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Hamiltonian Defined as a Graph Limit in a Simple System with an Infinite Renormalization

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The Hamiltonian of the neutral scalar field in interaction with a fixed source, when the infinite renormalization of vacuum energy is necessary, is defined by the graph limit of approximate Hamiltonians. Remarks are made on the asymptotic fields in this case.

§ 1. Introduction

The existence of the asymptotic fields in quantum field theory as strong limits of operators in the Hilbert space has been established by several authors.[1)~7)] However, all of the examples studied so far belong to the quantum field theory with the characteristics that they have no need of a renormalization.

Recently the first example of the asymptotic fields in the system which requires an infinite renormalization has been presented by Dimock in the Y_2 theory.[8)] It will be worthwhile to study in a similar way another example with infinite renormalization, always bearing in mind the extension of the method of asymptotic fields applicable to more complex and more realistic cases. The simplest model towards this direction is the neutral scalar field in interaction with a fixed source.

As is well known, the theory of neutral scalar field interacting with a fixed source in four-dimensional space-time shows its features distinct from both physical and mathematical points of view, according to the condition that the source function satisfies. Let $\omega(\boldsymbol{k})=(\boldsymbol{k}^2+m^2)^{1/2}$ be the energy of a single boson with a positive mass m, and let $\omega^{1/2}v(\boldsymbol{k})$ be proportional to the Fourier transform of the source function. Define the space G_n of the functions $v(\boldsymbol{k})$ equipped with the norm

$$\|v\|_n=\left[\int|v(\boldsymbol{k})|^2\omega^{-n+1}d\boldsymbol{k}\right]^{1/2}. \tag{1·1}$$

The Fock representation of the Hilbert space is possible so long as the function $v(\boldsymbol{k})$ belongs to G_3. Under the strongest condition $v\in G_1$ the total Hamiltonian is defined by the regular perturbation. The condition $v\in G_2$ is necessary and sufficient for the Hamiltonian to be defined by the regular perturbation in quadratic form, when the shift of the vacuum energy is still finite.[9)]

If $v\in G_3$, but $v\notin G_2$, the vacuum energy shifts by an infinite amount, so that we need an infinite renormalization of the Hamiltonian in order that the latter should be defined properly. The purpose of the present paper is to give a mathematically rigorous definition to the total Hamiltonian and to remark on the strong asymptotic limit of operators. The existence of asymptotic fields in this case in the general scheme has been suggested by a previous consideration by one of us.[10]

In defining the Hamiltonian we follow Glimm and Jaffe,[11] who used the graph limit to get the total Hamiltonian of the Y_2 theory with a space cutoff, but independent of momentum cutoff. Namely, in view of the fact that G_1 is dense in G_3 with respect to the norm $\|v\|_3$, we shall approximate the Hamiltonian H with any $v\in G_3$ by the sequence $\{H_n\}$ with $v_n\in G_1$ in the sense of graphs, thus giving the correct definition to the Hamiltonian H.

Together with definitions and notations, some mathematical tools are summarized in § 2. Important estimates are derived in § 3. The total Hamiltonian is defined and some remarks on the asymptotic fields are made in § 4.

§ 2. Preliminaries

The total Hamiltonian of the neutral scalar field interacting with a fixed source is formally given by

$$H=H_0+H_I+C \tag{2·1}$$

with

$$H_0=\int\omega(\boldsymbol{k})a^*(\boldsymbol{k})a(\boldsymbol{k})d\boldsymbol{k}\,,$$

$$H_I=\int(v(\boldsymbol{k})a^*(\boldsymbol{k})+\bar{v}(\boldsymbol{k})a(\boldsymbol{k}))d\boldsymbol{k}\,,$$

$$C=\int|v(\boldsymbol{k})|^2\omega(\boldsymbol{k})^{-1}d\boldsymbol{k}\,,$$

where H_0 is the free Hamiltonian and it is a densely defined self-adjoint operator in the Fock space $\mathscr{F}$. $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ are, as usual, annihilation and creation operators of the boson normalized by the relation

$$[a(\boldsymbol{k}),a^*(\boldsymbol{k}')]=\delta(\boldsymbol{k}-\boldsymbol{k}').$$

The function $[2\omega(\boldsymbol{k})]^{1/2}v(\boldsymbol{k})$ is the Fourier transform of the source function.*)

