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HAMILTONIAN DYNAMICS OF A RIGID BODY IN A CENTRAL GRAVITATIONAL FIELD

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ABSTRACT. This paper concerns the dynamics of a rigid body moving under the influence of a central gravitational field. Explicit account is taken of effects arising because of the finite extent of the body. The hamiltonian framework of the problem is exploited to elucidate questions concerning approximation, symmetry, Poisson reduction, relative equilibria, and associated stability problems.

1. INTRODUCTION.

In the study of the Newtonian (gravitational) many-body problem, it is customary to treat the bodies as point masses. See (Sternberg [40], Smale [39], and Abraham and Marsden [1]). However the proper accounting of stable planetary spins for instance, would seem to require the consideration of rigid (possibly nonhomogeneous) bodies of finite extent as a first approximation. The works of Duboshin [7], Ermenko [10], Elipe, Ferrer and Cid [8], [9], are concerned with the existence of special solutions (e.g. central configurations) in the Newtonian many-rigid-body problem. However, in these papers, the natural geometric and group-theoretic underpinnings of the problem are not exploited to the extent possible. We are not aware of a systematic program along these lines.

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In the design of large earth satellites, aerospace engineers have had to account for a "gravity-gradient torque" and its effect on the stability of earth-pointing satellite attitude. In the literature related to this problem, there are studies of relative equilibria and quasi-periodic motions based on various approximate models of the coupling between orbital motion and attitude motion of earth satellites. We refer the reader to the work of Beletskii [2], Duboshin [6], Roberson [34], [33], Longman [22], Meirovitch [27], Mohan, Breakwell and Lange [28], Likins [21], Sincarsin and Hughes [38], Pascal [30], and Sarychev [36]. The basic problem at hand is the dynamics of a rigid body or gyrostat in a central gravitational field.

In the present paper, we work out the noncanonical hamiltonian structure of the problem of motion of a rigid body in a central gravitational field. The group $SO(3)$ of three dimensional rotations appears as a symmetry group. Poisson reduction by the action of $SO(3)$ yields a nine-dimensional system that corresponds to observing the dynamics from a moving frame. In this body frame, the dynamics manifests the effect of a *fictitious torque* known as the gravity- gradient torque. There are Casimir functions that are conserved independently of the hamiltonian and hence of any *(convenient)* approximations to the Newtonian potential. We shall compute relative equilibria and determine their stability. All motions (whether exact or approximate) remain confined to the level sets of Casimir functions which are eight or six dimensional symplectic leaves.

It is noteworthy that the Poisson structure for the finite dimensional problem studied here is closely related to the one used by Krishnaprasad and Marsden [17] in their study of the dynamics of a rigid body with a flexible attachment (a physically distinct and infinite-dimensional problem). A key link is the geometry of Poisson reduction.

2. CONFIGURATION SPACE.

In Figure 1, let C denote a fixed gravitating body of mass M (with spherical symmetry) that influences the motion of a rigid body $\mathcal{B}$ of mass m. The inertial frame of reference (of the observer) is attached to C and a body frame is fixed on the rigid body $\mathcal{B}$ at its center of mass. A typical material particle Q in the rigid body is represented by the inertial vector $q = BQ + r$, where B is an element of $SO(3)$ (independent of the particle) and r is the vector from C to the center of mass of body $\mathcal{B}$. At any instant, the configuration of the rigid body $\mathcal{B}$ is determined uniquely from the pair $(B, r) \in SE(3)$, the special Euclidean group of rigid motions in $\mathbb{R}^3$.

In what follows, we will see that this is an example of a simple mechanical system with symmetry in the sense of Smale [39] (see also Abraham and Marsden [1]). Appendix 1 includes a short introduction to the abstract framework.

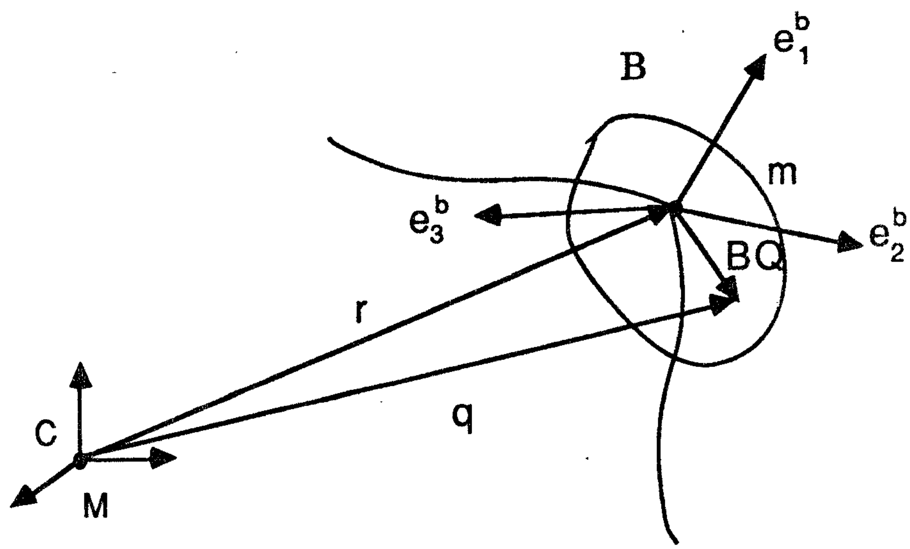

Figure 1. Rigid Body in Central Force Field

3. LAGRANGIAN.

The kinetic energy of the rigid body relative to the observer at C is,

$$T = \frac{1}{2} \int_{\mathcal{B}} | \dot{q} |^2 \, dm(Q)$$

where $dm\,(\cdot)$ denotes the mass measure of the body. Here onwards, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. It is an elementary fact that the above expression simplifies to the formula

$$T = \frac{1}{2} < \Omega, \mathbf{I}\Omega > + \frac{m}{2} | \dot{r} |^2 \tag{3.1}$$

where Ω is the body angular velocity vector of the rigid body , m is the total mass of the body and $\mathbf{I}$ is the moment of inertia tensor of $\mathcal{B}$ in the body frame.

Recall that the body angular velocity Ω is defined by

$$\dot{B} = B\hat{\Omega},$$

where

$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

is the skew-symmetric matrix associated to Ω.

The spatial angular velocity ω is defined by

$$\dot{B} = \hat{\omega} B,$$

and we have the relation

$$\omega = B\Omega.$$

In the notation of Appendix 1, we note that $K = 2T$ defines a riemannian metric on $SE(3)$ the configuration space.

The gravitational potential energy of the body $\mathcal{B}$ is given by,

$$V = - \int_{\mathcal{B}} \frac{GM}{| q |} \, dm(Q)$$

$$= - \int_{\mathcal{B}} \frac{GM \; dm(Q)}{| r + BQ |}, \tag{3.2}$$

where G is the universal gravitational constant.

The Lagrangian for the problem is then a function

$$L : T \, (SE(3)) \rightarrow \mathbb{R},$$

$$(B, r, \Omega, \dot{r}) \mapsto T - V.$$

4. SYMMETRY.

The inertial observer at C has the freedom to change his frame of reference to a new orientation. This corresponds to an $SO(3)$ action on the configuration space $\mathcal{C} = SE(3)$:

$$\Phi : \; SO(3) \times \mathcal{C} \mapsto \mathcal{C}$$

$$(P, (B, r)) \mapsto (PB, Pr).$$

It is easily checked that this action leaves invariant the kinetic energy T (riemannian metric on $\mathcal{C}$) and the potential V.

The hamiltonian $H = (T + V)$ is given by,

$$H = \frac{1}{2} \langle \Pi, \, \mathbf{I}^{-1} \, \Pi \rangle + \frac{| p |^2}{2m} - \int_{\mathcal{B}} \frac{GM}{| r + BQ |} dm \, (Q), \tag{4.1}$$

where $\Pi = \mathbf{I} \, \Omega$ is the body angular momentum of the rigid body $\mathcal{B}$, and $p = m\dot{r}$ is the spatial linear momentum of the body. One has also the formula,

$$
\begin{aligned}
\pi &= B\mathrm{I}\Omega + r \times m\dot{r} \\
&= B\Pi + r \times p
\end{aligned}
$$

for the *spatial angular momentum* of the rigid body.

It can be verified that $\pi = \pi\,(\Pi, B, r, p)$ is an Ad*-equivariant momentum mapping for the lifted action Φ^{T^*} on $T^*SE(3)$ and hence is a conserved quantity for the dynamics X_H . This is further equivalent to Euler's balance law. To see this, let $\mathcal{F}_{\text{resultant}}$ denote the force resultant on the rigid body. Then,

$$
\mathcal{F}_{\text{resultant}} = -\int_{\mathcal{B}} \frac{GM(r + BQ)}{|r + BQ|^3}\, dm(Q),
$$

and by linear momentum balance,

$$
\dot{p} = \mathcal{F}_{\text{resultant}}. \tag{4.2}
$$

On the other hand, the torque resultant,

$$
\begin{aligned}
\mathcal{T}_{\text{resultant}} &= -\int_{\mathcal{B}} \frac{(r + BQ) \times (r + BQ)\, GM}{|r + BQ|^3} dm(Q) \\
&= 0.
\end{aligned}
$$

Thus angular momentum (or Euler's) balance law yields:

$$
\dot{\pi} = 0. \tag{4.3}
$$

Collecting together the balance laws one can write the spatial form of the dynamics as

$$\begin{aligned}
\dot{\pi} &= 0, \\
\dot{p} &= -\int_{\mathcal{B}} \frac{GM\,(r + BQ)}{|\, r + BQ \,|^3} dm(Q), \\
\dot{B} &= \hat{\omega}\, B, \\
\dot{r} &= p/m.
\end{aligned} \tag{4.4}$$

Equivalently, in mixed body and space variables (Π, B, r, p) we get:

$$\begin{aligned}
\dot{\Pi} &= \Pi \times \mathbf{I}^{-1}\Pi + \int_{\mathcal{B}} \frac{GM(B^T r \times Q)}{|\, r + BQ \,|^3} dm(Q), \\
\dot{p} &= -\int_{\mathcal{B}} \frac{GM\,(r + BQ)}{|\, r + BQ \,|^3} dm(Q), \\
\dot{B} &= B\widehat{\mathbf{I}^{-1}\Pi}, \\
\dot{r} &= p/m.
\end{aligned} \tag{4.5}$$

Now, since H is $SO(3)$-invariant, one can induce a hamiltonian $\hat{H}$ on the quotient $T^*(SE(3))/SO(3)$ and express the dynamics $X_{\hat{H}}$ in terms of appropriate reduced variables (see Appendix 2 for the general framework). In the present context it is easy to determine the reduced variables. Note that

$$\Phi^{T^*} : SO(3) \times T^*SE(3) \to T^*SE(3)$$

$$(R, (\Pi, B, r, p)) \mapsto (\Pi, RB, Rr, Rp)$$

is the lifted action on $T^*SE(3)$. A representative for each equivalence class in $T^*SE(3)/SE(3)$ is given by

$$(\Pi, I, B^T r, B^T p).$$

Thus the reduced variables (or convected variables) are:

Π, the body angular momentum, $\lambda = B^T r$, the convected radius vector from C, and $\mu = B^T p$, the convected linear momentum.

In terms of these convected variables, the dynamics $X_{\hat{H}}$ takes the form

$$\begin{aligned}
\dot{\Pi} &= \Pi \times \mathbf{I}^{-1}\Pi + \int_{\mathcal{B}} \frac{GM\,(\lambda \times Q)}{|\lambda + Q|^3} dm(Q), \\
\dot{\lambda} &= \lambda \times \mathbf{I}^{-1}\Pi + \frac{\mu}{m}, \\
\dot{\mu} &= \mu \times \mathbf{I}^{-1}\,\Pi - \int_{\mathcal{B}} \frac{GM\,(\lambda + Q)}{|\lambda + Q|^3} dm(Q),
\end{aligned} \tag{4.6}$$

and the hamiltonian $\hat{H}$ is given by,

$$\hat{H} = \frac{1}{2}\langle \Pi, \mathbf{I}^{-1}\Pi \rangle + \frac{|\mu|^2}{2m} - \int_{\mathcal{B}} \frac{GM}{|\lambda + Q|} dm(Q). \tag{4.7}$$

Equations (4.6) with hamiltonian (4.7) are the Poisson reduced equations on $T^*SE(3)/SO(3) \simeq so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$. In terms of the Poisson tensor Λ on $so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$ derived in Appendix 2, these equations take the compact form,

$$\begin{pmatrix} \dot{\Pi} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix} \begin{pmatrix} \nabla_{\Pi}\hat{H} \\ \nabla_{\lambda}\hat{H} \\ \nabla_{\mu}\hat{H} \end{pmatrix} = \Lambda\,\nabla\hat{H}. \tag{4.8}$$

The Poisson structure is rank-degenerate, and there are nontrivial Casimir functions of Π, λ, μ. (Casimir functions are *kinematic* conserved quantities for equations of the form (4.8). See Appendix 2 for the precise definition.) In fact, any function C_ϕ of the form

$$C_\phi = \phi\,(|\,\Pi + \lambda \times \mu\,|^2),$$

is a Casimir function. Here $\phi : \mathbb{R} \to \mathbb{R}$ is any smooth scalar function. Moreover, these are the only Casimir functions defined on the open set of generic points of Λ .

