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Hamiltonian Approach to the Dynamics of Expanding Homogeneous Universes in the Brans-Dicke Cosmology*)

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In view of a grave importance of the problem of initial singularity in theoretical cosmology, the dynamical behavior of expanding homogeneous universes (without rotation) in the Brans-Dicke cosmology is studied by means of extending suitably the canonical formalism due to Arnowitt, Deser and Misner. It is shown that, even if the inertial (scalar) mode characteristic in their theory of gravitation is omitted, our Hamiltonian is somewhat different from Ryan's Hamiltonian in relativistic cosmology. This is due to the difference in the manipulation of the source (consisting of matter and radiation assumed as a perfect fluid whose total density and pressure are represented by ρ and p, respectively) Lagrangian, and it seems that our manipulation is superior to Ryan's. In spite of this, so far as $0\leq p\leq \rho/3$, there exists an extremely early stage at which the source term in our Hamiltonian becomes negligible in such a way that it is reduced to a generalized version of Misner's Hamiltonian derivable from Ryan's under the same approximation. If $\rho/3< p\leq \rho$, however, such a stage cannot exist because of some peculiar role of the inertial mode interacting with matter and radiation. Accordingly, the dynamical behavior of gravitational and inertial modes at the extremely early stage of both the Bianchi-type IX universe with $p=\rho/3$ and the Bianchi-type I universe with $p=\rho$ is analyzed in detail. The dynamical behavior is described as the three-dimensional motion of a world point in the presence of either the tri-angular potential walls (found by Misner) with gravitational origin or another potential field with inertial origin. It is shown in the former case that the inertial mode plays a significant role to modify Misner's bounce law for the collision of the world point with the potential walls, but is incompetent to eliminate the initial singularity of infinite density. On the other hand, in the latter case, the initial singularity may be formally removed under some condition which, however, contradicts with the requirement due to Brans-Dicke that the coupling constant ω must be larger than about 6.

§ 1. Introduction

In previous papers,[1] after examining the present status on the problem of initial singularity in any general relativistic model-universe compatible with the cosmological interpretation of the 3°K black-body radiation, we have proposed a a new approach on the basis of an extended version of the renormalized theory of gravitation due to Utiyama and DeWitt.[2] As a result, we have shown that there exists a homogeneous and isotropic model-universe which may bounce with a finite density and tends asymptotically to the usual Friedmann universe, provided

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that the temperature of cosmic matter and radiation at the instant of bounce is not lower than about 10^{12}°K and that temporal behavior of the model before and after the bounce is of a finite asymmetry. The motive of that approach was the recognition that the renormalized theory of gravitation may stand opposite in another manner to Misner's[3] conclusion that quantum effects do not significantly modify the nature of initial singularity in relativistic cosmology which has been studied in detail by Lifshitz and co-workers[4] on a classical level.

It is to be noticed, however, that Hoyle-Narlikar's[5] C-field cosmology may admit a bouncing model with homogeneity and isotropy,[6] if the C-field introduced originally for describing the creation of matter becomes a massless scalar field with negative energy-momentum but without clear physical meaning. On the other hand, there appears another type of a massless scalar field in Jordan's[7] and Brans-Dicke's[8] theories of gravitation for describing the variability of gravitation constant. As regards the latter theory, its possible significance on the problems of stellar collapse and galaxy formation has been studied by the author.[9]

In view of the present status on the problem of initial singularity in theoretical cosmology, it will be worthwhile to study what situation occurs at an early stage of expanding homogeneous universes in the Brans-Dicke cosmology. Of the two alternative approaches to the dynamics of homogeneous universes in relativistic cosmology, i.e., Lifshitz-Khalatnikov's approach[4] and Misner's Hamiltonian approach,[3] the latter seems to be more suitable to visualize the dynamical behavior of gravitational modes with two degrees of freedom, while the relation between them has recently been clarified by Belinski, Khalatnikov and Ryan.[10] Accordingly we shall seek for the dynamical behavior of gravitational and inertial (scalar) modes at an early stage of Bianchi-type (I$\sim$IX) universes[11] in terms of a generalized version of ADM's canonical formalism.[12] In order to do so, we must rely on Dicke's[13] field equations, which are derivable from Brans-Dicke's original ones[8] by means of the conformal regraduation (like ours[14]) $\tilde{g}_{\mu\nu}\rightarrow g_{\mu\nu}=(G\phi)\tilde{g}_{\mu\nu}$ and $\tilde{m}_e\rightarrow m_e=(G\phi)^{-1/2}\tilde{m}_e$ assuring the invariance of both the Planck constant and the light velocity. Here $\tilde{g}_{\mu\nu}$ and $\tilde{m}_e$ are the original metric tensor and electron mass, respectively, and ϕ is the inertial scalar field whose physical dimension is the same as that of G^{-1} (G is Newtonian gravitation constant).

In § 2, the action integral from which Dicke's regraduated field equations are derivable is cast into the $(3+1)$-dimensional form by means of ADM's procedure.[12] In § 3 the rewritten action integral is applied to a general Bianchi-type expanding universe whose constituent matter and radiation is assumed for simplicity to be a perfect fluid without vorticity. After elimination of four constraint equations (three of which become trivial in the case of no vorticity), the action integral is reduced to the desired form from which canonical equations of motion for the gravitational and inertial modes are derivable. Even if the inertial mode is omitted, our Hamiltonian is somewhat different from Ryan's[15]

Hamiltonian because of the difference in the manipulation of the matter Lagrangian. In spite of this, if the ratio p/ρ (p and ρ are the total pressure and density, respectively) is in the range $0\leq p/\rho\leq 1/3$, our Hamiltonian without inertial mode is reduced to Misner's Hamiltonian[3] in the case of homogeneous empty universes at an extremely early stage such that the matter part is negligible compared with the gravitational part. In §§ 4~5 the extremely early stage of Bianchi-type I and IX universes are dealt with in order to see how the inertial mode has influence upon Misner's result[3] concerning the bounce phenomena with the potential walls with gravitational origin. If, for instance, $p=\rho$ as envisaged by Zel'dovich,[16] the matter part of our Hamiltonian can survive even at an extremely early stage, in addition to its explicit dependence on the inertial mode. Accordingly the Bianchi-type I (for simplicity) universe consisting of matter with $p=\rho$ is dealt with in § 6. The Appendix is devoted to the illustration of various potential walls.

§ 2. The (3+1)-dimensional form of Dicke's regraduated field equations

After the conformal regraduation mentioned in § 1, the total Lagrangian density $\mathcal{L}$ in the scalar-tensor theory of gravitation is represented by

$$\mathcal{L}=(-{}^4g)^{1/2}\{{}^4R-(\omega+3/2)\,{}^4g^{\mu\nu}\psi_\mu\psi_\nu+16\pi GL_m\},^{*)} \tag{2·1}$$

where

$$\psi\equiv\ln(G\phi),\quad \psi_\mu\equiv\partial_\mu\psi\,,\qquad (\omega \text{ is the coupling constant}) \tag{2·2}$$

and L_m stands for the matter (plus radiation) Lagrangian. If the assemblage of matter and radiation consists of a perfect fluid, we may put**)

$$L_m=-\tfrac{1}{2}\{(\rho+p)\,{}^4g^{\mu\nu}u_\mu u_\nu+(\rho-p)\}\,,\qquad ({}^4g^{\mu\nu}u_\mu u_\nu=-1) \tag{2·3}$$

where ρ, p and $u_\mu\equiv{}^4g_{\mu\nu}u^\nu$ stand for the total density, the total pressure and the fluidal 4-velocity, respectively, which are connected with their counterparts $(\tilde\rho, \tilde p, \tilde u_\mu)$ before the regraduation by the following expressions:[14]

$$\rho=e^{-2\psi}\tilde\rho\,,\qquad p=e^{-2\psi}\tilde p\,,\qquad u_\mu=e^{\psi/2}\tilde u_\mu\,. \tag{2·4}$$

An application of the variational principle (due to $\delta\,{}^4g^{\mu\nu}$ and $\delta\psi$) to the action integral $I=(1/16\pi)\int\mathcal{L}d^4x$ leads to[13]

$${}^4G_{\mu\nu}={}^4R_{\mu\nu}-\tfrac{1}{2}{}^4R\,{}^4g_{\mu\nu}=(\omega+3/2)\,(\psi_\mu\psi_\nu-\tfrac{1}{2}\,{}^4g_{\mu\nu}\psi_\alpha\psi^\alpha)+8\pi G\,{}^4T_{\mu\nu} \tag{2·5}$$

and

$${}^4\Box\psi\equiv(-{}^4g)^{-1/2}\partial_\mu\{(-{}^4g)^{1/2}\,{}^4g^{\mu\nu}\partial_\nu\psi\}=\frac{8\pi G}{(3+2\omega)}{}^4T\,, \tag{2·6}$$

