HAMILTONIAN DYNAMICS ON CONVEX SYMPLECTIC MANIFOLDS

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Abstract. We study the dynamics of Hamiltonian diffeomorphisms on convex symplectic manifolds. To this end we first establish an explicit isomorphism between the Floer homology and the Morse homology of such a manifold, and then use this isomorphism to construct a biinvariant metric on the group of compactly supported Hamiltonian diffeomorphisms analogous to the metrics constructed by Viterbo, Schwarz and Oh. These tools are then applied to prove and reprove results in Hamiltonian dynamics. Our applications comprise a uniform lower estimate for the slow entropy of a compactly supported Hamiltonian diffeomorphism, the existence of infinitely many nontrivial periodic points of a compactly supported Hamiltonian diffeomorphism of a subcritical Stein manifold, new cases of the Weinstein conjecture, and, most noteworthy, new existence results for closed trajectories of a charge in a magnetic field on almost all small energy levels. We shall also obtain some new Lagrangian intersection results.

Contents

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1. Introduction and main results

Consider a $2n$-dimensional compact connected symplectic manifold (M, ω) with non-empty boundary ∂M. The boundary ∂M is said to be *convex* if there exists a Liouville vector field X (i.e., $\mathcal{L}_X \omega = d\iota_X \omega = \omega$) which is defined near ∂M and is everywhere transverse to ∂M, pointing outwards; equivalently, there exists a 1-form α on ∂M such that $d\alpha = \omega|_{\partial M}$ and such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form inducing the boundary orientation of $\partial M \subset M$.

Definition (cf. [14])**.** (i) A compact symplectic manifold (M, ω) is *convex* if it has non-empty convex boundary.

(ii) A non-compact symplectic manifold (M, ω) is *convex* if there exists an increasing sequence of compact convex submanifolds $M_i \subset M$ exhausting M, that is,

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots \subset M \quad \text{and} \quad \bigcup_i M_i = M.$$

A symplectic manifold (M, ω) is *exact* if $\omega = d\lambda$ and *weakly exact* if $[\omega]$ vanishes on $\pi_2(M)$.

Remarks on terminology 1. A "compact convex symplectic manifold" is called "compact symplectic manifold with contact type boundary" for instance in [4, 5].

2. By "weakly exact" we do not mean that also the first Chern class c_1 of (M, ω) vanishes on $\pi_2(M)$.

Examples 1. Cotangent bundles. Recall that every cotangent bundle T^*N over a smooth manifold N carries a canonical symplectic form $\omega_0 = -d\lambda$, where $\lambda = \sum p_i dq_i$ in canonical coordinates (q, p). The R-disc bundles

$$T_R^* N \;=\; \{(q,p) \in T^*N \mid |p| \leq R\}$$

over a closed Riemannian manifold N and $T^*N = \bigcup_{k \in \mathbb{N}} T_k^* N$ are examples of exact convex symplectic manifolds. A larger class of examples are

2. Stein manifolds. A *Stein manifold* is a triple (V, J, f) where (V, J) is an open complex manifold and $f \colon V \to \mathbb{R}$ is a smooth function which is exhausting and J-convex. "Exhausting" means that f is bounded from below and proper, and "J-convex" means that the 2-form

$$\omega_f \;=\; -d\,(df \circ J)$$

is a J-positive symplectic form, i.e., $\omega_f(v, Jv) > 0$ for all $v \in TV \setminus \{0\}$. We denote by $g_f(\cdot,\cdot) = \omega_f(\cdot, J\cdot)$ the induced Kähler metric on V, and by ∇f the gradient vector field of f with respect to g_f. We *do not* assume that ∇f is complete; in particular, (V, ω_f) can have finite volume. In any case,

$$\mathcal{L}_{\nabla f} \omega_f \;=\; d\iota_{\nabla f} \omega_f \;=\; -d\,(g_f(\nabla f, J\cdot)) \;=\; -d\,(df \circ J) \;=\; \omega_f. \tag{1}$$

A *Stein domain* in (V, J, f) is a subset $V_R = \{x \in V \mid f(x) \leq R\}$ for a regular value $R \in \mathbb{R}$. In view of (1), every Stein domain is an exact compact convex symplectic manifold, and so every Stein manifold is an exact convex symplectic manifold. We refer the reader to [12, 13, 14] for foundations of the symplectic theory of Stein manifolds.

3. (i) Let N be a closed oriented surface equipped with a Riemannian metric of constant curvature -1, and let σ be the area form on N. We endow the cotangent bundle $\pi \colon T^*N \to N$ with the twisted symplectic form $\omega_\sigma = \omega_0 - \pi^*\sigma$. It is shown in [48] that ω_σ is exact on $M = T^*N \setminus N$ and that M carries a vector field X such that $\mathcal{L}_X \omega_\sigma = \omega_\sigma$ and such that X is everywhere transverse to ∂M_i, pointing outwards, for each

$$M_i \;=\; \left\{ (q,p) \in T^*N \mid \tfrac{1}{i} \leq |p| \leq i \right\}, \qquad i \geq 2.$$

Since $H_3(M_i) = \mathbb{Z}$, the manifolds M_i, $i \geq 2$, are exact compact convex symplectic manifolds which are not Stein domains, and $M = \bigcup_{i \geq 2} M_i$ is an exact convex symplectic manifold which is not Stein. Smoothing the boundaries of k-fold products $\times_k M_i$, $i \geq 2$, we obtain such examples in dimension $4k$ for all $k \geq 1$.

(ii) Symplectically blowing up a Stein manifold of dimension at least 4 at finitely many points we obtain a convex symplectic manifold which is not weakly exact.

4. A product of convex symplectic manifolds does not need to be convex. Let N be a closed orientable surface different from the torus, and let σ be a 2-form on N. As we shall see in Lemma 12.6, the cotangent bundle T^*N endowed with the symplectic form $\omega_\sigma = \omega_0 - \pi^*\sigma$ is convex. For homological reasons, the product of (T^*N, ω_σ) with the convex symplectic manifold (T^*S^1, ω_0) is, however, convex only if σ is exact. We shall be confronted with such non-convex manifolds in our search for closed trajectories of magnetic flows on surfaces. We shall therefore develop our tools for symplectic manifolds which away from a compact subset look like a product of convex symplectic manifolds. $\diamond$

Throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$. Given any symplectic manifold (M, ω), we denote by $\mathcal{H}_c(M)$ the set of C^2-smooth functions $S^1 \times M \to \mathbb{R}$ whose support is compact and contained in $S^1 \times (M \setminus \partial M)$. The Hamiltonian vector field of $H \in \mathcal{H}_c(M)$ defined by

$$\omega\left(X_{H_t}, \cdot\right) \;=\; dH_t\left(\cdot\right)$$

generates a flow φ_H^t. The set of time-1-maps $\varphi_H = \varphi_H^1$ form the group

$$\mathrm{Ham}_c(M, \omega) \;:=\; \{\varphi_H \mid H \in \mathcal{H}_c(M)\}$$

of C^1-smooth compactly supported Hamiltonian diffeomorphisms of (M, ω). Many of our results will apply to those Hamiltonian diffeomorphisms whose support can be displaced from itself. We thus make the following definition.

Definition. A compact subset A of a symplectic manifold (M, ω) is *displaceable* if there exists $\varphi \in \mathrm{Ham}_c(M, \omega)$ such that $\varphi(A) \cap A = \emptyset$.

Example. Every compact subset of a symplectic manifold of the form $(M \times \mathbb{R}^2, \omega \times \omega_0)$ is displaceable.

Our main tools to study Hamiltonian systems on convex symplectic manifolds will be the Piunikhin–Salamon–Schwarz isomorphism and a spectral metric derived from it. Before explaining these tools, we describe their applications. While some applications recover or generalize well-known results, many are new; all of them, however, are straightforward consequences of the main tools. In this introduction we give samples of our applications, and we refer to Sections 9 to 13 for stronger results.

1. A lower bound for the slow length growth

Consider a weakly exact symplectic manifold (M, ω). For $H \in \mathcal{H}_c(M)$ the set of contractible 1-periodic orbits of φ_H^t is denoted by $\mathcal{P}_H$, and the symplectic action $\mathcal{A}_H(x)$ of $x \in \mathcal{P}_H$ is defined as

$$\mathcal{A}_H(x) \;=\; -\int_{D^2} \bar{x}^*\omega - \int_0^1 H(t, x(t))\, dt \tag{2}$$

where $\bar{x}\colon D^2 \to M$ is a smooth extension of x to the unit disc. Since $[\omega]|_{\pi_2(M)} = 0$, the integral $\int_{D^2} \bar{x}^*\omega$ does not depend on the choice of $\bar{x}$. The following result has been proved by Schwarz, [70], in the context of closed weakly exact symplectic manifolds (M, ω) whose first Chern class vanishes on $\pi_2(M)$.

Theorem 1. *Assume that (M, ω) is a weakly exact convex symplectic manifold. Then for every Hamiltonian function $H \in \mathcal{H}_c(M)$ generating a non-identical Hamiltonian diffeomorphism $\varphi_H \in \mathrm{Ham}_c(M, \omega)$ there exists $x \in \mathcal{P}_H$ such that $\mathcal{A}_H(x) \neq 0$.*

Theorem 1 is used in [25] to give a uniform lower bound for the slow length growth of Hamiltonian diffeomorphisms of exact convex symplectic manifolds $(M, d\lambda)$. Fix a Riemannian metric g on such a manifold and denote by Σ the set of smooth embeddings $\sigma\colon [0,1] \to M$. We define the slow length growth $s(\varphi) \in [0, \infty]$ of a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}_c(M, \omega)$ by

$$s(\varphi) \;=\; \sup_{\sigma \in \Sigma} \liminf_{n \to \infty} \frac{\log \mathrm{length}_g\left(\varphi^n(\sigma)\right)}{\log n}.$$

Notice that $s(\varphi)$ does not depend on the choice of g. We refer to [25] for motivations to consider this invariant. Following an idea of Polterovich, [64], we use Theorem 1 in [25] to show

Corollary 1. *Assume that $(M, d\lambda)$ is an exact convex symplectic manifold. Then $s(\varphi) \geq 1$ for any $\varphi \in \mathrm{Ham}_c(M, d\lambda) \setminus \{\mathrm{id}\}$.*

2. Infinitely many periodic points of Hamiltonian diffeomorphisms

We consider again a weakly exact convex symplectic manifold (M, ω). A *periodic point* of $\varphi_H \in \mathrm{Ham}_c(M, \omega)$ is a point $x \in M$ such that $\varphi_H^k(x) = x$ for some $k \in \mathbb{N}$. We say that a periodic point x is *trivial* if $\varphi_H^t(x) = x$ and $H_t(x) = 0$ for all $t \in \mathbb{R}$. Since $H \in \mathcal{H}_c(M)$, φ_H has many trivial periodic points. The *support* $\operatorname{supp} \varphi_H$ of a Hamiltonian diffeomorphism φ_H is defined as $\bigcup_{t \in [0,1]} \operatorname{supp} \varphi_H^t$. It has been proved by Schwarz, [70], in the context of closed weakly exact symplectic manifolds that if $\operatorname{supp} \varphi_H$ is displaceable, then φ_H has infinitely many nontrivial geometrically distinct periodic points. We shall prove an analogous result in our situation.