From the general properties of Casimir functions (see Appendix 2) we know that C_ϕ is an integral invariant for any hamiltonian vector field and in particular for $X_{\hat{H}}$. It is further important to note that replacing $\hat{H}$ by a suitable approximation (such as derived from series expansions of the Newtonian potential term) does not affect the integral invariance of C_ϕ. This is of some use in developing an analytic perturbation theory.

5. RELATIVE EQUILIBRIA.

The concept of relative equilibrium goes back to Poincaré. For simple mechanical systems with symmetry, there is an elegant characterization of relative equilibria due to Smale [39]. We discuss this below and use it in computing relative equilibria for a rigid body in a central force field.

Let (M, K, V, G) be a simple mechanical system with symmetry as defined in Appendix 1. Assume that the lifted action of G acts on T^*M freely and properly. Then the quotient space T^*M/G is a smooth manifold with an induced Poisson structure. Let $\{\cdot,\cdot\}_0$ be the canonical Poisson bracket on T^*M. Given $f, g \in C^\infty(T^*M/G)$, the induced Poisson bracket of f and g is defined by

$$\{f, g\} \circ \hat{\tau} = \{f \circ \hat{\tau}, g \circ \hat{\tau}\}_0$$

where $\hat{\tau} : T^*M \to T^*M/G$ is the canonical projection. For any G-invariant hamiltonian function H on T^*M, we have the induced function $\hat{H} : T^*M/G \to \mathbb{R}$ defined by,

$$\hat{H} \circ \hat{\tau}(x) = H(x).$$

In terms of the induced Poisson structure, and $\hat{H}$, the projected hamiltonian vector field $X_{\hat{H}}$ on T^*M/G is defined by the condition, for any $f \in C^\infty(T^*M/G)$,

$$X_{\hat{H}}[f] \;=\; \{f,\, \hat{H}\}.$$

DEFINITION.

$z_e \in T^*M$ is relative equilibrium for H if

$$X_{\hat{H}}\;(\hat{\tau}(z_e)) \;=\; 0.$$

For the dynamics $X_{\hat{H}}$ of a rigid body in a central gravitational field, the relative equilibria are determined by setting the time derivatives in equation (4.6) (or (4.8)) to zero. On the other hand, in general position, i.e. $\Pi \neq 0$, ∇C_ϕ spans the kernel of Λ. Thus we have the energy-Casimir characterization of relative equilibria in general position: (Π, λ, μ) is a relative equilibrium *iff*

$$\nabla \hat{H} \;=\; \nabla C_\phi, \qquad \text{for suitable } \phi, \tag{EC}$$

iff (Lagrange multiplier characterization)

$$\begin{pmatrix} \mathbf{I}^{-1}\Pi \\ \nabla_\lambda \hat{V} \\ \frac{\mu}{m} \end{pmatrix} \;=\; c \begin{pmatrix} \Pi + \lambda \times \mu \\ \mu \times (\Pi + \lambda \times \mu) \\ (\Pi + \lambda \times \mu) \times \lambda \end{pmatrix}. \tag{LM}$$

where $c \neq 0$ is a constant and

$$\hat{V}\;(\lambda) \;=\; -\int_B \frac{GM}{|\lambda + Q|}\, dm(Q).$$

Before we proceed to solve (EC), we note an alternate characterization of relative equilibria.

REMARK 5.1.

It can be shown [1] that z_e is a relative equilibrium *iff* there exists a $\xi \in \Im$ such that the flow of X_H,

$$F^t_{X_H}\;(z_e) \;=\; exp(t\hat{\xi})\;(z_e),$$

(i.e. the dynamical orbit is simply a group orbit). Thus if the observer were to be set in uniform motion according to the one-parameter group $exp(t\xi)$, then for such a moving observer, a relative equilibrium will appear to be stationary.

In the work of Smale [39], there appears a characterization of relative equilibria for simple mechanical systems with symmetry (see Appendix 1 for the relevant notation). We present this as an algorithm to determine relative equilibria.

Algorithm for relative equilibria

Step 0. Pick $\xi \in \Im$, the Lie algebra of G . Let ξ_M denote the corresponding vector field on M determined by the action Φ of G.

Step 1. Search for the critical points $q_e \in M$ of the function (*the augmented potential*)

$$V_\xi : M \to \mathbb{R}$$

$$V_\xi(q) = V(q) - \frac{1}{2} K \left(\xi_M(q), \xi_M(q)\right) \tag{5.1}$$

Step 2. Find the corresponding conjugate momentum p_e by the formula

$$p_e = K^\flat \left(\xi_M(q_e)\right). \tag{5.2}$$

Then $z_e = (q_e, p_e)$ is a relative equilibrium for the hamiltonian function

$$H(\alpha_q) = V(q) + \frac{1}{2} K \left((K^\flat)^{-1}\alpha_q, (K^\flat)^{-1}\alpha_q\right).$$

This principle was used in [41] to determine relative equilibria for the dynamics of two rigid bodies connected by a ball-in-socket joint. In what follows we apply this principle to find the relative equilibria for the problem of rigid body motion in a central force field. Here, the elements of a simple mechanical system with symmetry consist of

$$M = SE(3),$$

$$K\left((U_1, U_2), (W_1, W_2)\right) = tr(U_1\, I'\, W_1^T) + m < U_2, W_2 >_E,$$

$$V(B, r) = -\int_B \frac{GM}{| r + BQ |}\, dm(Q),$$

and

$$G = SO(3),$$

where $< \cdot, \cdot >_E$ denotes the Euclidean inner product on $\mathbb{R}^3$,

$$(U_1, U_2), (W_1, W_2) \in T_{(B, r)}SE(3),$$

and I' is the coefficient of inertia of the rigid body. The superscript ^{T} in W_i denotes matrix transpose.

For $\hat{\xi} \in so(3)$, the corresponding infinitesimal generator of the group action on M can be found as

$$\xi_M(B, r) = (\hat{\xi}B, \hat{\xi}r).$$

We then have

$$K(\xi_M(B, r), \xi_M(B, r)) = < B^T\xi, \mathbf{I}B^T\xi >_E + m \mid \xi \times r \mid^2,$$

and

$$V_\xi(B, r) = -\frac{1}{2} < B^T\xi, \mathbf{I}\, B^T\xi >_E - \frac{1}{2} m \mid \xi \times r \mid^2 - \int_{\mathcal{B}} \frac{GM}{\mid r + BQ \mid} dm(Q). \tag{5.3}$$

We then get the first order conditions for (B, r) to be a critical point:

$$(i) \qquad m\, \xi \times (\xi \times r) + \int_{\mathcal{B}} \frac{(r + BQ)\, GM}{\mid r + BQ \mid^3} dm(Q) = 0$$

$$(ii) \qquad \xi \times (B\mathbf{I}B^T\xi) - \int_{\mathcal{B}} \frac{(r \times BQ)\, GM}{\mid r + BQ \mid^3} dm(Q) = 0.$$

Next, we calculate p_e in Step 2. The map $K^\flat$ can be found as follows. For $(\hat{w}_1 B, w_2)$, $(\hat{u}_1 B, u_2) \in T_{(B,r)}SE(3)$,

$$K^\flat\ (\hat{w}_1 B,\ w_2)\ (\hat{u}_1 B,\ u_2)$$

$$= tr(\hat{w}_1 B\ I' B^T \hat{u}_1^T)\ +\ m\ <\ w_2,\ u_2 >_E$$

$$= <u_1,\ B\mathbf{I}B^T w_1 >_E\ +\ <u_2,\ mw_2 >_E\ .$$

Thus

$$K^\flat(\hat{w}_1 B,\ w_2)\ =\ ((\widehat{B\mathbf{I}B^T w_1})\ B,\ mw_2)\ \in\ T^*_{(B,r)}\ SE(3).$$

We then have

$$p_e\ =\ K^\flat(\xi_M(q_e)) = \left((\widehat{B\mathbf{I}B^T\xi})\ B,\ m\hat{\xi}r\right)$$

$$= \left(B(\widehat{\mathbf{I}B^T\xi}),\ m\hat{\xi}\ r\right).$$

Note that in the formula for p_e, the two components correspond to the angular momentum and linear momentum respectively. If we let μ denote the body representation of the linear momentum, we get

$$(iii) \qquad \mu\ =\ mB^T\hat{\xi}r.$$

Substituting $\Omega\ =\ B^T\ \xi,\ \lambda\ =\ B^T r$, conditions (i), (ii), (iii) read

$$(i')\quad m\ \Omega\times(\Omega\times\lambda)\ +\int_{\mathcal{B}} \frac{(\lambda\ +\ Q)GM}{|\ \lambda\ +\ Q\ |^3}\ dm(Q)\ =\ 0,$$

$$(ii')\quad \Omega\times\mathbf{I}\Omega\ -\ \int_{\mathcal{B}} \frac{(\lambda\times Q)\ GM}{|\ \lambda\ +\ Q\ |^3}\ dm(Q)\ =\ 0, \qquad (5.4)$$

and

$$(iii')\quad \mu\ =\ m\ (\Omega\times\lambda).$$

These conditions are identical to the conditions obtained from the reduced dynamics (4.6) and the definition of relative equilibrium.

Now, if we take the cross product with λ on both sides of (i'), we get

$$m\lambda_e \times (\Omega_e \times (\Omega_e \times \lambda_e)) + \int_B \frac{(\lambda_e \times Q)\, GM}{|\lambda_e + Q|^3} dm(Q) = 0.$$

(Here again the subscripts e refer to equilibrium.) Comparing it with (ii'), we obtain

$$m\,\lambda_e \times (\Omega_e \times (\Omega_e \times \lambda_e)) + \Omega_e \times \mathbf{I}\Omega_e = 0.$$

By standard identities in vector analysis, we get

$$\Omega_e \times (\mathbf{I} - m\lambda_e\lambda_e^T)\,\Omega_e = 0. \tag{5.5}$$

We conclude that Ω_e must be an eigenvector of the matrix $\mathbf{I} - m\lambda_e\lambda_e^T$.

Let k_e denote the corresponding eigenvalue. Then one can obtain the relative equilibrium characterization (LM) from (5.4) by setting,

$$c = \frac{1}{k_e + m\,|\lambda_e|^2} \tag{5.6}$$

Conversely, using the identity,

$$\begin{aligned}\lambda \times ((\Omega \times \lambda) \times \Omega) &= -\,\Omega \times (\lambda \times (\Omega \times \lambda)) \\ &= (\lambda \cdot \Omega)\mu,\end{aligned}$$

and a few further algebraic manipulations, one can derive (5.4) from the relative equilibrium characterization (LM). We leave the verification to the reader. Thus the two characterizations are equivalent. Of course, for simple mechanical systems with symmetry, the equivalence of the energy-Casimir characterization (EC) or (LM) and the variational characterization based on the augmented potential V_ξ holds in general.

Note that we fix ξ while searching for critical points of V_ξ. Thus $\Omega = B^T \xi$ is of fixed norm as B varies over SO(3).

Let

$$| \Omega |^2 = | \xi |^2 = \beta.$$

Define $\tilde{V}_\beta (\Omega, \lambda)$ to be $V_\xi(B, \gamma)$ expressed in the convected variables Ω, λ. Then,

$$\tilde{V}_\beta (\Omega, \lambda) = -\frac{1}{2} < \Omega, \mathbf{I}\Omega >_E - \frac{m}{2} | \Omega \times \lambda |^2 + \hat{V} (\lambda). \tag{5.7}$$

Clearly, the critical points of $\tilde{V}_\beta$ on the sphere $| \Omega |^2 = \beta$ satisfy the unconstrained variational principle,

$$d(\tilde{V}_\beta + \frac{1}{2c} | \Omega |^2) = 0, \tag{5.8}$$

where $1/2c$ is a Lagrange multiplier. The first order conditions associated to (5.8) are,

$$\mathbf{I}\, \Omega + m\, \lambda \times (\Omega \times \lambda) = \frac{1}{c} \Omega \tag{5.9a}$$

$$m\, (\Omega \times \lambda) \times \Omega = \nabla_\lambda \hat{V}. \tag{5.9b}$$

These are exactly the equations we get by eliminating $\mu = m(\Omega \times \lambda)$ in the relative equilibrium characterization (LM). The unconstrained variational principle (5.8), parametrized by c, and the associated first order conditions (5.9) appear to be most suited to the explicit computation of relative equilibria. Before we proceed with such specific computations we make some general geometric observations concerning relative equilibria.