We are primarily interested in the case when $v\in G_3$ and hence the counterterm C in the Hamiltonian is not finite. Quite formally it is known that this counterterm cancels completely the infinity appearing in the Hamiltonian. We shall see the correct meaning of such a renormalization cancellation in the course

*) For the point source we have $[2\omega(\boldsymbol{k})]^{1/2}v(\boldsymbol{k})=g$, the coupling constant, so that the corresponding function $v(\boldsymbol{k})$ is in G_4.

of defining the Hamiltonian by a limiting process. To this end we make use of the fact that the space G_1 is dense in the space G_3; for any $v \in G_3$, there exists a sequence of functions v_n such that $v_n \in G_1$ and

$$\lim_{n\to\infty} \|v_n - v\|_3 = 0 . \tag{2·2}$$

Correspondingly, let us introduce a sequence of approximate Hamiltonians H_n defined by

$$H_n = H_0 + H_{In} + C_n , \tag{2·3}$$

where

$$H_{In} = \int (v_n(\boldsymbol{k}) a^*(\boldsymbol{k}) + \bar{v}_n(\boldsymbol{k}) a(\boldsymbol{k})) d\boldsymbol{k} ,$$

$$C_n = \int |v_n(\boldsymbol{k})|^2 \omega^{-1} d\boldsymbol{k} .$$

Since $v_n \in G_1$, the total Hamiltonian H_n is a self-adjoint operator with the domain of definition identical with $D(H_0)$. As we shall see later, H_n is bounded from below uniformly in n.

The Hamiltonian H in (2·1) is given a mathematical meaning as a limit of the Hamiltonians H_n. We use the notions of the graph limit and the resolvent convergence.[12] Let $\{\theta_n\}$ be a sequence of vectors $\theta_n \in D(H_n)$ such that, putting $\psi_n = H_n \theta_n$, there exist $\theta = \lim_{n\to\infty} \theta_n$ and $\psi = \lim_{n\to\infty} \psi_n$. Denote the sets of such limit vectors $\{\theta\}$ and pair of limit vectors $\{\theta, \psi\}$ by $\mathbf{D}_\infty$ and $\mathbf{G}_\infty$, respectively. The denseness of $\mathbf{D}_\infty$ is necessary and sufficient for $\mathbf{G}_\infty$ to be the graph of a symmetric operator H_∞. The operator H_∞ is called the *graph limit* of $\{H_n\}$.

H_n is uniformly bounded from below, so that there is a sufficiently large negative number $-b (b > 0)$ which belongs to the union of the resolvent sets of all H_n. Therefore, in order to state the self-adjointness of the graph limit, we have only to prove the strong convergence of the resolvents $R_n(z) = (H_n - z)^{-1}$ for $z = -b$. It is easy to see that $\|(-b) R_n(-b)\|$ is bounded uniformly in n and b. In general, if the resolvents $R_n(z)$ converge strongly to an operator $R(z)$, which has a densely defined inverse, we speak of the *resolvent convergence* of $\{H_n\}$. In our case, that $R(-b)$ has a densely defined inverse is ensured by the existence of the graph limit and the uniform boundedness of $\|(-b) R_n(-b)\|$. $H = R(-b)^{-1} - b$ defines an operator which has a dense domain and is independent of b. The self-adjointness of H follows from the boundedness of $R(-b)$ which is evidently symmetric. We can prove that $H_\infty = H$.

In summary, central problems in defining the total Hamiltonian of the present model are, among others, to ascertain the uniform lower boundedness of H_n, the denseness of $\mathbf{D}_\infty$ and the strong convergence of $\{R_n(-b)\}$. Let us begin with the first problem.

§ 3. Estimates

Two estimates important in the following are

$$N \leqq \text{const}(H_n + b) \tag{3·1}$$

and

$$N^2 \leqq \text{const}(H_n + b)^2, \tag{3·2}$$

where $N = \int a^*(\boldsymbol{k})a(\boldsymbol{k})d\boldsymbol{k}$ is the number operator. Here and in the sequel, we shall use the identical letter to denote constants of the same character, and all the constants are independent of n unless stated explicitly.

The first order estimate (3·1) is valid on $D((H_n+b)^{1/2}) \times D((H_n+b)^{1/2})$ and shows that the approximate Hamiltonian H_n is uniformly bounded below in n as required. It seems certain that the estimate (3·1) can be verified directly by calculating the quadratic form, just as in a similar calculation by Kato and one of us (N.M) in the fixed-source model with $v \in G_2$.[2),9)] In order to avoid unnecessary complications, however, it will be convenient to use the dressing transformation, whose exact form is known in this case.