Theorem 2. *Consider a weakly exact convex symplectic manifold* (M, ω)*. If the support of* $\varphi_H \in \mathrm{Ham}_c(M, \omega) \setminus \{\mathrm{id}\}$ *is displaceable, then* φ_H *has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits.*

Theorem 2 covers Proposition 4.13 (2) of [71] stating that *any* non-identical compactly supported Hamiltonian diffeomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ has infinitely many nontrivial geometrically distinct periodic points, see also Theorem 11 in Chapter 5 of [41]. In fact, this is true for all subcritical Stein manifolds, which we are now going to define.

Example (Subcritical Stein manifolds). Let (V, J, f) be a Stein manifold. If $f \colon V \to \mathbb{R}$ is a *Morse*-function, then $\mathrm{index}_x(f) \leq \frac{1}{2} \dim_{\mathbb{R}} V$ for all critical points x of f. A Stein manifold (V, J, f) is called *subcritical* if f is Morse and $\mathrm{index}_x(f) < \frac{1}{2} \dim_{\mathbb{R}} V$ for all critical points x. The simplest example of a subcritical Stein manifold is $\mathbb{C}^n$ endowed with its standard complex structure J and the J-convex function $f(z_1, \dots, z_n) = |z_1|^2 + \dots + |z_n|^2$. ◇

It has been recently shown by Cieliebak, [3], that every subcritical Stein manifold is symplectomorphic to the product of a Stein manifold with $(\mathbb{R}^2, \omega_0)$, and so every compact subset of a subcritical Stein manifold is displaceable. We shall not use this difficult result but will combine Theorem 2 with a result from [1] to conclude

Corollary 2. *Any compactly supported non-identical Hamiltonian diffeomorphism of a subcritical Stein manifold has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits.*

3. The Weinstein conjecture

Another immediate application of our methods is a proof of the Weinstein conjecture for a large class of hypersurfaces of contact type. We recall the following definition.

Definition. A C^2-smooth compact hypersurface S without boundary of a symplectic manifold (M, ω) is called *of contact type* if there exists a Liouville vector field X which is defined in a neighbourhood of S and is everywhere transverse to S. A *closed characteristic* on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S \;=\; \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x S\}\,.$$

Theorem 3. *Consider a weakly exact convex symplectic manifold* (M, ω)*, and let* $S \subset M \setminus \partial M$ *be a displaceable* C^2*-smooth hypersurface of contact type. Then* S *carries a closed characteristic which is contractible in* M*.*

Theorem 3 implies a result first proved by Viterbo, [72].

Corollary 3. *Any* C^2*-smooth hypersurface of contact type in a subcritical Stein manifold* (V, J, f) *carries a closed characteristic which is contractible in* V*.*

We shall also obtain new existence results for closed characteristics nearby a given hypersurface. Roughly speaking, our methods allow to generalize the results which can be derived from the Hofer–Zehnder capacity for hypersurfaces in $\mathbb{R}^{2n}$ to displaceable hypersurfaces in weakly exact convex symplectic manifolds; in addition, the closed characteristics found are contractible, and their reduced actions are bounded by twice the displacement energy of the supporting hypersurface. We refer to Section 11 for the precise results.

4. Closed trajectories of a charge in a magnetic field

Consider a Riemannian manifold (N,g) of dimension at least 2. The motion of a unit charge on (N,g) subject to a magnetic field derived from a potential $A\colon N \to TN$ can be described as the Hamiltonian flow of the Hamiltonian $(p,q) \mapsto \frac{1}{2}|p-\alpha|^2$ on (T^*N,ω_0) where α is the 1-form g-dual to A and where again $\omega_0 = -d\lambda$ and $\lambda = \sum_i p_i dq_i$. The fiberwise shift $(q,p) \mapsto (q, p-\alpha(q))$ conjugates this Hamiltonian system with the Hamiltonian system

$$H\colon\ (T^*N,\omega_\sigma) \to \mathbb{R}, \qquad H(q,p) = \frac{1}{2}|p|^2, \tag{3}$$

where $\sigma = d\alpha$ and where the twisted symplectic form ω_σ is given by $\omega_\sigma = \omega_0 - \pi^*\sigma = -d(\lambda + \pi^*\alpha)$. The system (3) is a model for various other problems in classical mechanics and theoretical physics, see [52, 43].

A trajectory of a charge on (N,g) in the magnetic field σ has constant speed, and closed trajectories γ on N of speed $\sqrt{2c} > 0$ correspond to closed orbits of (3) on the energy level $E_c := \{H = c\}$. An old problem in Hamiltonian mechanics asks for closed orbits on a given energy level E_c, see [28]. We denote by $\mathcal{P}^\circ(E_c)$ the set of closed orbits on E_c which are contractible in T^*N; notice that $\mathcal{P}^\circ(E_c)$ is the set of closed orbits on E_c which project to contractible closed trajectories on N, and that if $\dim N \geq 3$, the orbits in $\mathcal{P}^\circ(E_c)$ are contractible in E_c itself.

Theorem 4.A. *Consider a closed manifold N endowed with a C^2-smooth Riemannian metric g and an exact 2-form σ which does not vanish identically. There exists $d > 0$ such that $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c \in\,]0,d[$.*

"Almost all" refers to the Lebesgue measure on $\mathbb{R}$. The number $d > 0$ has a geometric meaning: If the Euler characteristic $\chi(N)$ vanishes, d is the supremum of the real numbers c for which the sublevel set

$$H^c = \left\{(q,p) \in T^*N \mid H(q,p) = \tfrac{1}{2}|p|^2 \leq c\right\}$$

is displaceable in (T^*N,ω_σ), and if $\chi(N)$ does not vanish, d is defined via stabilizing (3) by the Hamiltonian system $(T^*S^1, dx\wedge dy) \to \mathbb{R}$, $(x,y) \mapsto \frac{1}{2}|y|^2$, see Section 12 for details. Theorem 4.A generalizes a result of Polterovich, [62], and Macarini, [48], who proved $\mathcal{P}^\circ(E_c) \neq \emptyset$ for a sequence $c \to 0$.

If the magnetic field on (N,g) cannot be derived from a potential, the motion of a unit charge in this field is still described by (3), where now σ is a closed but not exact 2-form on N, see [28] and again [52, 43] for further significance of such Hamiltonian systems. In this introduction we only consider the case that N is 2-dimensional. Since $H^2(N;\mathbb{R}) = 0$ if N is not orientable, we can assume that N is orientable.

Theorem 4.B. *Assume that N is a closed orientable surface of genus at least 1 endowed with a C^2-smooth Riemannian metric g and a closed 2-form $\sigma \neq 0$.*

(i) *There exists $d > 0$ such that $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c \in \,]0, d[$.*

(ii) *If N is a torus and σ is not exact, then $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c > 0$.*

The number $d > 0$ in Theorem 4.B (i) has the same geometric meaning as the number $d = d(g, \sigma)$ in Theorem 4.A. Theorem 4.B (i) was proved for symplectic magnetic fields by Arnold and Ginzburg for *all* sufficiently low energy levels, [28], and (ii) recovers a result of Lu, [47]. We refer to Section 12.3 for a result containing Theorems 4.A and 4.B as special cases. While the existence problem for closed trajectories of a charge in a magnetic field has been studied for about twenty years, see [28], the last five years brought considerable progress in this problem, [6, 9, 7, 10, 31, 32, 33, 42, 47, 48, 57, 62, 67]. We refer to Section 12.4 in [24] for the state of the art in March 2003, and to [67] for the state of the art in December 2004.

5. Lagrangian intersections

Our methods will provide a concise proof of a Lagrangian intersection result covering some well known as well as some new cases.

Theorem 5. *Consider a weakly exact convex symplectic manifold (M, ω), and let $L \subset M \setminus \partial M$ be a closed Lagrangian submanifold such that*

(i) *the injection $L \subset M$ induces an injection $\pi_1(L) \subset \pi_1(M)$;*

(ii) *L admits a Riemannian metric none of whose closed geodesics is contractible.*

Then L is not displaceable.

Note that the conclusion of Theorem 5 does not hold for a small circle L in a disc D^2, showing that condition (i) cannot be omitted. According to a theorem of Gromov, [35, 2.3.B_3'], the conclusion of Theorem 5 holds for any closed Lagrangian submanifold L of a geometrically bounded symplectic manifold (M, ω) for which $[\omega]|_{\pi_2(M,L)} = 0$. For two further results in this direction we refer to Remark 13.2.

The spectral metric

We shall derive the above results from a biinvariant spectral metric on the group $\mathrm{Ham}_c(M, \omega)$ of compactly supported Hamiltonian diffeomorphisms of a weakly exact compact convex symplectic manifold (M, ω). We recall that a symplectomorphism ϑ of (M, ω) is a diffeomorphism of M such that $\vartheta^*\omega = \omega$. We denote by $\mathrm{Symp}_c(M, \omega)$ the group of symplectomorphisms of (M, ω) whose support lies in $M \setminus \partial M$. We also recall that for any symplectic manifold (M, ω), Hofer's biinvariant metric d_H on $\mathrm{Ham}_c(M, \omega)$ is defined by

$$d_H(\varphi, \psi) = d_H\left(\varphi\psi^{-1}, \mathrm{id}\right), \quad d_H(\varphi, \mathrm{id}) = \inf\left\{\|H\| \mid \varphi = \varphi_H\right\},$$

where

$$\|H\| \;=\; \int_0^1 \left(\sup_{x\in M} H(t,x) - \inf_{x\in M} H(t,x) \right) dt. \tag{4}$$

It is shown in [45] that d_H is indeed a metric.

