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# Hamilton-Jacobi Equation for Brans-Dicke Theory and Its Long-Wavelength Solution

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The Hamilton-Jacobi equation for the Brans-Dicke theory is solved by using a long-wavelength approximation. We examine the non-linear evolution of the inhomogeneities in the dust fluid case and the cosmological constant case. In the case of dust fluid, it turns out that the inhomogeneities of space-time grow. In the case of cosmological constant, the inhomogeneities decay, which is consistent with the cosmic no hair conjecture. The inhomogeneities of the density perturbation and the gravitational constant behave in a manner similar to those of space-time.

## § 1. Introduction

It is generally believed that a consistent theory of quantum gravity is correctly described by superstring theory. The classical theory of gravity is nothing but a low energy effective theory of superstring theory. The classical theory, which is predicted by superstring theory, has the form of a scalar-tensor theory. The scalar field is the so-called dilaton. Therefore, it is necessary to consider the consequences of this extra scalar field at least in phenomena close to the Planck scale. If the dilaton field acquires a large mass due to an unknown dynamical mechanism, there will be no observable macroscopic difference between the superstring predicted theory and Einstein's general theory of relativity. However, recently, the possibility of the massless dilaton has been pointed out.[1] If so, it is important to study the scalar-tensor theory more seriously. The simplest scalar-tensor theory is the Brans-Dicke theory,[2] where the dilaton field acts like a dynamical gravitational constant.

On the other hand, to circumvent the graceful-exit problem of old inflation, the Brans-Dicke theory is renewed in the inflational universe scenario as extend inflation. Moreover, Bellido et al.[3] investigated the stochastic inflation formalism in the context of the Brans-Dicke theory. They described the inhomogeneous universe with fluctuations of the gravitational constant. The main idea of stochastic inflation is to solve the equations for the inhomogeneous fields in the de-Sitter space by separating both the gravitational and scalar fields into short wavelength quantum fluctuations, which oscillate on scales smaller than the Hubble radius, and long wavelength fluctuations which are treated as classical fields. Salopek and Bond[4] have developed a formalism to treat the long wavelength fluctuations in terms of the Hamilton-Jacobi equation, which is applicable not only to the inflational theory but also the late stage evolution of the density fluctuations as far as the typical scale of the fluctuation exceeds the Hubble radius.

The so-called long wavelength approximation has a rather long history dating back to Lifshitz and Khalatonikov.[5] Later, Tomita developed the above approximation as the Anti-Newtonian scheme.[6] Recently, Salopek and co-authors elegantly formulated the long wavelength approximation in the context of Hamilton-Jacobi theory.[7),8)] The direct method of Comer et al.[9] is also useful to calculate the higher order correction. Attempts to apply the formalism to the inflationary theory and the higher dimensional theory also exist.[10]

In this paper, we shall apply the long wavelength approximation to the Brans-Dicke theory. We are interested in the non-linear evolution of the long-wavelength inhomogeneities. As the inhomogeneities of the Brans-Dicke field imply the inhomogeneities of the gravitational constant, this is important for astrophysical phenomena. In our modest study, of course, we do not intend to make definite statements about astrophysics. However, it is important to investigate how the inhomogeneities of the gravitational coupling constant evolve. We start in § 2 by writing the Hamilton-Jacobi equation for the Brans-Dicke theory. The spatial gradient expansion is performed in § 3. As a matter field, the cosmological constant is interesting. This case is related to the cosmological no-hair conjecture. Section 4 is devoted to these subjects. In § 5, discussion of the various problems is presented. In the Appendix, the results of the direct method are explained.

## § 2. Hamilton-Jacobi equation for Brans-Dicke theory

For simplicity, we will consider dust fluid matter. The action for the Brans-Dicke theory with dust matter $\chi$ is given by

$$I=\int d^4x\sqrt{-g}\left[\phi\,{}^{(4)}R-\frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{n}{2m}(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi+m^2)\right], \tag{1}$$

where $n$ is the Lagrange multiplier and $m$ is the particle mass which is normalized to unity below. Here, $\omega$ is the parameter of the theory. The Brans-Dicke field $\phi$ is considered as the effective gravitational coupling constant. The Hamilton-Jacobi equation for the Brans-Dicke theory is obtained using the Arnowitt-Deser-Misner (ADM) formalism in which the space-time is foliated by a family of space-like hypersurfaces. In the ADM formalism, the metric is parametrized as

$$ds^2=-N^2dt^2+\gamma_{ij}(dx^i+N^idt)(dx^j+N^jdt), \tag{2}$$

where $N$ and $N^i$ are the lapse and shift functions, respectively, and $\gamma_{ij}$ is the 3-metric. Using the above metric, we obtain the Hamiltonian form of the action as

$$I=\int d^4x\{\pi^{ij}\dot\gamma_{ij}+\pi^\phi\dot\phi+\pi^\chi\dot\chi-NH-N^iH_i\}, \tag{3}$$

