlying observations, and does not represent the location of the majority of the data appropriately. That is, of course, caused by the fact that the expectation is known to be affected severely by erroneous data, and outliers, i.e., it is not robust. For univariate data, one can opt for the median in such situations. But, what is a median of

a multivariate dataset? Intuitively, the median should capture the location of the majority of observations, and should be little affected by errors, or other anomalies in the data. The median should be a point "deep" inside the data cloud. With the notion of the halfspace depth (the precise definition is given below), we consider the depth median being the point whose depth w.r.t. the random sample is the highest (brown star in Figure 1). The depth median is robust, i.e. it is much less affected by the four giant stars than the mean. It captures the center of the main bulk of data much better then the sample mean. Additionally, let us consider the Mahalanobis ellipse (for precise definition see (6) below) that corresponds to the sample mean, and the sample covariance of our dataset, and contains 25 % of the data points (red ellipse in Figure 1). This ellipse is intended to represent the scatter pattern of the data. As seen in Figure 1, it is also heavily biased towards the anomalous observations. On the other hand, the halfspace depth region that contains (roughly) 25 % of the deepest points (brown polygon), still represents the main modes of variation of the data quite reliably. Thus, both the halfspace median and the inner halfspace depth regions possess the property of robustness against measurement errors and outliers. The idea of robustness is of great importance in statistical inference, and a large body of literature on these topics exists; we mention monographs [76, 83, 122] and the references therein.
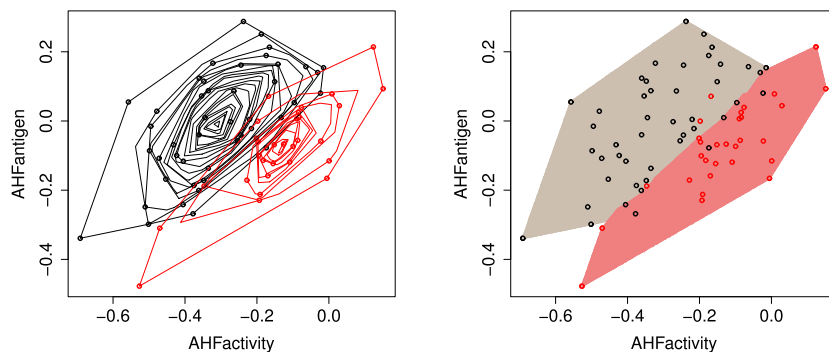
Fig 2. *Hemophilia data. The group of carriers of Hemophilia A (black points) is to be separated from the group of non-carriers (red points). Left panel: halfspace depth contours of the two groups; right panel: regions where one of the depths dominates the other — a new observation inside the light-gray polygon is classified into the black group, an observation inside the light-red polygon into the red group. Observations outside the polygons remain unclassified.*

For our second motivating example, the `hemophilia` data (available in [149]) are visualized in Figure 2. The dataset consists of bivariate measurements (AHF activity and AHF antigen) taken from blood samples of 75 women, out of whom 45 are known to be hemophilia A carriers (black dots on Figure 2). Our task here is classification — given a new datum with the two measured characteristics, decide whether the new patient is a potential hemophilia A carrier. The literature where problems of this type are studied in statistics is immense. One approach to this problem is to make use of the depth, and the ranking of the

observations. Firstly, compute the depth of the point w.r.t. both random samples, i.e. rank the new data point inside the group of carriers, and the group of non-carriers, respectively. Then, assign the new datum to that group for which it is more typical, reflected by its higher depth. Figure 2 shows the contours of the halfspace depth functions for both random samples (left panel), and the regions where one of the depths is larger than the other (the polygons on the right hand side). A new observation within the light-gray polygon would be assigned into the black cloud of points (carriers); a point inside the light-red polygon is assessed to come from a non-carrier patient[1]. This approach, sometimes called the maximum depth classification [62] and its other variants based on the depth such as the depth-depth classifiers [100], turned out be particularly appealing in the past years. Mainly due to their conceptual simplicity, versatility, and good robustness properties, depth-based classification rules have gained great importance over the past decade.

As we saw, data depth introduces ranking and ordering also for multivariate datasets. Other applications of the depth include multivariate extensions of the rank tests, L-statistics (linear combinations of order statistics), and many other nonparametric and robust procedures.

Let us now provide a rigorous definition of the halfspace depth. For $d \geq 1$, a point x in d-dimensional Euclidean space $\mathbb{R}^d$, and a probability distribution P on $\mathbb{R}^d$, the halfspace depth hD of x with respect to P is given by

$$hD(x; P) = \inf \left\{ P(H^-) : \ H \text{ is a hyperplane with } x \in H \right\},$$

where H^- denotes any of the two closed halfspaces associated with its boundary hyperplane H in $\mathbb{R}^d$. In other words, the depth hD is given as the smallest probability of a closed halfspace that contains x. Points outside the convex hull of the support of P have zero depth. More generally, any point x with $hD(x; P) < \delta$ can be separated from the main mass of P by a hyperplane cutting away both the point x, and a mass of probability at most δ. Points with rather high depth values can be seen as those lying at the center of the distribution, as no halfspace of small probability can separate them from the rest of P. This way, the depth hD acts as a mapping that orders the sample points in a center-outwards direction, with the ordering given subject to the distribution P.

Note that for a random sample $X_1, \ldots, X_n$ of n points in $\mathbb{R}^d$, the halfspace depth hD is usually computed with P the empirical measure of the observations, that is w.r.t. the empirical measure P_n that places a point mass of probability $1/n$ to each observation X_i.

One plausible statistical application of the depth is the possibility to introduce quantiles to multivariate data. Consider, for observations on the real line $\mathbb{R}$, the central quantile regions given as the intervals bounded by the α-, and

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[1]Note that there are points that remain unclassified. For instance, points outside both convex hulls of the sample points are not assigned to any of the clusters using our simple classification rule. More advanced depth-based techniques dealing with these problems are available in the literature; see, e.g., [97].

$(1-\alpha)$-quantiles of P, for $\alpha \in (0, 1/2]$. A natural multivariate analogue of these sets are the loci of points whose depth exceeds given thresholds. Such sets of points are called the *central regions* of P in $\mathbb{R}^d$ (given by the depth hD). As discussed in Section 3 below, the collection of all central regions of P consists of affine equivariant closed convex sets, monotone in the sense of set inclusion, see also the left panel of Figure 2, or Figure 3 below. For many distributions, the smallest non-empty set in this collection is a singleton — the most central point of $\mathbb{R}^d$ for P. This point is frequently recognized as a generalization of the median to $\mathbb{R}^d$-valued data.

The earliest contribution in statistics that deals with some form of the halfspace depth is believed to be [86] from 1955. There, a sign test for bivariate data is proposed and examined. Its test statistic takes the form of the depth hD at a single, given point x.

The seminal paper that introduced the depth in the sample case (that is, for datasets) is Tukey [186] from 1975. In that paper, the depth is proposed as a tool that enables efficient visualization of random samples in $\mathbb{R}^2$. It is in [186], where the word *depth* is used for the first time. The original formal definition of the halfspace depth for multivariate data can be found in Donoho [44] and Donoho and Gasko [45] (see also [177]).

Starting with the study of Donoho, much research has focused on data depth and related concepts. The prominent, loosely related simplicial depth for multivariate data was defined by Liu [102, 103], building upon the ideas presented in Oja [144]. The zonoid depth, based on the concept of the lift zonoid of a measure, was proposed in [93, 94]. Interestingly, just like floating bodies, also the theory of lift zonoids stands at the border of convex geometry and multivariate statistics; for an extensive study of lift zonoids and the associated depth see the monograph of Mosler [136], and [165, Section 3.5]. The idea of depth was also extended to data in non-linear spaces [177, 106], general metric spaces [35], observations on graphs [178], regression [157], or data taking values in functional spaces [56], and Banach spaces [43].

The general concept of data depth in $\mathbb{R}^d$ was formalized by Zuo and Serfling [196], Dyckerhoff [49], and Serfling [175], see also Mosler [137]. Nowadays, dozens of depth functions and related methods for all types of data can be found in the literature. It is, however, the halfspace depth hD, that is the single most important depth that continues to reappear as the prime representative of this idea.

Apart from statistics, halfspace depth gained considerable attention also in discrete and computational geometry (see [105]). There, the combinatorial nature of the sample version of hD provides a rich source of interesting problems, especially in connection with its computational aspects. For instance, the halfspace medians, i.e. points at which hD is maximized over $\mathbb{R}^d$, are closely related to the notion of *centerpoints* studied in discrete geometry (see [127, Section 1.4]). For a recent overview of data depth and its links to computational geometry see Rousseeuw and Hubert [158].

A notable article on the properties of the halfspace depth for general (probability) measures is Rousseeuw and Ruts [160]. Several interesting links between

the halfspace depth, and some sources outside mathematical statistics are outlined. In [160, Section 8] it is noted that the halfspace depth relates with the voting problem studied in Caplin and Nalebuff [33, 34] in the theory of social choice. Further, it is also observed that some results concerning the maximal depth of a point in $\mathbb{R}^d$ can be found already in Neumann [141], Rado [150], and Grünbaum [70], in the literature concerned with geometric properties of functions and sets.

In the present paper, we pursue this line of research, and point to the remarkable similarity of the notion of halfspace depth, and some concepts used in other fields of mathematics, especially in convex and affine geometry. For $\delta \geq 0$ given, the convex floating body of a (finite) Borel measure P on $\mathbb{R}^d$ is defined as the intersection of all closed halfspaces whose defining hyperplanes cut off sets of P-measure at most δ. Floating bodies are of great importance in convex geometry. It turns out that under a mild assumption on the support of P, the convex floating body of P is the same set as the halfspace depth central region of P of points whose depth is at least δ. Many implications of this observation will be explored in a considerable part of this article.

The paper is organized as follows. In Section 3 we introduce the notation, and give a brief overview of some of the most important properties of the halfspace depth. In Section 4 we follow the lead provided by Rousseeuw and Ruts [160], and trace a little known early precursor of the halfspace depth to be the so-called Winternitz measure of symmetry of convex bodies, a functional that dates back at least to Blaschke [17]. In Sections 5 and 6 we examine relations of the halfspace depth with the (convex) floating bodies. As demonstrated, the history of the halfspace depth is much longer than assumed: the earliest predecessors of the depth hD appear to be the floating bodies in $\mathbb{R}^d$, studied already by Dupin [47] in 1822. Later, floating bodies reappear in mathematics in 1923 in Blaschke [17], in connection with an affine invariant, the affine surface area of convex bodies, and other problems. As discussed in Section 5, the modern notion of the floating body, the convex floating body, defined independently by Bárány and Larman [12] and Schütt and Werner [170], plays a major role for the concept of affine surface area studied in geometry. We present extensions of this notion to log-concave measures and show its importance in questions of approximation of convex bodies by polytopes. It is also discussed in Section 6 that the convex floating body corresponds to the upper level sets of the halfspace depth as indicated above. Using this identity, we provide in Section 7 a surprising bound of the halfspace depth in terms of the Mahalanobis depth. Section 8 is devoted to the distribution-by-depth characterization problem, concerned with finding conditions under which no two probability measures can have the same depth on $\mathbb{R}^d$. It is shown that two important partial positive results to this problem [78, 91] are both special cases of a more general theorem, conveniently stated in terms of floating bodies of measures. As a corollary, we obtain some new classes of distributions characterized by their depth. Finally, in Section 9 we discuss some extensions of the depth to more exotic data. The survey is completed with a series of open problems relevant to the topics of halfspace depth and floating bodies.

3. Data depth: Notation and essential properties

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the probability space on which all random variables are defined. For a topological space $\mathcal{M}$, denote by $\mathcal{P}(\mathcal{M})$ the space of all Borel probability distributions on $\mathcal{M}$, and write $X \sim P$ for a random variable X with distribution $P \in \mathcal{P}(\mathcal{M})$. The support of P is denoted by $\mathrm{Supp}(P) \subseteq \mathcal{M}$. For $n \in \mathbb{N} = \{1, 2, \dots\}$ and a random sample $X_1, X_2, \dots, X_n$ from P, let $P_n \in \mathcal{P}(\mathcal{M})$ be the associated empirical measure, i.e. the uniform measure supported in the sample points. For $X \sim P$ and $\phi \colon \mathcal{M} \to \mathcal{M}$ measurable, write $P_{\phi(X)} \in \mathcal{P}(\mathcal{M})$ for the probability distribution of the transformed random variable $\phi(X)$. This way, $P \equiv P_X$.

The space $\mathbb{R}^d$ is equipped with the Euclidean norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. For $x \in \mathbb{R}^d$ and $r > 0$,

$$B^d(x, r) = \left\{ y \in \mathbb{R}^d \colon \|y - x\| \le r \right\}$$

is the closed Euclidean ball centered at x with radius r. B^d stands for the unit ball $B^d(0, 1)$ and $\mathbb{S}^{d-1} = \partial B^d$ denotes the unit sphere. ∂K stands for the topological boundary of $K \subset \mathbb{R}^d$. The Lebesgue measure of a measurable set K will be denoted also by $\mathrm{vol}_d(K)$.

A convex body is a convex, compact subset of $\mathbb{R}^d$ with non-empty interior. For $k \in \mathbb{N}$ it is said to have a C^k boundary if its boundary, locally parametrized as a function from $\mathbb{R}^{d-1}$, is k-times continuously differentiable. We denote the collection of all convex bodies in $\mathbb{R}^d$ by $\mathcal{K}^d$. For an interior point x_0 of a convex body K define the polar body K^{x_0} of K w.r.t. x_0 by

$$K^{x_0} = \left\{ y \in \mathbb{R}^d \colon \langle y, x - x_0 \rangle \le 1 \text{ for all } x \in K \right\}. \tag{1}$$

If $0 \in \mathrm{Int}(K)$, the interior of K, we write K° for the polar body (1) of K w.r.t. 0. A star body K is a compact subset of $\mathbb{R}^d$ with the property that there exists $x \in K$ such that the open line segment from x to any point $y \in K$ is contained in the interior of K. The Minkowski addition of $K, L \in \mathcal{K}^d$ is $K + L = \{x + y \colon x \in K, y \in L\}$. Likewise, for $\lambda \in \mathbb{R}$, $\lambda K = \{\lambda x \colon x \in K\}$.

We write

$$H_{u, \alpha} = \left\{ z \in \mathbb{R}^d \colon \langle z, u \rangle = \alpha \right\} \quad \text{for } u \in \mathbb{R}^d \setminus \{0\}, \alpha \in \mathbb{R},$$

for a hyperplane in $\mathbb{R}^d$, and denote for $H = H_{u, \alpha}$ the two closed halfspaces bounded by this hyperplane by $H^- = H^-_{u, \alpha}$ and $H^+ = H^+_{u, \alpha}$, respectively. By $\mathcal{H}$ we denote the set of all hyperplanes, and by $\mathcal{H}^-$ the set of all closed halfspaces in $\mathbb{R}^d$.

We say that the halfspace H^+ supports the set $K \subset \mathbb{R}^d$ if $K \cap H^- \ne \emptyset$ and $K \subseteq H^+$. A hyperplane $H \in \mathcal{H}$ is said to support K if either H^- or H^+ supports K. Recall that the boundary of a convex body $K \in \mathcal{K}^d$ is C^1 if and only if for any $x \in \partial K$ there exists a unique hyperplane $H \in \mathcal{H}$ that supports K with $x \in H$ [165, Theorem 2.2.4]. For a convex set $K \subset \mathbb{R}^d$, the support

function of K is defined as

$$h_K \colon \mathbb{S}^{d-1} \to \mathbb{R} \colon u \mapsto \sup \{\langle x, u \rangle \colon x \in K\}. \tag{2}$$

The centroid (or the barycenter) of a compact set $K \subset \mathbb{R}^d$ is the expectation of a random variable distributed uniformly on K.

3.1. Statistical depth for multivariate data

The first formal definition of the halfspace depth in $\mathbb{R}^d$ for general probability distributions can be found in Donoho [44].

Definition. Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ and $x \in \mathbb{R}^d$. The *halfspace depth* (or *Tukey depth*) of x w.r.t. P is defined as

$$hD\left(x; P\right) = \inf \left\{ P(H^-) \colon H \in \mathcal{H}, x \in H^- \right\}. \tag{3}$$

In (3), the infimum can be equally well taken only over those $H \in \mathcal{H}$ with $x \in H$ [160, Proposition 3].

In the study of the theoretical properties of hD, two regularity conditions imposed on $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ frequently play an important role. The first, a smoothness condition, appears in Dümbgen [46] and Mizera and Volauf [134], and reads

$$P(H) = 0 \text{ for all } H \in \mathcal{H}. \tag{4}$$

It is trivially satisfied if, for instance, P has a density (w.r.t. the d-dimensional Lebesgue measure) in $\mathbb{R}^d$.

The second requirement concerns the support of P. We say that $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ has *contiguous support* [134, 91] if there are no two disjoint halfspaces $H_1^-, H_2^- \in \mathcal{H}^-$ such that $P(H_1^-) > 0$ and $P(H_2^-) > 0$, but $P(H_1^-) + P(H_2^-) = 1$. In other words, the support of P cannot be separated by a slab between two parallel hyperplanes.

In Section 7 we demonstrate a surprising relation between the halfspace depth, and another renown depth function that can be found in the literature: the Mahalanobis depth. To this end, let us briefly recall its definition, and some elementary properties.

For any symmetric positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$, the *Mahalanobis distance* [119] of two points $x, y \in \mathbb{R}^d$ is defined as

$$\mathrm{d}_\Sigma(x, y) = \sqrt{(x - y)^\mathsf{T} \Sigma^{-1} (x - y)}. \tag{5}$$

It is a metric on $\mathbb{R}^d$. Based on this distance, Liu [104] proposed the following depth function.

Definition. Let $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be such that $\mathrm{E}\, X = \mu$ and $\mathrm{Var}\, X = \Sigma$ is positive definite. The *Mahalanobis depth* of x w.r.t. P is defined as

$$MD(x; P) = (1 + \mathrm{d}_\Sigma(x, \mu))^{-1}.$$

The Mahalanobis depth w.r.t. P takes the maximal value 1 at the expectation of P. Its upper level sets are concentric ellipsoids given, for $\delta \in (0, 1]$, by

$$\left\{x \in \mathbb{R}^d \colon MD(x; P) \geq \delta\right\} = \left\{x \in \mathbb{R}^d \colon \sqrt{(x - \mu)^\mathsf{T} \Sigma^{-1} (x - \mu)} \leq \frac{1 - \delta}{\delta}\right\}. \quad (6)$$

These ellipsoids are also called the *Mahalanobis ellipsoids* of the distribution P. Note that unlike the halfspace depth hD, the Mahalanobis depth MD is not defined for all $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, but rather it is restricted to distributions with finite second moments, and positive definite variance matrices.

3.2. *Properties of the halfspace depth*

In this section we collect some basic properties of the halfspace depth (3), and of its upper level sets, that will prove to be useful in the sequel.