*) This notation (such as ${}^4R\equiv{}^4g^{\mu\nu}\,{}^4R_{\mu\nu}$ for the scalar curvature) is due to ADM's procedure.[12]

**) We have used this form of the matter Lagrangian in the canonical treatment of the gravitational modes appearing in the Friedmann universe.[17]

where ${}^4T_{\mu\nu}$ is the energy-momentum tensor of the perfect fluid, i.e.,

$${}^4T_{\mu\nu}\equiv(\rho+p)\,u_\mu u_\nu+p\,{}^4g_{\mu\nu}\,,\qquad {}^4T\equiv{}^4g^{\mu\nu}\,{}^4T_{\mu\nu}=3p-\rho\,. \tag{2·7}$$

In the above derivation, it is to be noticed that (ρ, p, u_μ) are varied by the variation $\delta\psi$ via Eq. (2·4), while they are independent of $\delta\,{}^4g^{\mu\nu}$. Since ${}^4G_{\mu\nu}$ is divergence-free, it follows from Eqs. (2·5) and (2·6) that

$$u_\mu{}^4T^{\mu\nu}{}_{;\nu}=-\tfrac{1}{2}{}^4Tu^\mu\psi_\mu\,, \tag{2·8}$$

which does not contradict with the conservation of the total energy-momentum.

Following ADM's procedure,[12] let us introduce the following quantities:

$$\begin{aligned}
&g_{ij}\equiv{}^4g_{ij}\,,\qquad N\equiv(-{}^4g^{00})^{-1/2},\qquad N_i\equiv{}^4g_{0i}\,,\\
&\pi^{ij}\equiv(-{}^4g)^{1/2}({}^4\Gamma^0_{pq}-g_{pq}g^{rs}\,{}^4\Gamma^0_{rs})\,g^{ip}g^{jq},\\
&p_\psi\equiv(3+2\omega)\,N^{-1}g^{1/2}(\partial_t\psi-N^i\psi_i)\,,
\end{aligned} \tag{2·9}$$

where g^{ij} is the symmetric 3×3 matrix reciprocal to g_{ij}, i.e., $g^{ij}g_{ik}=\delta^j{}_k$, $N^i\equiv g^{ij}N_j$ and $g\equiv\det(g_{ij})=(-{}^4g)/N^2(>0)$. On inserting Eq. (2·9) into Eq. (2·1), we obtain

$$\begin{aligned}
\mathcal{L}=&-g_{ij}\partial_t\pi^{ij}+p_\psi\partial_t\psi+2\kappa(\rho+p)\,g^{1/2}\varpi u_0+NC^0+N_iC^i\\
&-2\{g^{1/2}N^{|i}+N_j(\pi^{ij}-\tfrac{1}{2}\pi^k{}_kg^{ij})\}_{|i}\,,
\end{aligned} \tag{2·10}$$

where

$$\left\{\begin{aligned}
C^0\equiv\,&g^{1/2}[{}^3R-(\omega+3/2)\,g^{ij}\psi_i\psi_j+2\kappa\{p-(\rho+p)\,\varpi^2\}]\\
&+g^{-1/2}[\tfrac{1}{2}(\pi^k{}_k)^2-\pi_{ij}\pi^{ij}-\tfrac{1}{2}p_\psi{}^2/(3+2\omega)]\,,\\
C^i\equiv\,&2\pi^{ij}{}_{|j}-p_\psi g^{ij}\psi_j-2\kappa g^{1/2}(\rho+p)\,\varpi u^i
\end{aligned}\right. \tag{2·11}$$

and

$$u_0=N\varpi+N^iu_i\,,\qquad \varpi\equiv-(g^{ij}u_iu_j+1)^{1/2}. \tag{2·12}$$

Here κ is an abbreviation of $8\pi G$, and the vertical bar indicates covariant differentiation in the 3-space whose metric tensor and scalar curvature are given by g_{ij} and ${}^3R\equiv g^{ij}\,{}^3R_{ij}$, respectively. On inserting Eq. (2·10) into the expression $I=(1/16\pi)\int\mathcal{L}d^4x$ and discarding divergence-terms, we arrive at the required (3+1)-dimensional form of the action integral

$$I=(1/16\pi)\int\{\pi^{ij}\partial_tg_{ij}+p_\psi\partial_t\psi+2\kappa(\rho+p)\,g^{1/2}\varpi u_0+NC^0+N_iC^i\}\,d^4x\,. \tag{2·13}$$

Varying I with respect to N and N_i, we obtain

$$C^0=0\quad\text{and}\quad C^i=0\,, \tag{2·14}$$

which, together with Eq. (2·11), are shown to be equivalent to four constraint equations in the 4-dimensional form, i.e., ${}^4G_{nn}$-eq. and ${}^4G^i_n$-eqs., where ${}^4G_{nn}\equiv n^\mu n^\nu\,{}^4G_{\mu\nu}$, ${}^4G^i_n\equiv n^\mu\,{}^4G_\mu{}^i$ and $n^\mu=(-1,N^i)/N$ being the unit normal to a space-like hyper-surface $x^0=t=$const. Varying I with respect to π^{ij} and g_{ij}, we obtain

$$\partial_t g_{ij} = 2Ng^{-1/2}(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi^k{}_k) + N_{i|j} + N_{j|i}\,, \tag{2·15}$$

$$\begin{aligned}\partial_t \pi^{ij} = &-Ng^{1/2}[{}^3R^{ij} - \tfrac{1}{2}{}^3Rg^{ij} - (\omega+3/2)(\psi^i\psi^j - \tfrac{1}{2}g^{ij}\psi_m\psi^m) \\ &-\kappa\{(\rho+p)u^iu^j + pg^{ij}\}] - 2Ng^{-1/2}(\pi^{ik}\pi^j{}_k - \tfrac{1}{2}\pi^{ij}\pi^k{}_k) \\ &+\tfrac{1}{2}Ng^{-1/2}\{\pi_{mn}\pi^{mn} - \tfrac{1}{2}(\pi^k{}_k)^2 + \tfrac{1}{2}p_\psi{}^2/(3+2\omega)\}g^{ij} \\ &+g^{1/2}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) + (N^m\pi^{ij})_{|m} - (N^i{}_{|m}\pi^{jm} + N^j{}_{|m}\pi^{im}),\end{aligned} \tag{2·16}$$

which are equivalent to ${}^4G_{ij}$-eqs. Similarly, it follows from the variations of I with respect to p_ψ and ψ, respectively, that

$$\partial_t\psi = Ng^{-1/2}p_\psi/(3+2\omega) + N^i\psi_i\,, \tag{2·17}$$

$$\partial_t p_\psi = (N^ip_\psi)_{|i} + (3+2\omega)g^{1/2}(N\psi^i)_{|i} + \kappa(\rho-3p)Ng^{1/2}, \tag{2·18}$$

which are equivalent to Eq. (2·6). We have regarded u_0 as a non-varied quantity in the derivation of Eqs. (2·15)$\sim$(2·17), but Eq. (2·18) relies on the following relation:

$$\varpi(\delta u_0/\delta\psi - N^i\delta u_i/\delta\psi) = N(1/2 + \varpi\delta\varpi/\delta\psi),$$

which is the (3+1)-dimensional form of ${}^4g^{\mu\nu}\delta(u_\mu u_\nu)/\delta\psi = -1$ derivable from Eq. (2·4). Moreover, we can reduce Eq. (2·8) to

$$\begin{aligned}\partial_t\rho + (\rho+p)\partial_t\{\ln(g^{1/2}|\varpi|)\} - (1/\varpi)\{(Nu^i + \varpi N^i)\partial_i\rho + (\rho+p)(Nu^i+\varpi N^i)_{|i}\} \\ = -\tfrac{1}{2}N(\rho-3p)\{g^{-1/2}p_\psi/(3+2\omega) - (1/\varpi)u^i\psi_i\}.\end{aligned} \tag{2·19}$$

A substitution of Eq. (2·14) in Eq. (2·13) gives

$$I = (1/16\pi)\int\{\pi^{ij}\partial_t g_{ij} + p_\psi\partial_t\psi + 2\kappa(\rho+p)g^{1/2}\varpi u_0\}d^4x\,, \tag{2·20}$$

in which all of g_{ij} and π^{ij} can no longer be independent of each other because of the existence of the constraint equations (2·14) with C^μ given by Eq. (2·11). Our task lies in reducing Eq. (2·20) to the following canonical form:

$$I = (1/16\pi)\int\{\pi^{ij\mathrm{TT}}\partial_t g_{ij}{}^{\mathrm{TT}} + p_\psi\partial_t\psi - \mathcal{H}(g_{ij}{}^{\mathrm{TT}}, \pi^{ij\mathrm{TT}}, \psi, p_\psi, \rho, p)\}d^4x\,, \tag{2·21}$$

by imposing a suitable set of coordinate conditions (in terms of which N and N_i are fixed without destroying the validity of Eqs. (2·15)$\sim$(2·18)) and solving Eq. (2·14) with respect to L(longitudinal)- and T(transverse)-parts of g_{ij} and π^{ij} as functions of their TT(transverse-traceless)-parts, the inertial mode (ψ, p_ψ) and the fluidal quantities (ρ, p).