Observe that, by taking the inner product of both sides of (iii') with λ_e, we get

$$< \lambda_e,\ \mu_e >_E = 0 \tag{5.10}$$

at a relative equilibrium ($\lambda_e,\ \mu_e,\ \Omega_e$). If ($r_e,\ B_e$) is a relative equilibrium configuration, then the dynamical motion is such that

$$\begin{aligned} r(t) &= e^{t\hat{\xi}}\ r_e \\ B(t) &= e^{t\hat{\xi}}\ B_e. \end{aligned} \tag{5.11}$$

This follows from Remark (5.1) that at a relative equilibrium the dynamical orbit is just a group orbit.

PROPOSITION 5.1.

In relative equilibrium, the radius vector $r(t)$ generates a right circular cone.

Proof

From (5.11),

$$< r(t),\ r(t) >_E = < e^{t\hat{\xi}}\ r_e,\ e^{t\hat{\xi}}\ r_e >_E = < r_e,\ r_e >_E .$$

Also

$$\begin{aligned} &< r - \frac{< r,\xi >_E\ \xi}{|\xi|^2},\ r - \frac{< r,\xi >_E\ \xi}{|\xi|^2} >_E \\ &= < r,\ r >_E - \frac{< r,\xi >_E^2}{|\xi|^2} \\ &= < r_e,\ r_e >_E - \frac{< e^{t\hat{\xi}}\ r_e,\ e^{t\hat{\xi}}\ \xi >_E^2}{|\xi|^2} \\ &= < r_e,\ r_e >_E - \frac{< r_e,\ \xi >_E^2}{|\xi|^2} \\ &= \text{constant}. \end{aligned}$$

Thus $r(t)$ is a circle of radius $\left(< r_e, r_e >_E - \frac{< r_e,\xi >_E^2}{|\xi|^2}\right)^{\frac{1}{2}}$ centered at $C' = \frac{< r,\xi >_E}{|\xi|^2}\xi$. See Figure 2.

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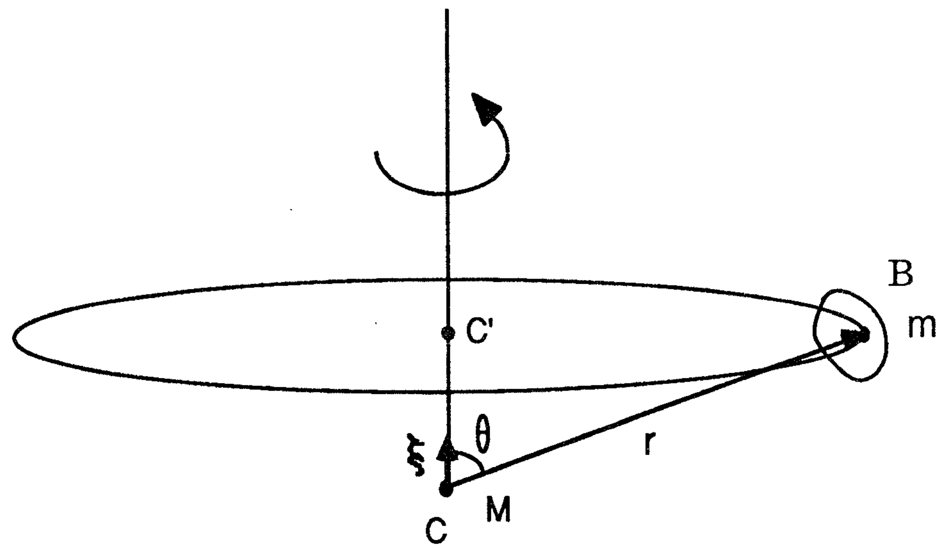

Figure 2. Cone generated by $r(t)$.

NON-GREAT CIRCLE MOTIONS.

For a rigid body of finite extent, *if* the center of (relative equilibrium) rotation C' *does not* coincide with the center C of the force field, then the stationary motion will be called a non-great circle motion. The existence of such motions is in question. See, e.g. the model problem below and also the gyrostat example in Rumyantsev [35].

From equation (4.5),

$$- \int_B \frac{GM\ (r\ +\ BQ)}{\mid r\ +\ BQ \mid^3} dm(Q) \ = \ \dot{p} \ = \ m\ \ddot{r}$$

$$= m \frac{d^2}{dt^2} \left(e^{t\hat{\xi}}\ r_e \right) \ = \ m\ e^{t\hat{\xi}}\ \hat{\xi}^2\ r_e .$$

Substituting $r = e^{t\hat{\xi}}\ r_e$ and $B = e^{t\hat{\xi}}\ B_e$ on the left hand side, we get,

$$- \int_B \frac{GM\ (r_e\ +\ B_e\ Q)}{\mid r_e\ +\ B_e Q \mid^3} dm(Q) \ = \ m\ \hat{\xi}^2\ r_e .$$

Taking the inner product of both sides with ξ, we get,

$$- \int_B GM\ \frac{< \xi,\ r_e >_E\ +\ < \xi,\ B_e\ Q >_E}{\mid r_e\ +\ B_e Q \mid^3} dm(Q) \ = \ m < \xi,\ \hat{\xi}^2\ r_e > = \ 0.$$

Hence

$$< \xi,\ r_e >_E \int\limits_{\mathcal{B}} \frac{dm(Q)}{\mid r_e\ +\ B_e Q \mid^3} \ = -\int\limits_{\mathcal{B}} \frac{< \xi,\ B_e\ Q >_E}{\mid r_e\ +\ B_e Q \mid^3} dm(Q).$$

Equivalently,

$$< \xi,\ r_e >_E = \ -\int\limits_{\mathcal{B}} \frac{< \xi,\ B_e\ Q >_E}{\mid r_e\ +\ B_e Q \mid^3} dm(Q)\ /\ \int\limits_{\mathcal{B}} \frac{dm(Q)}{\mid r_e\ +\ B_e Q \mid^3}.$$

The quantity $< \xi,\ r_e >_E$ is proportional to the $cos(\theta)$ (refer to Figure 2), and C and C' coincide *iff* $< \xi,\ r_e >_E = 0$. If the body $\mathcal{B}$ were a point mass, $Q = 0$ and hence $< \xi,\ r_e >_E = 0$. If for a rigid body of finite extent, the integral

$$\int\limits_{\mathcal{B}} \frac{< \xi,\ B_e\ Q >_E}{\mid r_e\ +\ B_e Q \mid^3} dm(Q) \neq\ 0,$$

then C, C' are not coincident.

Since

$$\begin{aligned} < \xi,\ r_e >_E\ &= < B_e^T \xi,\ B_e^T\ r_e >_E \\ &= \ \Omega_e \cdot \lambda_e, \end{aligned}$$

we conclude that a relative equilibrium $(\lambda_e,\ \Omega_e,\ \mu_e)$ determines a non-great circle motion *iff*

$$\Omega_e \cdot \lambda_e \neq\ 0. \tag{5.12}$$

One can test the non-vanishing condition (5.12) in various settings. We now demonstrate that there are examples which do not admit great circle relative equilibria. We first assume that the relative equilibrium is a great circle. Then the equilibrium can be found by solving (from (5.9) and $\Omega \cdot \lambda = 0$),

$$\begin{aligned} \Omega \times \mathbf{I}\Omega \ &= \ 0, \\ \int_{\mathcal{B}} \frac{GM(\lambda + Q)}{|\lambda + Q|^3}\ dm(Q) \ &= \ m|\Omega|^2 \lambda. \end{aligned} \tag{5.13}$$

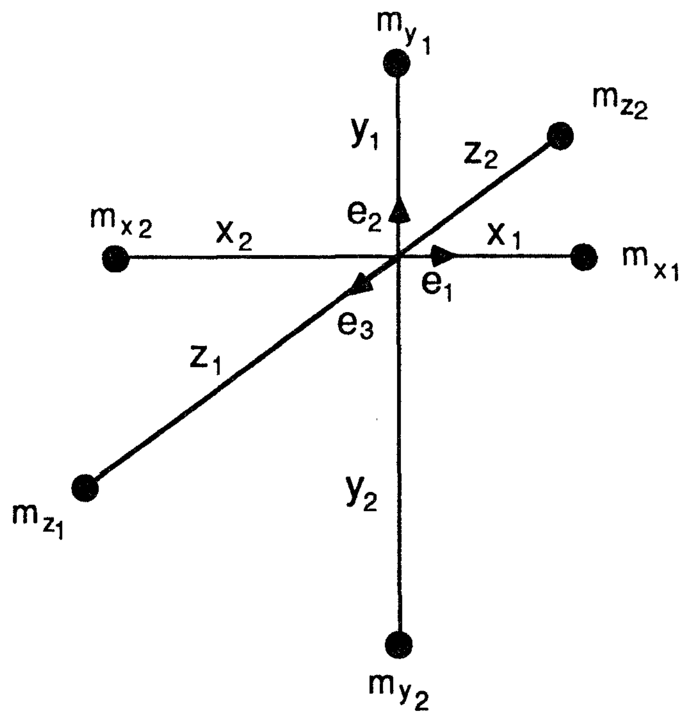

Figure 3. The "molecule"

We note that given the norm of Ω, the two equations above are decoupled and are equivalent to

1. Ω is an eigenvector of $\mathbf{I}$.
2. λ is a critical point of the function

$$\tilde{V} = \int_B \frac{GM}{|\lambda + Q|}\, dm(Q) + \frac{m}{2}|\Omega|^2|\lambda|^2.$$

Moreover, the second condition is equivalent to finding the critical points of

$$V^\dagger = \int_B \frac{GM}{|\lambda + Q|}\, dm(Q),$$

subject to

$$\frac{1}{2}|\lambda|^2 = \text{constant}.$$

with $m|\Omega|^2$ being the Lagrange multiplier.

Now we consider a model problem. The body is an asymmetric "molecule" consisting of six point masses, two on each principal axis. See Figure 3.

In this example, we know that **I** is diagonal and thus for a great circle solution, Ω must be along one axis. For a given set of data, for example,

$$m_{x1} = 101,\ m_{x2} = 1,\ m_{y1} = 100,\ m_{y2} = 1,\ m_{z1} = 99,\ m_{z2} = 1,$$

$$x_1 = 0.01,\ y_1 = 0.01,\ z_1 = 0.01,$$

the corresponding x_2, y_2, z_2 can then be determined such that $(0,0,0)$ is the center of mass. The corresponding function $V^\dagger$ on the sphere $|\lambda| = 400$ is shown in Figure 4. The coordinate system is the following. The sphere is parametrized by the usual spherical coordinates θ, $0 \le \theta < \pi$, and ϕ, $0 \le \phi < 2\pi$. In Figure 4, the function $V^\dagger$ is plotted above the disc of radius π, with (θ, ϕ) interpreted as planar polar coordinates.

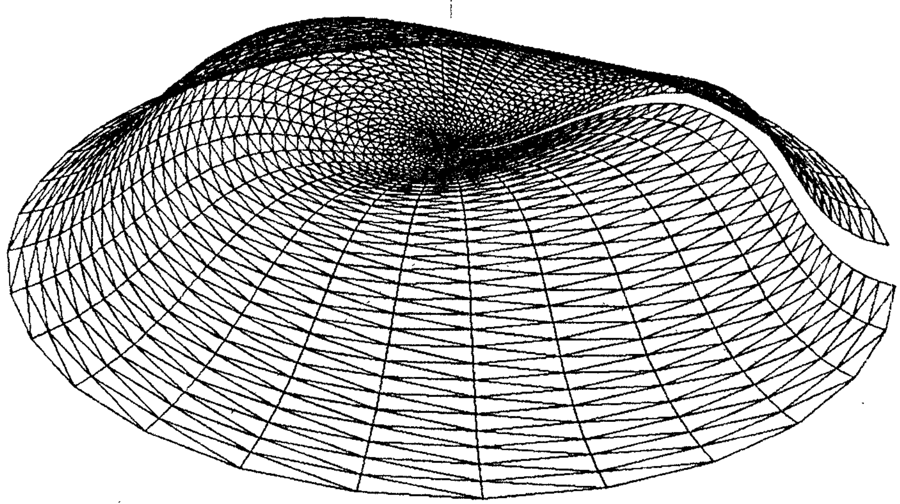

Figure 4. The function $V^\dagger$

The extremal critical points are determined numerically to be as follows.

$$\text{maxima} : \lambda = (-398.5, -33.7, -7.2),\ \ (399.3, -22.1, -10.6),$$

$$\text{minima} : \lambda = (13.7, 32.4, 398.5),\ \ (-4.0, 8.5, -399.9).$$

These extremals are found by using an optimization package CONSOLE [11] with the assistance of the 3-D graphical representation in Figure 4. From Figure 4, we note that there are also two saddles near the axis e_2. We therefore search for the maxima along that axis and check if they are also minima in the transverse direction. By this process, we can verify that no critical value of λ is perpendicular to a principal axis. Accordingly, we conclude that for this example, there are no great circle relative equilibria.

