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Inventiones mathematicae

Hall algebras and quantum groups

Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld, D-4800 Bielefeld 1, FRG

Dedicated to Jacques Tits on his sixtieth birthday

Let R be a finite-dimensional representation-finite hereditary algebra over some field. Let Δ be its type, this is a disjoint union of Dynkin diagrams [DR]. Let Φ^+ be the set of positive roots for Δ. Given $\alpha \in \Phi^+$, there is (up to isomorphism) a unique indecomposable R-module $M(\alpha)$ with dimension vector α. Given a function $a\colon \Phi^+ \to \mathbb{N}_0$, let $M(a)$ denote the direct sum of $a(\alpha)$ copies of the various $M(\alpha)$ with $\alpha \in \Phi^+$; in this way, the isomorphism classes of R-modules of finite length correspond bijectively to the functions $a\colon \Phi^+ \to \mathbb{N}_0$. Given a, $b, c\colon \Phi^+ \to \mathbb{N}_0$, we denote by $\varphi^{M(b)}_{M(a), M(c)} = \varphi^b_{ac}$ the corresponding Hall polynomial [R1], it is a polynomial with integer coefficients which counts (for finite R) the number of filtrations of $M(b)$ with factors $M(a)$ and $M(c)$. If Λ is an arbitrary commutative ring, and $q \in \Lambda$, we define the Hall algebra $\mathscr{H}(R, \Lambda, q)$ as the free Λ-module with basis $(u_{[M]})_{[M]}$ indexed by the isomorphism classes of R-modules of finite length, with multiplication

$$u_{[N]}\, u_{[N']} = \sum_{[M]} \varphi^M_{NN'}(q)\, u_{[M]},$$

in this way, we obtain a (usually non-commutative) associative ring with 1. In [R2], we have shown that we may identify $\mathscr{H}(R, \mathbb{C}, 1)$ with the universal enveloping algebra $U(\mathbf{n}_+)$ of $\mathbf{n}_+$, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of the semisimple complex Lie algebra of type Δ.

It would be of interest to find a natural enlargement of $\mathscr{H}(R, \mathbb{C}, 1)$ in order to obtain $U(\mathbf{g})$ itself. As we will show in Sect. 3, there is a canonical way for obtaining at least $U(\mathbf{b}_+)$, where $\mathbf{b}_+ = \mathbf{h} \oplus \mathbf{n}_+$ is the Borel algebra. Let $S_1, \ldots, S_s$ be a complete set of simple R-modules. If M is an R-module of finite length, let $(\dim M)_i$ be the Jordan-Hoelder multiplicity of S_i in M. Then the map δ_i of $\mathscr{H}(R, \Lambda, q)$ into itself defined by $\delta_i(u_{[M]}) = (\dim M)_i\, u_{[M]}$ is a derivation, so we may define the skew polynomial ring

$$\mathscr{H}'(R, \Lambda, q) = \mathscr{H}(R, \Lambda, q)\, [T_i, \delta_i]_i$$

in s variables $T_1, \ldots, T_s$. Since $\mathscr{H}(R, \mathbb{C}, 1)$ is isomorphic to $U(\mathbf{n}_+)$, it follows that $\mathscr{H}'(R, \mathbb{C}, 1)$ is isomorphic to $U(\mathbf{b}_+)$.

Instead of dealing with the degenerate Hall algebra $\mathscr{H}'(R, \mathbb{C}, 1)$, we are going to consider the generic Hall algebra $\mathscr{H}'(R, \mathbb{C}[q], q)$, where $\mathbb{C}[q]$ is the polynomial ring in the indeterminate q, or its completion

$$\widehat{\mathscr{H}'}(R) = \varprojlim_{m} \mathscr{H}'(R, \mathbb{C}[q]/(q-1)^m, q),$$

this is an algebra over the power series ring $\mathbb{C}[[q-1]]$. Our aim is to give a complete description of $\widehat{\mathscr{H}'}(R)$ by generators and relations.

In $\mathbb{C}[[q-1]]$, the element $\ln q = \sum_{m \geqq 1} (-1)^{m+1} \frac{1}{m} (q-1)^m$ is a multiple of $q-1$, thus, for $c \in \mathbb{C}$, the element $\exp(c \ln q) = \sum_{m \geqq 0} \frac{1}{m!} c^m (\ln q)^m$ is defined. We also will write q^c instead of $\exp(c \ln q)$, in particular, both $q^{\frac{1}{2}}$ and $q^{-\frac{1}{2}}$ are defined. We denote by $\begin{bmatrix} n \\ t \end{bmatrix}_q = \frac{\varphi_n}{\varphi_t \varphi_{n-t}}$ the Gauss polynomials, where $\varphi_n = (1-q)\dots(1-q^n)$.