where

$$H=\frac{1}{\phi\sqrt{\gamma}}\pi^{ij}\pi^{kl}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+3}\gamma_{ij}\gamma_{kl}\right]+\frac{1}{2(2\omega+3)}\frac{\phi}{\sqrt{\gamma}}\pi_\phi{}^2-\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\pi\pi_\phi$$
$$+\sqrt{1+\gamma^{ij}\chi_{,i}\chi_{,j}}\,\pi^\chi-\sqrt{\gamma}\,\phi R+\omega\frac{\sqrt{\gamma}}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi+2\sqrt{\gamma}\,\Delta\phi, \tag{4}$$

$$H_i=-2(\gamma_{ik}\pi^{kj})_{,j}+\pi^{lk}\gamma_{lk,i}\pi^{\phi}\phi_{,i}+\pi^{\chi}\chi_{,i}\,. \tag{5}$$

Here $\pi^{ij}$, $\pi^{\phi}$ and $\pi^{\chi}$ are conjugate to $\gamma_{ij}$, $\phi$ and $\chi$, respectively, and $R$ denotes the 3-dimensional scalar curvature. Variation with respect to the momentum yields the equations of motion

$$\frac{1}{N}(\dot{\phi}-N^i\phi_{,i})=\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}[\phi\pi_{\phi}-\pi]\,, \tag{6}$$

$$\frac{1}{N}(\dot{\chi}-N^i\chi_{,i})=\sqrt{1+\gamma^{ij}\chi_{,i}\chi_{,j}}\,, \tag{7}$$

$$\frac{1}{N}(\dot{\gamma}_{ij}-N_{i|j}-N_{j|i})=\frac{2}{\phi\sqrt{\gamma}}\pi^{kl}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+3}\gamma_{ij}\gamma_{kl}\right]-\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\pi_{\phi}\,. \tag{8}$$

Variation of the action (3) with respect to the field variables yields the evolution equations for the momentum. These are automatically satisfied provided that

$$\pi^{ij}=\frac{\delta S}{\delta\gamma_{ij}}\,,\quad \pi^{\phi}=\frac{\delta S}{\delta\phi}\,,\quad \pi^{\chi}=\frac{\delta S}{\delta\chi} \tag{9}$$

satisfy the constraint equations and provided that the evolution equations (6) hold. Here, instead of solving the equations of motion for the momentum fields, we will use the Hamilton-Jacobi method. The Hamilton-Jacobi equation is

$$\begin{aligned}&\frac{1}{\phi\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\gamma_{kl}}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+3}\gamma_{ij}\gamma_{kl}\right]\\&\quad+\frac{1}{2(2\omega+3)}\frac{\phi}{\sqrt{\gamma}}\left(\frac{\delta S}{\delta\phi}\right)^2-\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\phi}\\&\quad+\sqrt{1+\gamma^{ij}\chi_{,i}\chi_{,j}}\frac{\delta S}{\delta\chi}-\sqrt{\gamma}\,\phi R+\omega\frac{\sqrt{\gamma}}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi+2\sqrt{\gamma}\,\Delta\phi=0\,.\end{aligned} \tag{10}$$

The momentum constraint is a rather trivial condition which states that the generating functional is invariant under the spatial coordinate transformation.

Hamilton-Jacobi formalism has a great advantage in its intimate relation to quantum gravity. The Wheeler-DeWitt equation for the Brans-Dicke theory is given by

$$H\Psi=0\,,\quad H_i\Psi=0\,, \tag{11}$$

where the canonical commutation relations

$$[\gamma_{ij}(x),\,\pi^{kl}(y)]=\frac{i}{2}(\delta_i{}^k\delta_j{}^l+\delta_j{}^k\delta_i{}^l)\delta(x-y)\,, \tag{12}$$

$$[\phi(x),\,\pi^{\phi}(y)]=i\delta(x-y)\,, \tag{13}$$

$$[\chi(x),\,\pi^{\chi}(y)]=i\delta(x-y) \tag{14}$$

are assumed. If we consider the WKB approximation, we get Eq. (10) as the lowest order equation. Research in this direction from the point of view of the long-wavelength approximation is a project we will pursue in the future.

## § 3. Long wavelength solution

The heart of the long wavelength approximation is the following: For illustration, we take the metric in the synchronous form

$$ds^2 = -dt^2 + \gamma_{ij}(x^k, t)dx^i dx^j . \quad (i, j = 1, 2, 3) \tag{15}$$

At each point one can define a local scale factor $a$ and a local Hubble parameter $H$ by

$$a^2 \equiv (\det \gamma_{ij})^{1/3} , \quad H \equiv \frac{\dot{a}}{a} , \tag{16}$$

where the dot denotes the time derivative. The Hubble parameter leads to the characteristic proper time on which the metric evolves. The characteristic comoving length on which it varies is denoted by $L$: $\partial_i \gamma_{jk} \approx L^{-1} \gamma_{ij}$. The long wavelength approximation is the assumption that the characteristic scale of spatial variation is much longer than the Hubble radius, that is

$$\frac{1}{a} \partial_i \gamma_{jk} \ll \dot{\gamma}_{ij} \Leftrightarrow aL \gg H^{-1} . \tag{17}$$

Then we can drop the spatial curvature term in the Einstein equations in the lowest order. If we incorporate the curvature effect perturbatively, we obtain an understanding of the non-linear evolution of the inhomogeneities. This direct method is technically useful. We presented the analysis in the Appendix for the purpose of a check. For conceptual reasons, here, we take another approach, i.e., we consider the long-wavelength approximation in the context of the Hamilton-Jacobi formalism.