§ 3. Hamiltonian treatment of a general Bianchi-type expanding universe

Following Belinski, Khalatnikov and Ryan,[10] let us denote a general Bianchi-type expanding universe by the metric

$$ds^2 = -N^2(\Omega)d\Omega^2 + g_{ab}(\Omega)\sigma^a\sigma^b, \qquad (\sigma^a \equiv e^a_i dx^i) \tag{3·1}$$

where $x^0=\Omega$ is Misner's[3] time variable, and $e^a_i=e^a_i(x^k)$ stand for three $(a=1, 2, 3)$ covariant 3-vectors such that the exterior derivative of σ^a obeys the relation $d\sigma^a=C^a{}_{bc}\sigma^b\wedge\sigma^c$ in which $C^a{}_{bc}(=-C^a{}_{cb})$ are the structure constants of the group of motions specifying homogeneity of the 3-space $\Omega=\text{const}$, i.e.,

$$[X_a, X_b]=C^c{}_{ab}X_c\,, \qquad (X_a\equiv e^i_a\partial_i) \tag{3·2}$$

where e^i_a is the reciprocal of e^a_i such that

$$e^i_a e^b_i=\delta^b_a\,, \qquad e^i_a e^a_j=\delta^i_j\,. \tag{3·3}$$

Then we can introduce the following quantities including g_{ab} in Eq. (3·1):

$$\begin{aligned}
&g_{ab}(\Omega)\equiv e^i_a e^j_b g_{ij}(\Omega, x^k),\\
&\pi^a_b(\Omega)\equiv e^a_i e^j_b \pi^i{}_j(\Omega, x^k), \qquad \pi^{ab}{}_{|b}\equiv e^a_i\pi^{ij}{}_{|j}(\Omega, x^k),\\
&u_a(\Omega)\equiv e^i_a u_i(\Omega, x^k), \qquad \psi_a(\Omega)\equiv e^i_a\psi_i(\Omega, x^k),\\
&N_a(\Omega)\equiv e^i_a N_i(\Omega, x^k), \qquad N_{a|b}\equiv e^i_a e^j_b N_{i|j}(\Omega, x^k),
\end{aligned} \tag{3·4}$$

which are scalars or scalar densities with respect to a transformation of coordinates $x^i\to x'^i=f^i(x^j)$, but affine tensors or vectors with respect to a rotation of e^i_a and e^a_i preserving relation (3·3). In the universe under consideration, we must have

$$N_a=\psi_a=0\,, \tag{3·5}$$

in addition to the situation that p_ψ, ρ and p are functions of Ω alone.

Taking account of Eqs. (3·4) and (3·5), we can reduce Eq. (2·20) with $t=\Omega$ and $d^4x=d\Omega\wedge\sigma^1\wedge\sigma^2\wedge\sigma^3$ to

$$I=\int\{\pi^{ab}dg_{ab}+p_\psi d\psi+2\kappa(\rho+p)g^{1/2}\varpi u_0 d\Omega\}, \tag{3·6}$$

on the prescription that

$$\int\sigma^1\wedge\sigma^2\wedge\sigma^3=(4\pi)^2, \tag{3·7}$$

which is simply a normalization condition for the Bianchi-type IX universe with closed 3-space, but becomes a periodicity condition for an open universe (e.g., Bianchi-type I and V).[3] By using again Eqs. (3·4) and (3·5), we can transform Eq. (2·14) with C^μ given by Eq. (2·11) into

$$C^0\equiv g^{1/2}[{}^3R+2\kappa\{p-(\rho+p)\varpi^2\}]+g^{-1/2}\{\tfrac{1}{2}(\pi^c_c)^2-\pi^a_b\pi^b_a-\tfrac{1}{2}p_\psi{}^2/(3+2\omega)\}=0\,, \tag{3·8}$$

$$C^a\equiv 2\pi^{ab}{}_{|b}-2\kappa(\rho+p)g^{1/2}\varpi u^a=0\,, \tag{3·9}$$

where

$$u_0=N\varpi=-N(g^{ab}u_a u_b+1)^{1/2}. \tag{3·10}$$

Similarly, it follows from Eqs. (2·15) and (2·17) that

$$dg^{1/2}/d\Omega = -\tfrac{1}{2}N\pi^c_c\,, \tag{3·11}$$

$$d\psi/d\Omega = Ng^{-1/2}p_\psi/(3+2\omega)\,. \tag{3·12}$$

By the use of Eq. (3·12), we can reduce Eq. (2·19) to

$$d\rho/d\Omega + (\rho+p)\,d\{\ln(g^{1/2}|\varpi|)\}/d\Omega = -\tfrac{1}{2}(\rho-3p)\,d\psi/d\Omega\,. \tag{3·13}$$

Let us introduce here an auxiliary 3-space with the metric

$$dl^2 = \gamma_{ab}(\Omega)\,\sigma^a\sigma^b, \tag{3·14}$$

where

$$\begin{cases} \gamma_{ab} = (e^{2\beta})_{ab} \equiv R^{-2}g_{ab}\,, & (\beta_{aa}=0) \\ R = R_0 e^{-\Omega}, & (R_0 = \text{const} = 1/\sqrt{6\pi},\ \text{say}) \end{cases} \tag{3·15}$$

the last one showing that Ω is measured from future to past. If we denotes the scalar curvature of the auxiliary space by $K = R^2(\Omega)\,{}^3R$, we can derive the following relation:

$${}^3R = \tfrac{3}{2}(1-V)R^{-2}(\Omega)\,, \tag{3·16}$$

with

$$V \equiv 1 - 2K/3 = 1 - \tfrac{1}{3}[\{n^{ab}(e^{2\beta})_{ab}\}^2 - 2n^{ab}n^{pq}(e^{2\beta})_{ap}(e^{2\beta})_{bq}] + 4a_a a_b (e^{-2\beta})_{ab}\,, \tag{3·17}$$

where $n^{ab}(=n^{ba})$ and a_a are an affine tensor and vector, respectively, with constant values such that (cf. the second paper in Ref.4))

$$C^c_{ab} = e_{abd}n^{dc} + \delta^c_b a_a - \delta^c_a a_b\,, \qquad (n^{ab}a_b = 0) \tag{3·18}$$

and e_{abd} is Levi-Civita's anti-symmetric tensor. Moreover, it follows from Eqs. (3·11) and (3·15) that

$$N = 6R^3/\pi^c_c\,. \tag{3·19}$$

Until now we have considered a general Bianchi-type universe, but let us assume in what follows that the metric tensor $\gamma_{ab} = (e^{2\beta})_{ab}$ is diagonal and the σ^a-frame is comoving with the fluid matter. In this case, we may put[3)]

$$\begin{cases} \beta_{ab} = \mathrm{diag}(\beta_+ + \beta_-\sqrt{3},\ \beta_+ - \beta_-\sqrt{3},\ -2\beta_+)\,, & (\beta_{aa}=0) \\ p^a_b \equiv 2\pi(\pi^a_b - \delta^a_b\pi^c_c/3) = \tfrac{1}{6}\,\mathrm{diag}(p_+ + p_-\sqrt{3},\ p_+ - p_-\sqrt{3},\ -2p_+)\,, & (p^a_a = 0) \\ u_a = 0\,. \quad (\text{so that } \varpi = -1,\ u_0 = -N) & \end{cases} \tag{3·20}$$