For any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, consider the upper level sets of hD

$$P_\delta = \left\{y \in \mathbb{R}^d \colon hD(y; P) \geq \delta\right\} \quad \text{for } \delta \in [0, 1]. \quad (7)$$

Immediately from the definition we see that the collection of sets P_δ, $\delta \in [0, 1]$ is nested, that is decreasing in the sense of set inclusion, and $P_0 = \mathbb{R}^d$. The set P_δ is also called the *central region* of P corresponding to $\delta \in [0, 1]$.

Example 1. Let $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be the uniform distribution on the unit ball B^d. The marginal distribution function of the first coordinate of X is given by

$$F_1(s) = \begin{cases} \frac{\Gamma((d+2)/2)}{\Gamma((d+1)/2)\sqrt{\pi}} \int_{-1}^s \left(1 - t^2\right)^{(d-1)/2} \mathrm{d}\,t & \text{for } s \in (-1, 1), \\ 0 & \text{for } s \leq -1, \\ 1 & \text{for } s \geq 1. \end{cases}$$

It is not difficult to see that

$$hD(x; P) = F_1\left(-\|x\|\right) \quad \text{for all } x \in \mathbb{R}^d,$$

i.e. the central region P_δ of P is a ball with radius $-F_1^{-1}(\delta)$ for $\delta \in [0, 1/2]$ and F_1^{-1} the quantile function corresponding to F_1. For $d = 2$ we obtain

$$hD(x; P) = \begin{cases} \frac{1}{2} - \frac{1}{\pi} \left(\arcsin(\|x\|) + \|x\|\sqrt{1 - \|x\|^2}\right) & \text{if } \|x\| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

which agrees with Rousseeuw and Ruts [160, Section 5.6]. Uniform distributions on balls are a special case of spherically (and elliptically) symmetric distributions. Such distributions will be treated in Example 2 below.

For the uniform distribution $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ on the unit square $[0, 1]^2$, the halfspace depth and its central regions were computed by Rousseeuw and Ruts [160,

Section 5.4]

$$hD(x; P) = \begin{cases} 2 \min\{x_1, 1 - x_1\} \min\{x_2, 1 - x_2\} & \text{for } x = (x_1, x_2) \in [0, 1]^2, \\ 0 & \text{otherwise.} \end{cases}$$

The expression for the halfspace depth of $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ distributed uniformly on the equilateral triangle can be found in Rousseeuw and Ruts [160, Section 5.3]. Several central regions (7) of the halfspace depth for the latter two distributions centered at their halfspace medians are displayed in Figure 3. Exact expressions for the halfspace depth for the uniform distribution on a simplex, and a (hyper)-cube in $\mathbb{R}^d$ are much more involved for $d > 2$ than for $d = 2$. They can be obtained from [167, Lemma 1.3 and its proof].
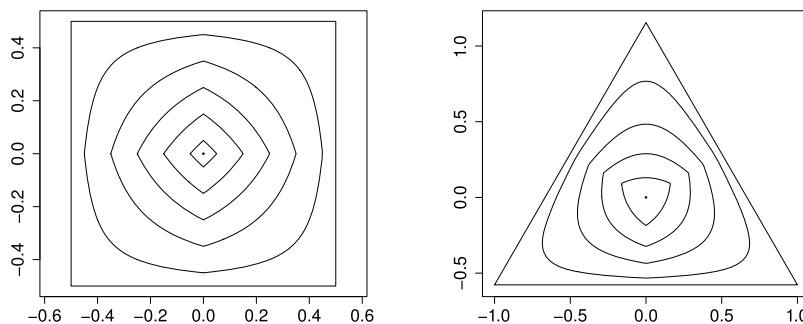
FIG 3. *Halfspace depth central regions* (7) *for* $\delta \in \{0.05, 0.15, 0.25, 0.35, 0.45\}$ *for the uniform distribution on the unit square (left panel), and an equilateral triangle (right panel). The black dots stand for the halfspace medians of these two distributions. Note that the central region for* $\delta = 0.45$ *is empty for* P *uniform on a triangle. In that case we have maximum depth* $\Pi(P) = 4/9 < 0.45$.

Example 2. In accordance with Fang, Kotz and Ng [53] we say that the distribution of $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is α-*symmetric*, $0 < \alpha \leq 2$, if for some continuous function $\phi \colon \mathbb{R} \to \mathbb{R}$ the characteristic function of the random vector X takes the form
$$\psi_X(t) = \mathrm{E}\, e^{\mathrm{i}\langle t, X \rangle} = \phi\left(\|t\|_\alpha\right) \quad \text{for all } t \in \mathbb{R}^d,$$
where i is the imaginary unit, and for $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ we set $\|t\|_\alpha = \left(\sum_{i=1}^d |t_i|^\alpha\right)^{1/\alpha}$ for $0 < \alpha < \infty$, and $\|t\|_\infty = \max_{i=1,\ldots,d} |t_i|$. For $\alpha = 2$ we obtain the collection of all spherically symmetric distributions, i.e. distributions invariant under all orthonormal rotations of the sample space [53, Chapter 2]. For instance, the uniform distribution on the unit ball B^d, the uniform distribution on the unit sphere $\mathbb{S}^{d-1}$, or the standard multivariate Gaussian distribution are all spherically symmetric. The multivariate probability distribution with independent Cauchy marginals is 1-symmetric. For $\phi(s) = e^{-s^\alpha}$ with $\alpha \in (0, 2]$ we obtain the multivariate symmetric stable laws.

α-symmetric distributions have been studied by many authors [32, 53, 89]. They are distinguished by the special property that all univariate projections of an α-symmetric measure $X = (X_1, \ldots, X_d) \sim P$ are multiples of the same univariate distribution

$$\langle u, X \rangle \stackrel{d}{=} \|u\|_\alpha X_1 \quad \text{for all } u \in \mathbb{R}^d,$$

where $\stackrel{d}{=}$ stands for "is equal in distribution" [53, Theorem 7.1]. This makes it possible to compute the depth $hD(\cdot; P)$ exactly. For an α-symmetric $P \in \mathcal{P}(\mathbb{R}^d)$ we have for all $x \in \mathbb{R}^d$

$$
\begin{aligned}
hD(x; P) &= \inf_{u \in \mathbb{R}^d \setminus \{0\}} \mathsf{P}\left(\langle u, X \rangle \leq \langle u, x \rangle\right) = \inf_{u \in \mathbb{R}^d \setminus \{0\}} \mathsf{P}\left(X_1 \leq \langle u, x \rangle / \|u\|_\alpha\right) \\
&= F_1\left(-\|x\|_{\alpha^*}\right),
\end{aligned}
\tag{8}
$$

where F_1 is the distribution function of X_1, and

$$
\alpha^* = \begin{cases} \alpha/(\alpha - 1) & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha \in (0, 1], \end{cases}
$$

is the conjugate exponent to α. The last equality in (8) is due to the (generalized) Hölder inequality (see, e.g., [38, Lemma A.1]). All central regions P_δ of an α-symmetric distribution are therefore the lower level sets of the norm $\|\cdot\|_{\alpha^*}$. In particular, for all spherically symmetric distributions the central regions are centered balls, and for all $\alpha \leq 1$ the central regions are centered (hyper)-cubes in $\mathbb{R}^d$. Apart from simple uniform distributions on convex bodies such as those in Example 1 and atomic distributions (see the left panel of Figure 2), α-symmetric distributions (and their affine images) appear to be the only class of probability distributions whose depth hD are we able to evaluate exactly. This was noticed by Massé and Theodorescu [126, Example (C)] and Chen and Tyler [38]. See also Figure 4.

Note that the α-symmetric distributions from Example 2 are for $\alpha \neq 2$ in no direct relation with distributions whose densities take the form $\psi(\|\cdot\|_\alpha)$ for some fixed function ψ. In relation to the halfspace depth, the latter collection of probability measures was studied in [48], where it was shown that for $\alpha \neq 2$ and ψ decreasing the central regions P_δ never coincide with the density level sets of P. For $\alpha = 2$ one obtains, using both approaches, collections of spherically symmetric distributions.

3.2.1. Affine invariance

For a non-singular matrix $A \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$, consider the affine transformation $T \colon \mathbb{R}^d \to \mathbb{R}^d \colon x \mapsto Ax + b$. The depth hD is invariant with respect to T

$$hD(x; P_X) = hD\left(T(x); P_{T(X)}\right) \quad \text{for all } x \in \mathbb{R}^d, P_X \in \mathcal{P}\left(\mathbb{R}^d\right).$$

This implies that central regions $P_\delta \equiv (P_X)_\delta$ are affine equivariant under affine transformations T of full rank, i.e. $T((P_X)_\delta) = (P_{T(X)})_\delta$ for any $\delta \in [0,1]$. Due to the affine invariance of hD and Example 2, the central regions P_δ of elliptically symmetric distributions (i.e. invertible affine images of spherically symmetric distributions, see [53, Chapter 2]) are concentric ellipsoids with the same center and orientation as the density level sets of P (if the density exists). In particular, this holds true for the central regions of any full-dimensional multivariate Gaussian distribution (see Figure 4).
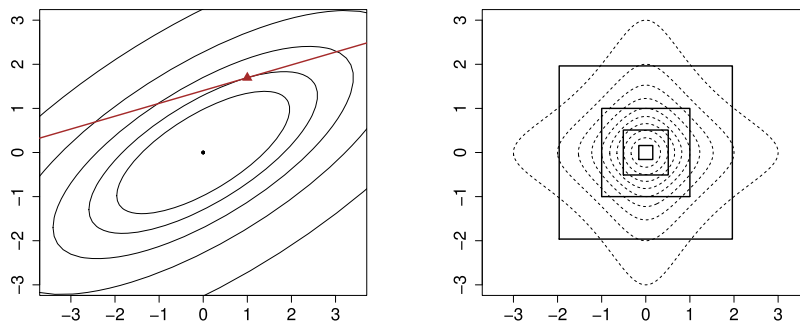
FIG 4. *Left panel: several contours of the density of a bivariate centered Gaussian measure P (thin black lines). For any $x \in \mathbb{R}^d \backslash \{0\}$ (brown triangle) the unique hyperplane $H \ni x$ such that $P(H^-) = hD(x; P)$ is the hyperplane (solid brown line) that supports the contour ellipsoid of the density of P passing through x. Thus, the depth central regions P_δ with $\delta \in (0, 1/2)$ are all concentric ellipsoids of the same shapes as the density contours. Right panel: several density contours (dashed lines) and the corresponding halfspace depth contours (solid lines) of the bivariate distribution P with independent Cauchy marginals. Since P is 1-symmetric, the central regions P_δ are all concentric squares.*

3.2.2. Quasi-concavity

The sets P_δ are all convex, which means that the mapping hD is quasi-concave in its first argument. Quasi-concavity of hD is essential for the construction of estimators based on the depth, such as the depth-trimmed means.

3.2.3. Maximality at the center

Denote the maximal depth value of a distribution by

$$\Pi(P) = \sup_{x \in \mathbb{R}^d} hD(x; P) \quad \text{for } P \in \mathcal{P}\left(\mathbb{R}^d\right). \tag{9}$$

By Rousseeuw and Struyf [161, Lemma 1],

$$\Pi(P) \leq \left(1 + \sup_{x \in \mathbb{R}^d} P\left(\{x\}\right)\right)/2,$$

and $\Pi(P) \leq 1/2$ for P that satisfies (4). As shown by Rousseeuw and Ruts [160, Proposition 7], for any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ the maximal depth is attained in $\mathbb{R}^d$. Therefore, it makes sense to define the *halfspace median* (or *depth median*) of P as any point $x_P \in \mathbb{R}^d$ such that

$$hD(x_P; P) = \Pi(P).$$

The halfspace median is not necessarily unique — consider, for instance, the uniform distribution P on the vertices of a simplex in $\mathbb{R}^d$, where any point in that simplex is a halfspace median of P. If $M(P) = \left\{x \in \mathbb{R}^d \colon hD\left(x; P\right) = \Pi(P)\right\}$ is not a singleton, some authors prefer to define the halfspace median as the barycenter of the region $M(P)$. In this paper, we do not follow that convention, and unless stated otherwise, we call all elements of the set $M(P)$ halfspace medians of P.

In general, the set of all halfspace medians of P can be shown to be non-empty, compact and convex. If (4) is true for P with contiguous support, then by Mizera and Volauf [134, Proposition 7] the halfspace median of P is unique. In any case, the central regions (7) are non-empty if and only if $\delta \in [0, \Pi(P)]$.

If the distribution P is (in some sense) symmetric around a point $x_P \in \mathbb{R}^d$, it is natural to require that the center of symmetry x_P is the unique halfspace median of P, i.e. the only point such that $\Pi(P) = hD(x_P; P)$.

Definition. The distribution $P \equiv P_X \in \mathcal{P}\left(\mathbb{R}^d\right)$ is said to be *centrally symmetric* around $x_P \in \mathbb{R}^d$, if

$$P_{X-x_P} = P_{x_P-X}.$$

P is centrally symmetric, if it is centrally symmetric around some $x_P \in \mathbb{R}^d$.

If P is centrally symmetric, the maximal depth value $\Pi(P)$ must be at least $1/2$, and this depth is attained only at the center of symmetry x_P. But centrally symmetric distributions are not the only ones for which the maximal depth is at least $1/2$. This leads to the following definition, due to Zuo and Serfling [197].

Definition. $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is *halfspace symmetric* around $x_P \in \mathbb{R}^d$, if

$$\Pi(P) = hD(x_P; P) \geq 1/2.$$

P is said to be halfspace symmetric, if it is halfspace symmetric around some $x_P \in \mathbb{R}^d$.

As discussed in Zuo and Serfling [197], the halfspace symmetry of measures in $\mathbb{R}^d$ is more general than the rather restrictive central symmetry, in the sense that any centrally symmetric distribution is also halfspace symmetric. To see that the converse does not hold true, consider the following example.

Example 3. Let $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ be the uniform distribution concentrated in the vertices $(\pm 1, \pm 1)$ of a centered square in $\mathbb{R}^2$. P is halfspace symmetric, and centrally symmetric around $x_P = 0 \in \mathbb{R}^2$. For any $\lambda > 0$, translate the point mass from $(1, 1)$ to (λ, λ). The resulting distribution P' is then still halfspace symmetric around the origin. Yet, for $\lambda \neq 1$, P' is not centrally symmetric.

Any univariate distribution is halfspace symmetric around its (univariate) median. For a comprehensive discussion on the subject of symmetry of multivariate probability distributions see Serfling [176].

3.2.4. Vanishing at infinity

Any random vector $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ lives with large probability inside a closed ball of finite diameter. Thus, it is reasonable to ask that also the depth associated to P assigns high values of hD only to points inside (big) closed balls. This property, often called the vanishing at infinity property of hD, can be expressed as

$$\lim_{M \to \infty} \sup \{hD(x; P) \colon \|x\| > M\} = 0 \quad \text{for all } P \in \mathcal{P}\left(\mathbb{R}^d\right).$$

For the halfspace depth this condition is satisfied (see, for instance, [196, Theorem 2.1]). The central regions (7) are therefore bounded for all $\delta > 0$.

3.2.5. Continuity of the depth

As observed by Donoho and Gasko [45, Lemma 6.1], the halfspace depth is upper semi-continuous in its first argument

$$\limsup_{x \to x_0} hD(x; P) \leq hD(x_0; P) \quad \text{for all } P \in \mathcal{P}\left(\mathbb{R}^d\right), x_0 \in \mathbb{R}^d. \qquad (10)$$

By [134, Proposition 1] if (4) holds true for P, then hD is also continuous in x. For the central regions (7) condition (10) means that each P_δ is a (convex) closed set for $\delta \in [0, \Pi(P)]$, and compact for $\delta \in (0, \Pi(P)]$ for any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$.

3.2.6. Continuity of the central regions

Consider now the set-valued mapping that for $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ given, to $\delta \in (0, \Pi(P))$ assigns its central region (7). This mapping is essential for understanding the properties of the depth, as the level sets of hD are usually of greater interest than individual depth values at fixed points in $\mathbb{R}^d$. The mapping $\delta \mapsto P_\delta$ takes values in the space $\mathcal{K}^d$ of convex bodies in $\mathbb{R}^d$. That space can be equipped with the Hausdorff distance (see, e.g., [165, Section 1.8])

$$\mathrm{d}_H(K, C) = \inf \{\varepsilon > 0 \colon K \subseteq C_\varepsilon \text{ and } C \subseteq K_\varepsilon\} \quad \text{for } K, C \in \mathcal{K}^d, \qquad (11)$$

where K_ε is the ε-neighborhood of K,

$$K_\varepsilon = K + B^d(0, \varepsilon) = \bigcup_{x \in K} B^d(x, \varepsilon) \quad \text{for } K \in \mathcal{K}^d \text{ and } \varepsilon > 0.$$

Continuity properties of the map $\delta \mapsto P_\delta$ were investigated by several authors. The following result was first stated by Massé and Theodorescu [126, Remark 3.6], and later refined by Mizera and Volauf [134, Theorem 6 and Proposition 7], and Dyckerhoff [50, Theorem 3.2 and Example 4.2]. In a slightly different context, it was also considered by Kong and Mizera [90].

Theorem 1. *Let* (4) *be true for $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with contiguous support. Then the map $\delta \mapsto P_\delta$ is continuous in the Hausdorff distance for $\delta \in (0, \Pi(P))$.*

3.2.7. Consistency, robustness and other statistical properties

In statistics, the true distribution $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is seldom known. Instead, one usually observes for $n \in \mathbb{N}$ only a random sample $X_1, \ldots, X_n$ of independent random variables with distribution P, and infers the properties of P from the empirical distribution P_n of that sample. As $n \to \infty$, the halfspace depth is universally consistent, which means that for any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ the depth hD based on the empirical distribution P_n (the sample depth) almost surely approaches the true depth evaluated w.r.t. P uniformly over $\mathbb{R}^d$

$$\sup_{x \in \mathbb{R}^d} |hD(x; P_n) - hD(x; P)| \xrightarrow[n \to \infty]{\text{a.s.}} 0 \quad \text{for any } P \in \mathcal{P}\left(\mathbb{R}^d\right).$$

This result was first established in Donoho and Gasko [45, p. 1817]. Interestingly, it does not require any properties of the distribution P. For P satisfying (4), it can be strengthened to the form that for any sequence of measures $\{P_\nu\}_{\nu=1}^{\infty} \subset \mathcal{P}\left(\mathbb{R}^d\right)$ weakly convergent to P,

$$\sup_{x \in \mathbb{R}^d} |hD(x; P_\nu) - hD(x; P)| \xrightarrow[\nu \to \infty]{} 0.$$

This property follows by an argument of Dümbgen [46, Corollary 2] applied to hD, see also [139, Theorem A.3], and is frequently called the uniform qualitative robustness property of hD. Further robustness properties of hD were studied by Romanazzi [155, 156], and Chen and Tyler [37, 38], among others.