The derivation of (3·2) from (3·1) without reference to the dressing transformation has already been given essentially in the previous work;[10)] the only difference of a c-number counterterm in the Hamiltonians causes no substantial change in the estimate. It is sufficient to remark that the constant in (3·2) first appears accompanying $\|v_n\|_3$, which is however regarded as uniformly bounded for large n, since $\|v_n\|_3 \leqq \|v_n - v\|_3 + \|v\|_3$ and $\|v_n - v\|_3$ can be made arbitrarily small. We omit the proof of the second order estimate. Instead we shall give in the Appendix, together with a detailed description of the proof for (3·1), an orientation for the proof of the higher order estimate

$$N^j \leqq \text{const}(H_n + b)^j \tag{3·3}$$

valid on $D((H_n+b)^{j/2}) \times D((H_n+b)^{j/2})$, where j is any positive integer.

§ 4. Definition of the Hamiltonian

Now we are going to give the Hamiltonian (2·1) with $v \in G_3$ a mathematical meaning as a graph limit of the approximate Hamiltonians (2·3) with $v_n \in G_1$ such that $\lim_{n\to\infty} \|v_n - v\|_3 = 0$. We shall show a stronger convergence of resolvents for H_n, that is, the convergence in norm. Let $R_n(-b) = (H_n + b)^{-1}$ be the resolvent of the Hamiltonian H_n, in which the constant b is to be taken large enough so that the first and second order estimates (3·1) and (3·2) hold.

Lemma 4. 1.

$$\|R_n(-b) - R_m(-b)\| \to 0 \quad \textit{as } n,\ m \to \infty. \tag{4·1}$$

Proof. Corresponding to $H_n = H_0 + H_{In} + C_n$, we have

$$R_n(-b)-R_m(-b)=-R_m(-b)\{\delta H_I+\delta C\}R_n(-b), \tag{4·2}$$

where

$$\delta H_I=\int(\delta v(\boldsymbol{k})a^*(\boldsymbol{k})+\delta\bar{v}(\boldsymbol{k})a(\boldsymbol{k}))d\boldsymbol{k},$$

$$\delta C=\int\delta|v(\boldsymbol{k})|^2\omega^{-1}d\boldsymbol{k},$$

$$\delta v(\boldsymbol{k})=v_n(\boldsymbol{k})-v_m(\boldsymbol{k}),\quad \delta\bar{v}(\boldsymbol{k})=\bar{v}_n(\boldsymbol{k})-\bar{v}_m(\boldsymbol{k}),$$

$$\delta|v(\boldsymbol{k})|^2=|v_n(\boldsymbol{k})|^2-|v_m(\boldsymbol{k})|^2=\delta\bar{v}(\boldsymbol{k})v_n(\boldsymbol{k})+\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k}).$$

The use of the pull through formula

$$a(\boldsymbol{k})R_n(-b)=R_n(-b-\omega)a(\boldsymbol{k})-R_n(-b-\omega)[a(\boldsymbol{k}),H_{In}]R_n(-b) \tag{4·3}$$

with

$$[a(\boldsymbol{k}),H_{In}]=v_n(\boldsymbol{k})$$

gives for the right-hand side of (4·2)

$$R_m(-b)\{\delta H_I+\delta C\}R_n(-b)=R_1+R_2+R_3+R_4, \tag{4·4}$$

where

$$\begin{aligned}
R_1&=\int d\boldsymbol{k}\delta v(\boldsymbol{k})a^*(\boldsymbol{k})R_m(-b-\omega)R_n(-b),\\
R_2&=R_m(-b)\int d\boldsymbol{k}\delta\bar{v}(\boldsymbol{k})R_n(-b-\omega)a(\boldsymbol{k}),\\
R_3&=R_m(-b)\int d\boldsymbol{k}\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k})[\omega^{-1}-R_m(-b-\omega)]R_n(-b),\\
R_4&=R_m(-b)\int d\boldsymbol{k}\bar{v}(\boldsymbol{k})v_n(\boldsymbol{k})[\omega^{-1}-R_n(-b-\omega)]R_n(-b).
\end{aligned} \tag{4·5}$$