Let $(a_{ij})_{ij}$ be the Cartan matrix of type Δ, and $(f_i)_i$ the (minimal) symmetrization of A (so that $f_i a_{ij} = f_j a_{ji}$). Let $q_i = q^{f_i}$. We will show that $\widehat{\mathscr{H}'}(R)$ is, as a complete $\mathbb{C}[[q-1]]$-algebra, generated by elements $H_1, \dots, H_s$, $X_1, \dots, X_s$ subject to the relations

$$[H_i, H_j] = 0,$$
$$[H_i, X_j] = a_{ij} X_j,$$
$$\sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_{q_i} q_i^{-\frac{t(n-t)}{2}} X_i^t X_j X_i^{n-t} = 0, \quad \text{with } n = 1 - a_{ij}, \quad \text{and } i \neq j.$$

This description shows that $\widehat{\mathscr{H}'}(R)$ is precisely the quantization $U_h(\mathbf{b}_+)$ of $U(\mathbf{b}_+)$ as described by Drinfeld in his Berkeley lecture [D] (with $h = \ln q$). In particular, it follows that $\widehat{\mathscr{H}'}(R)$ is a Hopf algebra.

The Hall algebra approach yields a rather natural interpretation of the awkward relations above. Consider besides

$$\rho_n(q, X, Y) = \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{-\frac{t(n-t)}{2}} X^t Y X^{n-t}$$

also the polynomials

$$\rho_n^+(q, X, Y) = \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{\binom{t}{2}} X^t Y X^{n-t},$$
$${}^+\rho_n(q, X, Y) = \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{\binom{t}{2}} X^{n-t} Y X^t.$$

Observe that $\mathscr{H}(R, \mathbb{C}[[q-1]], q)$ is a subring of $\widehat{\mathscr{H}'}(R)$. The elements $X_1, \dots, X_s$ of $\widehat{\mathscr{H}'}(R)$ are suitable multiples of the canonical generators $u_1 = u_{[S_1]}, \dots, u_s = u_{[S_s]}$ of $\mathscr{H}(R, \mathbb{C}[[q-1]], q)$. The relations which are satisfied by

$u_1, \ldots, u_s$ and which give rise to the relations above, depend on the orientation of Δ defined by R. So assume $\mathrm{Ext}^1(S_i, S_j) = 0$ for some pair $i \neq j$. We will show that

$$\rho^+_{1-a_{ij}}(q_i, u_i, u_j) = 0, \qquad \text{and} \quad {}^+\rho_{1-a_{ji}}(q_j, u_j, u_i) = 0,$$

and a simple substitution transforms these relations into the symmetric ones involving ρ instead of ρ^+ and ${}^+\rho$. The relations involving ρ^+ and ${}^+\rho$ will be shown in a quite general setting in Sect. 2. In order to do so, we will introduce in Sect. 1 the composition algebra $\mathscr{C}(R)$ for an arbitrary ring R.

The reader should be aware that q (and $q_i = q^{f_i}$) may denote an integer, or a variable, in different parts of the paper.

The author is endebted to R. Dipper, B. Pareigis, and L. Scott for helpful comments: they insisted that there should be a strong relationship between Hall algebras as presented in [R1] and [R2] and the recent advances on Hopf algebras and quantum groups.

1. Composition algebras

Let R be any ring, let $\mathscr{S}$ be the set of isomorphism classes of finite simple R-modules (where 'finite' means: having only a finite number of elements). Let $\mathscr{W}(R)$ be the free semigroup with basis $\mathscr{S}$, thus the elements of $\mathscr{W}(R)$ are words of the form $w = [S_1][S_2]\ldots[S_t]$, where $S_1, \ldots, S_t$ are finite simple R-modules, and $[S_i]$ denotes the isomorphism class of S_i; here, t is the length of the word w, and there is a unique word of length zero (denoted by 1). We denote by $\mathscr{A}(R)$ the free (associative) algebra with basis $\mathscr{S}$. Clearly, the additive group of $\mathscr{A}(R)$ is the free abelian group with basis $\mathscr{W}(R)$. Given an element $w \in \mathscr{W}(R)$, say $w = [S_1]\ldots[S_t]$, and an R-module M, let $\langle w | M \rangle$ denote the number of filtrations