Uniform consistency results hold true also for the depth level sets (7). In its full generality, the following result, recently established by Dyckerhoff [50, Theorem 4.5 and Example 4.2], unifies and completes the partial results from [126, 80, 198].

Theorem 2. *Let* (4) *be true for $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with contiguous support. Then for every compact interval $A \subset (0, \Pi(P)]$*

$$\sup_{\delta \in A} \mathrm{d}_H\left(P_\delta, (P_n)_\delta\right) \xrightarrow[n \to \infty]{\text{a.s.}} 0.$$

In Theorem 2, $(P_n)_\delta$ stands for the δ-central region (7) of the empirical measure P_n. Further valuable improvements of the statistical theory of the halfspace depth include the derivation of the rates of convergence of the depth and its central regions [87, 29, 28], and distributional asymptotics of these and related quantities [8, 194, 195, 123, 124, 125].

4. Description at the center: The Winternitz measure of symmetry

4.1. Maximal depth of a point

Several results on the maximal depth mapping Π from (9) can be found in literature much earlier than the definition of the halfspace depth (see [160,

Sections 3 and 4]). From these references, it appears that the behavior of the maximal depth relates to the degree of concavity of the measure P. Following Borell [19], see also Bobkov [18], let us first provide a rigorous definition of concave probability measures.

Definition. We say that $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is an *s-concave measure* for $-\infty \leq s < \infty$, if

$$P\left(\lambda A + (1 - \lambda)B\right) \geq \begin{cases} \min\left\{P(A), P(B)\right\} & \text{for } s = -\infty, \\ P(A)^\lambda P(B)^{1-\lambda} & \text{for } s = 0, \\ \left(\lambda P(A)^s + (1 - \lambda)P(B)^s\right)^{1/s} & \text{otherwise,} \end{cases}$$

for all non-empty Borel sets $A, B \subseteq \mathbb{R}^d$ and $\lambda \in [0, 1]$.

As noted by Bobkov [18], if P is not a Dirac measure, then $s \leq 1$. Further, a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is s-concave with $s \leq 1/d$ if and only if P has a density f that is supported on an open convex subset U of $\mathbb{R}^d$ and that is $s_d = s/(1 - ds)$-concave, i.e., for all $x, y \in U$, for all $\lambda \in [0, 1]$,

$$f\left(\lambda x + (1 - \lambda)y\right) \geq \begin{cases} \min\left\{f(x), f(y)\right\} & \text{for } s = -\infty, \\ f(x)^\lambda f(y)^{1-\lambda} & \text{for } s = 0, \\ \left(\lambda f(x)^{s_d} + (1 - \lambda)f(y)^{s_d}\right)^{\frac{1}{s_d}} & \text{otherwise.} \end{cases}$$

For $s = 0$, s-concave measures are also called log-concave measures, and represent a natural generalization of uniform measures on convex bodies. Indeed, any uniform measure on a convex body is log-concave.

We are ready to state a result that summarizes what is known about the maximal depth functional $\Pi(P)$ defined in (9).

Theorem 3. *The following inequalities hold true:*

(i) For any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$\left(1 + \sup_{x \in \mathbb{R}^d} P\left(\{x\}\right)\right)/2 \geq \Pi(P) \geq \frac{1}{d+1}.$$

(ii) For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ uniformly distributed on a convex body,

$$1/2 \geq \Pi(P) \geq \left(\frac{d}{d+1}\right)^d > e^{-1} > 0.367.$$

(iii) For an s-concave measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with $-1 < s \leq 1$,

$$1/2 \geq \Pi(P) \geq \begin{cases} e^{-1} & \text{for } s = 0, \\ \left(\frac{1}{s+1}\right)^{1/s} & \text{otherwise.} \end{cases} \tag{12}$$

As noted by Grünbaum [70, Section 4], the lower bounds in parts (i) and (ii) are sharp. In part (i) it is enough to take the uniform distribution in the vertices of a simplex in $\mathbb{R}^d$. For part (ii) one takes the uniform distribution on the simplex in $\mathbb{R}^d$.

Problem 1. Are the lower bounds in part (iii) of Theorem 3 sharp? That is, does there exist an s-concave probability measure P with equality on the right hand side of (12)?

The lower bounds in parts (i) and (ii) of Theorem 3 were proved by Neumann [141] for $d = 2$. In full generality, part (ii) was proved independently by Grünbaum [70], and Hammer [75]. Part (iii) can be found in Caplin and Nalebuff [34, Proposition 3], see also Bobkov [18, Theorem 5.2]. As discussed by Bobkov [18], the condition $s > -1$ implies the existence of the expectation $\mathrm{E}\,X$ of $X \sim P$. Actually, in all three parts of Theorem 3 in the proofs it is shown that $hD(\mathrm{E}\,X; P)$ is never smaller than the given lower bounds.

Problem 2. Is there a non-trivial lower bound for $\Pi(P)$ for all s-concave measures with $s \leq -1$?

4.2. Central and halfspace symmetry: Funk's theorem

For part (ii) of Theorem 3, there exists a remarkable converse.

Theorem 4. Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be uniformly distributed on a convex body $K \in \mathcal{K}^d$. Then P is halfspace symmetric around $x_P \in \mathbb{R}^d$ if and only if it is centrally symmetric around x_P.

The proof of Theorem 4 was first obtained in 1915 for $d = 2$ and $d = 3$ by Funk [59]. In its full generality the result was conjectured, among others, by Grünbaum [72, p. 251], but completely solved only in 1970 in Schneider [163, Satz 4.2] and Schneider [164, Theorem 1.5], see also Falconer [52]. For its modern version, including an extension to star convex bodies $K \subset \mathbb{R}^d$ see Groemer [66, Section 5.6].

By Theorem 4, the two notions of central and halfspace symmetry from Section 3.2.3 coincide for uniform distributions on (star) convex bodies in $\mathbb{R}^d$, see also Example 3. This suggests the following problem.

Problem 3. Under which conditions can Theorem 4 be generalized to probability measures?

A partial answer to Problem 3 can be found if one considers the notion of angular symmetry for random vectors, proposed by Liu [102, Section 2].

Definition. The distribution of a random vector $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is said to be *angularly symmetric* around $x_P \in \mathbb{R}^d$, if the random variables $(X - x_P)/\|X - x_P\|$ and $-(X - x_P)/\|X - x_P\|$ are identically distributed. P is angularly symmetric, if it is angularly symmetric around some $x_P \in \mathbb{R}^d$.

Angular symmetry can be shown to be an intermediate between the rather strong concept of central symmetry, and the halfspace symmetry, considered in Section 3.2.3. Any P that is centrally symmetric around x_P is angularly symmetric around x_P [197, Lemma 2.2], and any P angularly symmetric around x_P is also halfspace symmetric around x_P [197, Lemma 2.4]. None of these implications can be reversed. Though, a partial reverse to the second one was asserted in the statistical literature. For $d = 2$, Zuo and Serfling [197, Theorem 2.6] in 2000 and Dutta, Ghosh and Chaudhuri [48, Theorem 2] in 2011 independently proved that if P is absolutely continuous besides a possible atom at $x_P \in \mathbb{R}^d$ and halfspace symmetric around x_P, then P must be also angularly symmetric around x_P. Rousseeuw and Struyf [161, Theorems 1 and 2] in 2004 gave a complete proof for general $d \in \mathbb{N}$ in the following form.

Theorem 5. *The distribution $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is angularly symmetric around $x_P \in \mathbb{R}^d$ if and only if*

$$hD(x_P; P) = (1 + P(\{x_P\}))/2.$$

In particular,

(i) *any P halfspace symmetric around x_P with $P(\{x_P\}) = 0$ is angularly symmetric around x_P, and*

(ii) *for any P such that $\sup_{x \in \mathbb{R}^d} P(\{x\}) = 0$, halfspace symmetry and angular symmetry are equivalent notions.*

When P is the uniform distribution on a (centered) convex body $K \in \mathcal{K}^d$, Theorem 5 stands as a generalization of Funk's theorem to probability measures. Indeed, assume that P is halfspace symmetric around the origin $x_P = 0 \in \mathbb{R}^d$. Since P is absolutely continuous, by Theorem 5 it is also angularly symmetric around x_P. Because P is uniform, angular symmetry of P implies that the support function h_K from (2) must be an even function on $\mathbb{S}^{d-1}$, which in turn gives that K must be centrally symmetric around x_P.

Remarkably, Theorems 1 and 2 in Rousseeuw and Struyf [161] were discovered independently of the results in geometry. The proof in [161] makes use of the classical theorem of Cramér and Wold [40] from 1936, closely related to the Fourier transforms of measures. The known proofs of Theorem 4 employ techniques from spherical harmonics, or integral equations. Thus, all known proofs of Theorems 4 and 5 are non-trivial, but have in common the use of harmonic analysis.

4.3. Measures of symmetry

Characterization results like Theorem 4 for convex bodies stimulated much research in convex geometry. Eventually, these efforts led to *measures of symmetry for convex sets*, comprehensively covered by Grünbaum [72]. A measure of symmetry is a mapping $\mathfrak{S}\colon \mathcal{K}^d \to [0,1]$ such that

(i) $\mathfrak{S}(K) = 1$ if and only if K is (centrally) symmetric,

(ii) $\mathfrak{S}(K) = \mathfrak{S}(T(K))$ for any non-singular affine transformation $T\colon \mathbb{R}^d \to \mathbb{R}^d$, and

(iii) $\mathfrak{S}$ is continuous on $\mathcal{K}^d$ (equipped with a suitable topology[2]).

A variant of part (ii) in Theorem 3, that states that for any $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ uniformly distributed on a convex body

$$hD\left(\operatorname{E} X; P\right) \geq \left(\frac{d}{d+1}\right)^d,$$

is known since the 1910s as the Winternitz theorem (due to Artur Winternitz, according to [17]). This result gave rise to the following measure of symmetry, which is remarkably close to the halfspace depth.

Definition. Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be the uniform distribution on $K \in \mathcal{K}^d$. For $x \in K$ and $H \in \mathcal{H}$ with $x \in H$, let

$$w_K(x; H^-) = \frac{P(H^- \cap K)}{1 - P(H^- \cap K)},$$

and consider $w_K(x) = \min\{w_K(x; H^-)\colon H^- \in \mathcal{H}^-, x \in H\}$. The *Winternitz measure of symmetry* of K is then defined as

$$W(K) = \max\{w_K(x)\colon x \in K\}.$$

The measure of symmetry $W(K)$ was considered by many authors. For a historical account and the theoretical background on measures of symmetry see the seminal paper of Grünbaum [72, Section 6.2]. For a modern treatment of the topic see Toth [184].

Obviously, for $K \in \mathcal{K}^d$, the Winternitz measure of symmetry is equivalent with the maximal depth (9) attained w.r.t. the uniform measure P on K

$$\Pi(P) = \frac{W(K)}{1 + W(K)}.$$

The function $w_K\colon K \to [0, \infty]$ used in the definition of $W(K)$ links directly to hD via

$$hD(x; P) = \frac{w_K(x)}{1 + w_K(x)}.$$

For w_K, it was noted already by Grünbaum [72] in 1963 that its upper level sets are convex, and that its maximal value is always attained in K (cf. Sections 3.2.2 and 3.2.3 above).

Connections of the depth hD with results on partitions of convex bodies (Theorem 3 above) have already been noted by Rousseeuw and Ruts [160]. Though, as far as we know, no links between the measures of symmetry for convex bodies and the halfspace depth have yet been established in the statistical literature.

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[2]For details on possible choices of topology see Grünbaum [72].

In the other direction, some notions of depth can be found in the literature on the geometry of convex bodies. For instance, in Bose et al. [24] the "depth" for a convex body K is defined as the halfspace depth (3) of the associated uniform distribution, in connection with a generalized version of the Winternitz theorem. Nonetheless, precise links between the respective fields of mathematics appear to be still lacking.

4.3.1. The ray basis theorem

For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ and $x \in \mathbb{R}^d$ we say that a halfspace $H^- \in \mathcal{H}^-$ is *minimal* at x if $x \in H$ and $P(H^-) = hD(x; P)$. H is then called a *minimal hyperplane* of x. From the definition of the minimal halfspace it is easy to see that the following holds.

Proposition 6. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ have contiguous support and let $H^- \in \mathcal{H}^-$ be minimal at $x \in \mathbb{R}^d$ with $hD(x; P) = \delta$. Then the halfspace H^+ supports P_δ.*

An interesting characterization of the halfspace median of a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ in terms of minimal halfspaces was observed by Donoho and Gasko [45, pp. 1818–1819] in 1992 and Rousseeuw and Ruts [160, Propositions 8 and 12] in 1999. For P absolutely continuous, x is a halfspace median of P if and only if the union of the collection of minimal halfspaces at x is $\mathbb{R}^d$. In Rousseeuw and Ruts [160], this result is dubbed the ray basis theorem.

Theorem 7. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, and $x \in \mathbb{R}^d$ be such that the union of the collection of minimal halfspaces at x is $\mathbb{R}^d$. Then x is a halfspace median of P.*

Assume that P satisfies (4), *and let $x \in \mathbb{R}^d$ be a halfspace median of P. Then there exists a collection of minimal halfspaces at x of cardinality at most $d + 1$ whose union is $\mathbb{R}^d$.*

The smoothness condition (4) is important in Theorem 7. As noted by Massé [124, Example 4.3] it is possible to construct distributions $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, that violate (4), with a unique minimal halfspace at their halfspace median.

For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ uniformly distributed on a convex body $K \in \mathcal{K}^d$, a result similar to Theorem 7 was stated in Grünbaum [72, p. 251] in 1963 for the Winternitz measure of symmetry. There, it was asserted that it follows from a version of Helly's theorem that there must exist at least $d + 1$ different minimal halfspaces at the halfspace median x_P of P. The assumptions of that result appear, however, to be incomplete, as pointed out to us by M. Tancer [147].

Another interesting problem closely connected with the halfspace median and Theorem 7, is a conjecture of Grünbaum [71, p. 41] from 1961 that asks if for any convex body $K \in \mathcal{K}^d$ with $d \geq 2$ there exists a point $x \in K$ that is a centroid of at least $d + 1$ sections of K by different hyperplanes passing through x. For $d = 2$, the solution to this problem is straightforward, as noted already in [71]. For $d > 2$, this problem appears to be still open (see [181], [72, p. 251], and [41, Problem A8]). It is natural to conjecture that the halfspace median is such a point. Indeed, combine Theorem 7 with a theorem of Dupin

[47] (stated in part (ii) of Proposition 12 below) that says that for any $K \in \mathcal{K}^d$ the point $x \in K$ is the centroid of all minimal hyperplanes at x (w.r.t. the uniform distribution P on K) to obtain that if the minimal hyperplanes at x are in general position, then the halfspace median is a point as postulated in the conjecture. Here, a set of hyperplanes is said to be in general position if for all choices of at most d such distinct hyperplanes their normals are linearly independent. A further open question is if the conjecture holds true with x being the centroid of K.

Theorem 7 provides a useful characterization criterion for the depth-based extension of the median. Apart from its theoretical appeal, it promises applications in the computation of the depth, and the depth median.

4.3.2. Minimality and stability

An important question regarding the measures of symmetry concerns their minimality, i.e. characterization of $K \in \mathcal{K}^d$ such that $\mathfrak{S}(K) = \inf \left\{ \mathfrak{S}(K') \colon K' \in \mathcal{K}^d \right\}$. As remarked by Grünbaum [70, Section 4] in 1960, for the Winternitz measure of symmetry

$$\inf \left\{ W(K') \colon K' \in \mathcal{K}^d \right\} = \frac{d^d}{(d+1)^d - d^d},$$

and this value is attained if and only if K is a bounded cone in $\mathbb{R}^d$. This value corresponds to

$$\inf_{K' \in \mathcal{K}^d} \left\{ \Pi(P) \colon P \text{ is distributed uniformly on } K' \right\} = \left(d/(d+1) \right)^d,$$

see also Theorem 3. In a related question, Grünbaum [70] also determined the collection of measures $P \in \mathcal{P}\left(\mathbb{R}^d \right)$ such that

$$\Pi(P) = 1/(d+1) = \inf \left\{ \Pi(P') \colon P' \in \mathcal{P}\left(\mathbb{R}^d \right) \right\},$$

by showing that this can happen if and only if P is a uniform distribution on the vertices of a non-degenerate simplex in $\mathbb{R}^d$. In statistics, this result was observed independently by Donoho and Gasko [45, Lemma 6.3] in 1992 for hD.

In convex analysis, another desirable property of measures of symmetry is their stability. A measure of symmetry $\mathfrak{S}$ is said to have the *stability property* if for any $\varepsilon > 0$ and $K \in \mathcal{K}^d$ with $\mathfrak{S}(K) < \inf \left\{ \mathfrak{S}(K') \colon K' \in \mathcal{K}^d \right\} + \varepsilon$ there exists a constant $c > 0$ and $L \in \mathcal{K}^d$ such that $\mathfrak{S}(L) = \inf \left\{ \mathfrak{S}(K') \colon K' \in \mathcal{K}^d \right\}$, and $\delta(K, L) \leq c\varepsilon$. Here, δ stands for some metric on $\mathcal{K}^d$, and c may depend on d, as well as on some characteristic of K such as its volume, or diameter. An important stability theorem for the Winternitz measure of symmetry was derived by Groemer [67, Theorem 2].

Theorem 8. *Let $K \in \mathcal{K}^d$ and let $P \in \mathcal{P}\left(\mathbb{R}^d \right)$ be uniformly distributed on K. Let $\varepsilon \geq 0$. There exists a constant $\lambda > 0$ depending only on d such that*

$\Pi(P) \leq (d/(d+1))^d + \varepsilon$ *implies that* K *contains a bounded cone* $C \in \mathcal{K}^d$ *with*

$$\mathrm{d}_S(K, C) \leq \lambda \, \mathrm{vol}_d(K) \, \varepsilon^{1/2d^2},$$

where,

$$\mathrm{d}_S(K, C) = \mathrm{vol}_d(K \cup C) - \mathrm{vol}_d(K \cap C) \tag{13}$$

is the symmetric difference metric on $\mathcal{K}^d$.

As far as we are aware, no results corresponding to stability theorems can be found for probability measures and the halfspace depth.

Problem 4. Does a variant of a stability result such as Theorem 8 hold for probability measures and depth medians?

4.4. Affine invariant points

Symmetry is a key structural property of convex bodies relevant in many problems. A systematic study of symmetry was initiated by Grünbaum in his seminal paper [72] from 1963. A crucial notion in his work is that of affine invariant point. It allows to analyze the symmetry situation. In a nutshell: the more affine invariant points, the fewer symmetries.

Recall that the set $\mathcal{K}^d$ is equipped with the Hausdorff distance (11).

Definition. A map $p \colon \mathcal{K}^d \to \mathbb{R}^d$ is called an *affine invariant point*, if p is continuous and if for every non-singular affine map $T : \mathbb{R}^d \to \mathbb{R}^d$ one has,

$$p(T(K)) = T(p(K)).$$

We denote by $\mathcal{P}^d$ the set of all affine invariant points on $\mathbb{R}^d$.