Let us follow the method developed by Salopek and the co-authors. They expanded the generating functional in a series of terms according to the number of spatial gradients they contain:

$$S = S^{(0)} + S^{(2)} + S^{(4)} + S^{(6)} + \cdots . \tag{18}$$

As a result the Hamilton-Jacobi equation can be solved perturbatively as we will show. The lowest order Hamilton-Jacobi equation is

$$\frac{1}{\phi\sqrt{\gamma}} \frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \gamma_{kl}} \left[ \gamma_{ik}\gamma_{jl} - \frac{\omega+1}{2\omega+3} \gamma_{ij}\gamma_{kl} \right] + \frac{1}{2(2\omega+3)} \frac{\phi}{\sqrt{\gamma}} \left( \frac{\delta S^{(0)}}{\delta \phi} \right)^2$$
$$- \frac{1}{2\omega+3} \frac{1}{\sqrt{\gamma}} \gamma_{ij} \frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \phi} + \sqrt{1 + \gamma^{ij}\chi_{,i}\chi_{,j}} \frac{\delta S^{(0)}}{\delta \chi} = 0 . \tag{19}$$

It is difficult to obtain complete solutions of the Hamilton-Jacobi equation. Fortunately, we need only the growing mode in the late stage or inflationary stage. Hence, we assume the quasi-isotropic ansatz

$$S^{(0)} = -2 \int d^3x \sqrt{\gamma}\, \phi H(\chi) . \tag{20}$$

The diffeomorphism invariance is automatically satisfied by this ansatz. Substituting the ansatz (20) into Eq. (19), we obtain

$$H^2=-\frac{4\omega+6}{3\omega+4}\frac{\partial H}{\partial \chi}\,. \tag{21}$$

The solution of Eq. (21) becomes

$$H(\chi)=\frac{4\omega+6}{3\omega+4}\frac{1}{\chi}\,, \tag{22}$$

where we have ignored the integration constant, because it is a decaying mode. Hereafter, we choose the comoving gauge $\chi_{,i}=0$. In principle, we are free to choose the time slicing. For a dust fluid, a natural choice is the comoving gauge $\chi_{,i}=0$. Fortunately, in this case, we can also take the synchronous gauge, and $\chi$ has the meaning of time. Equations of motion give

$$\dot{\phi}=\frac{H}{2\omega+3}\phi\,, \tag{23}$$

$$\dot{\gamma}_{ij}=\frac{2\omega+2}{2\omega+3}H\gamma_{ij}\,. \tag{24}$$

Their solutions are

$$\phi=\chi^{2/(3\omega+4)}\tilde{\phi}(x)\,, \tag{25}$$

$$\gamma_{ij}=\chi^{(4\omega+4)/(3\omega+4)}h_{ij}(x)\,, \tag{26}$$

where $\tilde{\phi}(x)$ and $h_{ij}(x)$ are arbitrary functions of spatial coordinates, which we call the seed scalar and seed metric, respectively.

The second order Hamilton-Jacobi equation is given by

$$\frac{2\omega+2}{2\omega+3}\gamma_{ij}\frac{\delta S^{(2)}}{\delta\gamma_{ij}}+\frac{H}{2\omega+3}\phi\frac{\delta S^{(2)}}{\delta\phi}$$
$$+\frac{\delta S^{(2)}}{\delta\chi}-\sqrt{\gamma}\,\phi R+\omega\frac{\sqrt{\gamma}}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi+2\sqrt{\gamma}\,\Delta\phi=0\,. \tag{27}$$

To simplify the Hamilton-Jacobi equation, we utilize a conformal transformation of the three-metric and the Brans-Dicke scalar to define variables

$$f_{ij}=\Omega^{-2}\gamma_{ij}\,, \tag{28}$$

$$\psi=W^{-1}\phi\,, \tag{29}$$

where $\Omega$ and $W$ satisfy

$$\frac{\partial\Omega}{\partial\chi}=\frac{\omega+1}{2\omega+3}H\Omega\,, \tag{30}$$

$$\frac{\partial W}{\partial\chi}=\frac{1}{2\omega+3}HW\,. \tag{31}$$

And then,

$$\Omega = \chi^{(2\omega+2)/(3\omega+4)}, \tag{32}$$

$$W = \chi^{2/(3\omega+4)}. \tag{33}$$

Hence, the Hamilton-Jacobi equation reduces to

$$\left.\frac{\delta S^{(2)}}{\delta\chi}\right|_{f_{ij},\phi} = -W\Omega\left[-\sqrt{f}\,\phi R(f) + \omega\frac{\sqrt{f}}{\phi} f^{ij}\partial_i\phi\partial_j\phi\right]. \tag{34}$$