$$M = M_0 \supset M_1 \supset \ldots \supset M_t = 0$$

such that $M_{i-1}/M_i \cong S_i$. (The number of such filtrations always is finite: if M has at least one such filtration, then M is a finite module, and so has only finitely many submodules.) In general, given $\sum_{i=1}^{n} \lambda_i w_i \in \mathscr{A}(R)$, with $\lambda_i \in \mathbb{Z}$, $w_i \in \mathscr{W}(R)$, and an R-module M, we define

$$\langle \sum_{i=1}^{n} \lambda_i w_i | M \rangle = \sum_{i=1}^{n} \lambda_i \langle w_i | M \rangle.$$

Let $\mathscr{I}(R)$ be the set of all $a \in \mathscr{A}(R)$, with $\langle a | M \rangle = 0$ for all R-modules M. This is an ideal of $\mathscr{A}(R)$. (For $a \in \mathscr{A}(R)$ and S a finite simple R-module, $\langle [S]\, a | M \rangle = \sum_U \langle a | M \rangle$, where the summation ranges over all submodules U of M such that $M/U \cong S$; similarly, $\langle a[S] | M\rangle = \sum_V \langle a | M/V\rangle$, where the summation ranges over all submodules V of M with M/V isomorphic to S.) Define

$$\mathscr{C}(R) = \mathscr{A}(R)/\mathscr{I}(R),$$

the *composition algebra* of R. Note that $\langle - | - \rangle$ yields a bilinear form

$$\mathscr{C}(R) \times K(R\text{-fin}) \to \mathbb{Z}.$$

Assume that the ring R is finitary, so that the Hall algebra $\mathscr{H}(R)$ is defined. Consider the ring homomorphism $\eta\colon \mathscr{A}(R) \to \mathscr{H}(R)$ sending $[S]$ to $u_{[S]}$. Then $\mathscr{I}(R) = \ker \eta$. (For, $\eta([S_1]\dots[S_t]) = \sum_{[M]} F^M_{S_1,\dots,S_t} u_{[M]}$ and $F^M_{S_1,\dots,S_t} = \langle [S_1]\dots[S_t] | M\rangle$; therefore, given $a \in \mathscr{A}(R)$, we have $\eta(a) = \sum_{[M]} \langle a | M\rangle u_{[M]}$.) As a consequence, we can identify $\mathscr{C}(R)$ with the subring of $\mathscr{H}(R)$ generated by the elements of the form $u_{[S]}$ with $[S] \in \mathscr{S}$.

2. The fundamental relations

Let R be a finitary ring. Let $S_i (i \in I)$ be a complete set of finite simple R-modules (thus, they are pairwise non-isomorphic, and any finite simple R-module is isomorphic to one of them). We assume that $\operatorname{Ext}^1(S_i, S_i) = 0$ for all i. Let $q_i = |\operatorname{End}(S_i)|$. Let $i \neq j$ with $\operatorname{Ext}^1(S_i, S_j) = 0$, and

$$a_{ij} = -\dim \operatorname{Ext}^1(S_j, S_i)_{\operatorname{End}(S_i)},$$
$$a'_{ij} = -\dim_{\operatorname{End}(S_j)} \operatorname{Ext}^1(S_j, S_i),$$

thus $q_i^{a_{ij}} = q_j^{a'_{ij}}$.

Proposition. *Both elements $\rho^+_{1-a_{ij}}(q_i, [S_i], [S_j])$ and ${}^+\rho_{1-a'_{ij}}(q_j, [S_j], [S_i])$ belong to $\mathscr{I}(R)$.*

