Notre Dame Journal of Formal Logic
Volume 24, Number 4, October 1983

Halldén-Completeness by Gluing of Kripke Frames

J. F. A. K. van BENTHEM and I. L. HUMBERSTONE

1 Introduction We give in this paper a sufficient condition, cast in semantic terms, for Halldén-completeness in normal modal logics, a modal logic being said to be *Halldén-complete* (or 'Halldén-reasonable') just in case for any disjunctive formula provable in the logic, where the disjuncts have no propositional variables in common, one or other of those disjuncts is provable in the logic.

It is clear from this definition that the weakest normal modal logic, the system K, is not Halldén-complete, in view of the provability in K of the formula $\Box\bot \vee \Diamond\top$, since neither disjunct is a theorem of K and we may rewrite $\bot$ as $p \wedge \sim p$ and $\top$ as $q \vee \sim q$. (Notation and terminology are as in [5]; some historical and philosophical remarks on Halldén-completeness may be found in [3].) Thus the Halldén-complete normal logics are either extensions of $K + \Box\bot$ (the 'Absurd' system) or of $K + \Diamond\top$ (the system D). Amongst the latter systems are such familiar modal logics as T, $S4$, and $S5$, already known to be Halldén-complete. There are several proofs in the literature for these systems and other isolated cases of systems in this spectrum, though these have an *ad hoc* appearance about them in the sense that they tend to exploit rather specific properties of the individual systems concerned (see, e.g., [4]). We point to a common semantic principle which brings some order into the situation, and see how far it takes us in general. It turns out (Theorem 2, below) that a very simple semantic condition is sufficient for Halldén-completeness amongst the extensions of D.

2 P-morphic fusion The main theorem of the paper requires first a definition.

Received December 30, 1981; revised September 3, 1982

Definition A class $\mathcal{C}$ of frames is *closed under p-morphic fusion* iff for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}$, $w_1 \in W_1$, $w_2 \in W_2$ (where $\mathcal{F}_i = \langle W_i, R_i \rangle$) there exist $\mathcal{F} = \langle W, R \rangle$ and $w \in W$ with p-morphisms f_1 and f_2 from $\mathcal{F}$ to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, such that $f_1(w) = w_1$ and $f_2(w) = w_2$, and $\mathcal{F} \in \mathcal{C}$.

Theorem 1 *If a system S is determined by some class of frames which is closed under p-morphic fusion, then S is Halldén-complete.*

Proof: Let $A \vee B$ be some variable-disjoint disjunction such that $S \nvdash A$, $S \nvdash B$. We have to show that $S \nvdash A \vee B$, on the assumption that S is determined by some class $\mathcal{C}$ of frames closed under p-morphic fusion. Thus there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}$ with valuations V_1 and V_2 and points w_1 and w_2 such that $\langle \mathcal{F}_1, V_1 \rangle \nvDash_{w_1} A$ and $\langle \mathcal{F}_2, V_2 \rangle \nvDash_{w_2} B$. Since $\mathcal{C}$ is closed under p-morphic fusion, there is a frame $\mathcal{F} \in \mathcal{C}$ with $w \in W$ and p-morphisms f_1 and f_2 from $\mathcal{F}$ to $\mathcal{F}_1$, $\mathcal{F}_2$ with $f_1(w) = w_1$ and $f_2(w) = w_2$. Let V be any valuation on $\mathcal{F}$ with the property that for all $x \in W$, $V(p_i,x) = V_1(p_i,f_1(x))$ for all propositional variables p_i in A, and $V(p_i,x) = V_2(p_i,f_2(x))$ for propositional variables p_i in B. (These stipulations cannot conflict because no p_i occurs both in A and in B.) By the p-morphism theorem ([5], p. 37) we infer that for any formulas A' and B', all of whose variables occur in A, B, respectively, and any $x \in W$:

$$\langle \mathcal{F}, V \rangle \vDash_x A' \text{ iff } \langle \mathcal{F}_1, V_1 \rangle \vDash_{f_1(x)} A'$$

and

$$\langle \mathcal{F}, V \rangle \vDash_x B' \text{ iff } \langle \mathcal{F}_2, V_2 \rangle \vDash_{f_2(x)} B'$$

and so, in particular, that $\langle \mathcal{F}, V \rangle \nvDash_w A$, since $\langle \mathcal{F}_1, V_1 \rangle \nvDash_{f_1(w)} A$ and $\langle \mathcal{F}, V \rangle \nvDash_w B$ since $\langle \mathcal{F}_2, V_2 \rangle \nvDash_{f_2(w)} B$. Thus $\langle \mathcal{F}, V \rangle \nvDash_w A \vee B$ and since $\mathcal{F} \in \mathcal{C}$ and $\mathcal{C}$ determines S, $S \nvdash A \vee B$.