$\mathcal{P}^d$ is an affine subspace of $C(\mathcal{K}^d, \mathbb{R}^d)$, the space of continuous mappings from $\mathcal{K}^d$ to $\mathbb{R}^d$.

Examples of affine invariant points, already known to Grünbaum [72] are, e.g., the centroid of a convex body K (i.e. the expectation of the uniform distribution on K), the Santaló point (the unique point $s(K)$ in the interior of $K \in \mathcal{K}^d$ for which the minimum of the functional $\mathrm{vol}_d\left(K^{s(K)}\right)$ is attained, see also the important Blaschke-Santaló inequality in (19) below), and the center of the ellipsoid of maximal volume inside a convex body. Grünbaum [72] asked a number of questions about affine invariant points:

  (i) Is there a convex body K such that $\mathcal{P}^d(K) = \{p(K) \colon p \in \mathcal{P}^d\} = \mathbb{R}^d$?
 (ii) Is the space $\mathcal{P}^d$ infinite-dimensional?
(iii) Let K be a convex body and let $T \colon \mathbb{R}^d \to \mathbb{R}^d$ be an affine map with $T(K) = K$. We denote

$$\mathcal{F}^d(K) = \left\{x \in \mathbb{R}^d \colon \text{ for all } T \text{ such that } T(K) = K \text{ we have } Tx = x\right\}.$$

Do we have $\mathcal{F}^d(K) = \mathcal{P}^d(K)$?

One can argue that those convex bodies that have only one affine invariant point are the most symmetric convex bodies. This would include the simplex in $\mathbb{R}^d$ which is from another point of view the most non-symmetric convex body (see Theorem 3).

A convex body has only one affine invariant point, if it has enough symmetries. We say that an affine map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is a *symmetry of a convex body* K if $T(K) = K$. We say that a convex body has *enough symmetries* if the only affine maps commuting with all symmetries of K are multiples of the identity.

For a convex body K with enough symmetries the halfspace median coincides with the centroid of K.

The following theorems answer Grünbaum's questions (i) and (ii). They can be found in Meyer, Schütt and Werner [131].

Theorem 9. *For every $d \geq 2$ there is a body $K \in \mathcal{K}^d$ with*

$$\mathcal{P}^d(K) = \left\{ p(K)\colon p \in \mathcal{P}^d \right\} = \mathbb{R}^d.$$

Such convex bodies are actually dense in $\mathcal{K}^d$ with respect to the Hausdorff metric.

Theorem 10. *The space $\mathcal{P}^d$ is infinite-dimensional.*

In the proofs of these theorems, new classes of affine invariant points were introduced using convex floating bodies (see Section 5.2 below). We define $p_\delta \colon \mathcal{K}^d \to \mathbb{R}^d$ to be the mapping that sends K to the centroid of P_δ from (7) for P uniform on K.

Moreover, in Meyer, Schütt and Werner [131, Theorem 2] it was shown that for convex bodies K with $\dim(\mathcal{P}^d(K)) = d - 1$ a positive answer to Grünbaum's question (iii) above holds, i.e. $\mathcal{F}^d(K) = \mathcal{P}^d(K)$. It was settled in all dimensions by Mordhorst [135], based on work by Kučment [95] (see also [96]) where question (iii) of Grünbaum was almost proved already in 1972, with only a compactness argument missing.

Theorem 11. *For any $K \in \mathcal{K}^d$ we have that $\mathcal{F}^d(K) = \mathcal{P}^d(K)$.*

5. Description at the boundary: Convex floating bodies

Data depth is intimately related to the concept of *floating body* which we now introduce. We start with a brief discussion of differentiability properties of the boundary of convex bodies, since this will be essential in what follows.

5.1. Curvature of convex bodies

We take as a measure on the boundary ∂K of a convex body $K \in \mathcal{K}^d$ the restriction of the $(d-1)$-dimensional Hausdorff measure to ∂K. We call this measure the *boundary measure*, or the Lebesgue measure on ∂K, and denote it by $\mu_{\partial K}$. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^d$ and $f\colon \mathcal{U} \to \mathbb{R}$ be a twice continuously

differentiable function. Then the classical *Gauss-Kronecker curvature* at $x_0 \in \mathcal{U}$ is

$$\kappa(x_0) = \frac{\det\left(\nabla^2 f(x_0)\right)}{\left(1 + \|\nabla f(x_0)\|^2\right)^{\frac{d+2}{2}}},$$

where ∇f is the gradient of f and $\nabla^2 f$ the Hessian of f. The Gauss-Kronecker curvature of the boundary of a convex body is the curvature of a function parametrizing the boundary.

By a theorem of Rademacher (see, e.g., [23, Theorem 2.5.1]), a convex function on $\mathbb{R}^d$, and in particular the boundary of a convex body, is almost everywhere differentiable. There are, however, examples of convex functions and convex bodies that are not differentiable on a dense set of $\mathbb{R}^d$ and of the boundary of the convex body, respectively. Those examples do not have a second derivative at any point and thus the classical Gauss-Kronecker curvature κ does not exist at any point.

Therefore we use the generalized Gauss-Kronecker curvature as introduced by Busemann and Feller [30] in dimension $d = 3$ and Aleksandrov [2] in general. We present here only a short explanation of the generalized Gauss-Kronecker curvature and we refer to e.g., [173, Section 1.6] and [170] for a detailed account.

A cap of $K \in \mathcal{K}^d$ at $x \in \partial K$ is the intersection of a halfspace H^- with K such that there is a supporting hyperplane to K at x that is parallel to H. There may, of course, be points on the boundary of K having more than one supporting hyperplane. But, those points are of measure 0 and shall be of less importance in our discussion.

If K has a unique supporting hyperplane at $x \in \partial K$, we denote by $\Delta(x, \delta)$ the height of a cap with volume δ. The height of a cap is the distance of the supporting hyperplane at x to the parallel hyperplane cutting off a set of volume δ.

Definition. Let $K \in \mathcal{K}^d$ and $x \in \partial K$. Let $c_d = 2^{d+1}\left(\text{vol}_{d-1}\left(B^{d-1}\right)/(d+1)\right)^2$. Assume that K has at x a unique supporting hyperplane. We say that K has a *generalized Gauss-Kronecker curvature* if the limit

$$\lim_{\delta \to 0} c_d \, \frac{\Delta(x, \delta)^{d+1}}{\delta^2}$$

exists. In this case we define

$$\kappa(x) = \lim_{\delta \to 0} c_d \, \frac{\Delta(x, \delta)^{d+1}}{\delta^2} \tag{14}$$

to be the generalized Gauss-Kronecker curvature at x.

If the Gauss-Kronecker curvature exists, then it is equal to the generalized Gauss-Kronecker curvature. By a theorem of Busemann, Feller and Aleksandrov [30, 2] the generalized Gauss-Kronecker curvature of a convex body exists almost everywhere (in the sense of the boundary measure of K). Geometrically, the existence of the generalized Gauss-Kronecker curvature at x means that ∂K

can be "well" approximated by an ellipsoid, or ellipsoidal cylinder at x (see, e.g., [173, Section 1.6]).

The following example clarifies the difference between Gauss-Kronecker curvature and generalized Gauss-Kronecker curvature.

Example 4. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } |x| = 1/n \text{ and } n \in \mathbb{N}, \\ \frac{2n+1}{n(n+1)} |x| - \frac{1}{n(n+1)} & \text{if } \frac{1}{n+1} < |x| < 1/n \text{ and } n \in \mathbb{N}. \end{cases}$$

The function f is not differentiable at the points $x = \pm 1/n$ and therefore f is not twice differentiable at 0. Thus, the Gauss-Kronecker curvature of f does not exist at 0. On the other hand, it is not difficult to compute that f has a generalized Gauss-Kronecker curvature at 0 and this curvature is 2, see Figure 5.
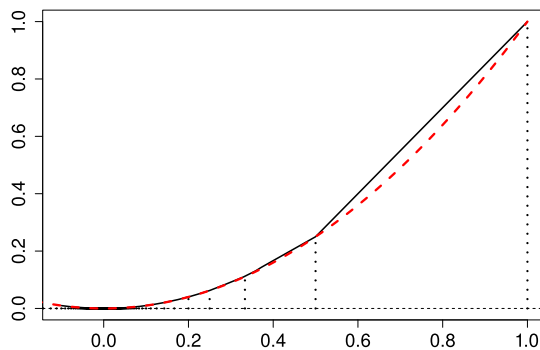

FIG 5. *The function f from Example 4 (black solid line) and the function $x \mapsto x^2$ (red dashed line) that approximates f around $x = 0$. Since f is not twice differentiable at $x = 0$, its Gauss-Kronecker curvature does not exist at 0. Its generalized Gauss-Kronecker curvature at 0 exists and is equal to 2, the Gauss-Kronecker curvature of $x \mapsto x^2$.*

5.2. Floating body and convex floating body

Earliest records on floating bodies can be traced back to the early 19th century work of Dupin [47] and are motivated by mechanics. By the Archimedean principle, a solid convex body $K \in \mathcal{K}^3$ of constant (volumetric mass) density that floats in water has always a set of the same volume above the water surface, regardless of its position.

This leads to the definition of floating bodies for convex bodies in $K \in \mathcal{K}^d$ according to Dupin.

Definition. A nonempty convex subset $K_{[\delta]}$ of $K \in \mathcal{K}^d$ is a *floating body* of K with index $\delta > 0$ if each supporting hyperplane to $K_{[\delta]}$ cuts off a set of volume δ from K.
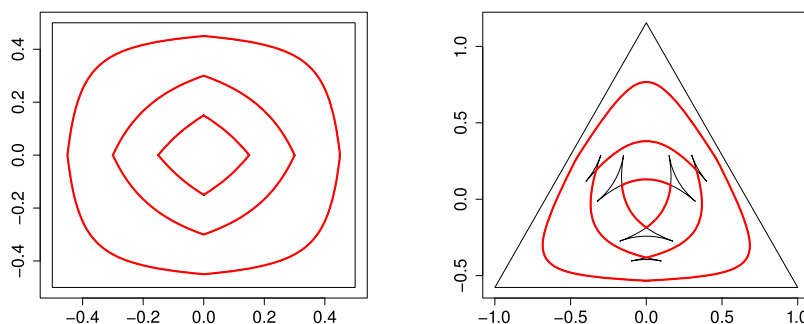
FIG 6. *Curves of barycenters of hyperplanes that cut off volume $\delta \in \{0.05, 0.20, 0.35\}$ from convex bodies (thin black lines), and the boundaries of convex floating bodies for the same values of δ (thick red lines), for the uniform distribution on a square (left panel) and a triangle (right panel). For the square, all (Dupin's) floating bodies exist, and coincide with the convex floating bodies. For the triangle, the boundaries of all convex floating bodies are proper sub-curves of the corresponding black curves (the difference is not visible in the plot for $\delta = 0.05$). The (Dupin's) floating bodies do not exist.*

Dupin observed that a support hyperplane H to $K_{[\delta]}$ touches the boundary of $K_{[\delta]}$ in exactly one point, the barycenter of $K \cap H$. It implies that if $K_{[\delta]}$ exists, its boundary is given by the surface of all barycenters of $H \cap K$ for hyperplanes H that cut off volume δ from K.

The floating body cannot exist for $\delta > \mathrm{vol}_d(K)/2$. Suppose it does exist. Then any two different parallel supporting hyperplanes of $K_{[\delta]}$ cut off disjoint sets of volume δ from K, and therefore $K_{[\delta]}$ is the empty set. As shown in the next example, the floating body $K_{[\delta]}$ may not exist even for small $\delta > 0$.

Example 5. Let $K \in \mathcal{K}^2$ be the equilateral triangle from Example 1. For all $\delta > 0$, the curve of barycenters of lines that cut off volume δ from K is not the boundary of a convex set. Some of these curves for various values of δ are displayed on the right panel of Figure 6. Therefore, in agreement with the observation of Leichtweiß [98, pp. 433–434], no floating body of a triangle exists. Compare this also to Example 1.

If $K \in \mathcal{K}^2$ is the unit square from Example 1, all floating bodies $K_{[\delta]}$ exist for $\delta \in (0, \mathrm{vol}_d(K)/2]$, and they coincide with the halfspace depth central regions (7).

If $K \in \mathcal{K}^d$ has a sufficiently smooth boundary, then $K_{[\delta]}$ exists by Leichtweiß [98, Satz 2], at least for small $\delta > 0$. However, in many applications (e.g., in Section 5.3 below), existence of floating bodies for all convex bodies is needed. Therefore a modified definition has been proposed, independently by Bárány and Larman [12] and Schütt and Werner [170], called the convex floating body.

Definition. Let K be a convex body in $\mathbb{R}^d$ and $\delta \geq 0$. The *convex floating body* is the intersection of all halfspaces whose defining hyperplanes cut off a set of

volume δ of K,

$$K_\delta = \bigcap_{\mathrm{vol}_d(K \cap H^-)=\delta} H^+,$$

where $H \in \mathcal{H}$ and H^+ and H^- are its associated halfspaces.

The convex floating body exists for all convex bodies since it is an intersection of halfspaces. For instance, the convex floating body of the triangle has a boundary described by the red curve in Figure 6. Note also that $K_0 = K$. It is easy to see that whenever $K_{[\delta]}$ exists, then $K_{[\delta]} = K_\delta$ [170]. Unlike the floating body, the convex floating body is allowed to be an empty set. This way, all convex floating bodies K_δ of K are well defined convex sets, but certainly $K_\delta = \emptyset$ if $\delta > \mathrm{vol}_d(K)/2$.

Properties of the convex floating body are stated in the next proposition.

Proposition 12. *Let $K \in \mathcal{K}^d$ and $\delta > 0$.*

(i) *Through every point of ∂K_δ there is at least one supporting hyperplane of K_δ that cuts off a set of volume δ from K.*

(ii) *A supporting hyperplane H of K_δ that cuts off a set of volume δ touches K_δ in exactly one point, the barycenter of $K \cap H$.*

(iii) *K_δ is strictly convex.*

(iv) *Let*

$$\delta_0 = \sup\{\delta \colon \mathrm{vol}_d(K_\delta) > 0\}. \tag{15}$$

Then K_{δ_0} consists of one point only, and for $\delta < \delta_0$ we have that K_δ is a convex body.

Most of Proposition 12 was proved in [172, Lemma 2]. Part (ii), in dimension $d = 3$, is due to Dupin [47], see also [98, p. 435]. In general, it is not true that all supporting hyperplanes to the convex floating body K_δ cut off a set of exactly volume δ from K. An example is the simplex, as can be seen also from Example 5. Not every point on the boundary of K_δ has a unique supporting hyperplane. An example is the cube, see Example 5.

Meyer and Reisner [129] show that for centrally symmetric convex bodies $K_{[\delta]}$ exists for any $\delta \in (0, \mathrm{vol}_d(K)/2]$. Moreover, in that case each $K_{[\delta]}$ is also (centrally) symmetric around the same center of symmetry as K. In an unpublished work, K. Ball gave a different proof of the existence result, see [130, Section 4].

Proposition 13. *Let $K \in \mathcal{K}^d$ be a convex body that is (centrally) symmetric with respect to the origin 0, i.e. $x \in K$ implies $-x \in K$. Then we have for any $\delta \in (0, \mathrm{vol}_d(K)/2)$*

(i) *The floating body $K_{[\delta]}$ of K exists, and is strictly convex.*

(ii) *If, in addition, K is strictly convex and has a C^1 boundary, then the floating body $K_{[\delta]}$ has a C^2 boundary.*

The next two results can be found in Schütt and Werner [171, Theorem 5.3 and Proposition 5.1], and describe the behavior of the volume of $K \setminus K_\delta$.

Proposition 14. *Let $K \in \mathcal{K}^d$, and let δ_0 be as in* (15)*. Then* $\mathrm{vol}_d\left(K \setminus K_\delta\right)$ *is a differentiable function of δ on $(0, \delta_0)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}\,\delta}\,\mathrm{vol}_d\left(K \setminus K_\delta\right) = \int_{\partial K_\delta} \frac{1}{\mathrm{vol}_{d-1}\left(K \cap H(x, N_{\partial K_\delta}(x))\right)}\,\mathrm{d}\,\mu_{\partial K_\delta}(x),$$

where $H(x, N_{\partial K_\delta}(x))$ is the hyperplane passing through x orthogonal to the normal of K_δ at x.

Proposition 15. *Let $K \in \mathcal{K}^d$ be a (centrally) symmetric convex body in $\mathbb{R}^d$. Then we have for all $\delta \in (0, \mathrm{vol}_d\left(K\right)/2)$*

$$\frac{\mathrm{d}}{\mathrm{d}\,\delta}\,\mathrm{vol}_d\left(K \setminus K_\delta\right) \leq \frac{d}{\delta}\,\mathrm{vol}_d\left(K \setminus K_\delta\right).$$

5.3. Affine surface area

An important affine invariant from affine convex geometry is the affine surface area. Applications of the affine surface area are numerous. We only name some in convex geometry [118, 61, 21, 109, 110, 73], in differential geometry [3, 4, 179, 85], approximation of convex bodies by polytopes (see Section 5.7), information theory [117, 189, 7, 190], and partial differential equations [115, 185].

Let K be a convex body in $\mathbb{R}^d$ with a C^2 boundary. Then for all $x \in \partial K$, the Gauss-Kronecker curvature $\kappa(x)$ exists and the (classical) *affine surface area*, introduced by Blaschke [17] in 1923 in dimensions two and three, is defined as

$$\mathrm{as}\left(K\right) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\,\mathrm{d}\,\mu_{\partial K}(x).$$

For a Euclidean ball with radius 1, the affine surface area equals its surface area. It is 0 for all polytopes. Blaschke [17] observed that for convex bodies in $\mathbb{R}^3$ with analytic boundary the following identity holds

$$\lim_{\delta \to 0} \frac{\mathrm{vol}_3(K) - \mathrm{vol}_3(K_{[\delta]})}{\delta^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi}} \int_{\partial K} \kappa(x)^{\frac{1}{4}}\,\mathrm{d}\,\mu_{\partial K}(x). \tag{16}$$

An important tool in the proof of this identity is the rolling theorem of Blaschke [17]: The floating body exists if a sufficiently small Euclidean ball rolls freely inside K, i.e., there is $r > 0$ such that for all $x \in \partial K$ there is $y \in K$ such that $\|x - y\| = r$ and $B^d(y, r) \subset K$.

It is natural to ask if formula (16) can be extended to all dimensions and all convex bodies using the convex floating body instead of the floating body. This is indeed the case and was achieved in Schütt and Werner [170], where now the function κ under the integral is the generalized Gauss-Kronecker curvature (14).