Theorem 6. *Assume that (M,ω) is a weakly exact compact convex symplectic manifold. There exists a function $\gamma\colon \mathrm{Ham}_c(M,\omega) \to [0,\infty[$ such that*

(i) $\gamma(\varphi) = 0$ *if and only if* $\varphi = \mathrm{id}$;
(ii) $\gamma(\varphi\psi) \le \gamma(\varphi) + \gamma(\psi)$;
(iii) $\gamma(\vartheta\varphi\vartheta^{-1}) = \gamma(\varphi)$ *for all* $\vartheta \in \mathrm{Symp}_c(M,\omega)$;
(iv) $\gamma(\varphi) = \gamma(\varphi^{-1})$;
(v) $\gamma(\varphi) \le d_H(\varphi, \mathrm{id})$.

In other words, γ is a symmetric invariant norm on $\mathrm{Ham}_c(M,\omega)$. The metric d_γ defined by

$$d_\gamma(\varphi,\psi) \;=\; \gamma\left(\varphi\psi^{-1}\right)$$

is thus a biinvariant metric on $\mathrm{Ham}_c(M,\omega)$ such that $d_\gamma \le d_H$. While the Hofer metric d_H is a Finsler metric, the metric d_γ is a spectral metric in the sense that $\gamma(\varphi)$ is the difference of two action values of φ. This property and the property that $\gamma(\varphi_H) \le 2\,\gamma(\psi)$ if ψ displaces the support of φ_H are crucial for our applications. Biinvariant metrics on $\mathrm{Ham}_c(M,\omega)$ with these properties have been constructed for $(\mathbb{R}^{2n},\omega_0)$ and for cotangent bundles over closed bases by Viterbo, [71], and for closed symplectic manifolds (M,ω) by Schwarz, [70], in the case that $[\omega]$ and c_1 vanish on $\pi_2(M)$, and by Oh, [54, 55], in general.

The main ingredient in the construction of the spectral metric d_γ is the Piunikhin–Salamon–Schwarz isomorphism (PSS isomorphism, for short) between the Floer homology and the Morse homology of a weakly exact compact convex symplectic manifold. Floer homology for weakly exact *closed* symplectic manifolds (M,ω) has been defined in Floer's seminal work [15, 16, 17, 18]. It is already shown there that the Floer homology of (M,ω) is isomorphic to the Morse homology of M and thus to the ordinary homology of M by considering time independent Hamiltonian functions. An alternative construction of this isomorphisms was described in [60]; it goes under the name PSS isomorphism. In the following three sections we establish the PSS isomorphism for weakly exact compact convex symplectic manifolds (M,ω). In Sections 5 to 7 we follow Schwarz, [70], and use our PSS isomorphism to construct the spectral metric d_γ on the group $\mathrm{Ham}_c(M,\omega)$. In Section 8 it is shown that the π_1-sensitive Hofer–Zehnder capacity is bounded from above by twice the displacement energy. The last five sections contain our applications.

Note added in June 2005: This work was finished in March 2003. Meanwhile, some of our applications were recovered or improved by further investigating the tools developed here, [30, 23], and by combining ideas of this work with methods from Hofer geometry, [67]. Moreover, exciting progress in the Morse theory for the free time action functional for convex Lagrangian systems, combined with Aubry–Mather theory, lead to new existence results for closed trajectories of a charge in a magnetic field, [7, 10, 56, 57], which overlap with our results.

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2. Convexity

In this section we follow Viterbo, [72], and prove a Maximum Principle for a perturbed Cauchy Riemann equation on a suitable completion of a compact convex symplectic manifold. It will be a main ingredient in the construction of Floer homology, of the PSS isomorphism, and of the spectral metric for weakly exact convex symplectic manifolds given in the subsequent sections.

We consider a compact convex $2n$-dimensional symplectic manifold (M, ω). Choose a smooth vector field X on M which points outwards along ∂M and is such that $\mathcal{L}_X\omega = d\iota_X\omega = \omega$ near ∂M. For the 1-form $\alpha := (\iota_X\omega)|_{\partial M}$ we then have $d\alpha = \omega|_{\partial M}$ and $\alpha\wedge(d\alpha)^{n-1}$ is a volume form inducing the boundary orientation of ∂M. Using X we can symplectically identify a neighbourhood of ∂M with

$$(\partial M \times (-2\epsilon, 0], d\,(e^r\alpha))$$

for some $\epsilon > 0$. Here, we used coordinates (x, r) on $\partial M \times (-2\epsilon, 0]$, and in these coordinates, $X(x, r) = \frac{\partial}{\partial r}$ on $\partial M \times (-2\epsilon, 0]$. We can thus view M as a compact subset of the non-compact symplectic manifold $(\widehat{M}, \widehat{\omega})$ defined as

$$\begin{aligned} \widehat{M} &= M \cup_{\partial M \times \{0\}} \partial M \times [0, \infty), \\ \widehat{\omega} &= \begin{cases} \omega & \text{on} \quad M, \\ d\,(e^r\alpha) & \text{on} \quad \partial M \times (-2\epsilon, \infty), \end{cases} \end{aligned}$$

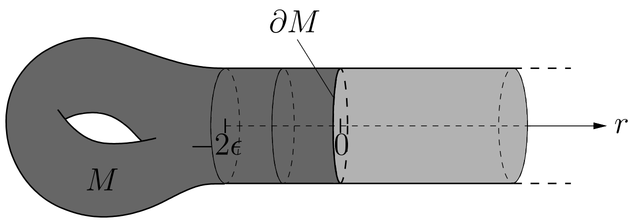

Figure 1. The completion $\widehat{M}$ of M.

see Figure 1, and X smoothly extends to $\widehat{M}$ by

$$\widehat{X}(x, r) := \frac{\partial}{\partial r}, \qquad (x, r) \in \partial M \times (-2\epsilon, \infty).$$

We denote the open "proboscis" $\partial M \times (-\epsilon, \infty)$ by P_ϵ. Let φ_t be the flow of $\widehat{X}$. Then $\varphi_r(x,0) = (x,r)$ for $(x,r) \in P_\epsilon$. We recall that an almost complex structure $\widehat{J}$ on $\widehat{M}$ is called *$\widehat{\omega}$-compatible* if

$$\langle \cdot, \cdot \rangle \equiv g_{\widehat{J}}(\cdot, \cdot) := \widehat{\omega}(\cdot, \widehat{J} \cdot)$$

defines a Riemannian metric on $\widehat{M}$. Following [2] we choose an $\widehat{\omega}$-compatible almost complex structure $\widehat{J}$ on $\widehat{M}$ such that

$$\widehat{\omega}\left(\widehat{X}(x), \widehat{J}(x)v\right) = 0, \qquad x \in \partial M,\ v \in T_x\partial M, \tag{5}$$

$$\widehat{\omega}\left(\widehat{X}(x), \widehat{J}(x)\widehat{X}(x)\right) = 1, \qquad x \in \partial M, \tag{6}$$

$$d\varphi_r(x,0)\widehat{J}(x,0) = \widehat{J}(x,r)d\varphi_r(x,0), \qquad (x,r) \in P_\epsilon. \tag{7}$$

We define $f \in C^\infty(P_\epsilon)$ by

$$f(x,r) := e^r, \quad (x,r) \in P_\epsilon. \tag{8}$$

Since $\mathcal{L}_{\widehat{X}}\widehat{\omega} = \widehat{\omega}$ on $\partial M \times (-2\epsilon, \infty)$, we have $\varphi_r^*\widehat{\omega} = e^r\widehat{\omega}$ on P_ϵ for all $r > -\epsilon$. This, (6) and (7) imply that

$$\left\langle \widehat{X}(p), \widehat{X}(p) \right\rangle = f(p), \quad p \in P_\epsilon. \tag{9}$$

Together with (5) this implies that

$$\nabla f(p) = \widehat{X}(p), \quad p \in P_\epsilon, \tag{10}$$

where ∇ is the gradient with respect to the metric $\langle \cdot, \cdot \rangle$. We shall need the following theorem of Viterbo, [72].

Theorem 2.1. *For $h \in C^\infty(\mathbb{R})$ define $H \in C^\infty(P_\epsilon)$ by*

$$H(p) = h(f(p)), \quad p \in P_\epsilon.$$

Let Ω be a domain in $\mathbb{C}$ and let $\widehat{J} \in \Gamma\big(\widehat{M} \times \Omega, \mathrm{End}\big(T\widehat{M}\big)\big)$ be a smooth section such that $\widehat{J}_z := \widehat{J}(\cdot, z)$ is an $\widehat{\omega}$-compatible almost complex structure satisfying (5)*,* (6) *and* (7)*. If $u \in C^\infty(\Omega, P_\epsilon)$ is a solution of Floer's equation*

$$\partial_s u(z) + \widehat{J}(u(z), z)\partial_t u(z) = \nabla H(u(z)), \quad z = s + it \in \Omega, \tag{11}$$

then

$$\Delta(f(u)) = \langle \partial_s u, \partial_s u \rangle + h''(f(u)) \cdot \partial_s(f(u)) \cdot f(u). \tag{12}$$

Proof. We abbreviate $d^c(f(u)) := d(f(u)) \circ i = \partial_t(f(u))ds - \partial_s(f(u))dt$. Then

$$-dd^c(f(u)) = \Delta(f(u))\, ds \wedge dt. \tag{13}$$

In view of the identities (9), (10) and (11) we can compute

$$
\begin{aligned}
(14) \qquad & -d^c(f(u)) \\
= \;& -(df(u)\partial_t u)\,ds + (df(u)\partial_s u)\,dt \\
= \;& -\Big(df(u)\big(\widehat{J}(u,z)\partial_t u\big)\Big)\,dt - \Big(df(u)\big(\widehat{J}(u,z)\partial_s u\big)\Big)\,ds \\
& + \Big(df(u)\big(\partial_s u + \widehat{J}(u,z)\partial_t u\big)\Big)\,dt + \Big(df(u)\big(\widehat{J}(u,z)\partial_s u - \partial_t u\big)\Big)\,ds \\
= \;& \widehat{\omega}\left(\widehat{X}(u), \partial_t u\right)dt + \widehat{\omega}\left(\widehat{X}(u), \partial_s u\right)ds \\
& + \langle \nabla f(u), \nabla H(u)\rangle\,dt + \left\langle \nabla f(u), \widehat{J}(u,z)\nabla H(u)\right\rangle ds \\
= \;& u^*\iota_{\widehat{X}}\widehat{\omega} + \left\langle \widehat{X}(u), h'(f(u))\widehat{X}(u)\right\rangle dt + 0 \\
= \;& u^*\iota_{\widehat{X}}\widehat{\omega} + h'(f(u))f(u)dt.
\end{aligned}
$$