6. APPROXIMATIONS.

For typical applications in the modeling of planets or artificial earth satellites, the nominal radius of the orbital motion is very large compared to the dimensions of the orbiting body. Accordingly, it seems appropriate to consider various approximations of the gravitational potential based on Taylor series in a neighborhood of $|\lambda| = \infty$ or equivalently $|r| = \infty$. Whilst such approximations are common in the literature cited in the Introduction, it is unclear whether the symmetries and conservation laws inherent in the problem are respected by the approximation process.

In the present paper, we take the Poisson reduced model (4.8) as the logical starting point for approximations. The hamiltonian $\hat{H}$ is approximated to various orders of $\epsilon =$ (nominal dimension of body) / (orbital radius), by the Taylor series expansion of the $\hat{V}(\lambda)$ potential term appearing in $\hat{H}$:

$$\begin{aligned}\hat{V}(\lambda) &= -\int_B \frac{GM}{|\lambda + Q|}\, dm(Q) \\ &= -\frac{GM}{|\lambda|}\int_B dm(Q)\left\{1 - \frac{<Q,\lambda>}{|\lambda|^2} - \frac{1}{2}\frac{|Q|^2}{|\lambda|^2} + \frac{3}{2}\frac{<Q,\lambda>^2}{|\lambda|^4} + o(|\lambda|^{-4})\right\} \\ &= \left[-\frac{GMm}{|\lambda|}\right] + \left[-\frac{GM}{2|\lambda|^3} tr(\mathbf{I}) + \frac{3}{2}\frac{GM}{|\lambda|^5} <\lambda, \mathbf{I}\lambda>\right] + o(|\lambda|^{-5}) \qquad (6.1)\end{aligned}$$

In (6.1) the first term in brackets is of the order ϵ^0 and the next term is of the order ϵ^2. The ϵ^1

term is absent due to the vanishing of $\int_B Q\ dm(Q)$.

We will therefore consider two approximate *model hamiltonians,*

$$\hat{H}_0 = \frac{1}{2} < \Pi,\ \mathbf{I}^{-1}\Pi > + \frac{|\mu|^2}{2m} - \frac{GMm}{|\ \lambda\ |}, \tag{6.2}$$

and

$$\hat{H}_2 = \frac{1}{2} < \Pi, \mathbf{I}^{-1}\Pi > + \frac{|\mu|^2}{2m} - \frac{GMm}{|\lambda|} - \frac{GM}{2|\lambda|^3}\, tr(\mathbf{I}) + \frac{3}{2}\frac{GM}{|\lambda|^5} < \lambda,\ \mathbf{I}\lambda > . \tag{6.3}$$

Upon substituting $\hat{H}_0$ and $\hat{H}_2$ respectively for $\hat{H}$ in the Poisson reduced dynamics (4.8), one obtains the *order zero* reduced dynamics;

$$\begin{aligned} \dot{\Pi} &= \Pi \times \mathbf{I}^{-1}\Pi \\ \dot{\lambda} &= \lambda \times \mathbf{I}^{-1}\Pi + \mu/m \\ \dot{\mu} &= \mu \times \mathbf{I}^{-1}\Pi - \frac{GMm}{|\ \lambda\ |^3}\lambda \end{aligned} \tag{6.4}$$

and the *order two* reduced dynamics;

$$\begin{aligned} \dot{\Pi} &= \Pi \times \mathbf{I}^{-1}\Pi + \frac{3GM}{|\ \lambda\ |^5}\hat{\lambda}\, \mathbf{I}\, \lambda \\ \dot{\lambda} &= \lambda \times \mathbf{I}^{-1}\, \Pi + \mu/m \\ \dot{\mu} &= \mu \times \mathbf{I}^{-1}\Pi - \frac{GMm}{|\ \lambda\ |^3}\lambda - \frac{3GM}{2|\ \lambda\ |^5}\, tr\ (\mathbf{I})\lambda \\ &\quad - \frac{3GM}{|\ \lambda\ |^5}\mathbf{I}\lambda + \frac{15}{2}\frac{GM}{|\ \lambda\ |^7} < \lambda,\ \mathbf{I}\lambda > \lambda \end{aligned} \tag{6.5}$$

As already noted at the end of section 4, all such approximations admit a common set of conserved quantities (Casimir Functions) of the form $C_\phi = \phi\,(|\ \Pi + \lambda \times \mu\ |^2)$. Since the order 0 dynamics is essentially decoupled, it has additional conserved quantities of the form $\psi(|\,\Pi\,|^2)$, and the spin energy $1/2 \ < \Pi,\ \mathbf{I}^{-1}\Pi >$. If the body is spherically symmetric, i.e., $\mathbf{I} \ = \ k\mathbf{1}$, then the *order two* approximation collapses to the *order zero* approximation. In general, the *order two* approximation displays nontrivial spin-orbit coupling.

6.1 ORDER ZERO RELATIVE EQUILIBRIA.

With the order zero approximation of $\hat{H}$, the relative equilibria $(\Omega_e,\ \lambda_e)$ satisfy

$$\mathbf{I}\,\Omega_e \ = \ k\,\Omega_e \tag{6.6a}$$

$$\frac{GMm}{|\,\lambda\,|^3}\,\lambda_e \ = \ m(\Omega_e \ \times \ \lambda_e) \ \times \ \Omega_e. \tag{6.6b}$$

By taking the inner product of both sides of (6.6b) with Ω_e, we conclude that $\lambda_e \cdot \Omega_e \ = \ 0$, i.e. all relative equilibria in the *order zero* approximation give rise to great-circle orbits. From 6.6(b) and the condition $\Omega_e \cdot \lambda_e \ = \ 0$, we get the Kepler frequency formula,

$$|\,\Omega\,| \ = \ \left(\frac{GM}{|\,\lambda\,|^3}\right)^{1/2}. \tag{6.7}$$

Summarizing, the only relative equilibria for the *order zero* approximation are

(a) Ω_e is a principal axis of $\mathbf{I}$;	
(b) λ_e is a vector perpendicular to Ω_e satisfying the Kepler formula (6.7);	(6.8)
(c) $\mu_e \ = \ m\,(\Omega_e \ \times \ \lambda_e)$ completes a triad.	

With the same assumptions as (6.8a) and (6.8b) above, it is possible verify the existence of "uniformly spinning solutions" to the *order zero* reduced dynamics:

$$\begin{aligned}
\Pi(t) &= \ \mathbf{I}\Omega(t) \equiv \ \mathbf{I}\Omega_e \\
\lambda(t) &= \ exp\left(t\frac{\omega}{|\,\Omega_e\,|}\ \hat{\Omega}_e\right)\ \lambda_e \\
\mu(t) &= \ m\ \left(1 \ + \ \frac{\omega}{|\,\Omega_e\,|}\right) exp\left(t\frac{\omega}{|\,\Omega_e\,|}\ \hat{\Omega}_e\right)\ \hat{\Omega}_e\ \lambda_e,
\end{aligned} \tag{6.9}$$

with the modified Kepler frequency formula,

$$(\omega + \mid \Omega_e \mid) = \left(\frac{GM}{\mid \lambda_e \mid^3}\right)^{1/2} . \tag{6.10}$$

The quantity ω measures the body spin relative to a moving Frenet- Serret frame at the center of mass of the body.

6.2 ORDER TWO RELATIVE EQUILIBRIA.

The first order conditions for the variational principle (5.8) take the form

$$\begin{aligned}(\mathbf{I} - m\,\lambda\lambda^T)\Omega &= (\frac{1}{c} - m \mid \lambda \mid^2)\,\Omega \\ m \mid \Omega \mid^2 \lambda - m\,(\Omega \cdot \lambda)\,\Omega &= \frac{GMm}{\mid \lambda \mid^3}\,\lambda + \frac{3GM}{2\mid \lambda \mid^5}\,tr(\mathbf{I})\lambda \\ &+ \frac{3GM}{\mid \lambda \mid^5}\,\mathbf{I}\lambda - \frac{15}{2}\,\frac{GM}{\mid \lambda \mid^7}\,(\lambda^T\,\mathbf{I}\,\lambda)\lambda\end{aligned} \tag{6.11}$$

The equations (6.11) admit a family of solutions (relative equilibria) corresponding to great circle motions:

(a) Ω_e is a principal axis (eigen-vector) of $\mathbf{I}$ with corresponding principal moment of inertia I_i, $i = 1, 2, 3$;

(b) λ_e is a principal axis (eigen-vector) of $\mathbf{I}$ perpendicular to Ω_e, with associated principal moment of inertia I_j;

(c) $\mu_e = m\,(\Omega_e \times \lambda_e)$;

and, the following modified Kepler frequency formula holds:

$$| \Omega_e | = \left(\frac{GM}{| \lambda_e |^3} \right)^{1/2} \left\{ 1 + \frac{3\,(I_i - 2I_j + I_k)}{2m \mid \lambda_e \mid^2} \right\}^{1/2} . \tag{6.12}$$

In the above relation i, j, k are distinct and takes values in $\{1,\ 2,\ 3\}$. Hence the correction term in (6.12) may be of either sign. It follows that for the *order two* approximation there are twenty-four 1 parameter families of relative equilibria (accounting for Ω being in each of the six directions parallel to the principal axes (with sign) and four directions for λ corresponding to each choice of Ω), the scalar parameter being $\beta = | \Omega |^2 = | \xi |^2$ as in Section 5.

This conclusion appears to be a classical result exhibited in different form. See for instance the book of Beletskii [2]. However, the hamiltonian point of view together with the approach of reduction has entirely eliminated the formidable mess of Euler angles and such.

In the following, we show that for practical parameter ranges, all the relative equilibria in the *order two* approximate model are great circle motions. Let

$$(\mathbf{I} - m\lambda\lambda^T)\Omega = \alpha\Omega,$$

or

$$\mathbf{I}\,\Omega - \alpha\Omega = m\lambda\lambda^T\Omega.$$

With the notation $\tau = \lambda^T\Omega$, we have

$$m\tau\lambda = \mathbf{I}\Omega - \alpha\Omega. \tag{6.13}$$

We note that $\tau \neq 0$ corresponds to solutions that are not great circles, while $\tau = 0$ implies a standard eigenvalue problem. The dot product of (6.13) with Ω then yields

$$\alpha = \frac{1}{|\Omega|^2}(\Omega^T\mathbf{I}\Omega - m\tau^2),$$

and substitution in (6.13) gives

$$\tau\lambda \;=\; \frac{1}{m|\Omega|^2}(|\Omega|^2 - \Omega\Omega^T)\mathbf{I}\Omega \;+\; \frac{1}{|\Omega|^2}\tau^2\Omega. \tag{6.14}$$

Taking the dot product with Ω of the second equation in (6.11), we get the following equation

$$(m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda)\tau \;+\; \frac{3}{|\lambda|^2}\Omega^T\mathbf{I}\lambda \;=\; 0. \tag{6.15}$$

Assuming $\tau \neq 0$, and multiplying (6.15) by τ, we have

$$(m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda)\tau^2 \;+\; \frac{3}{|\lambda|^2}\Omega^T\mathbf{I}\tau\lambda \;=\; 0.$$

With the expression for $\tau\lambda$ in (6.14), we obtain the equality,

$$(m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda + \frac{3}{|\lambda|^2|\Omega|^2}\Omega^T\mathbf{I}\Omega)\tau^2 \;=\; -\frac{3}{m|\lambda|^2|\Omega|^2}\left\{|\Omega|^2|\mathbf{I}\Omega|^2 - |\Omega^T\mathbf{I}\Omega|^2\right\}. \tag{6.16}$$

But we know that

$$|\Omega|^2|\mathbf{I}\Omega|^2 - |\Omega^T\mathbf{I}\Omega|^2 \;\geq\; 0,$$

thus (6.16) can have a solution with $\tau \neq 0$ only if

$$m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda + \frac{3}{|\lambda|^2|\Omega|^2}\Omega^T\mathbf{I}\Omega \;\leq\; 0,$$

which can be true only if

$$m - \frac{15}{2|\lambda|^4}\lambda^T \mathbf{I}\lambda \leq 0,$$

or

$$\frac{15}{2m}\tilde{\lambda}^T \tilde{\mathbf{I}}\tilde{\lambda} \geq 1, \tag{6.17}$$

where

$$\tilde{\lambda} = \frac{\lambda}{|\lambda|}, \qquad \tilde{\mathbf{I}} = \int \frac{|Q|^2}{|\lambda|^2} dm(Q)\, \mathbf{1} - \int \frac{QQ^T}{|\lambda|^2} dm(Q).$$

It is easy to see that for large λ, (6.17) is not satisfied. In particular, the ratio $\frac{|Q|^2}{|\lambda|^2}$ must be greater than $\frac{1}{15}$. But for typical artificial satellites, this ratio is approximately 10^{-10}. For the motion of moon around the earth, it is approximately 1.6×10^{-5}. Thus we have shown that for the practical case of large orbit radii, the 24 relative equilibria (for the *second order approximate model*) are the only relative equilibria. This conclusion is of special interest since we have constructed a numerical example (the "molecule" in Section 5) in which the *exact* model has *no* great circle relative equilibria.