Our aim is to show that

$$\|R_i\|\leqq\text{const}\,\|\delta v\|_3,\qquad i=1,2,3,4, \tag{4·6}$$

where the constant is independent of n and m for sufficiently large n and m. We shall see the renormalization cancellation in the estimate for $\|R_3\|$ and $\|R_4\|$. Indeed, by the resolvent equation

$$R_m(-b-\omega)=\omega(\boldsymbol{k})^{-1}-\omega(\boldsymbol{k})^{-1}(H_m+b)R_m(-b-\omega),$$

we obtain

$$\begin{aligned}
\|R_3\|&=\left\|R_m(-b)\int d\boldsymbol{k}\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k})\omega(\boldsymbol{k})^{-1}(H_m+b)R_m(-b-\omega)R_n(-b)\right\|\\
&\leqq\int d\boldsymbol{k}|\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k})\omega(\boldsymbol{k})^{-1}|\,\|R_m(-b-\omega)R_n(-b)\|\\
&\leqq\text{const}\,\|\delta v\|_3\|v_m\|_3.
\end{aligned}$$

We here remark again that $\|v_m\|_3$ is uniformly bounded for sufficiently large m. The estimate for $\|R_4\|$ can be carried out similarly.

In order to get the estimate for $\|R_1\|$ we use the unitary transformation

$$U_m=\exp\left\{\int(\bar{v}_m(\boldsymbol{k})a(\boldsymbol{k})-v_m(\boldsymbol{k})a^*(\boldsymbol{k}))\omega(\boldsymbol{k})^{-1}d\boldsymbol{k}\right\}, \tag{4·7}$$

which is well defined even for $v_m\in G_3$, and transforms $a(\bar{f})=\int a(\boldsymbol{k})\bar{f}(\boldsymbol{k})d\boldsymbol{k}$ with $f\in L_2$ into $U_m a(\bar{f})U_m^{-1}=a(\bar{f})+\int v_m(\boldsymbol{k})\bar{f}(\boldsymbol{k})\omega(\boldsymbol{k})^{-1}d\boldsymbol{k}$ and H_0 into H_m. Then we have

$$\begin{aligned}R_1&=\int d\boldsymbol{k}\delta v(\boldsymbol{k})a^*(\boldsymbol{k})U_m(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b)\\&=U_m\int d\boldsymbol{k}\delta v(\boldsymbol{k})a^*(\boldsymbol{k})(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b)\\&\quad-U_m\int d\boldsymbol{k}\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k})\omega(\boldsymbol{k})^{-1}(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b),\end{aligned}$$

and hence

$$\begin{aligned}\|R_1\|&\leqq\left\|\int d\boldsymbol{k}\delta v(\boldsymbol{k})a^*(\boldsymbol{k})(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b)\right\|\\&\quad+\left\|\int d\boldsymbol{k}\delta v(\boldsymbol{k})\bar{v}_m(\boldsymbol{k})\omega(\boldsymbol{k})^{-1}(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b)\right\|\\&\leqq\left[\int d\boldsymbol{k}|\delta v(\boldsymbol{k})|^2\|(N+1)^{1/2}(H_0+b+\omega)^{-1}U_m^{-1}R_n(-b)\|^2\right]^{1/2}\\&\quad+\int d\boldsymbol{k}|\delta v(\boldsymbol{k})||v_m(\boldsymbol{k})|\omega(\boldsymbol{k})^{-1}\|(H_0+b+\omega)^{-1}\|\,\|U_m^{-1}R_n(-b)\|\\&\leqq\text{const}\left[\int d\boldsymbol{k}|\delta v(\boldsymbol{k})|^2\omega(\boldsymbol{k})^{-2}\|(N+1)^{1/2}U_m^{-1}R_n(-b)\|^2\right]^{1/2}\\&\quad+\text{const}\int d\boldsymbol{k}|\delta v(\boldsymbol{k})||v_m(\boldsymbol{k})|\omega(\boldsymbol{k})^{-2}\\&\leqq\text{const}\|\delta v\|_3\|(N+1)U_m^{-1}R_n(-b)\|+\text{const}\|\delta v\|_3\|v_m\|_3.\end{aligned}$$

The first term is estimated separately as follows.