Theorem 16. *Let $K \in \mathcal{K}^d$. Then*

$$\lim_{\delta \to 0} \frac{\mathrm{vol}_d\left(K\right) - \mathrm{vol}_d\left(K_\delta\right)}{\delta^{\frac{2}{d+1}}} = \frac{1}{2}\left(\frac{d+1}{\mathrm{vol}_{d-1}\left(B^{d-1}\right)}\right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\,\mathrm{d}\,\mu_{\partial K}(x). \tag{17}$$

The expressions in the above theorem can thus be used to define the affine surface area for all convex bodies. Around the same time, different extensions of the affine surface area to arbitrary convex bodies were given by Leichtweiß [98] and Lutwak [112] and afterwards several more have been found, e.g., [82, 132, 187]. It has been shown that all those extensions coincide.

Expression (17) is called the affine surface area because of its similarity to Minkowski's definition of surface area

$$\mathrm{vol}_{d-1}\left(\partial K\right) = \lim_{\delta \to 0} \frac{1}{\delta} \left(\mathrm{vol}_d\left(K + \delta B^d\right) - \mathrm{vol}_d\left(K\right)\right),$$

and because for all affine maps $T \colon \mathbb{R}^d \to \mathbb{R}^d$, as $(T(K)) = |\det(T)|^{\frac{d-1}{d+1}}$ as (K). The latter equation follows easily from (17). Indeed,

$$(T(K))_\delta = T\left(K_{\frac{\delta}{|\det(T)|}}\right).$$

An important tool in the proof of Theorem 16 is a strengthening of Blaschke's rolling theorem. To achieve this, Schütt and Werner [170] introduce the rolling function. For $x \in \partial K$, the rolling function $r(x)$ is the supremum of all radii of Euclidean balls that contain x and that are contained in K, i.e. $r \colon \partial K \to \mathbb{R}$ is defined by

$$r(x) = \sup\left\{\|x - z\| : z \in K, B^d(z, \|x - z\|) \subseteq K\right\}.$$

If K does not have a unique normal at x then $r(x) = 0$. The following was shown by Schütt and Werner [170, Lemmas 4 and 5].

Proposition 17. *Let $K \in \mathcal{K}^d$ be such that $B^d \subset K$. Then we have for all t with $0 \le t \le 1$ that $\{x \in \partial K \colon r(x) \ge t\}$ is a closed set and*

$$(1 - t)^{d-1} \mathrm{vol}_{d-1}\left(\partial K\right) \le \mathrm{vol}_{d-1}\left(\{x \in \partial K \colon r(x) \ge t\}\right).$$

The inequality is optimal. In particular, the function $r^{-\alpha} \colon \partial K \to \mathbb{R}$ is Lebesgue integrable for all α with $0 \le \alpha < 1$.

Note that by taking $t = 0$ in Proposition 17 it follows that the boundary of a convex body is almost everywhere differentiable.

Affine invariance is a useful property as it lets us consider convex bodies independent of their position in space. Another extremely important property of the affine surface area is the affine isoperimetric inequality which says that for all convex bodies $K \in \mathcal{K}^d$,

$$\frac{\mathrm{as}\left(K\right)}{\mathrm{as}\left(B^d\right)} \le \left(\frac{\mathrm{vol}_d\left(K\right)}{\mathrm{vol}_d\left(B^d\right)}\right)^{\frac{d-1}{d+1}}, \tag{18}$$

with equality if and only if K is an ellipsoid (see, e.g., [165, Section 10.5]). The affine isoperimetric inequality is stronger than the classical isoperimetric inequality and provides solutions to many problems where ellipsoids are extrema [113, 172, 180, 192].

The affine isoperimetric inequality (18) is equivalent to another classical inequality from convex geometry, the Blaschke-Santaló inequality [17, 162]. For an interior point x_0 of a convex body K recall the definition of the polar body K^{x_0} of K w.r.t. x_0 from (1). The Blaschke-Santaló inequality states that for all convex bodies K in $\mathbb{R}^d$,

$$\mathrm{vol}_d\left(K\right)\mathrm{vol}_d\left(K^{s(K)}\right) \le \mathrm{vol}_d\left(B^d\right)^2, \tag{19}$$

where $s(K)$ is the *Santaló point* of K, i.e. the unique point for which the minimum is attained on the left hand side. This inequality and its counterpart, the reverse Blaschke-Santaló inequality (proved by Bourgain and Milman [26] and closely connected to the still-unsolved Mahler's conjecture, see e.g. Giannopoulos, Paouris and Vritsiou [63]), are helpful to estimate the volume of convex bodies in situations, when it is easier to compute the volume of the polar K^{x_0} of a convex body. These inequalities have important applications in convex geometry, functional analysis, Banach space theory, quantum information theory, operator theory and geometric number theory. For background including references, see e.g. the books [5, 60, 63, 89, 165].

To conclude this section, note that for a polytope $S \in \mathcal{K}^d$ we have a different behavior of the volume difference $\mathrm{vol}_d\left(S \setminus S_\delta\right)$ than that from Theorem 16. To describe it, we need the notion of flag. A *flag* of a polytope S is a d-tuple $(f_0, \ldots, f_{d-1})$ where f_i is an i-dimensional face of S with $f_i \subset f_{i+1}$. $\mathrm{fl}_d(S)$ denotes the number of flags of the polytope S.

Theorem 18. *Let S be a convex polytope with nonempty interior in $\mathbb{R}^d$. Then*

$$\lim_{\delta \to 0} \frac{\mathrm{vol}_d\left(S\right) - \mathrm{vol}_d\left(S_\delta\right)}{\delta\left(\log \frac{1}{\delta}\right)^{d-1}} = \frac{\mathrm{fl}_d(S)}{d!\, d^{d-1}}.$$

Theorem 18 was proved by Schütt [167, Theorem 1.2]. Recent extensions of this theorem can be found in Besau, Schütt and Werner [13].

5.4. L_p-affine surface area

The concept of affine surface area for convex bodies has been generalized to L_p-affine surface areas. Those are by now the cornerstones of the rapidly developing *L_p-Brunn-Minkowski theory*, initiated in the groundbreaking paper of Lutwak [114]. See also [165, Section 9.1] and, e.g., [146, 74, 116, 132]. The next definition was given by Lutwak [114] for $p > 1$, and Schütt and Werner [174] for all other p. See also Hug [84].

Definition. Let K be a convex body in $\mathbb{R}^d$ such that 0 is in the interior of K. Let $-\infty \le p \le \infty$, $p \neq -d$. The *L_p-affine surface area* of K is

$$\mathrm{as}_p\left(K\right) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{d+p}}}{\langle x, N_K(x)\rangle^{\frac{d(p-1)}{d+p}}}\, \mathrm{d}\,\mu_{\partial K}(x). \tag{20}$$

Here, $N_K(x)$ is the outer unit normal at $x \in \partial K$, $\mu_{\partial K}$ is the usual surface area measure on ∂K and κ is the generalized Gauss-Kronecker curvature at x.

For $p = 0$, $\mathrm{as}_0(K) = d \, \mathrm{vol}_d(K)$. For $p = \pm\infty$, the L_p-affine surface area is defined by the corresponding limit in (20)

$$\mathrm{as}_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa(x)}{\langle x, N_K(x) \rangle^d} \, \mathrm{d}\,\mu_{\partial K}(x),$$

which, for K sufficiently smooth, gives $\mathrm{as}_{\pm\infty}(K) = d \, \mathrm{vol}_d(K^\circ)$, where K° is the polar body (1) of K w.r.t. 0. For $p = 1$ we get the above mentioned affine surface area of K,

$$\mathrm{as}_1(K) = \mathrm{as}(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, \mathrm{d}\,\mu_{\partial K}(x).$$

Note that in general the L_p-affine surface area is not an affine invariant anymore, only a linear invariant. There exist geometric identities, analogous to (17), also for L_p-affine surface area. These use *weighted floating bodies* [188], *Santaló bodies* [132] and *surface bodies* [174]. We refer to those references for the details. Moreover, the corresponding L_p-affine isoperimetric inequalities hold true as well.

Theorem 19. *Let $K \in \mathcal{K}^d$ with the origin in its interior.*

(i) If $p \geq 0$, then

$$\frac{\mathrm{as}_p(K)}{\mathrm{as}_p(B^d)} \leq \left(\frac{\mathrm{vol}_d(K)}{\mathrm{vol}_d(B^d)} \right)^{\frac{d-p}{d+p}}.$$

(ii) If $-d < p < 0$, then

$$\frac{\mathrm{as}_p(K)}{\mathrm{as}_p(B^d)} \geq \left(\frac{\mathrm{vol}_d(K)}{\mathrm{vol}_d(B^d)} \right)^{\frac{d-p}{d+p}}.$$

Equality holds in (i) and (ii) if and only if K is an ellipsoid.

(iii) If K in addition has C^2 boundary with strictly positive Gauss-Kronecker curvature everywhere and if $p < -d$, then

$$c^{\frac{dp}{d+p}} \left(\frac{\mathrm{vol}_d(K)}{\mathrm{vol}_d(B^d)} \right)^{\frac{d-p}{d+p}} \leq \frac{\mathrm{as}_p(K)}{\mathrm{as}_p(B^d)}.$$

The constant c in (iii) is the constant from the reverse Blaschke-Santaló inequality due to Bourgain and Milman [26, Theorem 1].

Theorem 19 was proved by Lutwak [114] for $p > 1$ and by Werner and Ye [191, Theorem 4.2] for all other p.

5.5. Floating measures

Much effort has been devoted to extend the theory of convex bodies to a functional setting (e.g., [9, 6, 54]). Natural analogs of convex bodies in the realm of functions are log-concave functions, i.e. densities of log-concave measures. For such measures we present a notion of *floating measure*. Another approach will be shown in Section 6.

Let $\psi\colon \mathbb{R}^d \to \mathbb{R}$ be a convex function such that

$$0 < \int_{\mathbb{R}^d} e^{-\psi(x)}\,\mathrm{d}\,x < \infty. \tag{21}$$

In the general case, when ψ is neither smooth nor strictly convex, the gradient of ψ, denoted by $\nabla\psi$, exists almost everywhere by Rademacher's theorem [23, Theorem 2.5.1]. A theorem of Busemann and Feller [30] and Aleksandrov [2] guarantees the existence of the (generalized) Hessian, denoted by $\nabla^2\psi$, almost everywhere in $\mathbb{R}^d$ (for details see, e.g., [173, Section 1.6]). The Hessian is a quadratic form on $\mathbb{R}^d$, and if ψ is a convex function, for almost every $x \in \mathbb{R}^d$ one has, when $y \to 0$, that

$$\psi(x + y) = \psi(x) + \langle \nabla\psi(x), y \rangle + \frac{1}{2}\langle \nabla^2\psi(x)(y), y \rangle + o(\|y\|^2).$$

Let μ be a log-concave measure on $\mathbb{R}^d$, i.e. a measure with density $e^{-\psi}$, where $\psi\colon \mathbb{R}^d \to \mathbb{R}$ is a convex function. Note that we do not necessarily require that μ is a probability measure. Let

$$\mathrm{epi}(\psi) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}\colon y \geq \psi(x)\}$$

be the epigraph of ψ. Then $\mathrm{epi}(\psi)$ is a closed convex set in $\mathbb{R}^{d+1}$ and for sufficiently small δ we can define its floating set $\mathrm{epi}(\psi)_\delta$ as

$$\mathrm{epi}(\psi)_\delta = \bigcap_{\{H \in \mathcal{H}\colon\ \mathrm{vol}_d(H^- \cap\, \mathrm{epi}(\psi)) \leq \delta\}} H^+.$$

This was done in [101], where also the definition of a floating set was introduced for convex, not necessarily bounded subsets of $\mathbb{R}^d$.

It is easy to see that there exists a unique convex function $\psi_\delta\colon \mathbb{R}^d \to \mathbb{R}$ such that $(\mathrm{epi}(\psi))_\delta = \mathrm{epi}(\psi_\delta)$. Consequently, Li, Schütt and Werner [101] define the floating function of a convex function ψ and the floating measure of the (not necessarily probability) measure μ as follows.

Definition. Let $\psi\colon \mathbb{R}^d \to \mathbb{R}$ be a convex function. Let $\delta > 0$.

  (i) The *floating function* of ψ is defined to be the function ψ_δ such that

$$(\mathrm{epi}(\psi))_\delta = \mathrm{epi}\,(\psi_\delta)\,.$$

  (ii) Let μ be a measure with density $f(x) = e^{-\psi(x)}$. The *floating measure* of μ is the measure with density f_δ where

$$f_\delta(x) = e^{-\psi_\delta(x)} \quad \text{for } x \in \mathbb{R}^d.$$

Note that when ψ is affine, $\psi_\delta = \psi$ and, for $f = e^{-\psi}$, $f_\delta = f$.

5.6. Affine surface areas for log-concave measures

As far as we know, at present there are two approaches for a definition of affine surface area for log-concave measures. The first one is similar to the one discussed in Section 5.3 and uses the floating measure of Section 5.5 instead of the floating bodies K_δ. It was proposed in [101] and is inspired by the formula of Theorem 16. As in Section 5.5, we do not require that the log-concave measure μ with density $e^{-\psi}$ is a probability measure.

Theorem 20. *Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a convex function such that (21) holds true. Then*

$$
\lim_{\delta \to 0} \frac{\int_{\mathbb{R}^d} e^{-\psi(x)} \, \mathrm{d}\, x - \int_{\mathbb{R}^d} e^{-\psi_\delta(x)} \, \mathrm{d}\, x}{\delta^{2/(d+2)}}
$$
$$
= \frac{1}{2} \left(\frac{d+2}{\mathrm{vol}_d\left(B^d\right)} \right)^{\frac{2}{d+2}} \int_{\mathbb{R}^d} \left(\det \left(\nabla^2 \psi(x) \right) \right)^{\frac{1}{d+2}} \, e^{-\psi(x)} \, \mathrm{d}\, x.
$$

This theorem was proved in [101, Theorem 1]. Its comparison with convex bodies (see Theorem 16) led Li, Schütt and Werner [101] to call the right hand side integral of Theorem 20 the affine surface area of the measure μ.

Definition. For a log-concave measure μ on $\mathbb{R}^d$ with density $e^{-\psi}$ such that (21) holds true, the *affine surface area of the measure* μ is given by

$$
\mathrm{as}\,(\mu) = \int_{\mathbb{R}^d} \left(\det \left(\nabla^2 \psi(x) \right) \right)^{\frac{1}{d+2}} \, e^{-\psi(x)} \, \mathrm{d}\, x. \tag{22}
$$

This definition is further justified as the expression shares many properties of the affine surface area for convex bodies. For instance, it is invariant under affine transformations with determinant 1. For the standard Gaussian measure P we have that $\mathrm{as}\,(P) = 1$.

Another definition of affine surface area for log-concave measures was put forward in Caglar et al. [31]. Actually, an even more general approach was proposed, again for convex functions ψ such that (21) holds true. We put Ω_ψ to be the set of vectors in $\mathbb{R}^d$ at which $\nabla^2 \psi$ exists and is invertible.

Definition. For a log-concave measure μ on $\mathbb{R}^d$ with density $e^{-\psi}$ such that (21) holds true and $\lambda \in \mathbb{R}$, the *λ-affine surface areas* are

$$
\mathrm{As}_\lambda\,(\mu) = \int_{\Omega_\psi} e^{\lambda(2\psi(x) - \langle x, \nabla\psi(x) \rangle)} \left(\det \nabla^2 \psi(x) \right)^\lambda e^{-\psi(x)} \, \mathrm{d}\, x. \tag{23}
$$

We can replace Ω_ψ by $\mathbb{R}^d$ for $\lambda > 0$.

Differentiating with respect to α at $\alpha = 1$, we get in the case of 2-homogeneous convex functions ψ, that is $\psi(\alpha x) = \alpha^2 \psi(x)$, for any $\alpha > 0$ and $x \in \mathbb{R}^d$, that

$$
\langle x, \nabla\psi(x) \rangle = 2\psi(x).
$$

Thus, for 2-homogeneous functions ψ, formula (23) simplifies to

$$\mathrm{As}_\lambda(\psi) = \int_{\Omega_\psi} \left(\det \nabla^2 \psi(x)\right)^\lambda e^{-\psi(x)} \, \mathrm{d}\, x,$$

and definitions (22) and (23) agree for $\lambda = \frac{1}{d+2}$.

To understand why it is justified to name the quantities (23) affine surface areas, we recall the definition of the L_p-affine surface areas (20) for convex bodies K. It was noted in Caglar et al. [31] that the definition of λ-affine surface area for a log-concave density agrees with the definition of L_p-affine surface area for convex bodies if the function is the gauge function $\|\cdot\|_K$ of a convex body K with 0 in its interior,

$$\|x\|_K = \min\{\alpha \geq 0 \colon x \in \alpha K\}.$$

The next theorem is from Caglar et al. [31, Theorem 3].

Theorem 21. *Let K be a convex body in $\mathbb{R}^d$ that contains the origin in its interior. For any $p \geq 0$, let $\lambda = \frac{p}{d+p}$. Then*

$$\mathrm{As}_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) = \frac{(2\pi)^{\frac{d}{2}}}{d\,\mathrm{vol}_d(B^d)}\,\mathrm{as}_p(K).$$

Moreover, if the set of points of ∂K where the generalized Gauss-Kronecker curvature is strictly positive has full measure in ∂K, then the same relation holds true for every $p \neq -d$.

The L_p-affine isoperimetric inequalities for convex bodies of Theorem 19 have analogs for the λ-affine surface areas for log-concave measures. We only mention the case $\lambda \in [0, 1]$ and refer to Caglar et al. [31] for the other cases.

Proposition 22. *Let $\psi \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that (21) holds true and such that $\int_{\mathbb{R}^d} x e^{-\psi(x)} \,\mathrm{d}\, x = 0$. Then we have for all $\lambda \in [0, 1]$,*

$$\frac{\mathrm{As}_\lambda(\mu)}{\mathrm{As}_\lambda\left(\frac{\|\cdot\|^2}{2}\right)} \leq \left(\frac{\int_{\Omega_\psi} e^{-\psi(x)} \,\mathrm{d}\, x}{\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \,\mathrm{d}\, x}\right)^{1-2\lambda}.$$

In particular, if ψ is in addition 2-homogeneous, then

$$\frac{\mathrm{as}(\mu)}{\mathrm{as}\left(\frac{\|\cdot\|^2}{2}\right)} \leq \left(\frac{\int_{\Omega_\psi} e^{-\psi(x)} \,\mathrm{d}\, x}{\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \,\mathrm{d}\, x}\right)^{\frac{d}{d+2}}.$$

Equality holds in the inequalities if and only if there are $a \in \mathbb{R}$ and a positive definite matrix A such that for all $x \in \mathbb{R}^d$

$$\psi(x) = \langle Ax, x \rangle + a.$$

A main ingredient in the proof of this proposition is a functional version of the Blaschke-Santaló inequality. We refer to [9, 6, 54] for the details.