Proof. We first consider ρ^+. We are going to calculate

$$a_t(M) := \langle [S_i]^t [S_j] [S_i]^{n-t} | M\rangle$$

for an arbitrary module M. We may assume that M is of length $n+1$, with one composition factor S_j, the remaining ones of the form S_i. Since $\operatorname{Ext}^1(S_i, S_i) = 0 = \operatorname{Ext}^1(S_i, S_j)$, we can decompose $M = N \oplus dS_i$, with N indecomposable and some $0 \leqq d \leqq n$. The radical N' of N is isomorphic to $(n-d)S_i$, and $N/N' \cong S_j$. Since $\dim \operatorname{Ext}^1(S_j, S_i)_{\operatorname{End}(S_i)} = n-1$, it follows that $d \geqq 1$. Note that M does not have a factor module isomorphic to $(d+1)S$, thus $a_t(M) = 0$ for $t > d$. Therefore, we may assume $t \leqq d$. The composition series of M we are interested in are of the form

$$M = M_0 \supset M_1 \supset \dots \supset M_{n+1} = 0$$

with $M_t/M_{t+1} \cong S_j$. In particular, $N \subseteq M_t$, since $M/N \cong dS_i$. There are $\frac{v_d}{v_{d-t}}(q_i)$ possibilities for choosing chains

$$M = M_0 \supset M_1 \supset \dots \supset M_t \supseteq N$$

with M_i maximal in M_{i+1}, for $1 \leqq i \leqq t$, where $v_n = v_n(T) = \frac{\varphi_n(T)}{(1-T)^n}$. Always, M_t has a unique submodule M_{t+1} with $M_t/M_{t+1} \cong S_j$, and since $M_{t+1} \cong (n-t)\,S_i$, there are $v_{n-t}(q_i)$ composition series

$$M_{t+1} \supset M_{t+2} \supset \dots \supset M_n \supset M_{n+1} = 0.$$

Thus

$$a_t(M) = \frac{v_d\, v_{n-t}}{v_{d-t}}(q_i), \qquad \text{for all } t \leqq d.$$

We claim that for $1 \leqq d \leqq n$, we have

$$\sum_{t=0}^{d} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix} T^{\binom{t}{2}} \frac{v_d\, v_{n-t}}{v_{d-t}} = 0. \tag{$*$}$$

But the evaluation of this polynomial at q_i is just $\rho^+_{1-a_{ij}}(q_i, [S_i], [S_j])$, so this will finish the first part of the proof. We use

$$\begin{bmatrix} n \\ t \end{bmatrix} \frac{v_d\, v_{n-t}}{v_{d-t}} = \frac{\varphi_n}{\varphi_t\, \varphi_{n-t}} \cdot \frac{\varphi_d\, \varphi_{n-t}}{\varphi_{d-t}} \cdot \frac{1}{(1-T)^n} = v_n \begin{bmatrix} d \\ t \end{bmatrix},$$

in order to rewrite the left hand side $(*)$. We recall from [M] (I.2.Ex.3) that

$$E_d(T, X) := \sum_{t=0}^{d} \begin{bmatrix} n \\ t \end{bmatrix} T^{\binom{t}{2}} X^t = \prod_{i=0}^{d-1} (1 + T^i X).$$

Since $d \geqq 1$, the right hand side shows that $E_d(T, -1) = 0$, therefore

$$\sum_{t=0}^{d} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix} T^{\binom{t}{2}} \frac{v_d\, v_{n-t}}{v_{d-t}} = v_n \sum_{t=0}^{d} (-1)^t \begin{bmatrix} d \\ t \end{bmatrix} T^{\binom{t}{2}} = v_n\, E_d(T, -1) = 0.$$

In order to deal with ${}^+\rho$, we may use a corresponding calculation. Alternatively, we may argue as follows: Without loss of generality, we may assume that S_i, S_j are the only simple R-modules, thus R is a finite ring, and, in fact a k-algebra for some finite field k. We apply the previous considerations to the dual modules S_j^*, S_i^*, which we consider as R^{op}-modules. This is possible, since $\mathrm{Ext}^1_{R^{\mathrm{op}}}(S_j^*, S_i^*)=0$. Given an R-module M, we have

$$\langle [S_j^*]^t [S_i^*] [S_j^*]^{n-t} | M^* \rangle = \langle [S_j]^{n-t} [S_i] [S_j]^t | M \rangle,$$

this finishes the proof.

As a consequence, we see that $\mathscr{C}(R)$ always may be considered as a factor algebra of $\mathscr{A}(R)/\mathscr{I}(R)$.

3. Adjunction of $Hom_{\mathbb{Z}}(\mathbf{K}(\mathbf{R}), \mathbb{Z})$

Let R be a finitary ring. The class of all finite R-modules will be denoted by R-fin_0. Recall that a function d: R-$\mathrm{fin}_0 \to \mathbb{Z}$ is said to be additive on exact sequences provided $d(X)-d(Y)+d(Z)=0$ for any exact sequence $0\to X\to Y\to Z\to 0$ in R-fin_0.