$$\begin{aligned}&\|(N+1)U_m^{-1}R_n(-b)\|\\&=\left\|U_m^{-1}\left\{N+1+\|v_m\|_3^2+\int(v_m(\boldsymbol{k})a^*(\boldsymbol{k})+\bar{v}_m(\boldsymbol{k})a(\boldsymbol{k}))\omega(\boldsymbol{k})^{-1}d\boldsymbol{k}\right\}R_n(-b)\right\|\\&\leqq\|(N+1)R_n(-b)\|+\|v_m\|_3^2\|R_n(-b)\|+2\|v_m\|_3\|(N+1)^{1/2}R_n(-b)\|\\&\leqq\text{const}+\text{const}\|v_m\|_3^2+\text{const}\|v_m\|_3.\end{aligned}$$

In the last inequality we have used the second order estimate (3·2). In this way we obtain

$$\|R_1\| \leqq (c_1 + c_2 \|v_m\|_3 + c_3 \|v_m\|_3^2) \|\delta v\|_3 ,$$

and thus (4·6) is proved for R_1. The proof of (4·6) for R_2 is similar.

Summarizing, we have from (4·2), (4·4) and (4·6)

$$\|R_n(-b) - R_m(-b)\| \leqq \text{const} \|v_n - v_m\|_3$$

for large n and m. Since $\|v_n - v_m\|_3 \to 0$ as $n, m \to \infty$, the proof of Lemma 4.1 is completed.

We have used the unitary transformation to prove (4·6) for R_1 and R_2. It should be remarked that if $v \in G_2$ and the sequence $\{v_n\}$ with $v_n \in G_1$ approximates v in the norm $\|v_n - v\|_2$, then Lemma 4.1 can be proved by Glimm's standard estimates[13] without the aid of the unitary transformation.

Next we show the existence of a graph limit of the sequence $\{H_n\}$. For this purpose, it suffices for us to notice the fact that the unitary transformation is just the dressing transformation. We state the following propositions without proofs.

Lemma 4.2. *Let $\theta_n = U_n\phi$ and $\theta = U\phi$ for any $\phi \in \mathscr{F}$. Here U_n and U are unitary transformations given by the expression like (4·7) with $v_n \in G_1$ and $v \in G_3$, respectively, in place of v_m, and $\lim_{n\to\infty} \|v_n - v\|_3 = 0$. Then U_n converges strongly to U:*

$$\|\theta_n - \theta\| \to 0 \quad \text{as } n \to \infty .$$

Corollary 4.3. *Suppose ϕ in Lemma 4.2 be written as $\phi = H_0\chi$ with $\chi \in D(H_0)$, and let*

$$\psi_n = U_n H_0 \chi = H_n U_n \chi \quad \text{and} \quad \psi = U H_0 \chi .$$

Then we have

$$\|\psi_n - \psi\| \to 0 \quad \text{as } n \to \infty .$$

Lemma 4.4. *Let $\theta_n = U_n\chi$ and $\theta = U\chi$ with $\chi \in D(H_0)$. Let also ψ_n and ψ be given as in Corollary 4.3. Define $\mathsf{D}_\infty = \{\theta\}$ and $\mathsf{G}_\infty = \{\theta, \psi\}$. The D_∞ is dense in $\mathscr{F}$, and moreover G_∞ is the graph of a symmetric operator H_∞ having D_∞ as its domain.*

That D_∞ is dense in $\mathscr{F}$ is a direct consequence of the fact that $\{\chi\}$ is dense in $\mathscr{F}$.

With Lemmas 4.1 to 4.4, what we said in § 2 leads us to

Theorem 4.5. *Let $R_n(-b)$ be the resolvent of H_n with $b > 0$ large enough. Let $R(-b)$ be the strong limit of $R_n(-b)$, H_∞ the graph limit of H_n, both as $n \to \infty$. Then there exists $R(-b)^{-1}$ and H_∞ is given by $R(-b)^{-1} - b$. H_∞ is self-adjoint and its domain is D_∞.*

H_∞ is the operator that gives the Hamiltonian (2·1) a correct meaning. We shall suppress the suffix ∞ from here on. Clearly $H = UH_0U^{-1}$ and $D(H) = \{U\chi\}$

with $\chi \in D(H_0)$.