Moreover, let us put

$$\beta_0 \equiv \lambda^{-1}\psi\,, \quad p_0 \equiv \pi\lambda p_\psi\,. \qquad \left(\lambda \equiv \left(\frac{12}{3+2\omega}\right)^{1/2}\right) \tag{3·21}$$

A substitution of Eqs. (3·15), (3·20) and (3·21) in Eq. (3·13) gives

$$\rho' - 3(\rho + p) = -\tfrac{1}{2}\lambda(\rho - 3p)\beta_0', \tag{3·22}$$

which, together with an equation of state $p = p(\rho)$, leads to a definite relation among ρ, β_0 and Ω, where a dash denotes differentiation with respect to Ω in what follows. On inserting Eqs. (3·15), (3·16), (3·20) and (3·21) into Eq. (3·8) with $\kappa \equiv 8\pi G$ and solving the latter with respect to π^c_c, we obtain

$$2\pi\pi^c_c = \{p_+^2 + p_-^2 + p_0^2 + e^{-4\Omega}(V-1) + (16G/9)\rho e^{-6\Omega}\}^{1/2}. \tag{3·23}$$

The remaining constraints (3·9) are automatically satisfied in the case under consideration. Now it is an easy matter to transform the action integral (3·6) into the desired form

$$I = \int (p_+ d\beta_+ + p_- d\beta_- + p_0 d\beta_0 - H d\Omega), \tag{3·24}$$

where

$$H \equiv (2\pi\pi^c_c) - (8G/9)(\rho + p)e^{-6\Omega}/(2\pi\pi^c_c), \tag{3·25}$$

which, together with Eqs. (3·22) and (3·23), denotes the Hamiltonian of the system consisting of the gravitational modes $(\beta_\pm, p_\pm)$ and the inertial one (β_0, p_0).

Even if the inertial mode is omitted, our Hamiltonian is different from Ryan's Hamiltonian $H = 2\pi\pi^c_c$.[15] This is due to the situation that Ryan has discarded the last term on the right-hand side of Eq. (3·6) in his reduction. In spite of this, if there exists an extremely early stage such that the contribution of gravitational and inertial modes to H overwhelms that of matter, our Hamiltonian is approximated by

$$H = \{p_+^2 + p_-^2 + p_0^2 + e^{-4\Omega}(V-1)\}^{1/2}, \tag{3·26}$$

which is an extended version of Misner's Hamiltonian[3] (derivable from Ryan's under the same approximation). Because of a nonvanishing nature of the right-hand side (unless $p = \rho/3$) in Eq. (3·22), however, the existence of such a stage will be confined to the case where $0 \leq p \leq \rho/3$. Accordingly we shall derive the exact expressions for H in the typical two cases $p = \rho/3$ (suitable for the radiation dominant stage of a big-bang universe) and $p = \rho$ (envisaged by Zel'dovich[16]), for comparison.

The case $p = \rho/3$:

In this case, it follows from Eq. (3·22) that

$$3p = \rho = (\rho)_0 e^{4\Omega}, \tag{3·27}$$

where $(\rho)_0$ is the value of ρ at $\Omega = 0$. On inserting Eq. (3·27) into Eq. (3·25) with $2\pi\pi^c_c$ given by Eq. (3·23), we have

$$H = \{p_+^2 + p_-^2 + p_0^2 + e^{-4\Omega}(V-1) + (\mu/3)e^{-2\Omega}\}/\{p_+^2 + p_-^2 + p_0^2 + e^{-4\Omega}(V-1) + \mu e^{-2\Omega}\}^{1/2}, \tag{3·28}$$

where $\mu \equiv 16G(\rho)_0/9$. The above expression is reduced to Eq. (3·26) at an extremely early stage such that $e^{\Omega} \gg \{\mu/(p_+^2+p_-^2+p_0^2)\}^{1/2}$.

The case $p=\rho$:

In this case, Eq. (3·22) can be integrated as

$$p=\rho=\rho_* \exp(\lambda\beta_0+6\Omega), \tag{3·29}$$

where ρ_* is an integration constant. Then it follows from Eqs. (3·23) and (3·25) that

$$H=\{p_+^2+p_-^2+p_0^2+e^{-4\Omega}(V-1)\}/\{p_+^2+p_-^2+p_0^2+\mu_* e^{\lambda\beta_0}+e^{-4\Omega}(V-1)\}^{1/2}, \tag{3·30}$$

where $\mu_* \equiv 16G\rho_*/9$.

§ 4. Gravitational and inertial modes at an extremely early stage of the Bianchi-type I universe with $p=\rho/3$

Let us consider an extremely early stage of a big-bang homogeneous universe filled with matter and radiation obeying the relation $p=\rho/3$, so that the Hamiltonian for gravitational and inertial modes is represented by Eq. (3·26). The only model-dependent term in this expression for H is the potential term $e^{-4\Omega}(V-1)$, which vanishes only in the case of the Bianchi-type I universe specified by $n^{ab}=a_a=0$ (cf. Eq. (3·17)). This means that the Bianchi-type I universe serves as a kinematical background to analyze the dynamical behavior of gravitational and inertial modes in other Bianchi-type universes. Our subject of this section is to see what situation occurs by the presence of the inertial mode.

In this universe, we have

$$H=(p_+^2+p_-^2+p_0^2)^{1/2}. \tag{4·1}$$

Equations (3·24) and (4·1) lead to the following canonical equations of motion:

$$\beta_n'=\partial H/\partial p_n=p_n/H, \tag{4·2}$$

$$p_n'=-\partial H/\partial \beta_n=0 \quad \text{or} \quad p_n=\text{const}, \tag{4·3}$$

so that we obtain

$$H'=\partial H/\partial\Omega=0 \quad \text{or} \quad H=\text{const}, \tag{4·4}$$

where $n=+,\ -,\ 0$. By virtue of Eqs. (4·3) and (4·4), Eq. (4·2) can be integrated as

$$\beta_\pm=(p_\pm/H)\Omega, \qquad \beta_0=(\beta_0)_0+(p_0/H)\Omega, \tag{4·5}$$

where we have assumed without loss of generality that $\beta_\pm=0$ at $\Omega=0$. Moreover, we have

$$\beta'^2 \equiv (\beta_+')^2+(\beta_-')^2+(\beta_0')^2=1, \tag{4·6}$$

which, together with Eq. (4·5), shows that a world point in the β-space moves with unit Ω-velocity along some straight line.

To make clearer the role of the inertial mode in the universe under consideration, let us transform here from the (Ω, σ^a)-frame to the usual (t, x^i)-frame by means of the relations $dt = -Nd\Omega$ and $dx^i = e^i_a\sigma^a$. Taking account of Eqs. (3·15), (3·19) and (4·4) with $H = 2\pi\pi^c_c$ (at the extremely early stage), and $e^i_a = \delta^i_a$ (for the Bianchi-type I universe), we obtain

$$\Omega = -\tfrac{1}{3}\ln\tau\,, \qquad \sigma^a = dx^a, \tag{4·7}$$

where $\tau \equiv t/t_0$ and $t_0 \equiv (2/3H)R_0$. By making use of Eqs. (3·15), (3·20), (3·21), (4·5) and (4·7), we can reduce Eqs. (3·1) and (2·2) to

$$ds^2 = -dt^2 + R_0^{\,2}(\tau^{2p_1}dx^2 + \tau^{2p_2}dy^2 + \tau^{2p_3}dz^2)\,, \tag{4·8}$$

and

$$\phi = (\phi)_0\tau^{-q}, \tag{4·9}$$

where $(\phi)_0 \equiv G^{-1}\exp\{\lambda(\beta_0)_0\}$, and

$$\begin{cases} p_1 \equiv \dfrac{1}{3}\left\{1 - \dfrac{(p_+ + p_-\sqrt{3})}{H}\right\}, \quad p_2 \equiv \dfrac{1}{3}\left\{1 - \dfrac{(p_+ - p_-\sqrt{3})}{H}\right\}, \quad p_3 \equiv \dfrac{1}{3}\left(1 + \dfrac{2p_+}{H}\right), \\ q \equiv (\lambda/3)\,(p_0/H)\,. \end{cases} \tag{4·10}$$

It is easily seen that the above model-universe specified by Eqs. (4·8) ~ (4·10) is reduced to the Kasner universe[3),4)] in relativistic cosmology, if we discard the inertial mode, i.e., if we put $\beta_0 = p_0 = q = 0$ (so that we have $H = (p_+^{\,2} + p_-^{\,2})^{1/2}$). For the Kasner universe, the counterparts of (p_1, p_2, p_3) given by Eq. (4·10) satisfy the relations

$$p_1 + p_2 + p_3 = p_1^{\,2} + p_2^{\,2} + p_3^{\,2} = 1\,, \tag{4·11}$$

and it has been shown by Lifshitz and Khalatnikov[4)] that they are parametrized as follows:

$$p_1 = \frac{-u}{(u^2+u+1)}\,,\quad p_2 = \frac{u+1}{(u^2+u+1)}\,,\quad p_3 = \frac{u(u+1)}{(u^2+u+1)}\,, \tag{4·12}$$

which satisfy the inequalities $p_1 \leq p_2 \leq p_3$ for $u \geq 1$, and show further that the isotropic case $p_1 = p_2 = p_3$ is prohibited.