5.7. Applications of affine surface area: Approximation of convex bodies by polytopes

Approximation by polytopes is a central topic in convex geometry with numerous applications. There is a huge amount of literature on the subject. A (very incomplete) list is [20, 69, 152, 168, 68, 81, 111]. We present only one aspect of the subject, approximation by polytopes with a fixed number of vertices and refer to the literature for others.

5.7.1. Best and random approximation

Ideally, in approximation problems, one seeks a best approximating polytope in a given metric. One such result is given in the next theorem, where we consider all polytopes with at most N vertices that are contained in a convex body K. By compactness, there is a polytope P_N in this class with maximal volume. This means that the symmetric difference metric $d_S(K, P_N)$ from (13) is minimal. Such a polytope is called best approximating with respect to the symmetric difference metric.

Theorem 23. *Let K be a convex body in $\mathbb{R}^d$ with C^2-boundary ∂K and everywhere strictly positive Gauss-Kronecker curvature κ. For every $N \in \mathbb{N}$ let P_N be a best approximating polytope of K with at most N vertices. Then*

$$\lim_{N \to \infty} \frac{d_S(K, P_N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}} = \frac{1}{2} \operatorname{del}_{d-1} \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu_{\partial K}(x) \right)^{\frac{d+1}{d-1}}, \qquad (24)$$

where del_{d-1} is a constant depending only on the dimension d.

This theorem was proved by McClure and Vitale [128] in dimension 2 and by Gruber [69] for general dimension. It was shown by Mankiewicz and Schütt [120] that del_{d-1} is of the order of dimension, or more precisely,

$$\frac{d-1}{d+1} \operatorname{vol}_{d-1} \left(B^{d-1} \right)^{-\frac{2}{d-1}} \leq \operatorname{del}_{d-1}$$

$$\leq \frac{d-1}{d+1} \operatorname{vol}_{d-1} \left(B^{d-1} \right)^{-\frac{2}{d-1}} \frac{\Gamma\left(d+1+\frac{2}{d-1}\right)}{d!}. \qquad (25)$$

Note that $\frac{\Gamma\left(d+1+\frac{2}{d-1}\right)}{d!} \leq 1 + c\frac{\log d}{d}$, where c is an absolute constant.

On the right hand side of equation (24) we find the affine surface area of K from Section 5.3. It is natural that such a term should appear in approximation questions: Intuitively, we expect that more vertices of the approximating polytope should be put where the boundary of K is very curved, and fewer points where the boundary is flat, to get a good approximation in the d_S-metric.

However it is only in rare cases that a best approximating polytope can be singled out. Consequently, a common practice is to randomize: Choose N points at random in K with respect to a probability measure P on K. The convex hull

of these randomly chosen points is a random polytope. The expected volume of a random polytope of N points is

$$E(K, N) = \int_K \cdots \int_K \mathrm{vol}_d\left([x_1, \ldots, x_N]\right) \,\mathrm{d}\, P(x_1) \ldots \,\mathrm{d}\, P(x_N),$$

where $[x_1, \ldots, x_N]$ is the convex hull of the points $x_1, \ldots, x_N$. Thus the expression $\mathrm{vol}_d(K) - E(K, N)$ measures how close a random polytope and the convex body are in the symmetric difference metric.

We now compare best approximation with random approximation. The analog to Theorem 23 in the random case is the following theorem. There, the probability measure is the normalized Lebesgue measure on K.

Theorem 24. *Let K be a convex body in $\mathbb{R}^d$. Then*

$$\lim_{N \to \infty} \frac{\mathrm{vol}_d(K) - E(K, N)}{\left(\frac{\mathrm{vol}_d(K)}{N}\right)^{\frac{2}{d+1}}} = c(d) \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \,\mathrm{d}\, \mu_{\partial K}(x),$$

where $c(d)$ is a constant that depends only on d.

This theorem was proved by Rényi and Sulanke [153, 154] in dimension 2. Wieacker [193] settled the case of the Euclidean ball in dimension d. Bárány [11] proved the result for convex bodies with C^3-boundary and everywhere positive Gauss-Kronecker curvature. Finally, the general result for arbitrary convex bodies was proved by Schütt [168] and Böröczky, Fodor and Hug [22].

Notice that Theorem 24 does not give the optimal dependence on N for best approximation. One reason is that not all the points chosen at random from K appear as vertices of the approximating random polytope. Thus we now choose the points randomly from the boundary of K according to a measure with a density with respect to $\mu_{\partial K}$. We denote by $E(K, f, N)$ the expected volume of the corresponding random polytope. Which density is optimal? It turns out that it is, up to normalization, the $(d+1)$-root of the generalized Gauss-Kronecker curvature. The integral of this function is the affine surface area. The next theorem was shown by Schütt and Werner [173, Theorem 1.1], see also Reitzner [151].

Theorem 25. *Let K be a convex body in $\mathbb{R}^d$ such that there are $0 < r \leq R < \infty$ so that we have for all $x \in \partial K$*

$$B^d(x - rN_K(x), r) \subseteq K \subseteq B^d(x - RN_K(x), R),$$

for $N_K(x)$ an outer unit normal of K at x, and let $f \colon \partial K \to (0, \infty)$ be continuous with $\int_{\partial K} f(x) \,\mathrm{d}\, \mu_{\partial K}(x) = 1$. Then

$$\lim_{N \to \infty} \frac{\mathrm{vol}_d(K) - E(K, f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}} = c(d) \int_{\partial K} \frac{\kappa(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \,\mathrm{d}\, \mu_{\partial K}(x) \qquad (26)$$

where

$$c(d) = \frac{(d-1)^{\frac{d+1}{d-1}} \Gamma\left(d+1+\frac{2}{d-1}\right)}{2(d+1)!(\mathrm{vol}_{d-2}\left(\partial B^{d-1}\right))^{\frac{2}{d-1}}}.$$

The minimum at the right-hand side of (26) *is attained for the normalized affine surface area measure with density*

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{d+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, \mathrm{d}\, \mu_{\partial K}(x)} \quad \text{for } x \in \partial K.$$

Best approximation of Theorem 23 differs from random approximation of Theorem 25 only in the dimensional constants del_{d-1} and $c(d)$. Comparing those, using also (25), an amazing fact follows: with the density f_{as} random approximation is almost as good as best approximation,

$$\lim_{N\to\infty} \frac{\mathrm{d}_S(K,P_N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}} \leq \lim_{N\to\infty} \frac{\mathrm{vol}_d\left(K\right) - E(K, f_{as}, N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}}$$

$$\leq \left(1 + c\frac{\log d}{d}\right) \lim_{N\to\infty} \frac{\mathrm{d}_S(K,P_N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}},$$

where c is an absolute constant.

5.7.2. The floating body algorithm

Bárány and Larman [12, Theorem 1] established a relation between floating bodies and random polytopes for the uniform measure on the convex body: With high probability the volume of a random polytope is close to the volume of an appropriate floating body.

Theorem 26. *Let K be a convex body in $\mathbb{R}^d$. Then there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$*

$$c_1 \left(\mathrm{vol}_d\left(K\right) - \mathrm{vol}_d\left(K_{\frac{1}{N}\,\mathrm{vol}_d(K)}\right)\right) \leq \mathrm{vol}_d\left(K\right) - E(K, N)$$

$$\leq c_2 \left(\mathrm{vol}_d\left(K\right) - \mathrm{vol}_d\left(K_{\frac{1}{N}\,\mathrm{vol}_d(K)}\right)\right),$$

where c_1 and c_2 are constants that depend on d only.

Even more can be said about the connection between floating bodies and random polytopes. There is an algorithm, the floating body algorithm, where, for a given convex body K in $\mathbb{R}^d$, one uses floating bodies to construct a polytope P_N with as few vertices N as possible such that for a suitable δ, $K_\delta \subseteq P_N \subseteq K$ and such that P_N approximates the convex body K very well in the symmetric difference metric. It should be noted that we make no assumption on K.

We describe this algorithm: We are choosing the vertices $x_1, \ldots, x_N \in \partial K$ of the polytope P_N. Point x_1 is chosen arbitrarily. Having chosen $x_1, \ldots, x_{k-1}$ we choose x_k such that

$$\{x_1, \ldots, x_{k-1}\} \cap \text{Int}\left(K \cap H^-(x_k - \Delta_k N_K(x_k), N_K(x_k))\right) = \emptyset$$

where $N_K(x_k)$ denotes a (not necessarily unique) outer normal to ∂K at x_0, $\text{Int}(C)$ is the interior of a set $C \subset \mathbb{R}^d$, and Δ_k is determined by

$$\text{vol}_d\left(K \cap H^-(x_k - \Delta_k N_K(x_k), N_K(x_k))\right) = \delta.$$

The next theorem can be found in Schütt [169].

Theorem 27. *Let K be a convex body in $\mathbb{R}^d$. Then, for all δ with $0 < \delta \leq \frac{1}{4e^4} \text{vol}_d(K)$ there exists $N \in \mathbb{N}$ with*

$$\text{vol}_d\left(K \setminus K_\delta\right) \leq N \left(\frac{c}{4e^4}\right)^d \text{vol}_d\left(B^d\right) 4e^3\delta$$

where c is a universal constant, and there exists a polytope P_N that has at most N vertices and such that

$$K_\delta \subseteq P_N \subseteq K.$$

How well does this polytope approximate K? It follows from Theorem 27 that

$$\limsup_{N \to \infty} \frac{\text{d}_S(K, P_N)}{\left(\frac{1}{N}\right)^{\frac{2}{d-1}}} \leq c \, d^2 \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, \text{d}\,\mu_{\partial K}(x)\right)^{\frac{d+1}{d-1}}.$$

This should be compared to (24). Since del_{d-1} is of the order of d, both expressions differ only by a factor of the order of dimension d.

6. Floating bodies of measures

The definition of the (convex) floating body of a convex body $K \in \mathcal{K}^d$ discussed in Section 5 extends naturally also to general probability measures, in a manner different from that in Section 5.5. It is closely related to the halfspace depth. Analogously to the approach of Dupin [47], consider the following definition.

Definition. Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ and $\delta > 0$. We say that the nonempty convex set $P_{[\delta]}$ is the *floating body* of P with index δ if for each supporting halfspace H^+ of $P_{[\delta]}$ we have $P(H^-) = \delta$.

For P distributed uniformly on a convex body $K \in \mathcal{K}^d$ of unit volume, $P_{[\delta]} = K_{[\delta]}$. Therefore, the floating body $P_{[\delta]}$ does not exist for $\delta > 1/2$, and it may happen that it does not exist for any $\delta > 0$, see the example of (the uniform distribution on) a triangle from Example 5. Unlike in the situation with the floating body of $K \in \mathcal{K}^d$, even if the floating body $P_{[\delta]}$ of $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ exists, it may not be uniquely defined. Take, for instance, a distribution P on $\mathbb{R}$ whose support is not contiguous, such as that displayed in Figure 7. For $\delta = 1/4$,

and q_2 the $(1 - \delta)$-quantile of P, each interval $[q_1, q_2]$ for $q_1 \in [0, 1]$ is a floating body of P. Note that if P has contiguous support and $P_{[\delta]}$ exists, then it is unique.

To avoid these problems, let us consider, as in the case of convex bodies, the convex floating body of P, given by an intersection of halfspaces.

Definition. Let $P \in \mathcal{P}(\mathbb{R}^d)$. For $\delta \geq 0$, the convex floating body of P with index δ is defined as the intersection of all closed halfspaces whose defining hyperplanes cut off a set of probability content at most δ from P, i.e.

$$P_\delta^{FB} = \bigcap_{P(H^-) \leq \delta} H^+, \tag{27}$$

where $H \in \mathcal{H}$ and H^+ and H^- are its associated closed halfspaces.

Note that with the convention that the intersection of an empty collection of subsets of $\mathbb{R}^d$ is $\mathbb{R}^d$, convex floating bodies of a measure are always well defined, unique, convex subsets of $\mathbb{R}^d$. It can happen that $P_\delta^{FB} = \emptyset$, especially for larger values of δ. It is easy to see that for $P \in \mathcal{P}(\mathbb{R}^d)$ distributed uniformly on $K \in \mathcal{K}^d$ with $\mathrm{vol}_d(K) = 1$, $P_\delta^{FB} = K_\delta$ for any $\delta \geq 0$, and the convex floating bodies of measures generalize the convex floating bodies discussed throughout Section 5.

(Convex) floating bodies for general measures have already been considered in the literature, mainly due to the association of convex bodies and log-concave measures established by Ball [10]. The previous definitions were considered by Werner [188], Bobkov [18], Fresen [57, 58], and Brunel [28], among others. In connection with the halfspace depth, the floating bodies (27) were considered in Nolan [143], and Massé and Theodorescu [126]. In the latter paper, those regions are called the δ-trimmed regions of P.

The convex floating body of a measure P is very closely related to the depth central region P_δ, defined in (7) as the upper level set of the depth $hD(\cdot; P)$. Indeed, recall the characterization of Rousseeuw and Ruts [160, Proposition 6], who showed that for any $P \in \mathcal{P}(\mathbb{R}^d)$ and $\delta > 0$

$$P_\delta = \bigcap_{P(H^+) > 1 - \delta} H^+.$$

On the other hand, it is not difficult to see that the convex floating body (27) can be written also in the form

$$P_\delta^{FB} = \bigcap_{P(H^+) \geq 1 - \delta} H^+. \tag{28}$$

Now it is obvious that for all $\delta \geq 0$ we have that

$$P_\delta^{FB} \subseteq P_\delta,$$

and under the assumption of the contiguity of the support of P,

$$P_\delta = P_\delta^{FB}.$$

These results were noted by Kong and Mizera [90, Theorem 2] and Brunel [28, Lemma 1]. For general measures P it may happen that the convex floating body is a proper subset of the depth central region, see Figure 7.

It is interesting to investigate which results for convex bodies described in Section 5 carry over to measures.
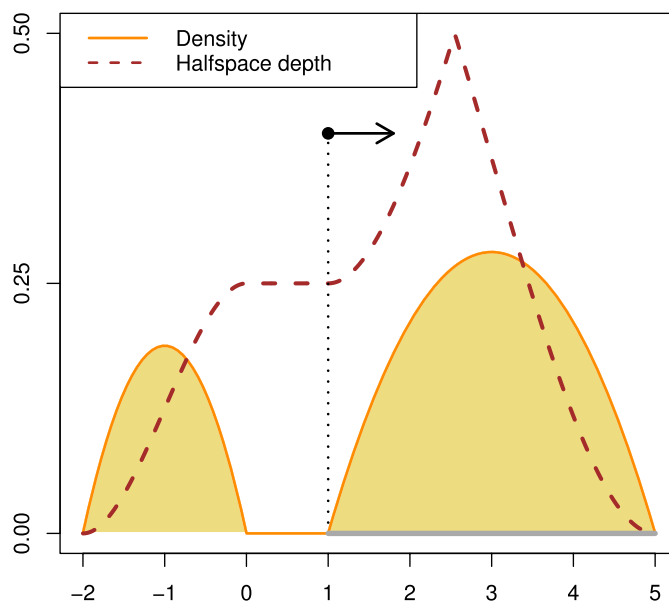

FIG 7. *For distributions whose support is not contiguous, the convex floating body* (27) *and the halfspace depth central region* (7) *may differ. In this example, the density of $P \in \mathcal{P}\left(\mathbb{R}^1\right)$, supported on disjoint intervals $[-2, 0]$ and $[1, 5]$, is displayed (orange line), along with its halfspace depth function (dashed brown line). For $\delta = 1/4$, the left endpoint of the interval P_δ is 0. But the complement of the halfline $[1, \infty)$ (black arrow) has probability $1/4$, and the left endpoint of the convex $1/4$-floating body of P is 1. Points in the interval $(0, 1)$ are not boundaries of any convex floating body of P.*

Let us first relate the floating body $P_{[\delta]}$ with the the convex floating body P_δ^{FB} and the central region P_δ. If a unique floating body $P_{[\delta]}$ of a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ exists, then the corresponding convex floating body P_δ^{FB} must be equal to $P_{[\delta]}$. For the sake of completeness, let us provide an elementary proof of this result.

Proposition 28. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ have contiguous support. Then $P_\delta^{FB} = P_\delta$ for any $\delta > 0$. If, in addition, also $P_{[\delta]}$ exists, then $P_\delta^{FB} = P_\delta = P_{[\delta]}$.*

Proof. Recall that if $P_{[\delta]}$ exists, then it is unique as P has contiguous support. For P with contiguous support, the proof of $P_\delta^{FB} = P_\delta$ can be found in [28, Lemma 1].

We show now that $P_{[\delta]} = P_\delta$. We show first that $P_\delta \subseteq P_{[\delta]}$. Let $x \notin P_{[\delta]}$. By the Hahn-Banach separation theorem there is a support hyperplane H_0 to

$P_{[\delta]}$ that strictly separates x and $P_{[\delta]}$, i.e., $x \in \text{Int}(H_0^-)$, $P_{[\delta]} \subset H_0^+$ and $P_{[\delta]} \cap H_0 \neq \emptyset$. Since H_0 is a supporting hyperplane to $P_{[\delta]}$, $P(H_0^-) = \delta$. Then, as $P_\delta = \bigcap_{P(H^-) \leq \delta} H^+$, $x \notin P_\delta$.

Now we show that $P_{[\delta]} \subseteq P_\delta$. Suppose not. Then there exists $x \in P_{[\delta]}$ such that, by (28), $x \notin H_1^+$ for some H_1 with $P(H_1^+) \geq 1 - \delta$. In that case there must exist a hyperplane H_2 with $x \in H_2^- \subsetneq H_1^-$. Because H_2^- lies completely in the open halfspace complementary to H_1^+, by the contiguity of P we know that $P(H_2^-) < \delta$. This contradicts $x \in P_{[\delta]}$, as for P contiguous the boundary hyperplane of any closed halfspace with probability δ must support $P_{[\delta]}$. □

Now we explore whether analogues of Propositions 12–15 stated for convex bodies in Section 5 hold true also for measures.

6.0.1. *Proposition 12 for measures*

A result analogous to part (i) of Proposition 12 would require that the infimum in the definition of the halfspace depth (3) can be replaced by a minimum, i.e. that a minimal halfspace of hD exists at each $x \in \mathbb{R}^d$ for any $P \in \mathcal{P}(\mathbb{R}^d)$. For measures that that do not satisfy (4) this is not true, as noted already by Rousseeuw and Ruts [160, Remark 1]. There, the following example is given.

Example 6. Let $P \in \mathcal{P}(\mathbb{R}^2)$ be a mixture of the standard bivariate Gaussian distribution and the Dirac measure at the point $(1, 1)$, with equal mixing proportions. Then, at $x = (0, 1)$, we have $hD(x; P) = \Phi(-1)/2$, where Φ is the distribution function of the standard univariate Gaussian distribution, see also Example 2. Yet, no minimal halfspace at x exists.

For a different example of the same phenomenon, see Massé [124, Section 2]. For distributions that satisfy (4), a minimal halfspace always exists for all $x \in \mathbb{R}^d$. That was shown, e.g., by Massé [124, Proposition 4.5 (i)].