This is easily integrated to

$$S^{(2)} = \frac{3\omega+4}{5\omega+8}\chi\int d^3x\sqrt{\gamma}\left[\phi R(\gamma) - \frac{\omega}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi\right], \tag{35}$$

where we have ignored the irrelevant homogeneous solution. Now the equation of motion becomes

$$\dot{\phi} = \frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\left[\phi\frac{\delta S}{\delta\phi} - \gamma_{ij}\frac{\delta S}{\delta\gamma_{ij}}\right], \tag{36}$$

$$= \frac{H}{2\omega+3}\phi + \frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\left[\phi\frac{\delta S^{(2)}}{\delta\phi} - \gamma_{ij}\frac{\delta S^{(2)}}{\delta\gamma_{ij}}\right]. \tag{37}$$

We expand $\phi$ as $\phi = \phi^{(0)} + \phi^{(2)} + \cdots$.

$$\dot{\phi}^{(2)} = \frac{H}{2\omega+3}\phi^{(2)} + \frac{3\omega+4}{5\omega+8}\frac{\chi}{2\omega+3}\left[\frac{1}{2}\tilde{\phi}R - \frac{1}{2}\frac{\omega}{\tilde{\phi}}\partial_k\tilde{\phi}\partial^k\tilde{\phi} + (2\omega+2)\Delta\tilde{\phi}\right]. \tag{38}$$

Up to second order, the solution is given by

$$\phi = \chi^{2/(3\omega+4)}\tilde{\phi} + \frac{(3\omega+4)^2}{(5\omega+8)(2\omega+3)(2\omega+4)}\chi^{(2\omega+6)/(3\omega+4)}F(h,\tilde{\phi}), \tag{39}$$

where

$$F(h,\tilde{\phi}) = \frac{1}{2}\tilde{\phi}R(h) - \frac{1}{2}\frac{\omega}{\tilde{\phi}}h^{ij}\partial_i\phi\partial_j\phi + (2\omega+2)h^{ij}\tilde{\phi}_{;ij}. \tag{40}$$

Similarly, the metric is solved as

$$\gamma_{ij} = \chi^{(4\omega+4)/(3\omega+4)}h_{ij} + \frac{(3\omega+4)^2}{(5\omega+8)(2\omega+4)}\chi^2 P(h,\tilde{\phi}), \tag{41}$$

where

$$P(h,\tilde{\phi}) = \frac{\omega+1}{2\omega+3}h_{ij}R(h) - 2R_{ij}(h) - \frac{\omega+1}{2\omega+3}h_{ij}\frac{\omega}{\tilde{\phi}^2}\partial_k\tilde{\phi}\partial_l\tilde{\phi}h^{kl}$$
$$+\frac{2\omega}{\tilde{\phi}^2}\partial_i\tilde{\phi}\partial_j\tilde{\phi} - \frac{2\omega+2}{2\omega+3}h_{ij}\frac{1}{\tilde{\phi}}h^{kl}\tilde{\phi}_{;kl} + \frac{2}{\tilde{\phi}}\tilde{\phi}_{;ij}.$$

The inhomogeneities of the space-time grow as $\chi^{(2\omega+4)/(2\omega+3)}$ in this case. It is interesting to see if this tendency will continue or not. Therefore, we proceed to higher order calculations. A higher order generating functional is obtained by

$$\frac{2\omega+2}{2\omega+3}H\gamma_{ij}\frac{\delta S^{(2n)}}{\delta\gamma_{ij}}+\frac{1}{2\omega+3}H\phi\frac{\delta S^{(2n)}}{\delta\phi}+\frac{\delta S^{(2n)}}{\delta\chi}+R^{(2n)}=0\,, \tag{42}$$

where

$$R^{(2n)}=\sum_{k=1}^{n-1}\frac{1}{\phi\sqrt{\gamma}}\frac{\delta S^{(2k)}}{\delta\gamma_{ij}}\frac{\delta S^{(2n-2k)}}{\delta\gamma_{kl}}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+1}\gamma_{ij}\gamma_{kl}\right]$$
$$+\sum_{k=1}^{n-1}\frac{1}{2(2\omega+3)}\frac{\phi}{\sqrt{\gamma}}\frac{\delta S^{(2k)}}{\delta\phi}\frac{\delta S^{(2n-2k)}}{\delta\phi}-\sum_{k=1}^{n-1}\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\frac{\delta S^{(2k)}}{\delta\gamma_{ij}}\frac{\delta S^{(2n-2k)}}{\delta\phi}\,. \tag{43}$$