On the contrary, the Brans-Dicke cosmology leads to a special isotropic model-universe such that

$$p_1 = p_2 = p_3 = 1/3\,,\quad q = \lambda/3\,. \qquad (\text{if } p_+ = p_- = 0) \tag{4·13}$$

If $p_\pm \neq 0$, however, the model must inevitably be anisotropic and the three constants (p_1, p_2, p_3) must satisfy the relations

$$p_1 + p_2 + p_3 = 1\,, \qquad p_1^{\,2} + p_2^{\,2} + p_3^{\,2} = 1 - 6(q/\lambda)^2, \tag{4·14}$$

the second one of which is permissible only when $|q|<\lambda/\sqrt{6}$. Then the counterpart of Eq. (4·12) is given by

$$\begin{cases} p_1=\dfrac{-u}{(u^2+u+1)}\,, \\ \begin{pmatrix} p_2 \\ p_3 \end{pmatrix}=\dfrac{(u+1)}{(u^2+u+1)}\left[\begin{pmatrix} 1 \\ u \end{pmatrix}+\begin{pmatrix} + \\ - \end{pmatrix}\dfrac{(u-1)}{2}\left\{1-\sqrt{1-\zeta^2\left(\dfrac{u^2+u+1}{u^2-1}\right)^2}\right\}\right], \end{cases} \tag{4·15}$$

where

$$\zeta\equiv(3+2\omega)^{1/2}q. \qquad \text{(cf. Eq. (3·21))} \tag{4·16}$$

In order that p_2 and p_3 given by Eq. (4·15) may be real, we must have

$$(1-\zeta)u^2-\zeta u-(1+\zeta)\geq 0\,. \tag{4·17}$$

If $0\leq\zeta<1$, (which is necessary for the inclusion of the Kasner universe to be specified by $\zeta=0$), it follows from Eq. (4·17) that

$$u\geq u_*\equiv(\zeta+\sqrt{4-3\zeta^2})/2(1-\zeta), \tag{4·18}$$

whose right-hand side is a monotonously increasing function of ζ such that $u_*(0)=1$ and $u_*(1)=\infty$. Moreover, we have

$$p_1(1/u)=p_1(u), \quad p_2(1/u)=p_3(u), \quad p_3(1/u)=p_2(u), \tag{4·19}$$

like (p_1, p_2, p_3) given by Eq. (4·12). Because of the above two relations (4·18) and (4·19), we see that $p_1<p_2\leq p_3$ for $u\geq u_*$, while $p_1<p_3<p_2$ for $u<u_*$. In addition, we have

$$\begin{aligned} &p_1(\infty)=0\,, \quad p_2(\infty)=\tfrac{1}{2}(1-\sqrt{1-\zeta^2}), \quad p_3(\infty)=\tfrac{1}{2}(1+\sqrt{1-\zeta^2}), \\ &p_1(1/u_*)=-\tfrac{1}{3}(\sqrt{4-3\zeta^2}-1), \quad p_2(1/u_*)=p_3(1/u_*)=\tfrac{1}{6}(2+\sqrt{4-3\zeta^2}). \end{aligned} \tag{4·20}$$

§ 5. Dynamical behavior of the gravitational and inertial modes in the Bianchi-type IX universe with $p=\rho/3$

Among the remaining Bianchi-type (II~IX) universes (for which respective forms of the potential $V=V(\beta_+,\beta_-)$ in Eq. (3·26) will be given in the Appendix), let us pick up the Bianchi-type IX universe which is a generalized version of the Friedmann universe with closed 3-space, for comparison with its general relativistic treatment due to Misner.[3]

The structure constants of this universe are given by $C^c_{ab}=\epsilon_{abc}$ (or $n^{ab}=\delta^{ab}$ and $a_a=0$ in Eq. (3·18)), so that Eq. (3·17) is reduced to

$$V=1+\tfrac{2}{3}e^{4\beta_+}\{\mathrm{ch}(4\sqrt{3}\beta_-)-1\}-\tfrac{4}{3}e^{-2\beta_+}\mathrm{ch}(2\sqrt{3}\beta_-)+\tfrac{1}{3}e^{-8\beta_+}, \tag{5·1}$$

which has originally been derived by Misner.[3] A substitution of the above potential function in Eq. (3·26) gives the Hamiltonian for the system of gravitational and inertial modes, which is available at an extremely early stage of the universe under consideration. Then, by the variation of the action integral I given by Eq. (3·24), the following canonical equations of motion arise:

$$\beta_\pm' = \partial H/\partial p_\pm = p_\pm/H\,, \tag{5·2·1}$$

$$p_\pm' = -\partial H/\partial \beta_\pm = -(e^{-4\Omega}/2H)\,\partial V/\partial\beta_\pm\,, \tag{5·2·2}$$

and

$$\beta_0' = \partial H/\partial p_0 = p_0/H\,, \tag{5·3·1}$$

$$p_0' = -\partial H/\partial \beta_0 = 0 \quad \text{or} \quad p_0 = \text{const}\,, \tag{5·3·2}$$

the last one of which is the only constant of motion, while we have four constants of motion (cf. Eqs. (4·3) and (4·4)) in the Bianchi-type I universe. On inserting Eqs. (5·2·1) and (5·3·1) into Eq. (3·26), we get

$$\beta'^2 \equiv (\beta_+')^2 + (\beta_-')^2 + (\beta_0')^2 = 1 - e^{-4\Omega}(V-1)H^{-2}, \tag{5·4}$$

which, together with $H' = \partial H/\partial\Omega$, leads to

$$(\ln H^2)' = 4(\beta'^2 - 1)\,. \tag{5·5}$$

The above two equations show that, unless V becomes sufficiently large for $\Omega \simeq \infty$, both $\beta'^2 \simeq 1$ and $H \simeq \text{const}$ for $\Omega \simeq \infty$ (just as in the Bianchi-type I universe) hold good in a good approximation. For the potential V given by Eq. (5·1), however, the required condition does not always hold. In fact, the above potential varies from $V \simeq 8\varpi_\beta^2$ for $\varpi_\beta \equiv (\beta_+^2 + \beta_-^2)^{1/2} \simeq 0$ to the following asymptotic form:

$$V \sim \tfrac{1}{3} e^{-8\beta_+} \qquad \text{for } \beta_+ \to -\infty, \tag{5·6}$$

which is valid if $|\beta_-| < -\sqrt{3}\beta_+$. A substitution of Eq. (5·6) in the expression $H^{-2}e^{-4\Omega}V \simeq 1$ (which assures a large discrepancy of β'^2 from unity) provides us with

$$\beta_+ \simeq (\varpi_\beta)_{\text{wall}} = -\tfrac{1}{2}\Omega - \tfrac{1}{8}\ln(3H^2)\,, \tag{5·7}$$

where $(\tilde{\omega}_\beta)_{\text{wall}}$ stands for an equi-potential wall in an ϖ_β-plane specified by $\beta_0 =$ const in the β-space. If the position of a world point is within the potential wall, i.e., $\varpi_\beta = (\beta_+^2 + \beta_-^2)^{1/2} < |(\varpi_\beta)_{\text{wall}}|$, there hold $|\beta'| = 1$, $H = \text{const}$ and $|(\varpi_\beta)'_{\text{wall}}| = 1/2$ in a good approximation. Accordingly, if $|\varpi_\beta'| \simeq (1-\beta_0'^2)^{1/2} > 1/2$, the world point collides soon after with the potential wall in an inelastic manner and changes into again the rectilinear motion specified by $|\beta'| = 1$ and $H = \text{const}$ (smaller than its initial value) till the subsequent collision with another potential wall. The difference of this picture from its general relativistic counterpart is the situation that the existence of the inertial mode makes the motion of a world point to be three-dimensional.