Using $d\iota_{\widehat{X}}\widehat{\omega} = \mathcal{L}_{\widehat{X}}\widehat{\omega} = \widehat{\omega}$ and again (11), we find

$$
\begin{aligned}
du^*\iota_{\widehat{X}}\widehat{\omega} \;=\; u^*\widehat{\omega} \;&=\; \widehat{\omega}\Big(\partial_s u, \widehat{J}(u,z)\partial_s u - \widehat{J}(u,z)\nabla H(u)\Big)\,ds\wedge dt \\
&= \big(\langle\partial_s u, \partial_s u\rangle - dH(u)\partial_s u\big)\,ds\wedge dt \\
&= \big(\langle\partial_s u, \partial_s u\rangle - \partial_s(h(f(u)))\big)\,ds\wedge dt.
\end{aligned}
$$

Together with (14) it follows that

$$
\begin{aligned}
-dd^c(f(u)) \;&=\; \big(\langle\partial_s u, \partial_s u\rangle - \partial_s(h(f(u))) + \partial_s(h'(f(u))f(u))\big)\,ds\wedge dt \\
&=\; \big(\langle\partial_s u, \partial_s u\rangle + h''(f(u))\cdot\partial_s(f(u))\cdot f(u)\big)\,ds\wedge dt,
\end{aligned}
$$

and so Theorem 2.1 follows in view of (13). □

Remark 2.2 (Time-dependent Hamiltonian)**.** Repeating the calculations in the proof of Theorem 2.1, one shows the following more general result. *Let* $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$ *and define* $H \in C^\infty(P_\epsilon \times \mathbb{R})$ *by*

$$H(p,s) = h(f(p), s), \quad p \in P_\epsilon,\; s \in \mathbb{R}.$$

If Ω *is a domain in* $\mathbb{C}$ *and if* $u \in C^\infty(\Omega, P_\epsilon)$ *is a solution of the time-dependent Floer equation*

$$\partial_s u(z) + \widehat{J}(u(z), z)\partial_t u(z) = \nabla H(u(z), s), \quad z = s + it \in \Omega, \tag{15}$$

then

$$\Delta(f(u)) = \langle\partial_s u, \partial_s u\rangle + \partial_1^2 h(f(u), s)\cdot\partial_s f(u)\cdot f(u) + \partial_1\partial_2 h(f(u), s)\cdot f(u).$$

In the following corollary we continue the notation of Theorem 2.1.

Corollary 2.3 (Maximum Principle)**.** *Assume that* $u \in C^\infty(\Omega, P_\epsilon)$ *and that one of the following conditions holds.*

(i) *u is a solution of Floer's equation* (11)*;*

(ii) *u is a solution of the time-dependent Floer equation* (15) *and* $\partial_1\partial_2 h \geq 0$.

If $f \circ u$ attains its maximum on Ω, then $f \circ u$ is constant.

Proof. Assume that u solves (11). We set

$$b(z) \;=\; -h''(f(u(z))) \cdot f(u(z)).$$

The operator L on $C^\infty(\Omega, \mathbb{R})$ defined by $L(v) = \Delta v + b(z)\partial_s v$ is uniformly elliptic on relatively compact domains in Ω, and according to Theorem 2.1, $L(f \circ u) \geq 0$. If $f \circ u$ attains its maximum on Ω, the strong Maximum Principle, [26, Theorem 3.5], thus implies that $f \circ u$ is constant. The other claim follows similarly from Remark 2.2 and the second part of [26, Theorem 3.5]. □

3. Floer homology

Floer homology was introduced by Floer for weakly exact closed symplectic manifolds in [15, 16, 17, 18], and we refer to [65] for a general exposition. The Floer chain complex of a Hamiltonian function is generated by the 1-periodic orbits of its Hamiltonian flow, and the boundary operator is defined, roughly speaking, by counting perturbed pseudo-holomorphic cylinders which converge at both ends to generators of the chain complex. In the presence of a contact type boundary ∂M the Hamiltonian has to be chosen appropriately near ∂M in order to insure that the Floer cylinders stay in the interior of M. Floer homologies for open subsets of $(\mathbb{R}^{2n}, \omega_0)$ and for weakly exact compact convex symplectic manifolds have been constructed in [20, 4, 5, 72]. These Floer homologies depend on the behaviour of the Hamiltonians near ∂M. In this section we define a Floer homology for weakly exact compact convex symplectic manifolds via Hamiltonians which increase slowly near ∂M. This Floer homology will be identified with the Morse homology of M via the PSS isomorphism in the next section.

The Reeb vector field R of $\alpha = (\iota_X \omega)|_{\partial M}$ on ∂M is defined by

$$\omega_x(v, R) = 0 \;\text{ and }\; \omega_x(X, R) = 1, \qquad x \in \partial M,\; v \in T_x\partial M. \tag{16}$$

By (5) and (6) we have $R = \widehat{J}X|_{\partial M}$. This and (10) imply that for $h \in C^\infty(\mathbb{R})$ the Hamilton equation $\dot{x} = X_H(x)$ of $H = h \circ f \colon P_\epsilon \to \mathbb{R}$ defined by $\omega\left(X_H(x), \cdot\right) = dH(x)$ restricts on ∂M to

$$\dot{x}(t) \;=\; -h'(1)\, R(x(t)). \tag{17}$$

Define $\kappa \in (0, \infty]$ by

$$\kappa := \inf \left\{ c > 0 \mid \dot{x}(t) = -c\, R(x(t)) \text{ has a 1-periodic orbit} \right\}.$$

We denote by $\widehat{\mathcal{H}}$ the set of smooth functions $\widehat{H} \in C^\infty(S^1 \times \widehat{M})$ for which there exists $h \in C^\infty(\mathbb{R})$ such that $0 \leq h'(\rho) < \kappa$ for all $\rho \geq 1$ and $\widehat{H}|_{S^1 \times \partial M \times [0,\infty)} = h \circ f$; with this choice of h the restriction of the flow $\varphi^t_{\widehat{H}}$ of $\widehat{H} \in \widehat{\mathcal{H}}$ to $\partial M \times [0, \infty)$ has no 1-periodic solutions. We introduce the set

$$\mathcal{H} := \left\{ H \in C^\infty(S^1 \times M) \mid H = \widehat{H}|_{S^1 \times M} \text{ for some } \widehat{H} \in \widehat{\mathcal{H}} \right\}$$

of *admissible Hamiltonian functions* on M. Moreover, we denote by $\widehat{\mathcal{J}}$ the set of smooth sections $\widehat{J} \in \Gamma\big(S^1 \times \widehat{M}, \mathrm{End}\big(T\widehat{M}\big)\big)$ such that for every $t \in S^1$ the section $\widehat{J}_t := \widehat{J}(t,\cdot)$ is an $\widehat{\omega}$-compatible almost complex structure which on $\partial M \times [0,\infty)$ is independent of the t-variable and satisfies (5), (6) and (7); and we introduce the set

$$\mathcal{J} := \left\{ J \in \Gamma\left(S^1 \times M, \mathrm{End}(TM)\right) \mid J = \widehat{J}|_{S^1\times M} \text{ for some } \widehat{J} \in \widehat{\mathcal{J}} \right\}$$

of *admissible almost complex structures* on TM. A well-known argument shows that the space $\widehat{\mathcal{J}}$ is connected, see [2, Remark 4.1.1]. Since the restriction map $\widehat{\mathcal{J}} \to \mathcal{J}$ is continuous, $\mathcal{J}$ is also connected.

For $H \in \mathcal{H}$ let $\mathcal{P}_H$ be the set of contractible 1-periodic orbits of the Hamiltonian flow of H. By "generic" we shall mean "belonging to a countable intersection of sets which are open and dense in the strong Whitney C^∞-topology". For generic $H \in \mathcal{H}$ for no $x \in \mathcal{P}_H$ the value 1 is a Floquet multiplier of x, i.e.,

$$\det\big(\mathrm{id} - d\varphi_H^1(x(0))\big) \neq 0. \tag{18}$$

Since M is compact, $\mathcal{P}_H$ is then a finite set. An admissible H satisfying (18) for all $x \in \mathcal{P}_H$ is called *regular*, and the set of regular admissible Hamiltonians is denoted by $\mathcal{H}_{\mathrm{reg}} \subset \mathcal{H}$. For $H \in \mathcal{H}_{\mathrm{reg}}$ we define $CF(M;H)$ to be the $\mathbb{Z}_2$-vector space consisting of formal sums

$$\xi \;=\; \sum_{x\in\mathcal{P}_H} \xi_x\, x, \qquad \xi_x \in \mathbb{Z}_2.$$

We assume first that the first Chern class $c_1 = c_1(\omega) \in H^2(M;\mathbb{Z})$ of the bundle (TM,J) vanishes on $\pi_2(M)$. In this case, the Conley–Zehnder index $\mu(x)$ of $x \in \mathcal{P}_H$ is well-defined, see [66]. We normalize μ in such a way that for C^2-small time-independent Hamiltonians,

$$\mu(x) = 2n - \mathrm{ind}(x)$$

for each critical point $x \in \mathrm{Crit}(H)$; here, $\mathrm{ind}(x)$ is the Morse index of H at x. The Conley–Zehnder index turns $CF(M;H)$ into the graded $\mathbb{Z}_2$-vector space $CF_*(M;H)$. For $x, y \in \mathcal{P}_H$ let $\mathcal{M}(x,y)$ be the moduli space of Floer connecting orbits from x to y, i.e., $\mathcal{M}(x,y)$ is the set of solutions $u \in C^\infty(\mathbb{R}\times S^1, M)$ of the problem

$$\begin{cases} \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \\ \lim\limits_{s\to-\infty} u(s,t) = x(t), \quad \lim\limits_{s\to\infty} u(s,t) = y(t). \end{cases} \tag{19}$$