Once the total Hamiltonian H has been given a correct meaning, it is rather trivial to analyze the asymptotic condition in this case. Consider the adjusted operator defined by

$$a_f(t) = e^{iHt} e^{-iH_0 t} a(\bar{f}) e^{iH_0 t} e^{-iHt}$$
$$= e^{iHt} a(\bar{f} e^{i\omega t}) e^{-iHt} \tag{4·8}$$

and similarly for its adjoint. Here

$$a(\bar{f}) = \int a(\boldsymbol{k}) \bar{f}(\boldsymbol{k}) d\boldsymbol{k}\,, \qquad f \in L_2$$

is a smeared annihilation operator. From the standard estimate $\|a(\bar{f})(N+1)^{-1/2}\| \leqq \text{const}\,\|f\|$ and the first order estimate (A·4), it follows that the adjusted operator (4·8) is well defined at least on $D(H^{1/2})$. More precisely, since (4·8) can be transformed into

$$a_f(t) = a(\bar{f}) + \int \bar{f}(\boldsymbol{k}) v(\boldsymbol{k}) \omega(\boldsymbol{k})^{-1} (e^{i\omega t} - 1) d\boldsymbol{k}\,, \tag{4·9}$$

the domain of $a_f(t)$ is identical with $D(a(\bar{f}))$. If v is locally L_1, the second term on the right-hand side of (4·9) tends to zero as $t \to \pm\infty$ by virtue of Riemann-Lebesgue's lemma.

It will be interesting at the pedagogical level to note the possibility of applying the general scheme of asymptotic fields to the present model. Define the approximate adjusted operator $a_f(t, n)$ by the expression like (4·8) with H replaced by H_n. The ascertainment of the two limiting processes

$$\|a_f(t)\theta - a_f(t, n)\theta_n\| \to 0 \quad \text{as } n \to \infty \tag{4·10}$$

and

$$\|a_f(t, n)\theta_n - a_f(t', n)\theta_n\| \to 0 \quad \text{as } t, t' \to \infty \tag{4·11}$$

allows us to establish $\|a_f(t)\theta - a_f(t')\theta\| \to 0$ as t, $t' \to \infty$ by an $\varepsilon/3$ argument. In these relations θ_n and θ are the vectors we have met in defining the graph G_∞. (4·11) should be verified independently of n. We first restrict the class of the functions involved. Let v and its first derivatives as well be in G_3. Then we can always find a sequence of functions v_n in G_1 which, together with first derivatives, tends to v in the G_3 norm. Consider also f such that it is in C_0^∞ and vanishes in a neighborhood of the origin of momentum space. For such functions, it is not difficult to verify the asymptotic condition via (4·1) and (4·11); the validity of (4·10) is easily seen from Lemma 4.2.

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Appendix

We shall prove the first order estimate (3·1). Let us define $N_n = U_n N U_n^{-1}$ and $a_n^\#(f) = U_n a^\#(f) U_n^{-1}$ to give

$$N = N_n - a_n(\bar{f}_n) - a_n^*(f_n) + \|f_n\|^2, \tag{A·1}$$

where $f_n = v_n\omega^{-1}$. In view of the relations $H_n = U_n H_0 U_n^{-1}$ and $N \leqq m^{-1} H_0$, we have

$$N_n \leqq m^{-1} H_n . \tag{A·2}$$

For every $\psi \in D((H_n + b)^{1/2})$ we consider the quadratic form of (A·2). The positivity of N and the use of Schwarz's inequality yield

$$(\psi, N\psi) \leqq (\psi, N_n\psi) + 2\|a_n(\bar{f}_n)\psi\|\|\psi\| + \|f_n\|^2\|\psi\|^2.$$

We use (A·2) and the standard estimate

$$\|a_n(\bar{f}_n) N_n^{-1/2}\| \leqq \|f_n\| = \|v_n\|_3 \tag{A·3}$$

to obtain

$$(\psi, N\psi) \leqq \frac{1}{m}(\psi, H_n\psi) + \frac{2}{\sqrt{m}}(\psi, H_n\psi)^{1/2}\|v_n\|_3\|\psi\| + \|v_n\|_3^2\|\psi\|^2$$

$$\leqq \frac{2}{m}(\psi, (H_n + m\|v_n\|_3^2)\psi).$$

Since $\|v_n\|_3 \leqq \|v_n - v\|_3 + \|v\|_3 \leqq$ const for large n, we have finally

$$(\psi, N\psi) \leqq \text{const}(\psi, (H_n + b)\psi)$$

for large b.

It is not difficult to prove the higher order estimate (3·3) by induction with the aid of (A·1) and (A·3). We here remark only that the constants in (3·3) may vary with j.

If U_n in the above is replaced by U, it is immediate to get

$$N \leqq \text{const}(H + b) \tag{A·4}$$

and similar higher order estimates.

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