7. STABILITY OF RELATIVE EQUILIBRIA IN THE APPROXIMATE MODELS.

In this section, we study the stability properties of the relative equilibria for the approximate models discussed in Section 6. For both cases, the triple (Π, λ, μ) is a relative equilibrium if the three vectors are along the three principal axes. Without loss of generality we let

$$\begin{aligned} \Pi_e &= |\Pi|\, e_1 = I_1|\Omega|e_1, \\ \lambda_e &= |\lambda|\, e_2, \\ \mu_e &= \frac{m|\Pi||\lambda|}{I_1}\, e_3 = m|\Omega|\lambda|\, e_3, \end{aligned} \tag{7.1}$$

where $|\Omega|$ and $|\lambda|$ are related through the appropriate form of the Kepler frequency formula, and

$$\begin{aligned}\mathbf{I}\, e_1 &= I_1 e_1,\\ \mathbf{I}\, e_2 &= I_2 e_2,\\ \mathbf{I}\, e_3 &= I_3 e_3.\end{aligned}$$

We shall examine the stability of this relative equilibrium in various cases determined by the relative magnitudes of the principal moments of inertia I_i.

7.1 Order Zero Approximate Model. (Instability Proof)

For the *order zero* reduced dynamics (6.4), the energy-Casimir method of Appendix 3 is inconclusive since the second variation of the energy-Casimir function is only positive semi-definite (has a zero eigenvalue). We linearize the system around the relative equilibrium (7.1). Let

$$\delta x = (\delta\Pi_1,\ \delta\Pi_2,\ \delta\Pi_3,\ \delta\lambda_1,\ \delta\lambda_2,\ \delta\lambda_3,\ \delta\mu_1,\ \delta\mu_2,\ \delta\mu_3)^T.$$

We have the linearized system

$$\dot{\delta x} = A\,\delta x,$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & (\frac{1}{I_1}-\frac{1}{I_3})|\Pi| & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & (\frac{1}{I_2}-\frac{1}{I_1})|\Pi| & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{|\lambda|}{I_3} & 0 & 0 & 0 & \frac{1}{m} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{|\Pi|}{I_1} & 0 & \frac{1}{m} & 0\\ 0 & 0 & 0 & 0 & -\frac{|\Pi|}{I_1} & 0 & 0 & 0 & \frac{1}{m}\\ 0 & -\frac{m|\Pi||\lambda|}{I_1 I_2} & 0 & -\frac{mGM}{|\lambda|^3} & 0 & 0 & 0 & 0 & 0\\ \frac{m|\Pi||\lambda|}{I_1^2} & 0 & 0 & 0 & \frac{2mGM}{|\lambda|^3} & 0 & 0 & 0 & \frac{|\Pi|}{I_1}\\ 0 & 0 & 0 & 0 & 0 & -\frac{mGM}{|\lambda|^3} & 0 & -\frac{|\Pi|}{I_1} & 0 \end{pmatrix}.$$

By the frequency formula (6.7), we can write A in the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{I_3-I_1}{I_3}|\Omega| & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{I_2-I_1}{I_2}|\Omega| & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{|\lambda|}{I_3} & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & |\Omega| & 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & 0 & -|\Omega| & 0 & 0 & 0 & \frac{1}{m} \\ 0 & -\frac{m|\Omega||\lambda|}{I_2} & 0 & -m|\Omega|^2 & 0 & 0 & 0 & 0 & 0 \\ \frac{m|\Omega||\lambda|}{I_1} & 0 & 0 & 0 & 2m|\Omega|^2 & 0 & 0 & 0 & |\Omega| \\ 0 & 0 & 0 & 0 & 0 & -m|\Omega|^2 & 0 & -|\Omega| & 0 \end{pmatrix}.$$

Denote the upper left 3×3 matrix by B and the lower right 6×6 matrix by C. It can be shown that

$$p_1(s) \triangleq det(sI - B) = s\left(s^2 + \frac{I_3 - I_1}{I_3}\frac{I_2 - I_1}{I_2}|\Omega|^2\right),$$

$$p_2(s) \triangleq det(sI - C) = s^2\left(s^2 + |\Omega|^2\right)^2.$$

The characteristic polynomial of A is $p(s) = p_1(s)p_2(s)$. It can be further verified that the minimal polynomial of A is

$$m(s) = s^2\left(s^2 + \frac{I_3 - I_1}{I_3}\frac{I_2 - I_1}{I_2}|\Omega|^2\right)\left(s^2 + |\Omega|^2\right).$$

The occurrence of a repeated root of the minimal polynomial at $s = 0$ implies linear instability of the relative equilibrium (7.1) for the *order zero* approximate model (See Gantmacher [12], Theorem 3, pp. 129). Alternatively, the one parameter family of "uniformly spinning solutions" given by (6.9) represents a perturbation of the relative equilibrium (7.1) that departs any small neighborhood of the relative equilibrium in finite time, and hence we have instability. We note that this conclusion is independent of the relative magnitudes of the I_i's.

REMARK.

The projection of (Π, λ, μ) to the space of Π projects the *order zero* dynamics to the usual rigid body dynamics. For this *projected dynamics,* the equilibria in which the vector Π is along the maximum or minimum principal axes are stable.

∎

7.2 Order Two Approximate Model. (Energy-Casimir Method)

We now study the stability of relative equilibria of the *order two* reduced dynamics (6.5). For the relative equilibrium (7.1), we have the following identity, (from (6.11) or (6.12))

$$m|\Omega|^2 = \frac{GM}{|\lambda|^3}\left(m + \frac{3}{2|\lambda|^2}tr(\mathbf{I}) - \frac{9}{2|\lambda|^2}I_2\right). \tag{7.2}$$

Now we discuss sufficient conditions for the stability of these 24 relative equilibria.

In general, for a hamiltonian system with hamiltonian H and Casimir C, we have the energy-Casimir type sufficient condition for stability (see Appendix 3). The general form of the energy-Casimir function for our case is

$$\tilde{H}_\phi = \frac{1}{2}\Pi^T\mathbf{I}^{-1}\Pi + \frac{1}{2}\frac{|\mu|^2}{m} - \frac{mGM}{|\lambda|} - \frac{GM}{2|\lambda|^3}tr(\mathbf{I}) + \frac{3GM}{2|\lambda|^5}\lambda^T\mathbf{I}\lambda + \phi\left(\frac{1}{2}|\Pi + \lambda \times \mu|^2\right).$$

The first variation of $\tilde{H}_\phi$ can be found as $\delta\tilde{H}_\phi(\Pi,\lambda,\mu) = \nabla\tilde{H}_\phi \cdot \delta x$, where

$$\nabla\tilde{H}_\phi(\Pi,\lambda,\mu) = \begin{pmatrix} \mathbf{I}^{-1}\Pi + \phi'\mathbf{n} \\ \frac{GM}{|\lambda|^3}\left(m + \frac{3tr\mathbf{I}}{2|\lambda|^2} - \frac{15\lambda^T\mathbf{I}\lambda}{2|\lambda|^4}\right)\lambda + \frac{3GM}{|\lambda|^5}\mathbf{I}\lambda + \phi'\hat{\mu}\mathbf{n} \\ \frac{\mu}{m} - \phi'\hat{\lambda}\mathbf{n} \end{pmatrix},$$

and δx is as in Section 7.1, and,

$$\mathbf{n} = \Pi + \lambda \times \mu.$$

The matrix representation of the second variation of $\tilde{H}_\phi$ is

$$\nabla^2 \tilde{H}_\phi(\Pi,\lambda,\mu) \triangleq$$

$$\begin{pmatrix} \mathbf{I}^{-1} + \phi'\mathbf{1} + \phi''\mathbf{n}\mathbf{n}^T & -\phi'\hat{\mu} - \phi''\mathbf{n}\mathbf{n}^T\hat{\mu} & \phi'\hat{\lambda} + \phi''\mathbf{n}\mathbf{n}^T\hat{\lambda} \\ \phi'\hat{\mu} + \phi''\hat{\mu}\mathbf{n}\mathbf{n}^T & \begin{array}{c} \frac{GM}{|\lambda|^3}\left(m + \frac{3tr\mathbf{I}}{2|\lambda|^2} - \frac{15\lambda^T\mathbf{I}\lambda}{2|\lambda|^4}\right)\mathbf{1} \\ -\frac{3GM}{|\lambda|^5}\left(m + \frac{5tr\mathbf{I}}{2|\lambda|^2} - \frac{35\lambda^T\mathbf{I}\lambda}{2|\lambda|^4}\right)\lambda\lambda^T \\ -\frac{15GM}{|\lambda|^7}\lambda\lambda^T\mathbf{I} - \frac{15GM}{|\lambda|^7}\mathbf{I}\lambda\lambda^T + \frac{3GM}{|\lambda|^5}\mathbf{I} \\ -\phi'\hat{\mu}\hat{\mu} - \phi''\hat{\mu}\mathbf{n}\mathbf{n}^T\hat{\mu} \end{array} & -\phi'\hat{\mathbf{n}} + \phi'\hat{\mu}\hat{\lambda} + \phi''\hat{\mu}\mathbf{n}\mathbf{n}^T\hat{\lambda} \\ -\phi'\hat{\lambda} - \phi''\hat{\lambda}\mathbf{n}\mathbf{n}^T & \phi'\hat{\mathbf{n}} + \phi'\hat{\lambda}\hat{\mu} + \phi''\hat{\lambda}\mathbf{n}\mathbf{n}^T\hat{\mu} & \frac{1}{m}\mathbf{1} - \phi'\hat{\lambda}\hat{\lambda} - \phi''\hat{\lambda}\mathbf{n}\mathbf{n}^T\hat{\lambda} \end{pmatrix}.$$

In the above formulae, ϕ' represents its value at $|\mathbf{n}|^2/2$, and the same convention is applied to ϕ''. Now we find the variations at $(\Pi_e, \lambda_e, \mu_e)$. By using (7.2), we have

$$\nabla \tilde{H}_\phi(\Pi_e,\lambda_e,\mu_e) = (1+\phi'K)\begin{pmatrix} |\Omega|e_1 \\ m|\lambda||\Omega|^2 e_2 \\ |\lambda||\Omega|e_3 \end{pmatrix},$$

where

$$K = I_1 + m|\lambda|^2.$$

Thus in order for the first variation to vanish, we require $1+\phi'K = 0$, or

$$\phi' = -\frac{1}{K}. \tag{7.3}$$

Substituting these values in the second variation formula, we get $F = \nabla^2\tilde{H}_\phi(\Pi_e,\lambda_e,\mu_e)$ as

$$\begin{pmatrix} \frac{R-I_1}{I_1 R} & 0 & 0 & 0 & -\frac{m|\Omega||\lambda|}{R} & 0 & 0 & 0 & -\frac{|\lambda|}{R} \\ 0 & \frac{K-I_2}{I_2 K} & 0 & \frac{m|\Omega||\lambda|}{K} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{K-I_3}{I_3 K} & 0 & 0 & 0 & \frac{|\lambda|}{K} & 0 & 0 \\ 0 & \frac{m|\Omega||\lambda|}{K} & 0 & F_{44} & 0 & 0 & 0 & 0 & 0 \\ -\frac{m|\Omega||\lambda|}{R} & 0 & 0 & 0 & \begin{matrix} -m|\Omega|^2(4+\frac{m|\lambda|^2}{R}) \\ +\frac{2mGM}{|\lambda|^3} \end{matrix} & 0 & 0 & 0 & -|\Omega|(1+\frac{m|\lambda|^2}{R}) \\ 0 & 0 & 0 & 0 & 0 & F_{66} & 0 & |\Omega| & 0 \\ 0 & 0 & \frac{|\lambda|}{K} & 0 & 0 & 0 & \frac{I_1}{mK} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & |\Omega| & 0 & \frac{1}{m} & 0 \\ -\frac{|\lambda|}{R} & 0 & 0 & 0 & -|\Omega|(1+\frac{m|\lambda|^2}{R}) & 0 & 0 & 0 & \frac{R-m|\lambda|^2}{mR} \end{pmatrix}$$