For later use we notice that by a standard computation,

$$\int_{\mathbb{R}\times S^1} |\partial_s u|^2\, ds dt \;=\; \mathcal{A}_H(x) - \mathcal{A}_H(y) \;\geq\; 0, \qquad u \in \mathcal{M}(x,y). \tag{20}$$

For generic $J \in \mathcal{J}$ the moduli spaces $\mathcal{M}(x,y)$ are smooth manifolds of dimension $\mu(x) - \mu(y)$ for all $x, y \in \mathcal{P}_H$, see [66]. Such a J is called *H-regular*, and a pair (H,J) is called regular if H is regular and J is H-regular. The group $\mathbb{R}$ acts on $\mathcal{M}(x,y)$ by translation, $u(s,t) \mapsto u(s+\tau,t)$ for $\tau \in \mathbb{R}$. Since $[\omega]$ vanishes on $\pi_2(M)$, there is no bubbling off of pseudo holomorphic spheres. It thus follows from Corollary 2.3 (i) and the usual compactness arguments that if $\mu(x) - \mu(y) = 1$, then the quotient $\mathcal{M}(x,y)/\mathbb{R}$ is a compact zero-dimensional manifold and hence a finite set. Set

$$n(x,y) := \#\{\mathcal{M}(x,y)/\mathbb{R}\} \mod 2.$$

For $k \in \mathbb{N}$ we define the Floer boundary operator $\partial_k \colon CF_k(M;H) \to CF_{k-1}(M;H)$ as the linear extension of

$$\partial_k x = \sum_{\substack{y \in \mathcal{P}_H \\ \mu(y)=k-1}} n(x,y)\, y$$

where $x \in \mathcal{P}_H$ and $\mu(x) = k$. Proceeding as in [16, 68] one shows that $\partial^2 = 0$. The complex $(CF_*(M;H), \partial_*)$ is called the Floer chain complex. Its homology

$$HF_k(M;H,J) := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$$

is a graded $\mathbb{Z}_2$-vector space which does not depend on the choice of a regular pair (H,J), see again [16, 68], and so we can define the Floer homology $HF_*(M)$ by

$$HF_*(M) := HF_*(M;H,J)$$

for any regular pair (H,J).

In case that $c_1(\omega)$ does not vanish, the moduli spaces $\mathcal{M}(x,y)$ for $x, y \in \mathcal{P}_H$ are still smooth manifolds for generic $J \in \mathcal{J}$, but now may contain connected components of different dimensions. We denote by $\mathcal{M}^1(x,y)$ the union of the 1-dimensional connected components of $\mathcal{M}(x,y)$. Since $[\omega]$ vanishes on $\pi_2(M)$, the space $\mathcal{M}^1(x,y)/\mathbb{R}$ is still compact, and we can define

$$n(x,y) := \#(\mathcal{M}^1(x,y)/\mathbb{R}) \mod 2.$$

Proceeding as above we define an ungraded Floer homology whose chain complex is generated again by the set $\mathcal{P}_H$ and whose boundary operator is the linear extension of

$$\partial x = \sum_{y \in \mathcal{P}_H} n(x,y)\, y$$

where $x \in \mathcal{P}_H$.

Products. As we pointed out in Example 4 of the introduction, the product of convex manifolds does not need to be convex. Nevertheless, the Floer homology of a product of weakly exact compact convex symplectic manifolds can still be defined. In fact, Floer homology can be defined for a yet larger class of compact symplectic manifolds with corners. We recall that an n-dimensional manifold with corners is a space locally modeled on $\mathbb{R}^k \times (\mathbb{R}_{\geq 0})^{n-k}$ for variable $k \in \{0, \dots, n\}$.

Definition 3.1. A compact symplectic manifold with corners (M,ω) is *split-convex* if there exist compact convex symplectic manifolds (M_j, ω_j), $j = 1, \dots, k$, and compact subsets $K \subset M \setminus \partial M$ and $K' \subset M_1 \times \cdots \times M_k$ such that

$$(M \setminus K, \omega) = ((M_1 \times \cdots \times M_k) \setminus K', \omega_1 \oplus \cdots \oplus \omega_k).$$

Consider a weakly exact compact split-convex symplectic manifold (M,ω), and let (M_j,ω_j), $j=1,\dots,k$, be as in Definition 3.1. For notational convenience, we assume $k=2$. We specify the set of admissible Hamiltonian functions $\mathcal{H}\subset C^\infty(S^1\times M)$ and the set of admissible almost complex structures $\mathcal{J}\subset\Gamma(S^1\times M,\mathrm{End}\,(TM))$ as follows. For $i=1,2$, let $\widehat{M_i}=M_i\cup_{\partial M_i\times\{0\}}\partial M_i\times[0,\infty)$ be the completion of M_i endowed with the symplectic form $\widehat{\omega}_i$ as in Section 2, and let $\widehat{\mathcal{H}}_i\subset C^\infty\big(S^1\times\widehat{M_i}\big)$ and $\widehat{\mathcal{J}}_i\subset\Gamma\big(S^1\times\widehat{M_i},\mathrm{End}\,\big(T\widehat{M_i}\big)\big)$ be the set of admissible functions and admissible almost complex structures on $\widehat{M_i}$. We define the completion $\big(\widehat{M},\widehat{\omega}\big)$ of (M,ω) as

$$\begin{aligned}\widehat{M} &= M\cup\Big(\big(\widehat{M_1}\times\widehat{M_2}\big)\setminus(M_1\times M_2)\Big),\\ \widehat{\omega} &= \begin{cases}\omega & \text{on}\quad M,\\ \widehat{\omega}_1\oplus\widehat{\omega}_2 & \text{on}\quad \big(\widehat{M_1}\times\widehat{M_2}\big)\setminus(M_1\times M_2)\,.\end{cases}\end{aligned}$$

We first define the set of admissible functions $\widehat{\mathcal{H}}\subset C^\infty\big(S^1\times\widehat{M}\big)$ as the set of functions $\widehat{H}\in C^\infty\big(S^1\times\widehat{M}\big)$ for which there exist $\widehat{H_i}\in\widehat{\mathcal{H}_i}$, $i=1,2$, such that

$$\widehat{H}|_{(\widehat{M_1}\times\widehat{M_2})\setminus(M_1\times M_2)} = (\widehat{H_1}+\widehat{H_2})\Big|_{(\widehat{M_1}\times\widehat{M_2})\setminus(M_1\times M_2)};$$

and we then define the set $\mathcal{H}$ of admissible functions on M as the set of functions $H\in C^\infty(S^1\times M)$ for which there exists $\widehat{H}\in\widehat{\mathcal{H}}$ such that

$$H = \widehat{H}|_M.$$

Similarly, we first define the set of admissible almost complex structures $\widehat{\mathcal{J}}$ as the set of $\widehat{J}\in\Gamma\Big(S^1\times\widehat{M},\mathrm{End}\,\big(T\widehat{M}\big)\Big)$ for which there exist admissible almost complex structures $\widehat{J}_i\in\widehat{\mathcal{J}}_i$, $i=1,2$, such that

$$\widehat{J}|_{(\widehat{M_1}\times\widehat{M_2})\setminus(M_1\times M_2)} = (\widehat{J}_1\times\widehat{J}_2)\Big|_{(\widehat{M_1}\times\widehat{M_2})\setminus(M_1\times M_2)};$$

and we then define the set $\mathcal{J}$ of admissible almost complex structures on M as the set of almost complex structures $J\in\Gamma(S^1\times M,\mathrm{End}\,(TM))$ for which there exists $\widehat{J}\in\widehat{\mathcal{J}}$ such that

$$J = \widehat{J}|_M.$$

Using the Maximum Principle Corollary 2.3 factorwise we define the Floer homology $HF(M)$ as above. If $c_1(\omega)$ vanishes on $\pi_2(M)$, then $HF(M)$ is graded by the Conley–Zehnder index.

4. The Piunikhin–Salamon–Schwarz isomorphism

We assume again that (M,ω) is a weakly exact compact convex symplectic manifold. We first assume that $c_1(\omega)$ vanishes on $\pi_2(M)$. Let $F\in C^\infty(M)$ be an *admissible Morse function*, i.e., F is a smooth Morse function for which there exists $\widehat{F}\in C^\infty(\widehat{M})$ such that

$$\widehat{F}|_M = F \;\text{ and }\; \widehat{F}(x,r)=e^{-r},\quad x\in\partial M,\; r\in[0,\infty).$$

The Morse chain complex $CM_*(M;F)$ of F is the $\mathbb{Z}_2$-vector space generated by the critical points of F and graded by the Morse index, and the boundary operator on $CM_*(M;F)$ is defined by counting flow lines of the negative gradient flow of F with respect to a generic Riemannian metric between critical points of index difference 1. The homology

$$HM_*(M) \;=\; HM_*(M;F)$$

of $CM_*(M;F)$ does not depend on the choice of F, cf. [68]. The Piunikhin–Salamon–Schwarz maps between the Morse chain complex and the Floer chain complex will induce explicit isomorphisms between the Morse homology $HM_*(M)$ of M and the Floer homology $HF_*(M)$ of (M,ω).