It may be useful to comment on the strategy involved in this proof. Intuitively, what we are doing is gluing together counterexamples to A and to B to get a counterexample for their disjunction. The frame $\mathcal{F}$ we arrive at is a p-morphic *pre*image of the frames of their falsifying models, however, and since the property of being a frame for a given logic is preserved by p-morphisms (in general) only in the 'forwards' direction, we need the condition of closure under p-morphic fusion in the theorem.

3 ***Applications*** In Section 1 we remarked that the Halldén-complete systems were extensions either of the Absurd system or of D. The first range of cases may be dealt with swiftly since the Absurd system has no proper consistent extensions, and that system is itself Halldén-complete: this is an immediate consequence of Theorem 1 and the fact that the system is determined by (the unit class of) a single-element frame. (The inconsistent system is Halldén-complete, of course: this follows trivially from the definition of Halldén-completeness.) We turn to the second range of cases. Here a particularly informative sufficient condition is available, as a corollary to Theorem 1. Before stating it, we recall the details of the salient frame construction (as in [6], [2]).

Definition The *direct product* of frames $\mathcal{F}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2 \rangle$ is the frame $\mathcal{F}_1 \otimes \mathcal{F}_2 = \langle W_1 \times W_2, R_1 \otimes R_2 \rangle$ where '×' is for Cartesian product, and:

$$\langle x_1, x_2 \rangle R_1 \otimes R_2 \langle y_1, y_2 \rangle \text{ iff } x_1 R_1 y_1 \text{ and } x_2 R_2 y_2.$$

Further, we describe a class of frames as *closed under direct products* when the direct product of any pair of frames in the class is itself in the class.

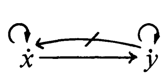
Theorem 2 *If a system S is determined by some class of serial frames which is closed under direct products, then S is Halldén-complete.*

Proof: Suppose S is determined by some class $\mathcal{C}$ of serial frames, and that $\mathcal{C}$ is closed under direct products. Then Halldén-completeness follows from Theorem 1 and the fact that $\mathcal{C}$ must be closed under p-morphic fusion: for, given $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}$, $w_1 \in W_1$, $w_2 \in W_2$, the frame $\mathcal{F}_1 \otimes \mathcal{F}_2$, by hypothesis a member of $\mathcal{C}$, has p-morphisms f_1, f_2 to $\mathcal{F}_1, \mathcal{F}_2$ with $f_1(w) = w_1$ and $f_2(w) = w_2$, where $w = \langle w_1, w_2 \rangle$: for it suffices to take as f_1 and f_2 the projections of $W_1 \times W_2$ onto W_1 and W_2, respectively. (Here we exploit the fact, mentioned in [6] and [2], that the projections from a direct product of *serial* frames are p-morphisms.)

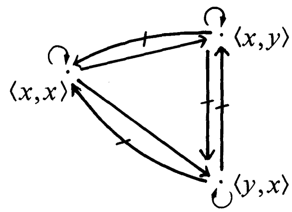
Many extensions of D are seen to be Halldén-complete with the aid of this theorem taken in conjunction with standard completeness results. These include D itself, T, $D4$, $S4$, $S5$, and many others. One may wonder whether, amongst the extensions of D, the condition cited in Theorem 2 is not only sufficient, but also necessary for Halldén-completeness. Since the first-order sentences which are preserved under general direct products are equivalent to so-called Horn sentences (see, e.g., [1], p. 328), the natural place to look for a counterexample to the envisaged converse to Theorem 2 would be amongst systems whose classes of frames are, although first-order definable, not definable by means of Horn sentences. The simplest kind of condition which is not guaranteed to be preserved in passing to direct products will be one with a single disjunction in the consequent of the propositional matrix, as in (piecewise) connectedness. Not surprisingly, then, we have the following counterexample.