In order to deal with the bounce phenomena of a world point with the tri-angular potential walls[3)] derivable from Eq. (5·1) in more detail, let us consider the effective Hamiltonian which is defined to be H given by Eqs. (3·26) and (5·6), i.e.,

$$H=\{p_+^2+p_-^2+p_0^2+\tfrac{1}{3}\exp(-8\beta_+-4\Omega)\}^{1/2}. \tag{5·8}$$

Then, from Eqs. (5·2) and (5·3), we can derive the following additional constants of motion (in the approximation adopted now):

$$p_-=\text{const}\,, \qquad J\equiv\tfrac{1}{2}p_++H=\text{const}\,. \tag{5·9}$$

Since the motion of a world point before and after its bounce with the potential wall specified by Eq. (5·7) is the same as that specified by Eqs. (4·5) and (4·6), we may put

$$\begin{aligned}
&(p_+/H)_i=(\beta_+')_i=-\sin\theta_i\cos\varphi_i\,, && (p_+/H)_f=(\beta_+')_f=\sin\theta_f\cos\varphi_f\,,\\
&(p_-/H)_i=(\beta_-')_i=\sin\theta_i\sin\varphi_i\,, && (p_-/H)_f=(\beta_-')_f=\sin\theta_f\sin\varphi_f\,,\\
&(p_0/H)_i=(\beta_0')_i=\cos\theta_i\,, && (p_0/H)_f=(\beta_0')_f=\cos\theta_f\,,
\end{aligned} \tag{5·10}$$

where (θ_i, φ_i) and (θ_f, φ_f) are the angular variables specifying the directions of motion before and after the bounce. A substitution of Eq. (5·10) in Eqs. (5·3·2) and (5·9) gives

$$H_i\sin\theta_i\sin\varphi_i=H_f\sin\theta_f\sin\varphi_f\,, \tag{5·11}$$

$$H_i(1-\tfrac{1}{2}\sin\theta_i\cos\varphi_i)=H_f(1+\tfrac{1}{2}\sin\theta_f\cos\varphi_f)\,, \tag{5·12}$$

$$H_i\cos\theta_i=H_f\cos\theta_f\,, \tag{5·13}$$

where we may assume that both θ_i and θ_f are smaller than $\pi/2$ and they are not equal to each other, because the case $\theta_i=\theta_f=\pi/2$ corresponds to the general relativistic model, and $H_i\neq H_f$ if $p_\pm\neq 0$. Eliminating H_i and H_f from Eqs. (5·11) $\sim$(5·13), we obtain

$$\sin\varphi_i/\sin\theta_i-\sin\varphi_i/\sin\theta_f=\tfrac{1}{2}\sin(\varphi_i+\varphi_f)\,, \tag{5·14}$$

$$\tan\theta_i\sin\varphi_i=\tan\theta_f\sin\varphi_f\,, \tag{5·15}$$

which, together with Eq. (5·13), are the bounce law in question. The bounce phenomenon obeying the above law is schematically shown in Fig. 1.

To make the required calculation simple, let us pick up a special case such that

$$\theta_i+\theta_f=\pi/2\,, \qquad \varphi_i+\varphi_f=\pi/3\,. \tag{5·16}$$

Then it follows from Eqs. (5·14) and (5·15) that

$$\sin\varphi_i=\frac{\sqrt{3}}{4}\sin\theta_f\cos\theta_i/(\cos\theta_f-\cos\theta_i)\,, \tag{5·17}$$

$$4(\cos\theta_f-\cos\theta_i)^2=\sin^2(\theta_i+\theta_f)-\tfrac{1}{4}\sin(2\theta_i)\sin(2\theta_f)\,. \tag{5·18}$$

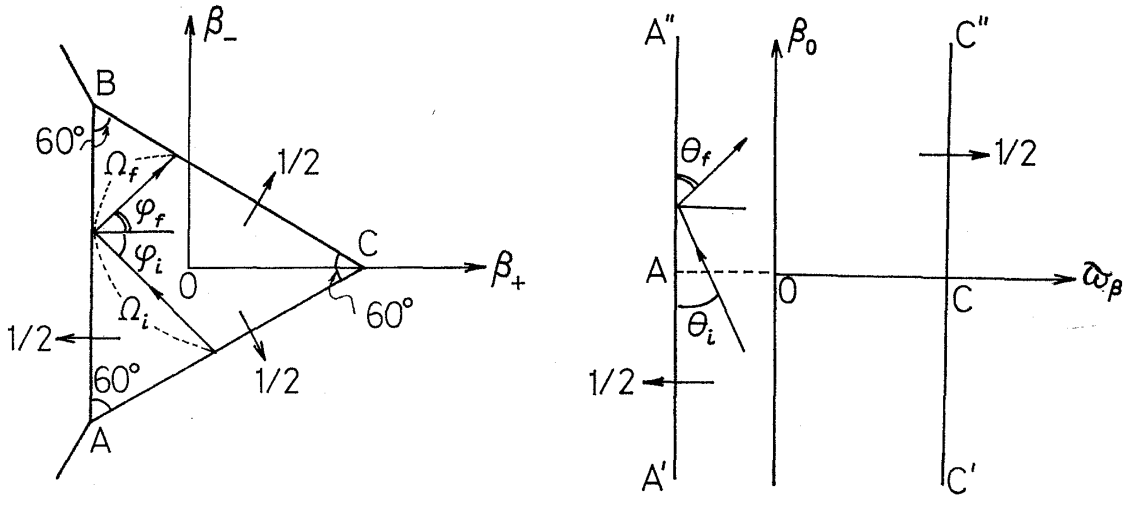

Fig. 1. Schematical representation of the bounce of a world point with one of the tri-angular potential walls moving with 0.5 Ω-velocity.
(a) Situation in the (β_+, β_-)-plane. (b) Situation in the $(\beta_0, \widetilde{\varpi_\beta})$-plane.

Since Eq. (5·18) is symmetrical with respect to the interchange of θ_i and θ_f $(=\pi/2-\theta_i)$, it has two solutions such that $\theta_i=\frac{1}{2}\arcsin(8-2\sqrt{13})=0.4545$ or $\theta_i=\pi/2-\frac{1}{2}\arcsin(8-2\sqrt{13})=1.116$. Of the two solutions, the latter alone makes the value of φ_i given by Eq. (5·17) positive. Thus we obtain

$$\begin{cases} \theta_i=1.116\,(63.95°), & \theta_f=0.4545\,(26.05°), \\ \varphi_i=0.1826\,(10.46°), & \varphi_f=0.8645\,(49.54°), \\ H_f/H_i=\cos\theta_i/\cos\theta_f=0.4686. \end{cases} \tag{5·19}$$

On the other hand, Misner's general relativistic result[3] (corresponding to the case where $\theta_i=\theta_f=\pi/2$ and $\varphi_i+\varphi_f=\pi/3$) is as follows:

$$\begin{cases} \varphi_i=15.5°, & \varphi_f=44.5°, \\ H_f/H_i=\sin\varphi_i/\sin\varphi_f=0.382\,. \end{cases} \tag{5·20}$$

A comparison of both results shows that, owing to the presence of the inertial mode, the damping rate of the "energy" per one bounce, i.e., $(H_i-H_f)/H_i$, is smaller about 14% and the incident angle φ_i in the ϖ_β-plane is smaller about 33% than their general relativistic counterparts. In spite of this, the inertial mode is useless to modify the relation

$$H_i\Omega_i=H_f\Omega_f\,, \tag{5·21}$$

whose quantum version led Misner[3] to the conclusion (cf. § 1) on the inevitability of the initial singularity in relativistic cosmology. Here Ω_i and Ω_f stand for the time durations during which a world point moves in the β-space before and after the bounce with one of the tri-angular potential walls, as shown in Fig. 1. This is due to the situation that $(H_i\sin\theta_i, H_f\sin\theta_f)$ and $(\Omega_i/\sin\theta_i, \Omega_f/\sin\theta_f)$ play the same roles as (H_i, H_f) and (Ω_i, Ω_f) in the derivation of Eq. (5·21).