Choose $H \in \mathcal{H}_{\text{reg}}$ and an admissible Morse function F. We first construct the Piunikhin–Salamon–Schwarz map

$$\phi\colon CM_*(M;F) \;\to\; CF_*(M;H).$$

By definition of $\mathcal{H}$, there exists $\widehat{H} \in \widehat{\mathcal{H}}$ such that $H = \widehat{H}|_{S^1\times M}$ and $\widehat{H}|_{S^1\times\partial M\times[0,\infty)} = h\circ f$ for some $h \in C^\infty(\mathbb{R})$ satisfying $0 \le h'(1) < \kappa$. For $s \in \mathbb{R}$ choose a smooth family $h_s \in C^\infty(\mathbb{R})$ such that

(h1) $h_s = 0, \quad s \le 0,$
(h2) $\partial_s h_s' \ge 0, \quad s \in \mathbb{R},$
(h3) $h_s = h, \quad s \ge 1,$

and then choose a smooth family $\widehat{H}_s \in C^\infty\big(S^1 \times \widehat{M}\big)$ such that

(H1) $\widehat{H}_s = 0, \quad s \le 0,$
(H2) $\widehat{H}_s|_{S^1\times\partial M\times[0,\infty)} = h_s \circ f, \quad 0 \le s \le 1,$
(H3) $\widehat{H}_s = \widehat{H}, \quad s \ge 1.$

We finally define the smooth family $H_s \in C^\infty(S^1 \times M)$ by

$$H_s \;:=\; \widehat{H}_s|_{S^1\times M}.$$

Theorem 4.1. *Let J^- be an H-regular admissible almost complex structure and J^+ be an arbitrary admissible almost complex structure. Consider the space $\mathcal{J}(J^-,J^+)$ of families of admissible almost complex structures $J_s \in \Gamma(S^1 \times M, \mathrm{End}(TM))$ for which there exists $s_0 = s_0(\widehat{J}_s) > 0$ such that $J_s = J^-$ for $s \le -s_0$ and $J_s = J^+$ for $s \ge s_0$. For a generic element $J_s \in \mathcal{J}(J^-,J^+)$ the moduli space of the problem*

$$\text{(21)} \qquad \begin{cases} u \in C^\infty(\mathbb{R}\times S^1, M), \\ \partial_s u + J_{s,t}(u)\big(\partial_t u - X_{H_{s,t}}(u)\big) = 0, \\ \int_{\mathbb{R}\times S^1} |\partial_s u|^2 < \infty, \end{cases}$$

is a smooth finite dimensional manifold. Here, $J_{s,t} = J_s(t,\cdot)$ and $H_{s,t} = H_s(t,\cdot)$.

Proof. We denote the moduli space of solutions of problem (21) by $\mathcal{M}_0$. Choose $\widehat{H}_s \in \widehat{\mathcal{H}}$ and $\widehat{J}_s \in \widehat{\mathcal{J}}$ satisfying $\widehat{H}_s|_M = H_s$ and $\widehat{J}_s|_M = J_s$. Instead of $\mathcal{M}_0$ we first consider the moduli space $\mathcal{M}$ of solutions of the problem

$$
\left\{
\begin{array}{l}
u \in C^\infty(\mathbb{R} \times S^1, \widehat{M}), \\
\partial_s u + \widehat{J}_{s,t}(u)(\partial_t u - X_{\widehat{H}_{s,t}}(u)) = 0, \\
\int_{\mathbb{R}\times S^1} |\partial_s u|^2 < \infty.
\end{array}
\right. \tag{22}
$$

The moduli space $\mathcal{M}_0$ consists of those $u \in \mathcal{M}$ whose image is entirely contained in M. We shall first prove that for generic choice of $\widehat{J}_s$, the moduli space $\mathcal{M}$ is a smooth finite dimensional manifold. We shall then use convexity to prove that the image of each $u \in \mathcal{M}$ is entirely contained in M and hence $\mathcal{M}_0 = \mathcal{M}$ is a smooth finite dimensional manifold.

We interpret solutions of (22) as the zero set of a smooth section from a Banach manifold $\mathcal{B}$ to a Banach bundle $\mathcal{E}$ over $\mathcal{B}$. To define $\mathcal{B}$ we first introduce certain weighted Sobolev norms. Choose a smooth cutoff function $\beta \in C^\infty(\mathbb{R})$ such that $\beta(s) = 0$ for $s < 0$ and $\beta(s) = 1$ for $s > 1$. Choose $\delta > 0$ and define $\gamma_\delta \in C^\infty(\mathbb{R})$ by

$$
\gamma_\delta(s) \;:=\; e^{\delta\beta(s)s}.
$$

Let Ω be a domain in the cylinder $\mathbb{R} \times S^1$. For $1 \le p \le \infty$ and $k \in \mathbb{N}_0$ we consider the standard Sobolev space $W^{k,p}(\Omega)$ as defined, e.g., in Appendix B.1 of [51], and define the $\|\cdot\|_{k,p,\delta}$-norm of $v \in W^{k,p}(\Omega)$ by

$$
\|v\|_{k,p,\delta} \;:=\; \sum_{i+j\le k} \left\|\gamma_\delta \cdot \partial_s^i \partial_t^j v\right\|_p .
$$

We introduce weighted Sobolev spaces

$$
\begin{aligned}
W^{k,p}_\delta(\Omega) \;&:=\; \left\{ v \in W^{k,p}(\Omega) \mid \|v\|_{k,p,\delta} < \infty \right\} \\
&=\; \left\{ v \in W^{k,p}(\Omega) \mid \gamma_\delta v \in W^{k,p}(\Omega) \right\},
\end{aligned}
$$

and we abbreviate

$$
L^p_\delta(\Omega) \;:=\; W^{0,p}_\delta(\Omega).
$$

The space $W^{k,p}_{\mathrm{loc}}(\Omega)$ is defined as the space (of equivalence classes) of locally p-integrable functions $v\colon \Omega \to \mathbb{R}$ whose restrictions to all bounded open subsets U of Ω are in $W^{k,p}(U)$. For $kp > 2$, the Sobolev spaces $W^{k,p}(\Omega, E)$, $W^{k,p}_\delta(\Omega, E)$ and $W^{k,p}_{\mathrm{loc}}(\Omega, X)$ of maps from Ω to a vector bundle E over a closed manifold or to a manifold X can be defined in terms of $W^{k,p}(\Omega)$, $W^{k,p}_\delta(\Omega)$ and $W^{k,p}_{\mathrm{loc}}(\Omega)$ by using local coordinate charts, see [51, Remark B.1.24].

Let $p > 2$ and fix a Riemannian metric g on $T\widehat{M}$. The Banach manifold $\mathcal{B} = \mathcal{B}^{1,p}_\delta(\widehat{M})$ consists of $W^{1,p}_{\mathrm{loc}}$-maps u from the cylinder $\mathbb{R} \times S^1$ to $\widehat{M}$ which satisfy the conditions

(B1) There exists a point $m \in \widehat{M}$, a real number $T_1 < 0$, and $v_1 \in W^{1,p}_\delta\big((-\infty, T_1) \times S^1, T_m\widehat{M}\big)$ such that

$$
u(s,t) \;=\; \exp_m(v_1(s,t)), \quad s < T_1.
$$

(B2) There exists $x \in \mathcal{P}_H \subset C^\infty\big(S^1, \widehat{M}\big)$, a real number $T_2 > 0$, and $v_2 \in W^{1,p}_\delta\big((T_2,\infty)\times S^1, x^*T\widehat{M}\big)$ such that

$$u(s,t) \;=\; \exp_{x(t)}(v_2(s,t)), \qquad s > T_2.$$

Here, the exponential map is taken with respect to g. Since $\widehat{M}$ has no boundary, $\mathcal{B}$ is a Banach manifold without boundary. Note that every solution of (22) lies in $\mathcal{B}$. Indeed, the finite energy assumption in (22) guarantees that solutions of (22) converge exponentially fast at both ends, see [22, Section 3.7]. Let $\mathcal{E}$ be the Banach bundle over $\mathcal{B}$ whose fibre over $u \in \mathcal{B}$ is given by

$$\mathcal{E}_u \;:=\; L^p_\delta(u^*TM).$$

We choose $\widehat{J}^-, \widehat{J}^+ \in \widehat{\mathcal{J}}$ such that $J^- = \widehat{J}^-|_M$ and $J^+ = \widehat{J}^+|_M$. For each smooth family $\widehat{J}_s$ for which there exists an $s_0 > 0$ such that $\widehat{J}_s = \widehat{J}^-$ for $s \le -s_0$ and $\widehat{J}_s = \widehat{J}^+$ for $s \ge s_0$ we define the section $\mathcal{F} = \mathcal{F}_{J_s} \colon \mathcal{B} \to \mathcal{E}$ by

$$\mathcal{F}(u) \;:=\; \partial_s u + \widehat{J}_{s,t}(u)(\partial_t u - X_{\widehat{H}_{s,t}}(u)).$$

If δ is chosen small enough, then the vertical differential $D\mathcal{F}$ is a Fredholm operator, see for example [22, Section 4.3]. One can prove that for generic choice of J_s the section $\mathcal{F}_{J_s}$ intersects the zero section transversally, see [21, Section 5] and [22, Section 4.5]. Hence,

$$\mathcal{M} \equiv \mathcal{M}_{J_s} \;:=\; \mathcal{F}_{J_s}^{-1}(0)$$

is a smooth finite dimensional manifold for generic J_s.

It remains to show that $\mathcal{M} = \mathcal{M}_0$, i.e., the image of every $u \in \mathcal{M}$ is contained in M. The finite energy condition in (22) and (H1) imply by removable singularities that $m := \lim_{s\to-\infty} u(s,t)$ is a point in $\widehat{M}$, and the finite energy condition and (H3) imply that $x(t) := \lim_{s\to\infty} u(s,t)$ is a 1-periodic orbit of the Hamiltonian flow of $\widehat{H}$, see [65, Section 2.7]. By our choice of $\widehat{H}$ we in fact have $x \in \mathcal{P}_H \subset M$. We claim that m is also contained in M. To see this, assume that $m \in \widehat{M} \setminus M$. Define $v \colon \mathbb{C} \to \widehat{M}$ by the conditions

$$v\left(e^{2\pi(s+it)}\right) = u(s,t), \qquad v(0) = m.$$

Since every admissible almost complex structure J restricted to $\widehat{M} \setminus M$ is independent of the t-variable, v is a pseudo holomorphic map in a neighbourhood of 0, which is not constant, since otherwise u were constant, contradicting $x \subset M$ and $m \in \widehat{M} \setminus M$. It follows from Corollary 2.3 (i) that $f \circ v$ does not have a local maximum at 0. In view of condition (h2) it follows from Corollary 2.3 (ii) that for every $(s,t) \in \mathbb{R} \times S^1$ for which $u(s,t) \in \widehat{M} \setminus M$, the function $f \circ u$, which is well-defined in a neighbourhood of (s,t), does not have a local maximum at (s,t). But then x cannot entirely lie in M. This contradiction proves $m \in M$. Now a similar reasoning as above, which uses again Corollary 2.3, shows that the whole image of u lies in M. We have shown that $\mathcal{M} = \mathcal{M}_0$, and so Theorem 4.1 is proved. $\square$

Define the evaluation map $\mathrm{ev}\colon \mathcal{M} \to M$ by

$$\mathrm{ev}(u) := \lim_{s\to-\infty} u(s,t).$$

Combining the techniques in [65, Section 2.7] and [22, Appendix C.2] one sees that for generic J_s the evaluation map ev is transverse to every unstable manifold of the Morse function $F \in C^\infty(M)$. Denote by $\mathrm{Crit}(F)$ the set of critical points of F and by $\mathrm{ind}(c)$ the Morse index of $c \in \mathrm{Crit}(F)$. Morse flow lines $\gamma\colon \mathbb{R} \to M$ are solutions of the ordinary differential equation

$$\dot\gamma(s) = -\nabla F(\gamma(s)) \tag{23}$$

where the gradient is taken with respect to a generic metric g on M. For generators $c \in \mathrm{Crit}(F) \subset M$ of the Morse chain complex and $x \in \mathcal{P}_H$ of the Floer chain complex, let $\mathcal{M}(c,x)$ be the moduli space of pairs (γ,u) such that $\gamma\colon (-\infty,0] \to M$ solves (23), u solves (21), and

$$\lim_{s\to-\infty}\gamma(s) = c, \quad \gamma(0) = \mathrm{ev}(u), \quad \lim_{s\to\infty} u(s,t) = x(t),$$

cf. Figure 2.