Lemma. *Let d: R-$\mathrm{fin}_0 \to \mathbb{Z}$ be additive on exact sequences. Define an additive function δ_d: $\mathscr{H}(R)\to\mathscr{H}(R)$ by $\delta_d(u_{[M]})=d(M)\,u_{[M]}$, for any finite R-module M. Then δ_d is a derivation.*

Proof. Let N, N' be finite R-modules. Then

$$\begin{aligned}\delta_d(u_{[N]}\,u_{[N']}) &= \delta\Big(\sum_{[M]} F^M_{N,N'}\,u_{[M]}\Big) = \sum_{[M]} F^M_{N,N'}\,d(M)\,u_{[M]}\\ &= \sum_{[M]} F^M_{N,N'}(d(N)+d(N'))\,u_{[M]}\\ &= d(N)\,u_{[N]}\,u_{[N']} + u_{[N]}\,d(N')\,u_{[N']}\\ &= \delta_d(u_{[N]})\,u_{[N']} + u_{[N]}\,\delta_d(u_{[N']}).\end{aligned}$$

As in the previous section, let S_i, $i\in I$ be a complete set of finite simple R-modules. For $i\in I$, and $M\in R$-fin_0, let $d_i(M)=(\dim M)_i$ be the Jordan-Hoelder multiplicity of S_i in M. Then d_i is additive on exact sequences (and $(d_i)_i$ is a basis of the free abelian group of all functions R-$\mathrm{fin}_0\to\mathbb{Z}$ which are additive on exact sequences). So we obtain a set of derivations $\delta_i=\delta_{d_i}$ of $\mathscr{H}(R)$.

Let $\mathscr{H}'(R)$ be obtained from $\mathscr{H}(R)$ by forming the skew polynomial ring

$$\mathscr{H}'(R)=\mathscr{H}(R)\,[T_i,\delta_i]_i$$

defined by the commutation rules

$$\begin{aligned}&[T_i, T_j]=0,\\ &[T_i, u_{[M]}]=\delta_i(u_{[M]})=(\dim M)_i\,u[M]\end{aligned}$$

for all $i,j\in I$, and all $M\in R$-fin_0.

Assume now that R is representation-directed, let Λ be an arbitrary commutative ring, and $q\in\Lambda$. Given a function d: R-$\mathrm{fin}_0\to\mathbb{Z}$ which is additive on exact sequences, we define δ_d: $\mathscr{H}(R,\Lambda,q)\to\mathscr{H}(R,\Lambda,q)$ by $\delta_d(u_{[M]})=d(M)\,u_{[M]}$, and again we see that δ_d is a derivation. In particular, we obtain the derivations δ_i with $\delta_i(u_{[M]})=(\dim M)_i\, u_{[M]}$, and we define

$$\mathscr{H}'(R,\Lambda,q)=\mathscr{H}(R,\Lambda,q)\,[T_i,\delta_i]_i$$

with the same commutation rules as above.

4. Completion

Let k be a finite field, let R be a finite-dimensional k-algebra with centre k which is representation-finite and hereditary. Let Δ be its type, it is a Dynkin diagram (since R is supposed to be connected). Let $S_1, \ldots, S_s$ be the simple R-modules, we assume that they are indexed in such a way that $\mathrm{Ext}^1(S_i,S_j)=0$ for $j<i$. We define $a_{ii}=2$, and, for $j<i$

$$a_{ij}=-\dim \mathrm{Ext}^1(S_j,S_i)_{\mathrm{End}(S_i)},$$
$$a_{ji}=a'_{ij}=-\dim_{\mathrm{End}(S_j)} \mathrm{Ext}^1(S_j,S_i).$$

Thus, $A=(a_{ij})_{ij}$ is the Cartan matrix of type Δ. Let $f_i=\dim_k \mathrm{End}(S_i)$, thus $(f_i)_i$ is the minimal symmetrization of A.

Let $\mathbb{C}[q]$ be the polynomial ring in the indeterminate q. We consider

$$\hat{\mathscr{H}}(R)=\varprojlim_m \mathscr{H}(R,\mathbb{C}[q]/(q-1)^m,q),$$

and the corresponding ring $\widehat{\mathscr{H}'}(R)$, both are algebras over the power series ring $\mathbb{C}[[q-1]]$. We are going to describe both algebras $\hat{\mathscr{H}}(R)$ and $\widehat{\mathscr{H}'}(R)$ by generators and relations. Let $u_i=u_{[S_i]}$ and $q_i=q^{f_i}$, for $1\leqq i\leqq s$.