where

$$\frac{1}{R} = \frac{1}{K} - \phi'' K^2 |\Omega|^2, \tag{7.4a}$$

$$F_{44} = m|\Omega|^2 \frac{I_1}{K} + \frac{3GM}{|\lambda|^5}(I_1 - I_2), \tag{7.4b}$$

$$F_{66} = m|\Omega|^2 + \frac{3GM}{|\lambda|^5}(I_3 - I_2). \tag{7.4c}$$

With the lower triangular matrix L defined by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{I_2 m|\Omega||\lambda|}{K-I_2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{m|\Omega||\lambda|I_1}{R-I_1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{I_3|\lambda|}{K-I_3} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{|\Omega|}{F_{66}} & 0 & 1 & 0 \\ L_{91} & 0 & 0 & 0 & L_{95} & 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$L_{91} = \frac{I_1|\lambda|}{R-I_1}\left(m|\Omega|L_{95}+1\right), \tag{7.5a}$$

$$L_{95} = \frac{U_{59}}{D_{55}}, \tag{7.5b}$$

$$U_{59} = -|\Omega|\left(1+\frac{K-I_1}{R-I_1}\right), \tag{7.5c}$$

$$D_{55} = m|\Omega|^2\left(4+\frac{K-I_1}{R-I_1}\right)-\frac{2mGM}{|\lambda|^3}, \tag{7.5d}$$

we can transform F into a diagonal matrix

$$D = L\,F\,L^T,$$

where

$$D = diag\left\{\frac{R-I_1}{I_1R},\ \frac{K-I_2}{I_2K},\ \frac{K-I_3}{I_3K},\ (I_1-I_2)D_{44},\ -D_{55}, F_{66},\right.$$
$$\left.\frac{I_1-I_3}{m(K-I_3)},\ (I_3-I_2)D_{88},\ \frac{U_{59}^2}{D_{55}}+\frac{R-K}{m(R-I_1)}\right\},$$

with

$$D_{44} = \frac{m|\Omega|^2}{K-I_2}+\frac{3GM}{|\lambda|^5}$$

$$D_{88} = \frac{3GM}{m|\lambda|^5F_{66}}.$$

(Since congruence transformations preserve the *matrix inertia*, we can read off the number of negative eigenvalues of F from D.)

We shall now consider the case in which,

$$I_1 > I_3 > I_2.$$

To have stability from the energy-Casimir method, we require that all the eigenvalues of F (equivalently of D) be positive. This holds if

$$\frac{R - I_1}{R} > 0, \tag{7.6a}$$

$$D_{55} < 0, \tag{7.6b}$$

and

$$\frac{U_{59}^2}{D_{55}} + \frac{R-K}{m(R-I_1)} > 0. \tag{7.6c}$$

For (7.4a), we have

$$\frac{R - I_1}{R} = 1 - \frac{I_1}{R} = \frac{m|\lambda|^2}{K} + \phi'' I_1 K^2 |\Omega|^2 .$$

Thus (7.6a) holds if $\phi'' > 0$. We next consider (7.6b). By the definition (7.5d) of D_{55} and the frequency formula (7.2), we get

$$D_{55} = m|\Omega|^2 \left(4 + \frac{K - I_1}{R - I_1} - \frac{2}{1 + \frac{3}{2m|\lambda|^2} tr\mathbf{I} - \frac{9}{2m|\lambda|^2} I_2} \right) .$$

Let

$$\epsilon \triangleq \frac{3}{2m|\lambda|^2} (I_1 + I_3 - 2I_2) .$$

For the case under consideration $\epsilon > 0$ and for $|\lambda|$ large compared to the typical dimensions of the body, ϵ is small. The other term,

$$\frac{K - I_1}{R - I_1} = \frac{m|\lambda|^2 \left(1/K - \phi'' K^2 |\Omega|^2\right)}{m|\lambda|^2/K + \phi'' I_1 K^2 |\Omega|^2} .$$

For $|\lambda|$ large and ϕ'' large enough,

$$\frac{K - I_1}{R - I_1} \simeq \frac{m|\lambda|^2 \left(-\phi'' K^2 |\Omega|^2\right)}{\phi'' I_1 K^2 |\Omega|^2} = -\frac{m|\lambda|^2}{I_1} \triangleq -\theta. \tag{7.7}$$

Thus

$$D_{55} \simeq m|\Omega|^2 \left(4 - \frac{2}{1+\epsilon} - \theta\right),$$

for $|\lambda|$ large and ϕ'' large enough. Since $\theta >> 4$, we have $D_{55} < 0$. This is (7.6b). Now we look at (7.6c). It is easy to see that *if we show*

$$U_{59}^2 + \frac{R-K}{m(R-I_1)} D_{55} < 0, \tag{7.8}$$

then together with (7.6b), we have (7.6c). From (7.7), for $|\lambda|$ large and ϕ'' large enough,

$$U_{59} \simeq -|\Omega|(1-\theta).$$

From the definition (7.4a) of R, we have

$$\frac{R-K}{R} = \phi'' K^3 |\Omega|^2.$$

Thus

$$\frac{R-K}{m(R-I_1)} = \frac{\phi'' K^3 |\Omega|^2}{\frac{m|\lambda|^2}{K} + \phi'' I_1 K^2 |\Omega|^2} \simeq \frac{K}{I_1} = 1+\theta,$$

for $|\lambda|$ large and ϕ'' large enough. Now we verify (7.8). Under the same condition,

$$\begin{aligned} U_{59}^2 + \frac{R-K}{m(R-I_1)} D_{55} &\simeq |\Omega|^2(1-\theta)^2 + \frac{1}{m}(1+\theta) m |\Omega|^2 \left(4 - \frac{2}{1+\epsilon} - \theta\right) \\ &= |\Omega|^2 \left(5 - \frac{2}{1+\epsilon} + (1 - \frac{2}{1+\epsilon})\theta\right) \\ &\simeq -|\Omega|^2 \theta \\ &< 0. \end{aligned}$$

Thus (7.4) hold for $|\lambda|$ large and ϕ'' large enough. We have the following theorem.

STABILITY THEOREM.

For the *order two* approximate model, the relative equilibrium

$$\mathbf{I}\,\Pi = I_1\Pi$$
$$\mathbf{I}\,\lambda = I_2\lambda$$
$$\mathbf{I}\,\mu = I_3\mu$$

is stable if $|\lambda|$ is sufficiently large and,

$$I_1 > I_3 > I_2.$$

∎

This shows that the relative equilibrium in which the body center of mass traverses a circular orbit, the angular velocity lies along the principal axis of the body with the largest associated moment of inertia (minor axis of the ellipsoid of inertia), and the radius vector is aligned to the principal axis with the least associated moment of inertia (major axis of the ellipsoid of inertia), is a stable relative equilibrium.

HISTORICAL REMARK.

A similar theorem appears in Beletskii's book [2], pp. 94–102. Beletskii uses a spatial/inertial model of the coupling between translational and rotational motion and presents arguments based on a Lyapunov–Chetayev approach [5], and uses in effect the variational equations about the stationary motion. In contrast, here we make consistent use of modern hamiltonian methods and reduced variables. The methods of this paper yield a nonlinear stability theorem and generalize to nonrigid and other complex configurations. See for instance the examples considered in [17], [23], [16], [18], [19], [32].

7.3 Order Two Approximate Model. (A Lagrange Multiplier Approach)

The previous section demonstrated that it is sometimes not straightforward to explicitly find an appropriate function ϕ in the energy-Casimir method. In Appendix 3, we describe a more classical characterization of relative equilibria as critical points of the constrained variational principle,

$$\begin{aligned} \min \qquad & \hat{H}_2(\Pi, \lambda, \mu) \\ \text{subject to} \quad C(\Pi, \lambda, \mu) &= \textit{constant} \end{aligned} \tag{7.9}$$

where $\hat{H}_2$ is the hamiltonian (6.3) and C is the Casimir $\frac{1}{2}|\Pi + \lambda \times \mu|^2$. The associated first-order conditions coincide with the characterization (LM) of relative equilibria, with the unknown constant c being interpreted as a Lagrange multiplier.

The Lagrangian (in the sense of optimization theory) associated with the above constrained variational principle is recovered if in Section 7.2 we take $\phi(x) = -c\,x$. Consequently the second variation can be recovered as a special case of that calculated in Section 7.2. When ϕ is linear, $\phi'' = 0$, and consequently, $R = K$. Therefore the second variation of $H - c\,C$ reduces to

$$F^c =$$

$$\begin{pmatrix}
\frac{K-I_1}{I_1K} & 0 & 0 & 0 & -\frac{m|\Omega||\lambda|}{K} & 0 & 0 & 0 & -\frac{|\lambda|}{K} \\
0 & \frac{K-I_2}{I_2K} & 0 & \frac{m|\Omega||\lambda|}{K} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{K-I_3}{I_3K} & 0 & 0 & 0 & \frac{|\lambda|}{K} & 0 & 0 \\
0 & \frac{m|\Omega||\lambda|}{K} & 0 & F_{44} & 0 & 0 & 0 & 0 & 0 \\
-\frac{m|\Omega||\lambda|}{K} & 0 & 0 & 0 & \begin{matrix} -m|\Omega|^2(4 + \frac{m|\lambda|^2}{K}) \\ + \frac{2mGM}{|\lambda|^3} \end{matrix} & 0 & 0 & 0 & -|\Omega|(1 + \frac{m|\lambda|^2}{K}) \\
0 & 0 & 0 & 0 & 0 & F_{66} & 0 & |\Omega| & 0 \\
0 & 0 & \frac{|\lambda|}{K} & 0 & 0 & 0 & \frac{I_1}{mK} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & |\Omega| & 0 & \frac{1}{m} & 0 \\
-\frac{|\lambda|}{K} & 0 & 0 & 0 & -|\Omega|(1 + \frac{m|\lambda|^2}{K}) & 0 & 0 & 0 & \frac{I_1}{mK}
\end{pmatrix}.$$

A comparatively simple Gaussian elimination then reveals that F^c is congruent to

$$D^c = diag\left\{\frac{K-I_1}{I_1K},\ \frac{K-I_2}{I_2K},\ \frac{K-I_3}{I_3K},\ (I_1-I_2)D_{44},\ -D^c_{55},\ F_{66},\right.$$
$$\left.\frac{I_1-I_3}{m(K-I_3)},\ (I_3-I_2)D_{88},\ \frac{4|\Omega|^2}{D^c_{55}}\right\},$$

where

$$D^c_{55} = 5m|\Omega|^2 - \frac{2mGM}{|\lambda|^3}$$

and the other coefficients are as defined previously. From the identity (7.2), we have further that

$$D^c_{55} = m|\Omega|^2\left(5 - \frac{2}{1+\epsilon}\right).$$

For $|\lambda|$ sufficiently large, the expressions for the various coefficients reveal that K is large and positive, and D_{44}, D^c_{55}, F_{66} and D_{88} are all positive. Consequently the signs of the entries of D^c are determined by the signs of the entries

$$\{+,\ +,\ +,\ (I_1-I_2),\ -,\ +,\ (I_1-I_3),\ (I_3-I_2),\ +\}.$$

We shall restrict attention to satellites in which the inertias are distinct so that F^c is nonsingular. Otherwise additional symmetries arise, and the analysis is slightly more complicated. There are six cases of distinct inertias. See the following table, in which the number of negative eigenvalues of F^c is shown in each case, and each case is assigned a reference number in parentheses.

	I_2 min	I_2 middle	I_2 max
$I_1 > I_3$	(1) 1	(2) 2	(3) 3
$I_1 < I_3$	(4) 2	(5) 3	(6) 4

Table 1.

According to Theorem A3.2 (in Appendix 3), it suffices to analyze whether the condition

$$\langle h,\ F^c h\rangle > 0, \quad \forall\ h \neq 0 \text{ and } \langle \nabla C(\Pi_e, \lambda_e, \mu_e),\ h\rangle = 0 \tag{7.10}$$

is satisfied. Because the subspace of admissible variations h has codimension 1, condition (7.10) cannot hold whenever F^c has two or more negative eigenvalues. Accordingly the only case in which (7.10) might hold is case (1), $I_1 > I_3 > I_2$, in which F^c is nonsingular and has precisely one negative eigenvalue.