Using the conformal transformation method, we obtain the recursion relation

$$\frac{\delta S^{(2n)}}{\delta\chi}+R^{(2n)}=0\,. \tag{44}$$

This leads to

$$S^{(2n)}=-\int d^3x\int_0^1 ds\chi R^{(2n)}[s\chi,\,\phi(x),\,f_{ij}(x)]$$
$$=-\frac{3\omega+4}{(2n+3)\omega+4n+4}\chi\int d^3xR^{(2n)}[\chi,\,\phi(x),\,\gamma_{ij}(x)]\,. \tag{45}$$

From this expression, we can guess the following formal expansion:

$$\gamma_{ij}=\sum_n C_n\chi^{(4\omega+4)/(3\omega+4)+(2\omega+4)/(3\omega+4)n}\,, \tag{46}$$

where $C_n$ can be, in principle, determined perturbatively. This formal expansion indicates the growing nature of the inhomogeneities of space-time. It should be noted that this expression (46) coincides with that of general relativity in the limit $\omega\to\infty$.

## § 4. Cosmological constant

It is possible to show that the inhomogeneities grow or decay, as time increases, depending on the equation of state for perfect fluid matter. As we have investigated dust matter which satisfies the strong energy condition, here, we will study the cosmological constant model as a typical one which does not satisfy the strong energy condition. The action is given by

$$I=\int d^4x\sqrt{-g}\left[\phi^{(4)}R-\frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-2\Lambda\right]. \tag{47}$$

As is explained in the previous section, we can obtain the Hamilton-Jacobi equation:

$$\frac{1}{\phi\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\gamma_{kl}}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+3}\gamma_{ij}\gamma_{kl}\right]+\frac{1}{2(2\omega+3)}\frac{\phi}{\sqrt{\gamma}}\left(\frac{\delta S}{\delta\phi}\right)^2$$
$$-\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\phi}-\sqrt{\gamma}\,\phi R+\omega\frac{\sqrt{\gamma}}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi+2\sqrt{\gamma}\,\Delta\phi+2\Lambda\sqrt{\gamma}=0\,. \tag{48}$$

It is our task to solve the above Hamilton-Jacobi equation using the long-wavelength approximation. The lowest equation becomes

$$\frac{1}{\phi\sqrt{\gamma}}\frac{\delta S^{(0)}}{\delta\gamma_{ij}}\frac{\delta S^{(0)}}{\delta\gamma_{kl}}\left[\gamma_{ik}\gamma_{jl}-\frac{\omega+1}{2\omega+3}\gamma_{ij}\gamma_{kl}\right]+\frac{1}{2(2\omega+3)}\frac{\phi}{\sqrt{\gamma}}\left(\frac{\delta S^{(0)}}{\delta\phi}\right)^2$$
$$-\frac{1}{2\omega+3}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\frac{\delta S^{(0)}}{\delta\gamma_{ij}}\frac{\delta S^{(0)}}{\delta\phi}+2\Lambda\sqrt{\gamma}=0\,. \tag{49}$$

Here, we seek the quasi-isotropic solution again. For this purpose, we assume the ansatz

$$S^{(0)}=-2\sqrt{\frac{2\omega+3}{3\omega}}\int d^3x\sqrt{\gamma}H(\phi)\,. \tag{50}$$

Substituting Eq. (50) into Eq. (49), we get

$$\frac{H^2}{\phi^2}=\frac{2}{3\omega}\left(\frac{\partial H}{\partial\phi}\right)^2-\frac{2}{\omega}\frac{H}{\phi}\frac{\partial H}{\partial\phi}+\frac{2\Lambda}{\phi}\,. \tag{51}$$

Then, we obtain

$$H=\sqrt{\frac{12\omega\Lambda}{6\omega+5}}\phi^{1/2}\,. \tag{52}$$

Using the equations of motion, we obtain the lowest solution

$$\phi^{(0)}=\frac{4\Lambda}{(2\omega+3)(6\omega+5)}t^2\,, \tag{53}$$

$$\gamma_{ij}^{(0)}=t^{2\omega+1}h_{ij}(x)\,. \tag{54}$$

If $\omega>1/2$, the space-time shows power-law expansion which is closely related to the extended inflationary scenario. The next order equation is

$$(2\omega+1)\sqrt{\frac{4\Lambda}{(2\omega+3)(6\omega+5)}}\phi^{-(1/2)}\gamma_{ij}\frac{\delta S^{(2)}}{\delta\gamma_{ij}}+2\sqrt{\frac{4\Lambda}{(2\omega+3)(6\omega+5)}}\phi^{1/2}\frac{\delta S^{(2)}}{\delta\phi}$$
$$-\sqrt{\gamma}\phi R+\sqrt{\gamma}\frac{\omega}{\phi}\gamma^{ij}\partial_i\phi\partial_j\phi+2\sqrt{\gamma}\Delta\phi=0\,. \tag{55}$$