Theorem. *As a complete $\mathbb{C}[[q-1]]$-algebra, $\hat{\mathscr{H}}(R)$ is generated by $u_1,\ldots,u_s$, with relations $\rho^+_{1-a_{ij}}(q_i,u_i,u_j)=0={}^+\rho_{1-a_{ji}}(q_j,u_j,u_i)$ for all $j<i$.*

Proof. Let $\mathscr{A}(R,\mathbb{C}[q])=\mathscr{A}(R)\otimes_{\mathbb{Z}}\mathbb{C}[q]$, the free $\mathbb{C}[q]$-algebra with generators $[S_1],\ldots,[S_s]$, and consider the algebra homomorphism

$$\eta\colon\ \mathscr{A}(R,\mathbb{C}[q])\to\mathscr{H}=\mathscr{H}(R,\mathbb{C}[q],q)$$

defined by $\eta([S_i])=u_i$. Let $\mathscr{J}$ be the ideal of $\mathscr{A}(R,\mathbb{C}[q])$ generated by the elements $\rho^+_{1-a_{ij}}(q_i,[S_i],[S_j])$, and ${}^+\rho_{1-a_{ji}}(q_j,[S_j],[S_i])$ for all $j<i$. According to Sect. 1, we see that $\mathscr{J}$ belongs to the kernel of η, thus we obtain an algebra homomorphism

$$\bar{\eta}\colon\ \bar{\mathscr{A}}=\mathscr{A}(R,\mathbb{C}[q])/\mathscr{J}\to\mathscr{H}.$$

We denote by

$$\bar{\eta}_m\colon\ \bar{\mathscr{A}}/(q-1)^m\bar{\mathscr{A}}\to\mathscr{H}/(q-1)^m\mathscr{H}$$

the induced map modulo $(q-1)^m$. According to [R2], the map $\bar{\eta}_1$ is bijective. We consider $\bar{\eta}_m$ as a map of Λ_m-modules, where $\Lambda_m=\mathbb{C}[q]/(q-1)^m$. Now,

$\mathscr{H}/(q-1)^m\mathscr{H}$ is a free A_m-module, thus with $\bar{\eta}_1$ also $\bar{\eta}_m$ is bijective. It follows that $\bar{\eta}$ induces an isomorphism

$$\varprojlim_m \bar{\mathscr{A}}/(p-1)^m\bar{\mathscr{A}} \to \varprojlim_m \mathscr{H}/(q-1)^m\mathscr{H} = \hat{\mathscr{H}}(R).$$

Corolary. *As a complete $\mathbb{C}[[q-1]]$-algebra, $\widehat{\mathscr{H}'}(R)$ is generated by the elements $T_1, \ldots, T_s, u_1, \ldots, u_s$ subject to the relations*

$$[T_i, T_j]=0, \qquad [T_i, u_j]=\delta_{ij}u_j, \qquad \textit{for all } i,j,$$

and

$$\rho^+_{1-a_{ij}}(q_i, u_i, u_j)=0={}^+\rho_{1-a_{ji}}(q_j, u_j, u_i), \qquad \textit{for all } j<i.$$

Here, δ_{ij} is the Kronecker delta: $\delta_{ii}=1$, $\delta_{ij}=0$, for $i \neq j$.

5. Revision of the relations

We keep the assumptions of the last section. We want to change the generators of $\widehat{\mathscr{H}'}(R)$ in order to obtain more familiar relations. First of all, let

$$H_i := \sum_{j=1}^{s} a_{ij} T_j.$$

Since the Cartan matrix $A=(a_{ij})_{ij}$ is invertible, the $\mathbb{C}$-space of $\widehat{\mathscr{H}'}(R)$ generated by $H_1, \ldots, H_s$ is the same as that generated by $T_1, \ldots, T_s$. Also, $[T_i, T_j]=0$ for all i,j is equivalent to requiring $[H_i, H_j]=0$ for all i, j. Similarly, $[T_i, u_j]=\delta_{ij}u_j$ for all i, j is equivalent to requiring $[H_i, u_j]=a_{ij}u_j$ for all i, j.