To analyze (7.10) in case (1), we shall apply a general result bearing upon families of extremals to variational principles. Notice that (7.1) actually defines a one-parameter family of relative equilibria which can be regarded as being parametrized by the magnitude of the radius of the orbit, i.e. $|\lambda|$. But the multiplier c is related to $|\lambda|$ through $c = \frac{1}{I_1+m|\lambda|^2}$ (cf. (7.3)), so the family can also be parametrized by the multiplier c. Along this family the Casimir can be written, using (7.1) and (7.2), as

$$\begin{aligned} \frac{1}{2}|\Pi + \lambda \times \mu|^2 &= \frac{1}{2}|\Omega|^2(I_1 + m|\lambda|^2)^2 \\ &= \frac{GM(I_1 + m|\lambda|^2)^2}{m|\lambda|^5}\left(m|\lambda|^2 + \frac{3}{2}(I_1 - 2I_2 + I_3)\right). \end{aligned} \tag{7.11}$$

Consequently, for $|\lambda|$ large the Casimir is an increasing function of $|\lambda|$ along the family of relative equilibria, and consequently a *decreasing* function of c. We may now apply the aforementioned result.

LEMMA. (Maddocks [23], Lemma 5.2, pp. 316)

Suppose a family of variational principles of the type (7.9) have a family of critical points $x_e(c)$ parametrized by the multiplier c. Moreover, suppose that the second variation at a particular extremal is nonsingular with one negative eigenvalue. Then the second-order sufficient conditions (7.10) at that extremal are satisfied if and only if the constraint C is a *decreasing* function of the multiplier c at that parameter value.

COROLLARY.

Solutions (7.1) in case (1), $I_1 > I_3 > I_2$ are Lyapunov stable for all $|\lambda|$ sufficiently large.

Proof

It has been shown that the hypotheses of the previous Lemma hold, and that $C(c)$ is decreasing for relative equilibria with $|\lambda|$ sufficiently large. Thus condition (7.10) holds and Theorem A3.2 in Appendix 3 then applies to provide the desired result.

∎

Accordingly we have rederived the Stability Theorem proved in Section 7.2. As a final remark, we observe that we have not proven instability in the *order two* model. Actually the results of Maddocks [24](Section 5) can be applied to show that for large $|\lambda|$ the relative equilibria in any of the cases (2),(4),(6) in Table 1, are dynamically unstable. An outline of the analysis is that when F^c has an even number of negative eigenvalues and $C(c)$ is a decreasing function, then the linearized dynamics must possess an unstable real eigenvalue.

8. CONCLUSIONS

This paper represents a first step in our program to understand the geometry and dynamics of motion in a central gravitational field. Treating the rigid body as a model problem, we adopt a modern approach to hamiltonian mechanics to address questions concerning approximation, symmetry and reduction, Poisson structure, relative equilibria, and associated nonlinear stability problems. Our methods should extend naturally to problems of current interest such as the dynamics of tether-connected (and other) multibody systems in orbit, the dynamics of elastic shells, etc. A common thread in the program would be the use of geometrically exact models (i.e. models that respect natural groups of symmetries).

Among the new results of this paper we note the reduced model(s), examination of non-great circle solutions, the instability proof for the *order zero* approximation, and the rigorous proof of a nonlinear stability theorem in the *order two* approximation.

The geometric framework adopted here should be helpful in understanding and further exploring some of the deep and exciting questions that have emerged in recent years such as:

spin-orbit coupling and the problem of Hyperion [43] [13] [14] [31] [35]; the stable resonances of Markeev and Sokolskii [25]; the work of Beletskii [3].

Further questions concerning bifurcations and instability are of some interest and appear to be worthy of careful study. In addition to the methods used in the present paper, such investigations may be facilitated by some of the new techniques based on the energy-momentum method [37]. We hope to report on these problems at a later date.

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APPENDIX 1. (Simple Mechanical Systems with Symmetry)

Let (M, K) be a riemannian manifold and let G be a Lie group with associated action,

$$\Phi : G \times M \rightarrow M$$
$$(g, q) \mapsto \Phi_g(q)$$

where Φ_g in an isometry for each $g \in G$. The riemannian metric induces a vector bundle isomorphism,

$$K^\flat : TM \rightarrow T^*M$$

defined by

$$K^\flat (v_q) \cdot w_q = K (v_q, w_q),$$

for all $v_q, w_q \in TM_q$. The canonical symplectic structure $\omega = -d\theta_0$ on T^*M can be pulled back to

$$\Omega = (K^\flat)^* (\omega),$$

which is also an exact symplectic structure on TM. One can verify that the action Φ lifts to symplectic actions Φ^T and Φ^{T^*} on TM and T^*M respectively.

Let $V : M \rightarrow \mathbb{R}$ be a $G-$invariant (potential) function on M. The hamiltonian $H : T^*M \rightarrow \mathbb{R}$, is defined by,

$$H (\alpha_q) = \frac{1}{2} K \left((K^\flat)^{-1} \alpha_q, (K^\flat)^{-1} \alpha_q \right) + V \circ \tau_M^* (\alpha_q)$$

where $\tau_M : T^*M \rightarrow M$ is the canonical projection.

The hamiltonian vector field X_H on T^*M determined by H, is given by the condition,

$$dH(Y) = \omega(X_H, Y),$$

for all vector fields Y on T^*M. The hamiltonian system (T^*M, ω, X_H) or equivalently the quadruple (M, K, V, G) is a simple mechanical system with symmetry. It admits a momentum mapping in a natural way. To see this, let $\Im$ denote the Lie algebra of G and $\Im^*$ the dual space of G. The symplectic action Φ^{T^*} on T^*M, defines a Lie algebra homomorphism of $\Im$ into hamiltonian vector fields on T^*M; we denote this correspondence as $\xi \mapsto \xi_{T^*M}$. Then the map,

$$J : T^*M \rightarrow \Im^*$$

defined by,

$$J(\alpha_q) \cdot \xi = (i_{\xi_{T^*M}} \theta_0) \; (\alpha_q), \; \xi \in \Im$$

is an Ad^*-equivariant momentum mapping. Hence J is a conserved quantity of the system (T^*M, ω, X_H). (See Abraham and Marsden [1] for proofs.)

APPENDIX 2. (Poisson Structure and Reduction)

In this appendix we outline the essentials of Poisson reduction and derive the Poisson bracket applicable to the problem of rigid body motion in a central force field. A good source for Poisson structures is the book by Libermann and Marle [20]. See also Olver [29], the papers of Weinstein [42], Marsden and Ratiu [26], and Krishnaprasad and Marsden [17].

A Poisson manifold P is a smooth manifold equipped with an $\mathbb{R}$-bilinear map (Poisson structure) on the space of smooth functions,

$$\{\cdot,\cdot\}_P \, : \, C^\infty(P) \, \times \, C^\infty(P) \to C^\infty(P)$$

satisfying the axioms, for $f, g \in C^\infty(P)$,

(i) $\{f,g\}_P \; = \; - \, \{g,f\}_P$

(ii) $\{fg,h\}_P \; = \; g\{f,h\}_P \; + \; f\{g,h\}_P$

(iii) $\{f, \, \{g,h\}_P \,\}_P \; + \; \{g,\{h,f\}_P\}_P \; + \; \{h,\{f,g\}_P\}_P \; = 0.$

Associated to a Poisson structure, there is a unique twice contravariant skew-symmetric, smooth tensor field Λ on P such that

$$\{f,g\}_P \; = \; \Lambda \, (df, dg),$$

where df, dg are differentials of f, g, respectively. The tensor field Λ defines a vector-bundle morphism,

$$\begin{aligned} \Lambda^\# : T^*P \; &\to \; TP \\ \alpha_x \; &\mapsto \; \Lambda^\#(\alpha_x) \; \in \; T_xP \end{aligned}$$

satisfying,

$$\beta_x \; (\Lambda^\#(\alpha_x)) \; = \; \Lambda \; (x) \; (\alpha_x, \beta_x) \text{ for all } \beta_x \; \in \; T_x^*P.$$

Let G be a Lie group and let $\Psi : G \, \times \, P \to \, P, \;\; (g,x) \mapsto \Psi_g(x),$ be a group action such that $\Psi_g(\cdot)$ is a Poisson morphism for every $g \, \in \, G$. Suppose that the action is proper and free. Then there exists a good quotient P/G that carries a Poisson structure $\{\cdot,\cdot\}_{P/G}$ induced from the one on P satisfying, for $f, g \in C^\infty(P/G)$,

$$\{f, g\}_{P/G} \circ \pi \ = \ \{f \circ \pi, g \circ \pi\}_P.$$

Here $\pi : P \to P/G$ is the canonical projection. By construction, it is a Poisson morphism.

G-equivariant dynamics on P induce dynamics on P/G. Suppose $h : P \to \mathbb{R}$ is a G-invariant hamiltonian function on P, i.e.,

$$h(\Psi_g(x)) \ = \ h(x) \ \ \forall \, g \in G.$$

Define a vector field X_h by

$$X_h[f] \ = \ \{f, h\}_P \ \ \forall \, f \in C^\infty(P).$$

The hamiltonian h descends to $\hat{h} : P/G \to \mathbb{R}$ and determines a *reduced* dynamics $\hat{X}_{\hat{h}}$ on P/G by

$$\hat{X}_{\hat{h}}\,[\hat{f}] \ = \ \{\hat{f}, \hat{h}\}_{P/G} \ \ \forall \hat{f} \in C^\infty\,(P/G).$$

Here $\hat{h}\,([x]) \ = \ h(x)$ for an equivalence class $[x]$ in P/G.

In what follows, we work out the Poisson reduction of $T^*SE(3)$ by $SO(3)$. The resulting bracket captures the geometry of the central force field problem studied in this paper.

Let $< \cdot, \cdot >$ denote the pairing between $T^*SO(3)$ and $TSO(3)$ defined by

$$T^*SO(3) \times TSO(3) \to \mathbb{R}$$

$$< \alpha_A \ , \ W_A > \equiv \frac{1}{2}\, tr(\ \alpha_A^T \ W_A \ ).$$

$P = T^*SE(3)$ carries a canonical symplectic structure and hence a Poisson structure $\{\cdot,\cdot\}_P$ given by

$$\{\bar{f}, \bar{g}\}_P(B, B\hat{\Pi}, r, p) \ = < D_B\bar{f}, \frac{\partial \bar{g}}{\partial B\hat{\Pi}} > - < D_B\bar{g}, \frac{\partial \bar{f}}{\partial B\hat{\Pi}} > + \frac{\partial \bar{f}}{\partial r} \cdot \frac{\partial \bar{g}}{\partial p} - \frac{\partial \bar{g}}{\partial r} \cdot \frac{\partial \bar{f}}{\partial p},$$

where $\frac{\partial \bar{f}}{\partial r} \cdot \frac{\partial \bar{g}}{\partial p}$ denotes the natural pairing, i.e. the Euclidean inner product on $\mathbb{R}^3$.

The group $G = SO(3)$ acts on $SE(3)$ by left multiplication. This action lifts to a symplectic action on $T^*SE(3)$ given by

$$(R, (B, B\hat{\Pi}, r, p)) \mapsto (RB, RB\hat{\Pi}, Rr, Rp).$$

Thus a representative for an equivalence class in $P/SO(3)$ is given by

$$(\hat{\Pi}, B^T r, B^T p).$$

Let $\lambda = B^T r$, $\mu = B^T p$. We will compute the reduced Poisson structure on $T^*SE(3)/SO(3) \simeq so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$. Since $so^*(3) \simeq \mathbb{R}^3$, the question is equivalent to finding a Poisson structure on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$.