Direct integration yields

$$S^{(2)}=\frac{2}{(2\omega+7)A}\int d^3x\sqrt{\gamma}\left[\phi^{3/2}R-\frac{(\omega-1)}{\phi^{1/2}}\gamma^{ij}\partial_i\phi\partial_j\phi\right], \tag{56}$$

where

$$A=\sqrt{\frac{4\Lambda}{(2\omega+3)(6\omega+5)}}\,. \tag{57}$$

From the generating functional, the next order correction is calculated as

$$\phi^{(2)}=-\frac{4\Lambda}{(\omega-1)(2\omega+7)(2\omega+3)^2(6\omega+5)}t^{3-2\omega}R(h)\,, \tag{58}$$

$$\gamma_{ij}^{(2)}=t^2\left[-\frac{2\omega+1}{2(\omega-1)(2\omega+3)(2\omega+7)}h_{ij}R(h)+\frac{4}{(2\omega-1)(2\omega+7)}R_{ij}(h)\right]. \tag{59}$$

In contrast to the dust case, the inhomogeneities will decay if $\omega>1/2$. This case corresponds to power law inflation. Recursive calculation gives higher order solutions. It is not difficult to guess the expansion form for further correction:

$$\phi=\sum t^{2+n(1-2\omega)}\bar{C}_n . \tag{60}$$

Therefore, the inhomogeneities of the gravitational constant will decay in the power law inflation case.

## § 5. Discussion

It is known that the Brans-Dicke theory is the most permissible alternative to general relativity in the sense that it appears as a low energy theory of superstring theory naturally and it passed the classical tests such as the equivalence principle. Here, let us comment on the experimental constraints. Solar-system experiments yield the constraint $\omega\geq 500$.[11] Recently, Fujii discussed the possibility of the quantum violation of the weak equivalence principle to constrain $\omega$ to be $\omega\geq 10^6$ for a certain force-range.[12] Even if it were true locally, inhomogeneities of the gravitational constant would be allowed. In these situations, the perturbative argument may not be applicable. Anyway, this direction of thought is still tentative although it is interesting and important. On the other hand, the time variation rate of the gravitational constant is constrained experimentally:[11]

$$\frac{\dot{G}}{G}\leq 10^{-12}.$$

In the isotropic and homogeneous flat Universe, the time variation of the gravitational constant becomes

$$\frac{\dot{G}}{G}=-\frac{H(t)}{\omega+1}\propto\frac{1}{t},$$

where $H$ is the Hubble parameter. The constraint on $\omega$ from the variation rate is not so severe in comparison to the solar-system experiments.

Now we shall discuss the evolution of the gravitational constant in the inhomogeneous case. First, let us consider the dust model and define the local Hubble,

$$\begin{aligned}\tilde{H}&=\frac{1}{6}\gamma^{ij}\dot{\gamma}_{ij}\\&=\frac{2\omega+2}{3\omega+4}\frac{1}{\chi}\left\{1-\frac{(\omega+3)(3\omega+4)^2}{4(5\omega+8)(2\omega+3)(\omega+1)}\chi^{(2\omega+4)/(3\omega+4)}U\right\},\end{aligned} \tag{61}$$

where

$$U=R(h)-\frac{\omega}{\tilde{\phi}^2}h^{kl}\partial_k\tilde{\phi}\partial_l\tilde{\phi}+\frac{2\omega}{\omega+3}\frac{1}{\tilde{\phi}}h^{kl}\tilde{\phi}_{;kl}. \tag{62}$$

Then, within the accuracy of the present approximation, the rate of variation of the gravitational constant is given by

$$\frac{\dot{G}}{G}=-\frac{\tilde{H}}{\omega+1}\left[1+\frac{(3\omega+4)^2}{6(5\omega+8)(\omega+1)}\chi^{(2\omega+2)/(3\omega+4)}Q(h,\tilde{\phi})\right], \tag{63}$$

where

$$Q(h,\tilde{\phi})=R(h)-\frac{\omega}{\tilde{\phi}^2}h^{ij}\partial_i\tilde{\phi}\partial_j\tilde{\phi}+\frac{3\omega+2}{\tilde{\phi}}h^{ij}\tilde{\phi}_{;ij}. \tag{64}$$

Due to the local curvature and the inhomogeneities of the Brans-Dicke field, the evolution of the effective coupling constant at that point is altered as in Eq. (62). This effect becomes dominant in the late stages. For example, in the case $\tilde{\phi}=0$, the positive curvature enhanced the decreasing effect, and the negative curvature supressed the decreasing effect. In general, the inhomogeneities of the gravitational constant itself influence the evolution rate of $G$. Thus the inhomogeneities become significant. Let us look at the evolution of the density of the dust:

$$\rho\sim\gamma^{-(1/2)}=\frac{1}{\sqrt{h}}\chi^{-(6\omega+6)/(3\omega+4)}\left\{1+\frac{(\omega+3)(3\omega+4)^2}{4(5\omega+8)(2\omega+3)(\omega+2)}\chi^{(2\omega+4)/(3\omega+4)}U\right\}. \tag{65}$$

This shows that the high density region and the small gravitational constant region coincides. This may have some interesting consequences for astrophysics.