Theorem 3 *S4.3 is a Halldén-complete system not determined by any class of serial frames that is closed under direct products.*

Proof: We show first that no class of frames which determines $S4.3$ is closed under direct products. If $S4.3$ is determined by $\mathcal{C}$, then $\mathcal{C}$ must contain at least one frame $\mathcal{F}$ in which there are points x and y whose interrelations are as illustrated (the arrows indicating the relation of the frame):



For otherwise, each instance of the B-schema $A \rightarrow \Box\Diamond A$ would be valid over $\mathcal{C}$, and hence a theorem of $S4.3$, which is not the case. Now if $\mathcal{C}$ were closed under direct products, $\mathcal{F} \otimes \mathcal{F}$ would belong to $\mathcal{C}$, but in this frame the relations amongst the pairs $\langle x,x \rangle$, $\langle x,y \rangle$, and $\langle y,x \rangle$ are as pictured here:



But this is impossible, since on such a frame the linearity axiom of $S4.3$ can always be refuted.

Next, we show that $S4.3$ is nevertheless Halldén-complete. Theorem 1 can be applied here, once we recall (from [5]) that $S4.3$ is determined by $\{\langle \mathbf{Q},\leqslant\rangle\}$, $\mathbf{Q}$ being the rationals. For suppose that, for formulas A and B we have $\langle \mathbf{Q},\leqslant,V_1\rangle \nvDash_{w_1} A$ and $\langle \mathbf{Q},\leqslant,V_2\rangle \nvDash_{w_2} B$. Then where $\langle \mathbf{Q}_1,\leqslant_1,V_1\rangle$ and $\langle \mathbf{Q}_2,\leqslant_2,V_2\rangle$ are the submodels of these models generated respectively by the points w_1 and w_2, we have $\langle \mathbf{Q}_1,\leqslant_1,V_1\rangle \nvDash_{w_1} A$ and $\langle \mathbf{Q}_2,\leqslant_2,V_2\rangle \nvDash_{w_2} B$. Now if A and B share no propositional variables, these models give rise to an obvious counterexample to $A \vee B$ on the rationals, since $\langle \mathbf{Q}_1,\leqslant_1\rangle$ and $\langle \mathbf{Q}_2,\leqslant_2\rangle$ are isomorphic.

The second part of this proof brings out a general fact. As in [6], we say a frame $\langle W,R\rangle$ is *homogeneous* when for any $x, y \in W$ there exists an automorphism f of $\langle W,R\rangle$ with $f(x) = y$. Then Theorem 1 implies that any system determined by $\{\mathscr{F}\}$ for homogeneous $\mathscr{F}$ is Halldén-complete. Again, one may wonder as to the converse. The situation is that the converse does not hold. By an argument that we shall not give here, the system $S4.3Grz$ is Halldén-complete, without being 'homogeneously complete'.

4 Further Questions (1) Does the converse of Theorem 1 hold for extensions of D? That is, are all such systems which are Halldén-complete determined by classes of frames closed under p-morphic fusion? (We have no reason for thinking the answer to be *yes*.) (2) In view of the close kinship between Halldén-completeness and the interpolation property, can broadly 'fusion' based methods such as those used here throw any light on the latter property?

REFERENCES

[1] Chang, C. C. and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[2] Goldblatt, R., "Diodorean modality in Minkowski space-time," *Studia Logica*, vol. 39 (1980), pp. 219-236.

[3] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968

[4] McKinsey, J. C. C., "Systems of modal logic not unreasonable in the sense of Halldén," *The Journal of Symbolic Logic*, vol. 18 (1953), pp. 109-113.

[5] Segerberg, K., *An Essay in Classical Modal Logic*, Filosofiska Studier, Uppsala, 1971.

[6] van Benthem, J. F. A. K., *The Logic of Time*, Groningen, 1980 (to appear in *Synthese Library*, D. Reidel, Dordrecht, 1982).

Filosofisch Instituut
Ryksuniversiteit
Westersingel 1g
8718 CA Groningen
The Netherlands

Department of Philosophy
Monash University
Clayton, Victoria
Australia 3168