An extension of Dupin's theorem (part (ii) of Proposition 12) to probability distributions was stated in Hassairi and Regaieg [78, Theorem 3.1]. Here we provide a version of that result with a slightly modified set of assumptions. The proof of the proposition follows very closely the original proof from [78, Theorem 3.1], and is omitted.

Proposition 29. *Let $X \sim P \in \mathcal{P}(\mathbb{R}^d)$ be absolutely continuous with contiguous support $\text{Supp}(P)$ and let $x \in \mathbb{R}^d$ be such that $hD(x; P) > 0$. Denote by f_u the density of the random variable given by $\langle X - x, u \rangle$ with $u \in \mathbb{S}^{d-1}$. Suppose that $f_u(y)$ is continuous as a function of $u \in \mathbb{S}^{d-1}$ and y in a neighborhood of $0 \in \mathbb{R}$. Let $H^- \in \mathcal{H}^-$ be a minimal halfspace at x, i.e. $x \in H$ and $P(H^-) = hD(x; P)$. Then*

$$x = \frac{\int_H y f(y) \, \mathrm{d}\, y}{\int_H f(y) \, \mathrm{d}\, y},$$

i.e. x is the conditional expectation of P given H. The integrals in the formula above are taken with respect to the $(d-1)$-dimensional Lebesgue measure on H.

One has to be careful with the statement of Proposition 29. Without the required continuity properties of the marginal densities f_u, the conditional expectation of P given a hyperplane H, may not even be well defined. To illustrate our point, we give an example that was brought to our attention by M. Tancer [183].

Example 7. Let $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ be distributed uniformly on the union of two squares with vertices $(1,0)$, $(1,2)$, $(-1,2)$, $(-1,0)$, and $(2,0)$, $(2,-4)$, $(-2,-4)$, $(-2,0)$, respectively, see the left panel of Figure 8. Consider $x = (\varepsilon, 0)$ for $-1/2 \le \varepsilon \le 1/2$. A simple computation shows that $hD(x; P) = 1/5$, and the unique minimal halfspace at all such points x is the halfspace H^+ that cuts off the smaller square from P. A direct analogue of Dupin's theorem would now assert that the conditional expectation of $H = \partial H^+$ is not unique — any x on the line segment L that joins $(-1/2, 0)$ and $(1/2, 0)$ would be a candidate for the barycenter of P given H. The problem here, of course, is due to the discontinuity of the marginal density of P at H. For this particular H, the conditional expectation of P given H is not properly defined.

Problem 5. Does a version of Dupin's theorem (i.e. a variant of Proposition 29) hold true also under weaker conditions on measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$?
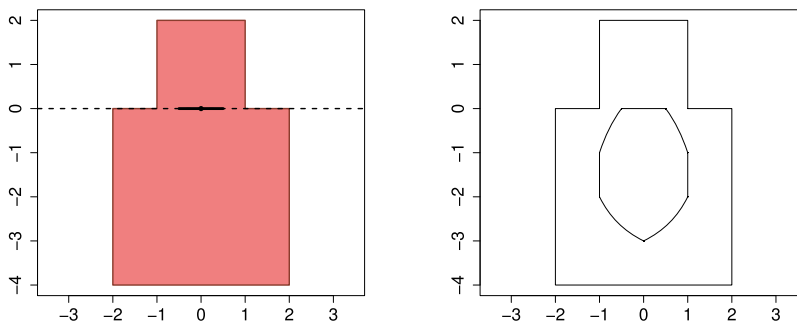

FIG 8. *Left panel: polygon where the measure P is supported in Example 7, line segment L (thick solid line), and the unique minimal hyperplane at $x = (\varepsilon, 0)$ for $|\varepsilon| \le 1/2$ (dashed line). Right panel: central region P_δ of P for $\delta = 0.2$. This region is not strictly convex, as it contains the line segment L between the points $(-1/2, 0)$ and $(1/2, 0)$.*

To see that the strict convexity of the central regions (part (iii) of Proposition 12) does not hold true for all measures, it is enough to return to Example 7. Indeed, due to the considerations made there, the line segment L lies on the boundary of the central region $P_\delta = P_\delta^{FB}$ for $\delta = 1/5$, and $P_{1/5}$ is not strictly convex, see also the right panel of Figure 8. For another example where the strict convexity of P_δ is violated, recall the collection of α-symmetric distributions from Example 2 for $\alpha \le 1$, and the right panel of Figure 4. In Example 7, the problem appears to stem from discontinuity of the density of P at the boundary of a minimal halfspace. For α-symmetric distributions the problem is that the expectation of P is not defined.

Problem 6. Under which conditions are the central regions P_δ and the convex floating bodies P_δ^{FB} strictly convex?

An extension of part (iv) of Proposition 12 was given by Mizera and Volauf [134, Proposition 7], who stated that if (4) is true for P with contiguous support, then the halfspace median of P is a unique point.

6.0.2. Proposition 13 for measures

The (Dupin's) floating body $P_{[\delta]}$ of a general measure P may not exist. Sufficient conditions for the existence of floating bodies of probability measures appear to be a challenging problem of great importance in mathematical statistics, and the theory of data depth (see, e.g., [28, Open question 1], or [124, 125]). Many theoretical results on the behavior of the depth and its central regions hold true only under the assumption of existence of floating bodies of P, see also the discussion in Section 8 below. Brunel [28, Open question 2] asks a question that can be rephrased as follows:

Is it true that for any log-concave measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ all floating bodies $P_{[\delta]}$ for $\delta > 0$ small enough exist?

From the example of the uniform distribution on a triangle (Example 5), we see that the answer to the above question is negative. Though, under the additional assumption of central symmetry of P, similar properties have been investigated by Meyer and Reisner [129] for convex bodies (see Proposition 13 above), and extended to certain probability measures by Bobkov [18, Section 6]. In the latter paper, it is shown that all $P_{[\delta]}$ exist for centrally symmetric s-concave measures with $s \geq -1$. As far as we are aware, the following theorem from [18, Theorem 6.1] is, up to date, the most general result on the existence of floating bodies of measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$.

Theorem 30. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be a centrally symmetric s-concave measure with $s \geq -1$ such that $\mathrm{Supp}(P)$ is a d-dimensional subset of $\mathbb{R}^d$. Then $P_{[\delta]}$ exists for all $\delta \in (0, 1/2]$.*

As remarked by Bobkov [18], it is not known whether the restriction $s \geq -1$ can be dropped.

Problem 7. Do the floating bodies of all centrally symmetric s-concave measures with full-dimensional support exist?

Problem 8. Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be centrally symmetric with a sufficiently smooth density f that is positive on $\mathbb{R}^d$. Suppose that all the upper level sets of the density $\{x \in \mathbb{R}^d \colon f(x) \geq t\}$ are (strictly) convex. Does this imply that the floating bodies of P exist for $\delta \in (0, 1/2]$?

As shown in the following theorem, there exists a close connection between the question of existence of floating bodies, and the problem of smoothness of the boundaries of the central regions P_δ.

Proposition 31. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ satisfy* (4), $\delta \in (0, 1/2)$, *and let P_δ be a convex body whose boundary is C^1. Then P_δ is a floating body.*

Proof. Let $x \in \partial P_\delta$. Since, under (4), the depth $hD(\cdot; P)$ is continuous on $\mathbb{R}^d$ ([134, Proposition 1], or Section 3.2.5 above), $hD(x; P) = \delta$. Using [124, Proposition 4.5 (i)] there exists a minimal halfspace $H^- \in \mathcal{H}^-$ at x, and H^+ then must support P_δ at x by Proposition 6. Because of the smoothness of the boundary of P_δ, there is only a single supporting halfspace of P_δ at each $x \in \partial P_\delta$. Thus, we have shown that for any supporting halfspace H^+ of P_δ, $P(H^-) = \delta$, and P_δ is a floating body of P. □

Smoothness of boundaries of P_δ was recognized to be crucial in establishing theoretical properties of hD already by Nolan [143], and Massé and Theodorescu [126]. Many theoretical results stated for the halfspace depth in statistics rely on that condition. For instance, as shown by Massé [124, Theorem 2.1], the asymptotic distribution of the sample halfspace depth at x is Gaussian if the boundary of P_δ passing through x has a unique minimal halfspace. For another application of the smoothness of boundaries of floating bodies see Section 8 below.

Despite being of critical importance, so far the only examples of distributions with smooth contours of hD are the (full-dimensional affine images of) α-symmetric distributions with $\alpha > 1$, see Example 2. As discussed in [64], apart from those distributions, no other multivariate measure with smooth depth contours is known in statistics. In that paper, it is also shown that simple distributions such as mixtures of multivariate Gaussian distributions, and distributions with smooth centrally symmetric, or smooth strictly quasi-concave densities, may have points at which the boundary of P_δ is not smooth. It is therefore remarkable that Meyer and Reisner [129, Theorem 3] (part (ii) of Proposition 13 above) showed that for certain (centrally) symmetric convex bodies, the boundaries of K_δ exhibit a high degree of smoothness. We are not aware of any result giving sufficient conditions for higher order differentiability of the boundary of the depth central regions, or convex floating bodies of measures, in statistics.

Problem 9. Under which conditions have the central regions P_δ and the convex floating bodies P_δ^{FB} boundaries of type C^1 or C^2?

6.0.3. Propositions 14 and 15 for measures

Problem 10. Are there analogues of Proposition 14 and Proposition 15 for measures?

6.1. Application: Multivariate extremes and depth

The intimate connections of floating bodies with the approximation problems described in Section 5.7 have analogues for probability measures. If $X_1, \ldots, X_n$ is a random sample from distribution $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, one can ask how fast does the

random polytope given by the convex hull of these random points grow to the convex hull of the support of P. In conjunction with the advances for uniform measures on convex bodies outlined in Section 5.7, it is not surprising that the halfspace depth and floating bodies of measures play a prominent role in these problems.

The following theorem, called the multivariate Gnedenko law of large numbers, can be found in Fresen [58, Theorem 2].

Theorem 32. *Let $q > 0$ and $p > 1$, and let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be a probability measure with a density of the form $f(x) = ce^{-g(x)^p}$ where $g \colon \mathbb{R}^d \to [0, \infty)$ is a convex function and $c > 0$. Then there exist constants $c_1, c_2 > 0$ such that for a random sample $X_1, \ldots, X_n$ of any size $n \in \mathbb{N}$ with $n \geq d + 2$ from P, it holds true that*

$$\mathsf{P}\left(\mathrm{d}_H\left([X_1, \ldots, X_n], P_{1/n}\right) \leq c_1 \frac{\log \log n}{(\log n)^{1-1/p}}\right) \geq 1 - c_2\left(\log n\right)^{-q}, \qquad (29)$$

where $[X_1, \ldots, X_n]$ is the closed convex hull of the points $X_1, \ldots, X_n$, and $P_{1/n}$ is the depth central region P_δ with $\delta = 1/n$.

Distributions P from Theorem 32 are sometimes called p-log-concave measures. For usual log-concave measures, an inequality only slightly weaker than (29) is given in Fresen [58, Theorem 1].

Theorem 32 asserts that, with large probability, convex hulls of large random samples from P behave as the halfspace depth central regions P_δ for very small values of δ. This observation opens a whole new field of applications of the depth in multivariate extreme value theory. Indeed, by now, data depth has been used in statistics predominantly as a robust tool that identifies the central parts of the probability mass of distributions, and little attention was paid to its behavior near the tails. Theorem 32 gives a probabilistic interpretation also to the boundaries of those depth regions that correspond to the extreme depth-quantiles. It is also interesting to compare Theorem 32 with the recent advances of Einmahl, Li and Liu [51] and He and Einmahl [79]. There, the authors employ extreme value theory in order to estimate P_δ for low values of δ reliably from the data. It will be interesting to see what can be obtained by a proper combination of the estimation techniques from the latter papers, and the asymptotic representations of Fresen [58].

In a further analogue with the exposition from Section 5, one may study the limit behavior of the quantity

$$P\left(P_0\right) - P\left(P_\delta\right) = 1 - P\left(P_\delta\right) \qquad (30)$$

as $\delta \to 0$ from the right. More specifically, assume that the difference (30) is scaled properly, so that the resulting limit is a finite, non-negative number $\Omega(P)$. The characteristic $\Omega(P)$, together with the sequence of its scaling constants, is then an affine invariant on $\mathcal{P}\left(\mathbb{R}^d\right)$. $\Omega(P)$ is not a generalized notion of the affine surface area such as the functional from Section 5.6, but it is interesting in its own right. From the viewpoint of statistics, $\Omega(P)$ may serve as an index of

heavy-tailedness of the distribution P, where not only the size of the tails is evaluated, but also "the complexity of the boundary" of $\mathrm{Supp}(P)$ is taken into account.

7. Mahalanobis ellipsoids and the halfspace depth

Let $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be distributed uniformly on $K \in \mathcal{K}^d$. The body K is said to be *isotropic*, or in the *isotropic position*, if $\mathrm{vol}_d(K) = 1$, $\mathrm{E}\,X = 0$, and $\mathrm{Var}\,X = L_K^2 I_d$ where $L_K > 0$ is a constant and I_d the $d \times d$ identity matrix. Geometrically, this means that the barycenter of K is at the origin and that the ellipsoid of inertia of K, or equivalently, all Mahalanobis ellipsoids of P from (6), are Euclidean balls. The constant

$$L_K = \frac{1}{d} \ \left(\int_K \|x\|^2 \,\mathrm{d}\,x \right)^{\frac{1}{2}}$$

is called the *isotropic constant* of K.

The isotropic constant plays an important role in the analysis of convex bodies. We refer to e.g. Milman and Pajor [133] and the book of Brazitikos et al. [27, Chapter 3]. The conjecture that for all $K \in \mathcal{K}^d$ the constant L_K is bounded from above by an absolute constant independent of the dimension d is one of the major open problems in geometric analysis. The best known upper estimate so far, due to Klartag [88], is that $L_K \leq c\, d^{\frac{1}{4}}$ for an absolute constant c, improving an earlier estimate by Bourgain [25] by a logarithmic factor. The conjecture is equivalent to the *hyperplane conjecture*, first formulated by Bourgain, which asks whether every centered convex body of volume 1 has a hyperplane section through the origin whose $(d-1)$-dimensional volume is greater than an absolute positive constant, independent of dimension d. We refer to e.g. [27, Section 3.1] for details.

For any K there exists an affine transformation T such that $T(K)$ is isotropic, and the isotropic position is uniquely determined up to orthogonal transformations. The isotropic constant of a general body $K \in \mathcal{K}^d$ is then defined as the isotropic constant of the corresponding isotropic body $T(K)$.

Similarly, we can define the isotropic constant for probability measures P with log-concave density f. A measure that corresponds to $X \sim P$ is *isotropic* if it is centered, i.e. $\mathrm{E}\,X = 0$, and if for all $u \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^d} \langle x, u \rangle^2 f(x) \,\mathrm{d}\,x = 1,$$

or, equivalently, $\mathrm{Var}\,X = I_d$. Then

$$L_P = \left(\sup_{x \in \mathbb{R}^d} f(x) \right)^{\frac{1}{d}}$$

is the *isotropic constant* of P. The isotropic constant of a general probability measure P with log-concave density is, again, given as the isotropic constant

of an affine image of P that is isotropic [27, Chapter 2]. Note that a convex body $K \in \mathcal{K}^d$ of volume 1 is isotropic, if and only if the density of the uniform distribution on the convex body K/L_K is an isotropic log-concave density.

For bodies and log-concave measures in isotropic position, many important geometrical results are known. In this section we state one that relates to the subjects of data depth and floating bodies.

Proposition 33. *The following holds true:*

(i) For any isotropic convex body $K \in \mathcal{K}^d$ and any $\delta \in \left(0, \frac{1}{e}\right)$

$$\left(\frac{1}{e} - \delta\right) L_K B^d \subseteq K_\delta \subseteq 17 \log\left(\frac{1}{\delta}\right) L_K B^d.$$

(ii) For any isotropic measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with a log-concave density

$$\left(\frac{1}{e} - \delta\right) L_P B^d \subseteq P_\delta \subseteq 17 \log\left(\frac{1}{\delta}\right) L_P B^d.$$

This proposition was proved by Milman and Pajor [133, Proposition in the Appendix], and re-stated by Fresen [57] who also gave the formulation in part (ii) for isotropic log-concave measures. Its further extension to centrally symmetric s-concave measures with $s > -\infty$ can be found in Bobkov [18, Theorem 5.1]. Part (i) of this proposition is a special case of more general relations between floating bodies and *p-centroid bodies* which can be found in [146, Theorem 2.2].

Proposition 33 has important implications for the theory of halfspace depth. By affine equivariance of the halfspace depth central regions P_δ, for any log-concave measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with expectation $\mu \in \mathbb{R}^d$ and a positive definite variance matrix $\Sigma \in \mathbb{R}^{d \times d}$,

$$\left\{x \in \mathbb{R}^d \colon \, \mathrm{d}_\Sigma(x, \mu) \leq \left(\frac{1}{e} - \delta\right) L_P\right\} \subseteq P_\delta$$

$$\subseteq \left\{x \in \mathbb{R}^d \colon \, \mathrm{d}_\Sigma(x, \mu) \leq 17 \log\left(\frac{1}{\delta}\right) L_P\right\}$$

where d_Σ is the Mahalanobis distance from (5). Therefore, all central regions of the halfspace depth for $\delta < 1/e$ of log-concave measures are, up to a constant that depends only on δ and L_P, isomorphic to the Mahalanobis ellipsoids given by the covariance structure of P. This corroborates the findings from statistics, where it has been long observed that the depth central regions P_δ tend to take more "ellipsoidal" shapes than the level sets of the densities, see also Figures 3 and 4 above. Results in this section provide quantitative statements that support those claims.

Problem 11. Is it possible to state an analogue of Proposition 33 also for more general probability measures?

8. Characterization of distributions

One of the most important open questions connected with the halfspace depth is the halfspace depth characterization problem. It has been conjectured (e.g. [42, p. 2306] and [90, p. 1598]) that for each distribution $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ there exists a unique depth surface $\left\{hD(x; P) \colon x \in \mathbb{R}^d\right\}$, i.e., that all probability distributions are determined by their halfspace depth. Such a result would be invaluable in statistics, as it would assert that just as the distribution function or the characteristic function of a random vector, also the halfspace depth could be used as a complete representative of any probability distribution.

Recently, the depth characterization conjecture was disproved in [138], where an example of two different probability distributions with the same depth at all points in $\mathbb{R}^d$, $d \geq 2$, was given. The example employs collections of different α-symmetric distributions with $\alpha \leq 1$ whose projections coincide along some directions.