In the case of the cosmological constant, the time evolution of the gravitational constant is given by

$$\frac{\dot{G}}{G}=-\frac{4}{2\omega+1}\tilde{H}\left[1+\frac{(4\omega-1)(4\omega+3)}{6(\omega-1)(2\omega+1)(2\omega+3)(2\omega+7)}t^{1-2\omega}R(h)\right], \tag{66}$$

where

$$\tilde{H}=\frac{2\omega+1}{2t}\left[1-\frac{2\omega(\omega+2)}{3(\omega-1)(2\omega+1)(2\omega+3)(2\omega+7)}t^{1-2\omega}R(h)\right]. \tag{67}$$

In the case of power law inflation, the homogenization of the time variation rate of the gravitational constant occurs.

## § 6. Conclusion

As a fundamentally important theory, we have studied the Brans-Dicke theory using the long-wavelength approximation. First of all, we presented the Hamilton-Jacobi equation for the Brans-Dicke theory which has a different structure from that of general relativity. The Hamilton-Jacobi equation thus obtained is considered as the semi-classical equation corresponding to the Wheeler-DeWitt equation for the Brans-Dicke theory. Applying the method of Salopek and co-authors, we have obtained approximate solutions up to the first order of curvature correction in the case of dust fluid and the cosmological constant. As the Brans-Dicke scalar can be regarded as the effective gravitational constant, the non-linear evolution of the inhomogeneities of the Brans-Dicke field is interesting. In the above two cases, we investigated this problem. From the results, we can conclude that the inhomogeneities of the gravitational constant will decay in the case of inflationary matter and grow in the case of ordinary matter which satisfies the dominant energy condition. We have also calculated the approximate solutions using the direct method of Comer et al. These calculations appear in the Appendix.

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## Appendix
### — *Anti-Newtonian Formalism* —

The equations of motion for the Brans-Dicke theory are

$$G_{\mu\nu}=T_{M\mu\nu}+\frac{1}{\phi}(\phi_{;\mu\nu}-g_{\mu\nu}\Box\phi)+\frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi\right), \tag{68}$$

$$\Box\phi=\frac{8\pi}{2\omega+3}T_M, \tag{69}$$

where $T_{M\mu\nu}$ is the energy momentum tensor of the matter. Hereafter we will consider irrotational dust fluid. In the synchronous gauge, we can write the above equations in the following way:

$$\frac{\partial}{\partial t}K+K_k{}^iK_i{}^k=\frac{1}{\phi}\left(-\frac{3}{2}\ddot{\phi}-\frac{1}{2}K\dot{\phi}+\frac{1}{2}\Delta\phi\right)-\frac{\omega}{\phi^2}\dot{\phi}^2-\frac{4\pi}{\phi}\rho, \tag{70}$$

$$-K_{i|k}^k+K_{|i}=-\frac{1}{\phi}\dot{\phi}_{,i}+\frac{1}{\phi}K_i{}^k\phi_{,k}-\frac{\omega}{\phi^2}\dot{\phi}\partial_i\phi, \tag{71}$$

$$KK_k{}^i+\frac{\partial}{\partial t}K_k{}^i+{}^{(3)}R_k{}^i=\frac{1}{\phi}\left[\phi\,{}^{|i}_{|k}-K_k{}^i\dot{\phi}+\frac{1}{2}\delta_k{}^i(-\ddot{\phi}-K\dot{\phi}+\Delta\phi)\right]$$
$$+\frac{\omega}{\phi^2}\partial^i\phi\partial_k\phi+\frac{4\pi}{\phi}\rho\delta_k{}^i, \tag{72}$$

$$-\ddot{\phi}-K\dot{\phi}+\Delta\phi=-\frac{8\pi}{2\omega+3}\rho, \tag{73}$$

where $\rho$ is the energy density of the dust fluid.