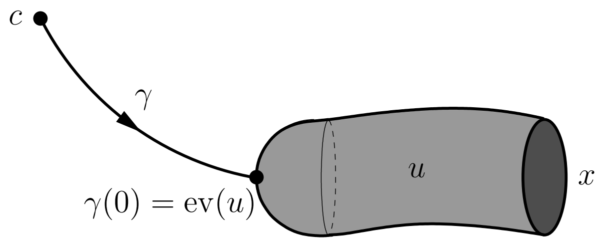

FIGURE 2. An element of $\mathcal{M}(c,x)$.

If $\mathrm{ind}(c) = \mu(x)$, then $\mathcal{M}(c,x)$ is a compact zero-dimensional manifold, see [60]. We can thus set

$$n(c,x) := \#\mathcal{M}(c,x) \mod 2.$$

The Piunikhin–Salamon–Schwarz map $\phi\colon CM_*(M;F) \to CF_*(M;H)$ is defined as the linear extension of

$$\phi(c) = \sum_{\substack{x\in\mathcal{P}_H\\ \mathrm{ind}(c)=\mu(x)}} n(c,x)\,x, \quad c \in \mathrm{Crit}(F).$$

By the usual gluing and compactness arguments one proves that ϕ intertwines the boundary operators of the Morse complex and the Floer complex and hence induces a homomorphism

$$\Phi\colon HM_*(M) \to HF_*(M),$$

see [51, 69]. To prove that Φ is an isomorphism we construct its inverse. We first define the Piunikhin–Salamon–Schwarz map $\psi\colon CF_*(M;H) \to CM_*(M;F)$. Let

$$U := \bigcup_{c\in\mathrm{Crit}(F)} W^s_F(c)$$

be the union of the stable manifolds of F. Since F is admissible, the stable manifolds of the critical points of F are entirely contained in the interior of M, i.e.,

$$\overline{U} \subset M \setminus \partial M.$$

Choose an open neighbourhood V of $\overline{U}$ in $M \setminus \partial M$ and a smooth family of admissible Hamiltonian functions H_s for which there exists $s_0 > 0$ such that

$$H_s = H \;\text{ if }\; s \leq -s_0 \qquad \text{and} \qquad H_s|_V = 0 \;\text{ if }\; s \geq s_0.$$

Recall that the Hamiltonian functions H_s are restrictions of Hamiltonian functions $\widehat{H}_s \in \widehat{\mathcal{H}}$ such that $\widehat{H}_s|_{S^1 \times \partial M \times [0,\infty)} = h_s \circ f$, where $h_s \in C^\infty(\mathbb{R})$ are such that $0 \leq h_s'(\rho) < \kappa$ for all $\rho \geq 1$. We assume in addition that $h_s = h$ is independent of the s-variable and that

$$h'(1) > 0.$$

Choose a smooth family $J_s \in \mathcal{J}(J^+, J^-)$ of admissible almost complex structures. For $x \in \mathcal{P}_H$ and $c \in \mathrm{Crit}(F)$ let $\mathcal{M}(x, c)$ be the moduli space of pairs (u, γ) such that u solves (21), $\gamma\colon [0, \infty) \to M$ solves (23), and

$$\lim_{s\to-\infty} u(s,t) = x(t), \qquad \lim_{s\to\infty} u(s,t) = \gamma(0), \qquad \lim_{s\to\infty} \gamma(s) = c,$$

cf. Figure 3.

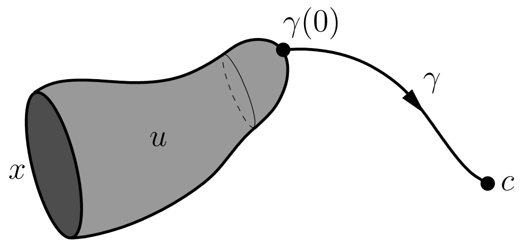

FIGURE 3. An element of $\mathcal{M}(x, c)$.

By our assumption on $\widehat{H}_s$ it follows from Corollary 2.3 (i) that every solution of problem (22) is entirely contained in M and hence solves problem (21). Arguing as in the proof of Theorem 4.1 we can thus show that for generic choice of J_s the moduli space $\mathcal{M}(x, c)$ is a finite dimensional manifold of dimension

$$\dim \mathcal{M}(x,c) \;=\; \mu(x) - \mathrm{ind}(c).$$

In case that $\mu(x) = \mathrm{ind}(c)$, the moduli space is compact, and we define

$$n(x,c) \;:=\; \#\,\{\mathcal{M}(x,c)\} \quad \mod 2.$$

The Piunikhin–Salamon–Schwarz map $\psi\colon CF_*(M;H) \to CM_*(M;F)$ is defined as the linear extension of

$$\psi(x) \;=\; \sum_{\substack{c\in\mathrm{Crit}(F)\\ \mu(x)=\mathrm{ind}(c)}} n(x,c)\,x, \qquad x \in \mathcal{P}_H.$$

Again, ψ intertwines the boundary operators in the Floer complex and the Morse complex and hence induces a homomorphism

$$\Psi\colon HF_*(M) \to HM_*(M).$$

One can prove that

$$\Psi \circ \Phi = \mathrm{id} \qquad \text{and} \qquad \Phi \circ \Psi = \mathrm{id},$$

cf. [60], and so Φ and Ψ are isomorphisms, called the PSS isomorphisms.

If $c_1(\omega)$ does not vanish on $\pi_2(M)$, we proceed in the same way and obtain the PSS isomorphisms between the ungraded homologies $HM(M)$ and $HF(M)$.

Products. Proceeding as above and applying the Maximum Principle Corollary 2.3 factorwise we construct PSS isomorphisms also for weakly exact compact split-convex symplectic manifolds.

5. The selector c

Let (M,ω) be a weakly exact compact split-convex symplectic manifold. We do not assume that $c_1(\omega)$ vanishes on $\pi_2(M)$ and shall work with ungraded chain complexes and homologies. For a regular admissible Hamiltonian $H \in \mathcal{H}_{\text{reg}}$ and $a \in \mathbb{R}$ let $CF^a(M;H)$ be the linear subspace of $CF(M;H)$ consisting of those formal sums

$$\xi = \sum_{x \in \mathcal{P}_H} \xi_x \, x, \qquad \xi_x \in \mathbb{Z}_2,$$

for which $\xi_x = 0$ if $\mathcal{A}_H(x) > a$. In view of (20), the Floer boundary operator ∂ preserves $CF^a(M;H)$ and thus induces a boundary operator ∂^a on the quotient $CF(M;H)/CF^a(M;H)$. We denote the homology of the resulting complex by $HF^a(M;H)$. Since the projection $CF(M;H) \to CF(M;H)/CF^a(M;H)$ intertwines ∂ and ∂^a, it induces a map $j^a\colon HF(M;H) \to HF^a(M;H)$. Choose a generic admissible Morse function $F \in C^\infty(M)$ which attains its maximum in only one point, say m. Let $[\max] \in HM(M)$ be the homology class represented by m. Following [71], [53] and in particular [70] we define

$$c(H) := \inf \{ a \in \mathbb{R} \mid j^a(\Phi([\max])) = 0 \} \tag{24}$$

where $\Phi\colon HM(M) \to HF(M;H)$ is the PSS isomorphism. Using convexity and the natural isomorphism $HF(M;H) \cong HF(M;K)$ for $H,K \in \mathcal{H}_{\text{reg}}$ one shows as in [70, Section 2] that

$$|c(H) - c(K)| \le \|H-K\| \qquad \text{for all } H,K \in \mathcal{H}_{\text{reg}} \tag{25}$$

where $\|\cdot\|$ denotes the Hofer norm defined in (4). In particular, c is C^0-continuous on $\mathcal{H}_{\text{reg}}$. Let $\mathcal{H}_c(M)$ be the set of C^2-smooth functions $S^1 \times M \to \mathbb{R}$ whose support is contained in $S^1 \times (M \setminus \partial M)$, and let $\mathcal{H}_c^\infty(M)$ be the set of C^∞-smooth functions in $\mathcal{H}_c(M)$. Since $\mathcal{H}_{\text{reg}}$ is C^∞-dense in $\mathcal{H}$ and since $\mathcal{H}_c^\infty(M)$ is C^2-dense in $\mathcal{H}_c(M)$, we can first C^∞-continuously extend c to a map $\mathcal{H} \to \mathbb{R}$ and can then C^2-continuously extend its restriction to $\mathcal{H}_c^\infty(M)$ to a map $\mathcal{H}_c(M) \to \mathbb{R}$ which we still denote by c. By (25),

$$|c(H) - c(K)| \le \|H-K\| \qquad \text{for all } H,K \in \mathcal{H}_c(M). \tag{26}$$

For $H \in \mathcal{H}$ or $H \in \mathcal{H}_c(M)$ we denote by $\mathcal{P}_H$ the set of contractible 1-periodic orbits of φ_H^t and by Σ_H the action spectrum

$$\Sigma_H \;=\; \{\mathcal{A}_H(x) \mid x \in \mathcal{P}_H\}\,.$$

The following property of c is basic for everything to come.