Let $f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$, and define $\bar{f}, \bar{g} \in C^\infty(T^*SE(3))$ as

$$\bar{f}(B, B\hat{\Pi}, r, p) = f(\Pi, B^T r, B^T p).$$

By the definition of reduced Poisson structure, we have

$$\{f, g\}_{P/G}(\Pi, \lambda, \mu) = \{\bar{f}, \bar{g}\}_P(B, B\hat{\Pi}, B\lambda, B\mu).$$

(The right hand side is the canonical bracket in $T^*SE(3)$.) Then, by the canonical bracket on $T^*SE(3)$,

$$\{f, g\}_{P/G}(\Pi, \lambda, \mu) = < D_B \bar{f}, \frac{\partial \bar{g}}{\partial B\hat{\Pi}} > - < D_B \bar{g}, \frac{\partial \bar{f}}{\partial B\hat{\Pi}} > + \frac{\partial \bar{f}}{\partial r} \cdot \frac{\partial \bar{g}}{\partial p} - \frac{\partial \bar{g}}{\partial r} \cdot \frac{\partial \bar{f}}{\partial p}.$$

Instead of computing each element in the above formula individually, we compute the *differential* of $\bar{f}$. Let $W = (B\hat{v}_1, B(\hat{v}_1\hat{\Pi} + \hat{v}_2), v_3, v_4) \in T_{(B, B\hat{\Pi}, r, p)} T^*SE(3)$. It generates the curve

$$\left(Be^{t\hat{v}_1}, Be^{t\hat{v}_1}(\hat{\Pi} + t\hat{v}_2), r + tv_3, p + tv_4\right) \subset T^*SE(3)$$

Thus the differential is given by,

$$d\bar{f}(B, B\hat{\Pi}, r, p) \cdot W = \left.\frac{d}{dt}\right|_{t=0} \bar{f}\left(Be^{t\hat{v}_1}, Be^{t\hat{v}_1}(\hat{\Pi} + t\hat{v}_2), r + tv_3, p + tv_4\right)$$

$$\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} f\left(\Pi + tv_2, e^{t\hat{v}_1^T} B^T(r + tv_3), e^{t\hat{v}_1^T} B^T(p + tv_4)\right) \\
&= \frac{\partial f}{\partial \Pi} \cdot v_2 + \frac{\partial f}{\partial \lambda} \cdot (\hat{v}_1^T B^T r + B^T v_3) + \frac{\partial f}{\partial \mu} \cdot (\hat{v}_1^T B^T p + B^T v_4) \\
&= \left(-\lambda \times \frac{\partial f}{\partial \lambda} - \mu \times \frac{\partial f}{\partial \mu}\right) \cdot v_1 + \frac{\partial f}{\partial \Pi} \cdot v_2 + B\frac{\partial f}{\partial \lambda} \cdot v_3 + B\frac{\partial f}{\partial \mu} \cdot v_4.
\end{aligned}$$

Let the elements in $T^*_{(B,B\hat{\Pi},r,p)} T^*SO(3)$ be denoted as

$$\left(B(\hat{b}\hat{\Pi} + \hat{a}), B\hat{b}, c, d\right).$$

We have

$$\boxed{\begin{aligned}
a &= -\lambda \times \frac{\partial f}{\partial \lambda} - \mu \times \frac{\partial f}{\partial \mu}, \\
b &= \frac{\partial f}{\partial \Pi}, \\
c &= B\frac{\partial f}{\partial \lambda}, \\
d &= B\frac{\partial f}{\partial \mu}.
\end{aligned}}$$

Thus we obtain

$$
\begin{aligned}
D_B\bar{f} &= B\left(\widehat{\frac{\partial f}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial f}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial f}{\partial \mu}})\right), \\
D_{B\hat{\Pi}}\bar{f} &= B\widehat{\frac{\partial f}{\partial \Pi}}, \\
\frac{\partial \bar{f}}{\partial r} &= B\frac{\partial f}{\partial \lambda}, \\
\frac{\partial \bar{f}}{\partial p} &= B\frac{\partial f}{\partial \mu}.
\end{aligned}
$$

The reduced bracket can then be derived as

$$
\begin{aligned}
&\{f,g\}_{P/G}(\Pi, \lambda, \mu) \\
&= < B\left(\widehat{\frac{\partial f}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial f}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial f}{\partial \mu}})\right), B\widehat{\frac{\partial f}{\partial \Pi}} > + B\frac{\partial f}{\partial \lambda} \cdot B\frac{\partial g}{\partial \mu} \\
&\quad < B\left(\widehat{\frac{\partial g}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial g}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial g}{\partial \mu}})\right), B\widehat{\frac{\partial f}{\partial \Pi}} > + B\frac{\partial g}{\partial \lambda} \cdot B\frac{\partial f}{\partial \mu}, \\
&= \frac{1}{2}tr\left(\hat{\Pi}^T \widehat{\frac{\partial f}{\partial \Pi}}^T \widehat{\frac{\partial g}{\partial \Pi}}\right) - \frac{1}{2}tr\left(\hat{\Pi}^T \widehat{\frac{\partial g}{\partial \Pi}}^T \widehat{\frac{\partial f}{\partial \Pi}}\right) + \frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu} \\
&\quad + \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu}\right) - \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu}\right), \\
&= -\Pi \cdot (\frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Pi}) + \frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu} \\
&\quad + \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu}\right) - \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu}\right) \\
&= (\frac{\partial f}{\partial \Pi}^T \quad \frac{\partial f}{\partial \lambda}^T \quad \frac{\partial f}{\partial \mu}^T) \begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial \Pi} \\ \frac{\partial g}{\partial \lambda} \\ \frac{\partial g}{\partial \mu} \end{pmatrix}.
\end{aligned}
$$

In terms of the notation introduced before, the matrix form for the Poisson tensor Λ is

$$\begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix}.$$

REMARK A2.1.

The reduced Poisson structure derived here is very closely related to the one derived by Krishnaprasad and Marsden for the dynamics of a rigid body with a flexible attachment [17]. The key link is the geometry. In [17] the unreduced phase space is infinite dimensional and is given by

$$P_\infty = T^*SO(3) \times T^*C$$

where $C = \{f : [0, L] \to \mathbb{R}^3 \mid f \text{ is smooth}\}$ is the configuration space for the shear beam attachment. In the present paper the unreduced phase space is

$$P = T^*SE(3) = T^*SO(3) \times T^*\mathbb{R}^3.$$

In both settings, the reduction is by $SO(3)$. In [17], the Poisson bracket takes the form

$$\{f, g\}_{P_\infty/G} = -\,\Pi \cdot (\frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Pi}) + \int_0^L \left(\frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu} \right) \, ds$$
$$+ \int_0^L \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu} \right) \, ds \; - \int_0^L \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu} \right) \, ds,$$

where the convected variables λ, μ are $\mathbb{R}^3$ valued functions on $[0, L]$. As we let the flexible attachment become vanishingly small ($L \to 0$) with infinitely large density, $\{\cdot, \cdot\}_{P_\infty/G}$ "collapses" to $\{\cdot, \cdot\}_{P/G}$.

REMARK A2.2. (Casimir functions)

A function f on a Poisson manifold $(P, \{\cdot, \cdot\}_P)$ is said to be a *Casimir function* if

$$\{\, f, \, \psi \,\}_P \; = \; 0 \qquad \forall \, \psi \in C^\infty(P).$$

Note that a Casimir finction is automatically a conserved quantity for *any* hamiltonian vector field X_h on P. If $\{\cdot,\cdot\}_P$ is induced from a symplectic structure and P is connected, then the only Casimir functions are the constant functions. In the present context, on $(P/G,\ \{\cdot,\cdot\}_{P/G})$ there are *nontrivial* Casimir functions. If $(\Pi,\ \lambda,\ \mu\ )$ is a generic point on P/G (i.e. a point where the matrix Λ has maximal rank $(= 8)$), then on a neighborhood of this point any Casimir function is of the form,

$$C_\phi \ = \ \phi(\ |\ \Pi \ + \ \lambda \ \times \ \mu\ |^2\ )$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary real-valued function.

This follows from the observation that $\nabla\ C_\phi$ is in the null space of Λ.

APPENDIX 3. (Stability Characterization)

We recall the Energy-Casimir theorem (see Holm, Marsden, Ratiu, and Weinstein [16]). Consider a finite dimensional Poisson system with a hamiltonian H

$$\dot{x} \;=\; \{x,\; H\}(x) \;\triangleq\; \Lambda(x)\, \nabla H(x). \tag{A3.1}$$

Assume the null space of Λ is not empty and is spanned by $\nabla\tilde{C}$, where $\tilde{C}$ is some (fixed) Casimir function. Then x_e is an equilibrium of (A3.1) iff $\exists\, \lambda$ such that,

$$\nabla H(x_e) \;=\; \lambda\, \nabla\tilde{C}(x_e). \tag{A3.2}$$

Notice that if $\tilde{C}$ is a Casimir, so is any smooth function C of $\tilde{C}$, and ∇C also spans the null space of Λ. So (A3.2) can be rewritten as

$$\nabla(H + C)(x_e) \;=\; 0,$$

for a whole class of Casimirs. Note that $H + C$ is a conserved quantity along trajectories of (A3.1). We have the following theorem.

THEOREM A3.1. (Energy-Casimir)

If there exists a Casimir function C such that

$$\nabla(H + C)(x_e) \;= 0, \tag{A3.3a}$$

$$\text{(second variation)} \quad \nabla^2(H + C)(x_e) \;> 0, \quad (\text{or } < 0) \tag{A3.3b}$$

then x_e is a Lyapunov stable equilibrium of (A3.1).

Proof

Define

$$V(x) \;=\; (H + C)(x) - (H + C)(x_e).$$

By assumption $\nabla^2(H+C)(x_e)$ is positive definite, so we know that x_e is a strict local minimum. Thus there exists a neighborhood U of x_e such that

$$V(x_e) = 0,$$

$$V(x) > 0, \quad \forall x \in U - \{x_e\}.$$

Since $H + C$ is a conserved quantity along trajectories of the given system, we have also

$$\dot{V}(x) = 0, \quad \forall x \in U - \{x_e\}.$$

We may therefore conclude that x_e is Lyapunov stable by a standard lemma (see e.g. Hirsch & Smale [15], pp. 193). A similar argument can be applied in the case that $\nabla^2(H + C)(x_e)$ is negative definite.

∎

Next we describe alternate approach to obtain a stability theorem. Consider the constrained variational problem

$$\begin{aligned} \min \quad & H(x), \\ \text{subject to} \quad & \tilde{C}(x) = b, \end{aligned} \qquad (A3.4)$$

where b is a constant representing prescribed data. The *Lagrangian* corresponding to this optimization problem can be written as

$$L(x, \lambda) = H(x) - \lambda \tilde{C}(x), \qquad (A3.5)$$

with $\lambda \in \mathbb{R}$. The first order necessary conditions for (A3.4) then coincide with (A3.2). We now recall the following lemma.

LEMMA. (See e.g. Bertsekas [4], p. 68)

Let P be a symmetric matrix and Q a positive semidefinite symmetric matrix, both of dimension $n \times n$. Assume that

$$\langle x,\ Px \rangle > 0, \quad \forall\ x \neq 0, \text{ with } \langle x,\ Qx \rangle = 0,$$

then there exists a (large, positive) scalar α such that

$$P \, + \, \alpha \, Q \, > \, 0,$$

i.e. $P + \alpha Q$ is positive definite.

∎

We can now state the stability criterion as follows.

THEOREM A3.2.

Suppose that x_e and $\lambda_e \in \mathbb{R}$ are such that

$$\nabla_x L(x_e, \lambda_e) \; = \; 0, \tag{A3.6a}$$

and, moreover,

$$\langle h, \; \nabla_x^2 L(x_e, \lambda_e) \; h \rangle \; > \; 0, \quad \forall \, h \neq 0 \text{ with } \langle \nabla \tilde{C}(x_e), \; h \rangle = 0. \tag{A3.6b}$$

Then x_e is a Lyapunov stable equilibrium of (A3.1).

Proof

Let

$$\begin{aligned} P \; &= \nabla_x^2 L(x_e, \lambda_e), \\ Q \; &= \nabla C(x_e) \; \nabla C(x_e)^T, \end{aligned}$$

so that by hypothesis P and Q satisfy the conditions of the previous lemma. Thus we can find $\alpha \in \mathbb{R}$ such that $P + \alpha Q$ is a positive definite matrix. Now, with the notation $b = \tilde{C}(x_e)$, define the *augmented Lagrangian* by,

$$L_\alpha(x, \lambda) \; = \; H(x) - \lambda \tilde{C}(x) + \frac{1}{2} \alpha \left(\tilde{C}(x) - b \right)^2 .$$

Then

$$
\begin{aligned}
\nabla_x L_\alpha(x_e, \lambda_e) &= \nabla H(x_e) + \lambda_e \nabla \tilde{C}(x_e) + \alpha \left(\tilde{C}(x_e) - b\right) \nabla \tilde{C}(x_e) \;=\; 0, \\
\nabla_x^2 L_\alpha(x_e, \lambda_e) &= \nabla^2 H(x_e) + \left(\lambda_e + \alpha \left(\tilde{C}(x_e) - b\right)\right) \nabla^2 \tilde{C}(x_e) + \alpha \; \nabla \tilde{C}(x_e) \nabla \tilde{C}(x_e)^T \\
&= P \;+\; \alpha\, Q \; > 0.
\end{aligned}
$$

Thus the augmented Lagrangian satisfies the requirement of a Lyapunov function and Theorem A3.1 can be applied to conclude that x_e is a Lyapunov stable equilibrium of system (A3.1).

∎

REMARKS.

(a) Conditions (A3.6) form a set of sufficient conditions for x_e to be a constrained local minimizer of (A3.4).

(b) In application of Theorem A3.1, we would search for a suitable Casimir C to fulfill the condition (A3.3). However, in application of Theorem A3.2 we can fix a particular Casimir $\tilde{C}$ and a scalar $\tilde{\lambda}$ satisfying (A3.6a), and then attempt to verify (A3.6b). The analysis in Sections 7.2 and 7.3 illustrate the differences between the two schemes.

(c) With appropriate hypotheses, both Theorem A3.1 and A3.2 can be generalized to cases in which there are n independent Casimirs, and the underlying space is infinite dimensional.