Define

$$K_j{}^i=\Sigma_j{}^i+\frac{1}{3}\delta_j{}^iK. \tag{74}$$

We can write the above equations in the following way:

$$-\Sigma^{k}_{i|k}+\frac{2}{3}K_{|i}=-\frac{1}{\phi}\dot{\phi}_{,i}+\frac{1}{\phi}K_i^{k}\phi_{,k}-\frac{\omega}{\phi^2}\dot{\phi}\partial_i\phi\,, \tag{75}$$

$$\frac{\partial}{\partial t}K+\frac{1}{2}K^2+\frac{3}{4}\Sigma_k{}^i\Sigma_i{}^k+\frac{1}{4}{}^{(3)}R=-\frac{1}{\phi}\left(\frac{3}{2}\ddot{\phi}+K\dot{\phi}\right)+\frac{1}{\phi}\Delta\phi$$
$$-\frac{3}{4}\frac{\omega}{\phi^2}\dot{\phi}^2+\frac{\omega}{4}\frac{1}{\phi^2}\partial^k\phi\partial_k\phi\,, \tag{76}$$

$$\frac{\partial}{\partial t}K+\Sigma_k{}^i\Sigma_i{}^k+\frac{1}{3}K^2=-\frac{\omega}{\phi^2}\dot{\phi}^2-(\omega+3)\frac{\ddot{\phi}}{\phi}$$
$$-(\omega+2)K\frac{\dot{\phi}}{\phi}+(\omega+2)\frac{1}{\phi}\Delta\phi\,, \tag{77}$$

$$\frac{\partial}{\partial t}\Sigma_k{}^i+K\Sigma_k{}^i-\Sigma_k{}^i\frac{\dot{\phi}}{\phi}=R_k{}^i-\frac{1}{3}\delta_k{}^iR+\frac{1}{\phi}\left(\phi{}^{|i}_{|k}-\frac{1}{3}\delta_k{}^i\Delta\phi\right)$$
$$+\frac{\omega}{\phi^2}\left(\partial^i\phi\partial_k\phi-\frac{1}{3}\delta_k{}^i\partial^m\phi\partial_m\phi\right). \tag{78}$$

We now consider these equations order by order in the gradient expansion. At lowest order, we neglect the terms which have two spatial derivatives, i.e., ${}^{(3)}R$, $\Delta\phi$, etc. Furthermore, we impose a quasi-isotropic nature $\Sigma_j{}^i=0$. In this order, we obtain the solutions

$$\gamma_{ij}=t^{(4\omega+4)/(3\omega+4)}h_{ij}(x)\,, \tag{79}$$

$$\phi=\tilde{\phi}(x)t^{2/(3\omega+4)}\,, \tag{80}$$

$$\rho=\frac{\tilde{\phi}(x)}{8\pi}\frac{4\omega+6}{3\omega+4}t^{-(6\omega+6)/(3\omega+4)}\,. \tag{81}$$

Here, $h_{ij}$ and $\tilde{\phi}$ depend on the spatial coordinates. Substituting (79)~(81) into Eqs. (75)~(78) and keeping the next order terms, we obtain

$$\Sigma_j{}^i=\frac{3\omega+4}{5\omega+8}t^{-\omega/(3\omega+4)}\left[-\left(R_j{}^i-\frac{1}{3}\delta_j{}^iR\right)+\frac{1}{\phi}\left(\phi{}^{|i}_{|j}-\frac{1}{3}\delta_j{}^i\Delta\phi\right)\right.$$
$$\left.+\frac{\omega}{\phi^2}\left(\partial^i\phi\partial_j\phi-\frac{1}{3}\delta_j{}^i\partial^k\phi\partial_k\phi\right)\right], \tag{82}$$

$$\phi^{(2)}=\frac{(3\omega+4)^2}{(5\omega+8)(2\omega+4)(2\omega+3)}t^{(2\omega+6)/(3\omega+4)}\left[\frac{1}{2}\phi R+2(\omega+1)\Delta\phi-\frac{\omega}{2\phi}\partial^k\phi\partial_k\phi\right], \tag{83}$$

$$K_j{}^{i(2)}=\frac{3\omega+4}{5\omega+8}t^{-\omega/(3\omega+4)}\left[-R_j{}^i+\frac{\omega+1}{2(2\omega+3)}\delta_j{}^iR+\frac{1}{\phi}\phi{}^{|i}_{|j}-\frac{\omega+1}{2\omega+3}\delta_j{}^i\frac{1}{\phi}\Delta\phi\right.$$
$$\left.+\frac{\omega}{\phi^2}\partial^i\phi\partial_j\phi-\frac{\omega+1}{2(2\omega+3)}\delta_j{}^i\frac{\omega}{\phi^2}\partial^k\phi\partial_k\phi\right], \tag{84}$$

$$\gamma_{ij}^{(2)}=\frac{(3\omega+4)^2}{(5\omega+8)(\omega+2)}t^2\left[-R_j{}^i+\frac{\omega+1}{2(2\omega+3)}\delta_j{}^iR+\frac{1}{\phi}\phi{}_{|}{}^{i}{}_{|j}-\frac{\omega+1}{2\omega+3}\delta_j{}^i\frac{1}{\phi}\Delta\phi\right.$$

$$\left.+\frac{\omega}{\phi^2}\partial^i\phi\partial_j\phi-\frac{\omega+1}{2(2\omega+3)}\delta_j{}^i\frac{\omega}{\phi^2}\partial^k\phi\partial_k\phi\right]. \tag{85}$$

The above results completely agree with the Hamilton-Jacobi calculation.

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