Even though the general characterization conjecture turned out to be false, important partial positive results to the characterization problem can be found in the literature. Thanks to the results of Struyf and Rousseeuw [182], Koshevoy [92], and Hassairi and Regaieg [77] we know that if $P, Q \in \mathcal{P}\left(\mathbb{R}^d\right)$ are distributions whose supports are finite subsets of $\mathbb{R}^d$, then $hD(x; P) = hD(x; Q)$ for all $x \in \mathbb{R}^d$ implies $P = Q$. For non-atomic distributions, two results can be found in the literature in the papers of Hassairi and Regaieg [78], and Kong and Zuo [91]. In this section we show that the last two theorems are special cases of the following theorem.

Theorem 34. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ have contiguous support, and let $x_P \in \mathbb{R}^d$ be the halfspace median of P. Then the following are equivalent:*

(**FB$_1$**) *For each $\delta \in (0, 1/2)$ the floating body $P_{[\delta]}$ of P exists.*
(**FB$_2$**) *(4) holds true, and*

$$P(H^-) = \begin{cases} \sup_{x \in H} hD(x; P) & \text{for any } H \in \mathcal{H} \text{ with } x_P \notin H^-, \\ 1 - \sup_{x \in H} hD(x; P) & \text{for any } H \in \mathcal{H} \text{ with } x_P \in H^-. \end{cases}$$
(31)

*Consequently, if (**FB$_1$**) is true, there is no other probability distribution that satisfies (**FB$_1$**) with the same depth at all points in $\mathbb{R}^d$.*

Proof. Assume first that (**FB$_1$**) is true. We show first that (4) holds. Suppose it does not hold. Then there exists a hyperplane H such that $P(H) > 0$. Without loss of generality we can assume that $P(H^-) \leq P(H^+)$. We put $\delta = P(H^-) - \frac{3}{4}P(H)$. Then $0 < \delta < 1/2$. We claim that the floating body $P_{[\delta]}$ does not exist, for if it does exist, then there is a supporting hyperplane H_1 to $P_{[\delta]}$ parallel to H such that $P(H_1^-) = \delta$. Note that $P(H^-) = \delta + \frac{3}{4}P(H) > \delta$, and it must be that $H_1^- \subsetneq H^-$. But, in that case, $P(H_1^-) \leq P(H^-) - P(H) < \delta$, a contradiction.

Take now any hyperplane $H \in \mathcal{H}$, and define $\psi(H^-) = \sup_{x \in H} hD(x; P)$. For any $x \in H$ we have

$$hD(x; P) = \inf\{P(G^-) \colon G^- \in \mathcal{H}^-, x \in G\} \leq P(H^-) \tag{32}$$

since the halfspace $H^- \in \mathcal{H}^-$ belongs to the collection over which the infimum is taken. Because (32) is valid for any $x \in H$, $\psi(H^-) \leq P(H^-)$.

To prove the other inequality, assume that $\delta = P(H^-) > 0$. Otherwise, trivially $\psi(H^-) \geq P(H^-) = 0$. Further, it is possible to assume that $\delta \leq 1/2$. If this is not the case, take $H^+ \in \mathcal{H}^-$, the closed halfspace complementary to H^-, and proceed with H^+ (note that in the latter case, we know by (4) that $P(H^+) \leq 1/2$ and $P(H^+) + P(H^-) = 1$). We first treat the case $\delta < 1/2$. Because all floating bodies of P are assumed to exist and because $P(H^-) = \delta$, the hyperplane H supports the floating body $P_{[\delta]}$ of P. That is, there must exist a point $x_H \in H \cap P_{[\delta]}$. As $x_H \in P_{[\delta]} = P_\delta = \{y \in \mathbb{R}^d : hD(y; P) \geq \delta\}$,

$$P(H^-) = \delta \leq hD(x_H; P) \leq \sup_{x \in H} hD(x; P) = \psi(H^-).$$

Thus (31) holds for $\delta < 1/2$. By continuity, it also holds for $\delta = 1/2$. Hence $(\mathbf{FB_1})$ implies that the probability of halfspaces is characterized by their depth as in (31).

For the opposite implication, assume that $(\mathbf{FB_2})$ is true and let $\delta \in (0, 1/2)$. Consider the depth level set P_δ. This is a convex compact set. From (31) with H^- such that $P(H^-) = 1/2$ and the continuity of the depth $hD(\cdot; P)$ guaranteed by (4), we see that P_δ must be non-empty for all $\delta \in (0, 1/2)$. Take any $H \in \mathcal{H}$ such that $P(H^-) = \delta$, and consider the family $\mathcal{G} \subset \mathcal{H}$ of all hyperplanes parallel to H. Then P_δ must be supported by some $G \in \mathcal{G}$ with $G^- \subseteq H^-$ or $G^- \supseteq H^-$. If $P(G^-) = \delta' > \delta$, $(\mathbf{FB_2})$ cannot be true as $P_{\delta'} \subset P_\delta$ by the nestedness and convexity of the central regions, and the continuity of hD. Indeed, because G supports P_δ, for all $x \in G$ either $x \in \partial P_\delta$ or $x \notin P_\delta$. In both cases $hD(x; P) \leq \delta$, since, using the continuity of hD again, $hD(x; P) = \delta$ for any $x \in \partial P_\delta$. By (31) this means that we have $\delta' = P(G^-) = \sup_{x \in G} hD(x; P) \leq \delta$, a contradiction. If $\delta' \leq \delta$, then there must exist $x_0 \in G \cap P_\delta$. But then $\delta \leq hD(x_0; P) \leq P(G^-) = \delta' \leq \delta$, and necessarily $P(G^-) = \delta' = \delta$. Because P has contiguous support, this means that $G = H$, and P_δ is supported by H. As this is true for any $H \in \mathcal{H}$ such that $P(H^-) = \delta$, $P_\delta = P_{[\delta]}$, and $(\mathbf{FB_2}) \implies (\mathbf{FB_1})$.

The characterization of P follows from $(\mathbf{FB_2})$ by a theorem of Cramér and Wold [40], see also [16, p. 383]. □

Note that a further minor extension of Theorem 34 can be obtained if P is allowed to have a single atom at its halfspace median x_P, with obvious modifications to the statement and the proof of this theorem.

By Theorem 34 and Proposition 31 we obtain that all α-symmetric distributions with $\alpha > 1$, and their full-dimensional affine images, satisfy $(\mathbf{FB_1})$. This array of examples complements the known examples of s-concave centrally symmetric measures with $s \geq -1$ from Theorem 30, for which $(\mathbf{FB_1})$ is true. As far

as we know, there are no further examples of measures satisfying $(\mathbf{FB_1})$ known at this time.

To see that there exist distributions $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ that satisfy $(\mathbf{FB_1})$, but not the assumptions of Proposition 31, take $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ to be the uniform distribution on a square in $\mathbb{R}^2$ from Example 5. For P it is known [98, pp. 433–434] that $(\mathbf{FB_1})$ is true, yet each floating body $P_{[\delta]}$ for $\delta \in (0, 1/2)$ contains four non-smooth points at its boundary, see also the left panel of Figure 3.

Condition $(\mathbf{FB_1})$ is, however, still rather strict. Not only does it impose (4) on P, but also it means that P must be halfspace symmetric. For (uniform measures on) convex bodies, this was noted by Meyer and Reisner [129, Lemma 4]. The next proposition extends that result to probability measures. Its proof follows closely the arguments of Meyer and Reisner [129, Lemma 4], and is omitted.

Proposition 35. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ have contiguous support. If $(\mathbf{FB_1})$ is true for P, then P must be halfspace symmetric.*

Problem 12. Describe the collection of all probability measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ whose halfspace depth is unique, i.e. there is no $Q \neq P$ with $hD(x;P) = hD(x;Q)$ for all $x \in \mathbb{R}^d$. Is the existence of the expectation $\mathrm{E}\,X$ for $X \sim P$ sufficient for the halfspace depth of P to be unique? Is the uniform distribution on a simplex in $\mathbb{R}^d$ characterized by its halfspace depth?

Problem 13. If condition $(\mathbf{FB_1})$ is not satisfied, how can one reconstruct the probability content of all halfspaces $P(H^-)$ from the depth $hD(x;P)$ for all $x \in \mathbb{R}^d$ only?

8.1. Characterization theorem of Kong and Zuo (2010)

In [91, Theorem 3.2] it is shown that if, for $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with contiguous support,

$$\text{for all } \delta \in (0, 1/2) \text{ the boundary of the central region } P_\delta \text{ is } C^1, \qquad (33)$$

and (4) holds, then (31) is true, and P is characterized by its halfspace depth. In Theorem 34 we provide a generalization of this result. Indeed, by Proposition 31 above, if (33) and (4) are true, then the floating body $P_{[\delta]}$ of P exists for all $\delta \in (0, 1/2)$, and Theorem 34 can be used.

8.2. Characterization theorem of Hassairi and Regaieg (2008)

Let us state a characterization result for the halfspace depth for absolutely continuous distributions that can be found in [78, Theorem 3.2]. For this, we define for any $x \in \mathbb{R}^d$ the *halfspace function*

$$\phi_x \colon \mathbb{S}^{d-1} \to [0, 1] \colon u \mapsto P\left(H^-_{u, \langle x, u \rangle}\right),$$

where $H^-_{u, \langle x, u \rangle} \in \mathcal{H}^-$ is the closed halfspace in $\mathbb{R}^d$ whose outer normal is parallel to u, and $x \in H_{u, \langle x, u \rangle}$.

Theorem 36. *Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be as in Proposition 29, and suppose that*

> *for all $x \in \mathbb{R}^d$, if ϕ_x has a local minimum at $u = u(x) \in \mathbb{S}^{d-1}$,*
> *then $\phi_x(u) = hD(x; P)$.* $\qquad$ (34)

Then (31) *holds true, and P is characterized by its halfspace depth.*

In [78, Theorem 3.2], condition (34) is formulated in a slightly different manner in terms of derivatives of functions related to ϕ_x. It is easy to see that for P that satisfies the conditions from Proposition 29, (34) and the corresponding condition from [78] are equivalent.

If P satisfies (4), then for any $x \in \mathbb{R}^d$ the function ϕ_x is continuous on $\mathbb{S}^{d-1}$ [124, Proposition 4.5]. Thus, it must attain a global minimum over its domain. Condition (34) therefore means that there cannot exist any local minimum of ϕ_x that is not global.

Suppose for a moment that (4) is valid for P. By Theorem 36, (34) implies the characterization result (31) which is, by Theorem 34, equivalent with (**FB₁**). Therefore, given that (4) is true, Condition (34) implies (**FB₁**), and the characterization of Hassairi and Regaieg [78] is a special case of Theorem 34 above[3].

8.3. Homothety conjecture

In convex geometry, the following open question, similar in nature to the depth characterization conjecture, was posed in [172]:

> Let a convex body $K \in \mathcal{K}^d$ and one of its convex floating bodies K_δ be homothetic, i.e. $K_\delta = \lambda K + x$ for some $\lambda > 0$ and $x \in \mathbb{R}^d$. Is then K necessarily an ellipsoid?

Schütt and Werner [172] showed that if K is homothetic to a sequence of its floating bodies K_{δ_n} with $\delta_n \to 0$, then K must be an ellipsoid. Stancu [180] demonstrated that for K with a sufficiently smooth boundary, K is homothetic to K_δ for a single small δ also implies that K is an ellipsoid. The latter result was later refined in [192].

Problem 14. Does the homothety conjecture hold true? More generally, which convex bodies are characterized by any of their convex floating bodies?

9. Conclusions and further perspectives

In this survey, we discussed little known relations of the concept of halfspace depth, studied extensively in statistics, and paradigms well known in functional

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[3]Nonetheless, it must be noted that smoothness of the density of P is not sufficient for (34) to hold true (cf. [78, p. 2312]). To see this, consider the uniform distribution P on a triangle from Example 1. This distribution has a smooth density in the interior of $\mathrm{Supp}(P)$, yet $\Pi(P) = 4/9$, and $\sup_{x \in H} hD(x; P) \leq 4/9 < 1/2$ for any $H \in \mathcal{H}$. Thus, the probability of halfspaces cannot be recovered as in (31), at least not for $H \in \mathcal{H}$ with $1/2 \geq P(H^-) > 4/9$. It is easy to see that (34) is violated for P, too.

analysis and geometry. In Section 4 we saw that the depth of the halfspace median is a particular example of a more general concept of measures of symmetry. In Sections 5 and 6 we focused on the floating body and its possible generalizations towards (probability) measures. These little explored junctions of mathematical statistics and geometry are, however, hardly limited only to the halfspace depth hD defined in finite-dimensional linear spaces $\mathbb{R}^d$. In this concluding section of our paper our intention is to outline, and properly refer to, a few further links between the statistics of depth functions, and current research in pure mathematics.

9.1. Depth in non-linear spaces

By directional data one understands data that live on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ [121]. Each observation can be interpreted as a direction of a non-zero vector in $\mathbb{R}^d$. Such data appear quite naturally, and it is of great interest to find depth functions suitable also for this kind of observations. Several definitions of depth have been proposed for directional data [177, 106, 1, 99, 145]. The following depth, proposed by Small [177], is an analogue of the halfspace depth for directional data.

Definition. Let $P \in \mathcal{P}\left(\mathbb{S}^{d-1}\right)$ and $x \in \mathbb{S}^{d-1}$. The *angular halfspace depth* (or *angular Tukey depth*) of x w.r.t. P is defined as

$$AhD\left(x; P\right) = \inf\left\{P(H^-)\colon H \in \mathcal{H}_0, x \in H\right\},$$

where $\mathcal{H}_0$ denotes the set of hyperplanes $H \in \mathcal{H}$ in $\mathbb{R}^d$ such that $0 \in H$.

It is natural to consider the collection $\mathcal{H}_0$ in the definition of AhD, as $\mathcal{H}_0 \cap \mathbb{S}^{d-1}$ is the collection of all closed hemispheres of $\mathbb{S}^{d-1}$. Therefore, it is not surprising that also for spherical convex bodies, concepts similar to floating bodies have been investigated. Recall that for $K \subset \mathbb{S}^{d-1}$, K is said to be *spherically convex* if the radial extension of K, given by

$$\operatorname{rad} K = \left\{\lambda x \colon x \in K, \lambda \geq 0\right\},$$

is a convex set in $\mathbb{R}^d$. A closed spherically convex subset of $\mathbb{S}^{d-1}$ such that the interior of $\operatorname{rad} K$ is nonempty is called a *spherical convex body*. Analogues of floating bodies and convex floating bodies for spherical convex bodies were studied by Besau and Werner [14].

Definition. For a spherical convex body $K \subset \mathbb{S}^{d-1}$ and $P \in \mathcal{P}\left(\mathbb{S}^{d-1}\right)$ uniformly distributed on K take $\delta \geq 0$. The *spherical convex floating body* of K is defined as

$$\bigcap_{H \in \mathcal{H}_0 \colon P(H^-) \leq \delta} H^+.$$

Just as in Section 6 it is possible to define floating bodies, and convex floating bodies also for general probability measures on $\mathbb{S}^{d-1}$, and it is easy to see that

the spherical convex floating body coincides with the central regions of the angular halfspace depth for uniform distributions on spherical convex bodies. Some results in the spirit of those discussed in Section 5 can be obtained also for spherical convex floating bodies [14]. In another paper, Besau and Werner [15] provide extensions of those results also to certain Riemannian manifolds. Research in this direction in the statistics of data depth is still only in its beginnings [55].

9.2. Depth for infinite-dimensional data

In statistics, since the work of Liu and Singh [107] and Fraiman and Muniz [56], considerable attention has focused also on devising depth functions applicable to data from high-dimensional, and infinite-dimensional (functional) spaces. Direct applications of the halfspace depth are known to be inadequate [48], but many other depth functions that are suited for functional data can be found in the literature [43, 108, 137, 39, 36, 140, 142, 65]. In geometry, some advances that appear to be related are the floating functions [101] considered in Section 5.5 above. Solid connections between these two areas of research appear to be uncharted.

9.3. Centroid body and simplicial volume depth

Apart from the halfspace depth, the simplicial depth, and the Mahalanobis depth mentioned above, there exists an abundance of other depth functions defined in $\mathbb{R}^d$ in statistics. A comprehensive survey on some of those is [196], where, based on the ideas of Oja [144], also the following depth function can be found. Recall that by $[x_1, \ldots, x_{d+1}]$ we mean the closed convex hull of $x_1, \ldots, x_{d+1} \in \mathbb{R}^d$.

Definition. Let $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be such that $\operatorname{Var} X = \Sigma$ is a positive definite matrix and $x \in \mathbb{R}^d$. The *simplicial volume depth* (or *Oja depth*) of x w.r.t. P is defined as

$$svD(x; P) = \left(1 + \operatorname{E} \frac{\operatorname{vol}_d\left([x, X_1, \ldots, X_d]\right)}{\sqrt{\det \Sigma}}\right)^{-1}, \tag{35}$$

where $X_1, \ldots, X_d \sim P$ are independent.

The factor $\sqrt{\det \Sigma}$ ensures the affine invariance of svD. Similarly as the Mahalanobis depth MD, also svD is not defined for all $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, but only for distributions with finite second moments, and positive definite variance matrices.

For (a uniform distribution on) a compact (possibly non-convex) set $K \subset \mathbb{R}^d$ with $\operatorname{vol}_d(K) > 0$, a concept closely related to $x \mapsto \operatorname{vol}_d\left([x, X_1, \ldots, X_d]\right)$, that is central in (35), is that of the centroid body of K. The *centroid body* of K is a convex body $Z \in \mathcal{K}^d$ defined via its support function (2)

$$h_Z(u) = \frac{1}{\operatorname{vol}_d(K)} \int_K |\langle x, u \rangle| \, \mathrm{d}\, x.$$

If K is (centrally) symmetric around around the origin, ∂Z is the locus of centroids of all intersections of halfspaces $H^- \in \mathcal{H}^-$ such that $0 \in H$ with K. As discussed in [60, Section 9.1], this body was defined by Petty [148], but its earlier predecessors can be traced back to the work of Dupin [47]. The volume of the centroid body Z of K determines the simplicial volume depth svD of $0 \in \mathbb{R}^d$ with respect to the the uniform distribution on K. The next theorem can be found in Gardner [60, Theorem 9.1.5]. Extensions not listed here can be found in [148, 166]. For star bodies $K \subset \mathbb{R}^d$ a version of this theorem is given in [165, Section 10.8].

Theorem 37. *Let $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be uniformly distributed on a compact set $K \subset \mathbb{R}^d$ with $\mathrm{vol}_d(K) > 0$. Denote $\mathrm{Var}\, X = \Sigma$. Let Z_x be the centroid body of $K - x$. Then*

$$
svD(x; P) = \left(1 + \frac{2^d}{\mathrm{vol}_d(K)^d} \frac{\mathrm{vol}_d(Z_x)}{\sqrt{\det \Sigma}} \right)^{-1}.
$$

Centroid bodies have been the subject of numerous studies in geometry and functional analysis. We only refer here to [165, Section 10.8] and [27, Section 5.1] and the references therein for a comprehensive account of results that can be found in the literature on centroid bodies and their extensions.

Problem 15. Is it possible to extend Theorem 37 also to more general probability measures?

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