In order to rewrite the relations ρ^+ and ${}^+\rho$, we will replace the elements u_i by suitable multiples $c_i u_i$, with c_i invertible in $\widehat{\mathscr{H}'}(R)$. Given an element $b \in \widehat{\mathscr{H}'}(R)$, the element $\exp(b \ln q) = \sum_{m \geqq 0} (-1)^m \frac{1}{m!} b^m (\ln q)^m \in \widehat{\mathscr{H}'}(R)$ is defined, since $\ln q$ is a multiple of $q-1$. If $b_1, b_2 \in \widehat{\mathscr{H}'}(R)$ commute, then $\exp((b_1+b_2)\ln q) = \exp(b_1 \ln q)\exp(b_2 \ln q)$; in particular, any $\exp(b \ln q)$ is invertible in $\widehat{\mathscr{H}'}(R)$, with inverse $\exp(-b \ln q)$.

For $1 \leqq i \leqq s$, let

$$X_i := \exp\Big(-\frac{1}{2}\sum_{j=1}^{i-1} f_i a_{ij} T_j \ln q\Big) u_i.$$

Theorem. *As a complete $\mathbb{C}[[q-1]]$-algebra, $\widehat{\mathscr{H}'}(R)$ is generated by the elements $H_1, \ldots, H_s, X_1, \ldots, X_s$, subject to the relations*

$$[H_i, H_j]=0,$$
$$[H_i, X_j]=a_{ij}X_j,$$
$$\rho_{1-a_{ij}}(q_i, X_i, X_j)=0, \qquad \textit{for } i \neq j.$$

Proof. For $1 \leqq j < i \leqq s$, let $c_{ij} = \exp(-\frac{1}{2} f_i a_{ij} T_j \ln q)$, and $c_i = c_{i1} \dots c_{i,j-1}$ (with $c_1 = 1$), thus $X_i = c_i u_i$. For $j \neq s$, we have $c_{ij} u_s = u_s c_{ij}$, since $[T_j, u_s] = 0$. On the other hand, $T_i u_i = u_i T_i + u_i = u_i(T_i + 1)$ implies by induction that $T_i^m u_i = u_i(T_i + 1)^m$ for all $m \geqq 1$. Therefore, for $c \in \mathbb{C}$

$$\begin{aligned}\exp(c T_i \ln q)\, u_i &= \sum_{m \geqq 0} \frac{1}{m!} (c T_i \ln q)^m u_i = \sum_{m \geqq 0} \frac{1}{m!} c^m (\ln q)^m u_i (T_i + 1)^m \\ &= u_i \exp(c(T_i + 1) \ln q) = u_i \exp(c T_i \ln q) \exp(c \ln q) \\ &= q^c \cdot u_i \exp(c T_i \ln q),\end{aligned}$$

thus we see that

$$c_{ij} u_j = q_i^{-\frac{1}{2} a_{ij}} u_j c_{ij}.$$

For $j < i$, it follows that

$$c_i u_i = u_i c_i, \qquad c_i u_j = q_i^{-\frac{1}{2} a_{ij}} u_j c_i, \qquad c_j u_i = u_i c_j, \qquad c_j u_j = u_j c_j,$$

and therefore, for all $0 \leqq t \leqq n$,

$$\begin{aligned} q_i^{\frac{1}{2} a_{ij} t} X_i^t X_j X_i^{n-t} &= u_i^t u_j u_i^{n-t} c_j c_i^n, \\ q_j^{\frac{1}{2} a_{ji} t} X_j^{n-t} X_i X_j^t &= u_j^{n-t} u_i u_j^t c_j^n c_i, \end{aligned}$$

where we have used that $f_i a_{ij} = f_j a_{ji}$, thus $q_i^{a_{ij}} = q_j^{a_{ji}}$. We assume now that $n = 1 - a_{ij}$. Then $\binom{t}{2} + \frac{a_{ij} t}{2} = \frac{t(t-1)}{2} + \frac{(1-n)t}{2} = -\frac{t(n-t)}{2}$, and therefore

$$\begin{aligned} \rho_{1-a_{ij}}(q_i, X_i, X_j) &= \rho^+_{1-a_{ij}}(q_i, u_i, u_j)\, c_j c_i^n, \\ \rho_{1-a_{ji}}(q_j, X_j, X_i) &= {}^+\rho_{1-a_{ji}}(q_j, u_j, u_i)\, c_j^n c_i. \end{aligned}$$

This finishes the proof.

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