To rewrite the bounce law (5·13)$\sim$(5·15) in terms of the Lifshitz and Khalatnikov's parameter u appearing in Eq. (4·15), let us insert Eqs. (4·10) and (4·16) into Eq. (5·10). Then we can rewrite Eqs. (5·14) and (5·15) as

$$\{(p_2)_i-(p_1)_i\}/\{3(p_3)_i+1\}=\{(p_2)_f-(p_1)_f\}/\{3(p_3)_f+1\}, \tag{5·22}$$

$$\{(p_2)_i-(p_1)_i\}/\zeta_i=\{(p_2)_f-(p_1)_f\}/\zeta_f, \tag{5·23}$$

where $(p_a)_i=p_a(u_i, \zeta_i)$ and $(p_a)_f=p_a(u_f, \zeta_f)$ should be given by Eq. (4·15). Even in the general relativistic case, however, the relation between u_i and u_f thus derived is not the simple law $u_f=u_i-1$,[4] but its rewritten form $2u_iu_f+u_i+u_f=0$.[18] Taking account of such a situation, let us assume that

$$(p_a)_i=p_a\left(-\frac{1}{u_i+1}, \zeta_i\right), \qquad (p_a)_f=p_a(1/u_f, \zeta_f), \tag{5·24}$$

where $p_a (a=1, 2, 3)$ are given by Eq. (4·15). Then it follows from Eqs. (5·13), (5·22) and (5·23) that

$$H_f/H_i=\zeta_i/\zeta_f=\left(\frac{u_f^2+u_f+1}{u_i^2+u_i+1}\right)\left(\frac{u_i}{u_f+1}\right)^2, \tag{5·25}$$

and

$$2(u_i+u_f+1)(u_iu_f+u_f+1)=(u_f+1)^2\{(u_i^2+2u_i)^2-\zeta_i^2(u_i^2+u_i+1)^2\}^{1/2}$$
$$-u_i^2(u_f+1)\{(u_f-1)^2-\zeta_i^2(u_f+1)^2(u_i^2+u_i+1)^2\}^{1/2}, \tag{5·26}$$

which is much complicated than $u_f=u_i-1$ corresponding to the limiting case such that $\zeta_i\to 0$, $\zeta_f\to 0$ and $\zeta_i/\zeta_f\to(u_f^2+u_f+1)/(u_i^2+u_i+1)$. This is the reason why we have mainly relied on the Hamiltonian approach rather than on the Lifshitz-Khalatnikov approach.

§ 6. Dynamical behavior of the gravitational and inertial modes in the Bianchi-type I universe with $p=\rho$

As pointed out in § 3, there exists an extremely early stage such that the full Hamiltonian (3·25) can be approximated by the simplified form (3·26) only when the constituent matter and radiation of the universe satisfy the inequalities $0\leq p\leq\rho/3$. This is clearly seen from an application of Eqs. (4·3), (4·4), (4·5) and (4·10) to the Bianchi-type I universe with $p=\rho$ (but not $p=\rho/3$), because Eq. (3·30) with $V=1$ leads to $H\simeq(G(\phi)_0\mu_*)^{-1/2}(p_+^2+p_-^2+p_0^2)\exp(-3q\Omega/2)\to 0$ in contradiction with Eq. (4·4). Accordingly, we shall deal with the Bianchi-type I universe with $p=\rho$ in what follows.

In this universe, the exact Hamiltonian is of the form

$$H=(p_+^2+p_-^2+p_0^2)/(p_+^2+p_-^2+p_0^2+\mu_*e^{\lambda\beta_0})^{1/2}, \tag{6·1}$$

which shows that there are three constants of motion

$$p_{\pm}=\text{const}\,,\qquad H=\text{const}\,. \tag{6·2}$$

If we put

$$v\equiv(p_+{}^2+p_-{}^2+p_0^2)/H^2,\qquad v_*\equiv(p_+{}^2+p_-{}^2)/H^2=\text{const}\,, \tag{6·3}$$

it follows from Eqs. (6·1) and (3·29) that

$$\mu_* e^{\lambda\beta_0}=H^2 v(v-1) \tag{6·4}$$

and

$$p=\rho=(9H^2/16G)\,v(v-1)\,e^{6\Omega}. \tag{6·5}$$

Moreover, the remaining canonical equations of motion $\beta_n{}'=\partial H/\partial p_n$ $(n=+,\,-,\,0)$ are reduced to

$$\varepsilon\lambda d\Omega=\frac{v}{(v-1)}\cdot\frac{dv}{\sqrt{v-v_*}}\,, \tag{6·6}$$

$$d\beta_{\pm}/d\Omega=(p_{\pm}/H)\frac{(2v-1)}{v^2}\,, \tag{6·7}$$

where $p_0=\varepsilon H\sqrt{v-v_*}$ and $\varepsilon=\pm1$. An integration of Eq. (6·6) gives

$$\varepsilon\lambda\Omega=\begin{cases}2\sqrt{v-v_*}+\dfrac{2}{\sqrt{v_*-1}}\arctan\sqrt{\dfrac{v-v_*}{v_*-1}}\,, & (\text{if } v_*>1)\\[2ex] 2(\sqrt{v-v_*}-\sqrt{1-v_*})+\dfrac{1}{\sqrt{1-v_*}}\ln\left|\dfrac{\sqrt{v-v_*}-\sqrt{1-v_*}}{\sqrt{v-v_*}+\sqrt{1-v_*}}\right|, & (\text{if } 0<v_*<1)\end{cases} \tag{6·8}$$

where we have chosen an integration constant suitably. Similarly, it follows from Eqs. (6·7) and (6·8) that

$$\frac{\varepsilon\lambda\beta_{\pm}}{(p_{\pm}/H)}=\begin{cases}\dfrac{2}{\sqrt{v_*}}\arctan\sqrt{\dfrac{v-v_*}{v_*}}+\dfrac{2}{\sqrt{v_*-1}}\arctan\sqrt{\dfrac{v-v_*}{v_*-1}}\,, & (\text{if } v_*>1)\\[2ex] \dfrac{2}{\sqrt{v_*}}\left(\arctan\sqrt{\dfrac{v-v_*}{v_*}}-\arctan\sqrt{\dfrac{1-v_*}{v_*}}\right) & \\[2ex] \qquad+\dfrac{1}{\sqrt{1-v_*}}\ln\left|\dfrac{\sqrt{v-v_*}-\sqrt{1-v_*}}{\sqrt{v-v_*}+\sqrt{1-v_*}}\right|. & (\text{if } 0<v_*<1)\end{cases} \tag{6·9}$$

(i) *The case* $v_*>1$:

Since the right-hand side of Eq. (6·8) is a monotonously increasing function of v becoming infinitely large when $v\to\infty$, the range of Ω is either $(-\infty,0)$ or $(0,\infty)$ according as $\varepsilon=-1$ or 1. In addition, irrespective of the sign of ε, we have

$$v \simeq (\lambda^2/4)\Omega^2 \qquad \text{for } v \gg v_* . \tag{6·10}$$

A substitution of the above expression in Eq. (6·5) gives

$$\rho \simeq (9/16G)\,(\lambda^2 H/4)^2 \Omega^4 e^{6\Omega} \qquad \text{for } \varepsilon\Omega \simeq \infty, \tag{6·11}$$

which shows that $\rho \to \infty$ or 0 for $\Omega \to \infty\,(\varepsilon=1)$ or $-\infty\,(\varepsilon=-1)$. This means that, so far as $v_*>1$, the initial singularity of infinite density is inevitable. Moreover, it follows from Eqs. (6·8) and (6·9) that

$$\beta_\pm \simeq \begin{cases} (p_\pm/H)\left(\dfrac{2v_*-1}{v_*^2-1}\right)\Omega & \text{for } \Omega \simeq 0\,, \\ (p_\pm/H)\,(\varepsilon\pi/\lambda)\,(1/\sqrt{v_*}+1/\sqrt{v_*-1}) & \text{for } \varepsilon\Omega \simeq \infty. \end{cases} \tag{6·12}$$

(ii) *The case* $0<v_*<1$:

Equation (6·8) shows that $\varepsilon\Omega \to -\infty$ or ∞ according as $v \to 1$ or ∞. Accordingly we have

$$v \simeq \begin{cases} (\lambda^2/4)\Omega^2 & \text{for } \Omega \simeq -\infty, \\ 1+4(1-v_*)\exp(-\lambda\sqrt{1-v_*}\,\Omega) & \text{for } \Omega \simeq \infty, \end{cases} \tag{6·13}$$

provided that $p_0 = -H\sqrt{v-v_*}$ or $\varepsilon=-1$. On inserting the second expression of Eq. (6·13) into Eq. (6·5), we obtain

$$\rho \simeq (9H^2/16G)\,(1-v_*)\exp\{(6-\lambda\sqrt{1-v_*}\,)\Omega\} \qquad \text{for } \Omega \simeq \infty. \tag{6·14}$$

In order that the above ρ may be of a finite value even when $\Omega \to \infty$, the parameter $\lambda \equiv \{12/(3+2\omega)\}^{1/2}$ (cf. Eq. (3·21)) must be equal to $6/\sqrt{1-v_*}$, so that we have

$$\omega = -(8+v_*)/6\,. \qquad (0<v_*<1) \tag{6·15}$$

On the other hand, the scalar-tensorial versions divided by the Einstein formulae for the so-called 3 tests of general relativity are represented by (the perihelion advance of Mercury) $=(4+3\omega)/(6+3\omega)$, (the gravitational deflection of light) $=(3+2\omega)/(4+2\omega)$ and (the gravitational shift of the wave length of light) $=1$, respectively.[8] A substitution of Eq. (6·15) in the first formula gives (the perihelion advance of Mercury) $=v_*/(v_*-4)<0$ which contradicts with the experimental evidence. If $\omega \gtrsim 6$ as insisted upon by Brans-Dicke,[8] we have $\lambda<6/\sqrt{1-v_*}$ showing again the inevitability of infinite density for $\Omega \to \infty$. Moreover, it follows from Eqs. (6·8) and (6·9) with $\varepsilon=-1$ that

$$\beta_\pm \simeq \begin{cases} -(p_\pm/H)\,(\pi/\lambda\sqrt{v_*}) & \text{for } \Omega \simeq -\infty, \\ (p_\pm/H)\Omega & \text{for } \Omega \simeq \infty. \end{cases} \tag{6·16}$$

As is seen from Eqs. (6·12) and (6·16), the functional form of $\beta_\pm$ or the metric component g_{ab} (cf. Eq. (3·15)) in the case $p=\rho$ varies sensibly with the lapse of time, while its counterpart (cf. Eq. (4·5)) in the case $p=\rho/3$ is independent of time. In spite of this, it has been shown that the initial singularity of infinite density cannot be eliminated, by virtue of the situation that Eq. (6·15) is incompatible with the requirement of $\omega \gtrsim 6$.

Remark The dynamical system specified by the Hamiltonian H given by Eq. (6·1) can be quantized by replacing (β_n, p_n) $(n=+, -, 0)$ with the operators satisfying the commutation relations $\beta_m p_n - p_n \beta_m = i\hbar\delta_{mn}$. Then we must replace the factor $v(v-1)$ in Eq. (6·5) with its vacuum expectation value. Even if the expectation value of ρ becomes finite for $\Omega \to \infty$, the metric component g_{ab} itself becomes singular because of the factor $R^2 = (1/6\pi) e^{-2\Omega}$.

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Appendix

The potential walls in other Bianchi-type universes

As was shown in § 3, the canonical description of Bianchi-type expanding universes in the Brans-Dicke cosmology is represented by the motion of a world point in the three dimensional β-space in the presence of the potential walls with gravitational origin, as well as of another potential with inertial origin (effective in the case of $p=\rho$). In §§ 4∼5 we have analyzed in what way the former potential has influence upon the motion in the case of Bianchi-type IX universe, while the role of the latter potential in the Bianchi-type I universe has been studied in § 6. The subject of this appendix is to derive various potential walls (with gravitational origin) corresponding to other Bianchi-type universes, by the use of Eqs. (3·17), (3·18) and (3·20).

We can prove that the asymptotic potential walls (in the ϖ_β-plane specified by $\beta_0 = \text{const}$) corresponding to Bianchi-types II and IV are a straight-lines and two half-straight-lines intersecting with the corner angle $\pi/3$, respectively. The straight-line and one of the two half-straight-lines move with 0.5 Ω-velocity, so that they may bounce with the world point only one time. Moreover, the asymptotic potential walls corresponding to Bianchi-types III and VII are shown to be of the same tri-angular form as that of the Bianchi-type VI. Accordingly, we shall call attention to the universes of Bianchi-type V, VI and VIII, respectively.

a) *Bianchi-type V (specified by $n^{ab}=0$, $a_a=(1, 1, 1)/\sqrt{3}$):*

For this universe, Eq. (3·17) is reduced to

$$V = 1 + \tfrac{8}{3} e^{-2\beta_+} \operatorname{ch}(2\sqrt{3}\beta_-) + \tfrac{4}{3} e^{4\beta_+}. \tag{A·1}$$

The above potential gives rise to the asymptotic tri-angular form shown in Fig. 2,

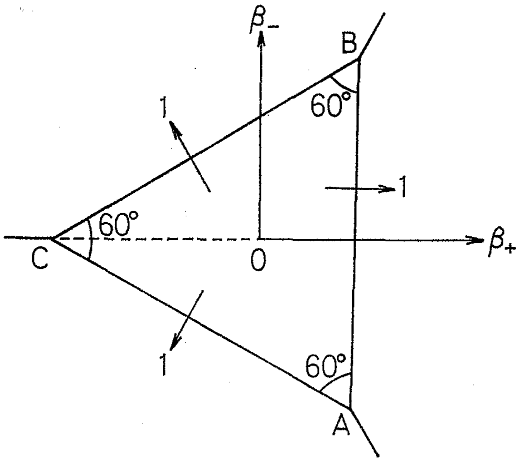

Fig. 2. The tri-angular potential walls in the Bianchi-type V universe.

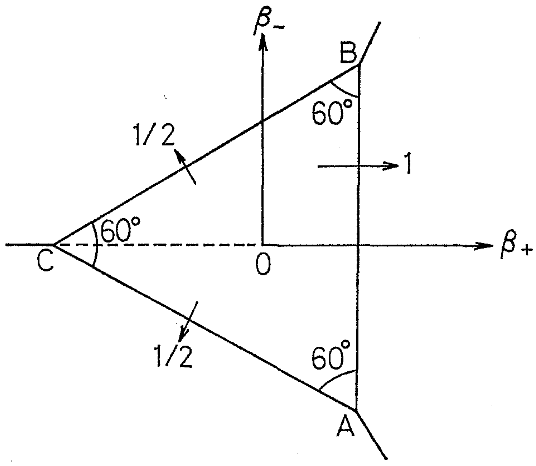

Fig. 3. The tri-angular potential walls in the Bianchi-type VI universe.

derivable from the condition that the potential becomes effective, i.e., $H^{-2}e^{-4\Omega}V \simeq 1$ (cf. Eq. (3·26)).

b) *Bianchi-type VI (specified by* $n^{11}=-n^{22}=1$, $n^{33}=a_a=0$*):*

For this universe, we have

$$V=1+\tfrac{2}{3}e^{4\beta_+}\{\mathrm{ch}(4\sqrt{3}\beta_-)+1\}, \qquad (\mathrm{A}\cdot 2)$$

whose asymptotic form is shown in Fig. 3.

c) *Bianchi-type VIII (specified by* $n^{11}=n^{22}=-n^{33}=1, a_a=0$*):*

For this universe, we have

$$V=1+\tfrac{2}{3}e^{4\beta_+}\{\mathrm{ch}(4\sqrt{3}\beta_-)-1\}+\tfrac{4}{3}e^{-2\beta_+}\,\mathrm{ch}(2\sqrt{3}\beta_-)+\tfrac{1}{3}e^{-8\beta_+}, \qquad (\mathrm{A}\cdot 3)$$

which is of the same form as Eq. (5·1), except for the plus sign in front of the third term on the right-hand side. However, the third term does not contribute to the asymptotic form, so that the latter is identical with the one shown in Fig. 1, in spite of the situation that the Bianchi-type VIII universe is an open universe.

Among the above three universes, the first two provide us with the bounce laws different from that discussed in § 5. In the Bianchi-type V universe, the bounce cannot at all occur, because each of the three walls AB, BC and CA moves with unit Ω-velocity, just as the world point. On the other hand, in the Bianchi-type VI universe, the world point may collide only with the two walls BC and CA whose Ω-velocities are 1/2.

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