Proposition 5.1. *For every $H \in \mathcal{H}_c(M)$ it holds that $c(H) \in \Sigma_H$.*

Proof. For $H \in \mathcal{H}_{\mathrm{reg}}$ it follow from definition (24) that $c(H) \in \Sigma_H$. For $H \in \mathcal{H}_c(M)$ we choose a sequence H_n, $n \geq 1$, in $\mathcal{H}_{\mathrm{reg}}$ converging to H in C^2 and choose $x_n \in \mathcal{P}_{H_n}$ such that $c(H_n) = \mathcal{A}_{H_n}(x_n)$. Using that M is compact we find a subsequence n_j, $j \geq 1$, such that $x_{n_j}(0) \to x_0 \in M$ as $j \to \infty$. Since the Hamiltonians H_{n_j} converge to H in C^2, it follows that $x(t) := \varphi_H^t(x_0)$ belongs to $\mathcal{P}_H$, and together with (26),

$$c(H) \;=\; \lim_{j\to\infty} c\left(H_{n_j}\right) \;=\; \lim_{j\to\infty} \mathcal{A}_{H_{n_j}}\left(x_{n_j}\right) \;=\; \mathcal{A}_H(x).$$

Therefore, $c(H) \in \Sigma_H$. □

The set $\mathcal{H}_c(M)$ forms a group with multiplication and inverse given by

$$H_t \Diamond K_t = H_t + K_t\left((\varphi_{H_t}^t)^{-1}\right), \quad H_t^- = -H_t \circ \varphi_{H_t}^t, \quad H_t, K_t \in \mathcal{H}_c(M).$$

It is shown in [70] that c satisfies the triangle inequality

$$c(H \Diamond K) \;\leq\; c(H) + c(K), \quad H, K \in \mathcal{H}_c(M). \tag{27}$$

The proof of (27) uses the product structure on Floer homology given by the pair of pants product, which is well-defined by convexity, and a sharp energy estimate for the pair of pants.

In the remainder of this section we give an upper bound for $c(H)$ and compute $c(H)$ for simple Hamiltonians.

5.1. An upper bound for $c(H)$.

Proposition 5.2. *Let (M, ω) be a weakly exact compact split-convex symplectic manifold, and let $H \in \mathcal{H}_c(M)$. Then*

$$c(H) \;\leq\; -\int_0^1 \inf_{x\in M} H_t(x)\, dt. \tag{28}$$

In particular, $c(H)) \leq \|H\|$.

Proof. Since c is C^2-continuous, it suffices to prove (28) for $H \in \mathcal{H}_{\mathrm{reg}}$. Let $\widehat{H} \in \widehat{\mathcal{H}}$ be such that $H = \widehat{H}|_{S^1\times M}$. We can choose the family $\widehat{H}_s \in C^\infty\big(S^1 \times \widehat{M}\big)$ used in the construction of the PSS map $\phi\colon CM(M;F) \to CF(M;H)$ of the form $\widehat{H}_s = \beta(s)\,\widehat{H}$ where $\beta\colon \mathbb{R} \to [0,1]$ is a smooth cut off function such that

$$\beta(s) = 0,\; s \leq 0; \quad \beta'(s) \geq 0,\; s \in \mathbb{R}; \quad \beta(s) = 1,\; s \geq 1. \tag{29}$$

In view of the construction of ϕ and the definition (24) of $c(H)$ we find $x^+ \in \mathcal{P}_H$ such that $\mathcal{A}_H(x^+) = c(H)$ and a solution $u \in C^\infty(\mathbb{R} \times S^1, M)$ of the problem (21) such that

$\lim_{s\to\infty} u(s,t) = x^+(t)$. Since the energy of u is finite, there exists $p \in M$ such that $\lim_{s\to-\infty} u(s,t) = p$. Using the Floer equation in (21) we compute

$$\begin{aligned}
0 &\leq \int_0^1 \int_{-\infty}^{\infty} |\partial_s u|^2 \, ds\, dt \\
&= -\int_0^1 \int_{-\infty}^{\infty} \left\langle \partial_s u, J_{s,t}(u) \left(\partial_t u - X_{H_{s,t}}(u)\right)\right\rangle \, ds\, dt \\
&= \int_{\mathbb{R}\times S^1} u^*\omega + \int_0^1 \int_{-\infty}^{\infty} \omega\left(X_{H_{s,t}}(u), \partial_s u\right) \, ds\, dt \\
&= \int_{\mathbb{R}\times S^1} u^*\omega + \int_0^1 \int_{-\infty}^{\infty} d(H_{s,t}(u))\partial_s u \, ds\, dt \\
&= \int_{\mathbb{R}\times S^1} u^*\omega + \int_0^1 \int_{-\infty}^{\infty} \frac{d}{ds}\Big(H_{s,t}(u)\Big) \, ds\, dt \\
&\quad - \int_0^1 \int_{-\infty}^{\infty} \beta'(s) H_t(u) \, ds\, dt \\
&\leq \int_{\mathbb{R}\times S^1} u^*\omega + \int_0^1 H_t\left(x^+(t)\right)\, dt \\
&\quad - \left(\int_{-\infty}^{\infty} \beta'(s)\, ds\right)\left(\int_0^1 \inf_{x\in M} H_t(x)\, dt\right) \\
&= -\mathcal{A}_H\left(x^+\right) - \int_0^1 \inf_{x\in M} H_t(x)\, dt.
\end{aligned}$$

The proof of Proposition 5.2 is complete. □

5.2. **A formula for $c(H)$.** For a class of simple Hamiltonians the distinguished action value $c(H)$ can be explicitly computed. The following theorem will be the main ingredient in showing that the spectral metric defined in Section 7 is non-degenerate, and together with Proposition 5.2 it will imply the energy-capacity inequality for the π_1-sensitive Hofer–Zehnder capacity given in Section 8.

Theorem 5.3. *Consider a weakly exact compact split-convex symplectic manifold (M,ω), and assume that $H \in \mathcal{H}_c(M)$ has the following properties.*

- (H1) *There exists $p \in \operatorname{Int} M$ such that $H_t(p) = \min_{x\in M} H_t(x)$ for every $t \in [0,1]$.*
- (H2) *The Hessian $\operatorname{Hess}(H)(p)$ of H at p with respect to an ω-compatible Riemannian metric satisfies*

$$\|\operatorname{Hess}(H_t)(p)\| < 2\pi \quad \textit{for all } t \in [0,1].$$

- (H3) *Every nonconstant contractible periodic orbit of the flow φ_H^t has period greater than 1.*

Then

$$c(H) = -\int_0^1 H_t(p)\,dt. \tag{30}$$

Proof. It follows from assumptions (H1) and (H3) that the constant orbit p is a critical point of the action functional $\mathcal{A}_{\lambda H}$ for every $\lambda \in [0,1]$ and that for any other critical point y of $\mathcal{A}_{\lambda H}$,

$$\mathcal{A}_{\lambda H}(y) \le \mathcal{A}_{\lambda H}(p) = -\lambda \int_0^1 H_t(p)\,dt, \quad \lambda \in [0,1]. \tag{31}$$

We choose a sequence of regular admissible Hamiltonians $H_n \in \mathcal{H}_{\text{reg}}$ such that $H_n \to H$ in C^2 and such that each H_n satisfies (H1), (H2) and (31) for the same point p. Since c is C^2-continuous, it suffices to prove (30) for each H_n. We fix n and from now on suppress n in the notation. We choose an admissible Morse function $F \in C^\infty(M)$ whose single maximum is attained at p, and as in the previous paragraph we choose the family $H_s \in C^\infty(S^1 \times M)$ of the form $H_s = \beta(s)\,H$ where $\beta\colon \mathbb{R} \to [0,1]$ satisfies (29). Let c_p be the generator in $CM(M;F)$ represented by the maximum p of F, and let x_p be the generator of $CF(M;H)$ represented by p. In view of the definition (24) of $c(H)$ and the construction of the PSS map $\phi\colon CM(M;F) \to CF(M;H)$, formula (30) follows if we can show that for generic choice of a smooth family J_s of admissible almost complex structures which are independent of s for $|s| \ge s_0$ large enough, the matrix coefficient

$$n(c_p, x_p) = \#\mathcal{M}(c_p, x_p) \mod 2$$

is odd. Equivalently, we are left with showing

Lemma 5.4. *For generic choice of the smooth family J_s of admissible almost complex structures independent of s for $|s| \ge s_0$ large enough, the number of solutions $u \in C^\infty(\mathbb{R} \times S^1, M)$ of the problem*

$$\begin{cases} \partial_s u + J_{s,t}(u)\left(\partial_t u - X_{H_{s,t}}(u)\right) &= 0, \\ \lim_{s\to-\infty} u(s,t) &\in W_F^u(p), \\ \lim_{s\to\infty} u(s,t) &= p, \\ c_1(u) &= 0, \end{cases} \tag{32}$$

is odd. Here, $W_F^u(p)$ denotes the unstable manifold of F at p.

Proof. We choose a smooth family of smooth families of admissible almost complex structures J_s^λ, $s \in \mathbb{R}$, $\lambda \in [0,1]$, such that $J_s^\lambda = J^{\lambda,\pm}$ is independent of s if $|s| \ge s_0$ is large enough, and consider for every $\lambda \in [0,1]$ the problem

$$\begin{cases} \partial_s u + J_{s,t}^\lambda(u)\left(\partial_t u - \lambda X_{H_{s,t}}(u)\right) &= 0, \\ \lim_{s\to-\infty} u(s,t) &\in W_F^u(p), \\ \lim_{s\to\infty} u(s,t) &= p, \\ c_1(u) &= 0. \end{cases} \tag{33}$$

Assumption (H2) guarantees that for each $\lambda \in]0,1]$ the fixed point p of $\varphi^1_{\lambda H}$ is regular in the sense of (18), and hence for generic choice of J_s^λ the space $\mathcal{M}_{\rm tot}$ of pairs (u,λ) solving (33) for some $\lambda \in [0,1]$ is a smooth 1-dimensional manifold. The boundary $\partial\overline{\mathcal{M}}_{\rm tot}$ of its compactification $\overline{\mathcal{M}}_{\rm tot}$ contains an even number of elements,

$$\#\partial\overline{\mathcal{M}}_{\rm tot} \;=\; 0 \mod 2. \tag{34}$$

For generic choice of the family J_s^λ transversality theory implies that $\partial\overline{\mathcal{M}}_{\rm tot}$ consists of three types of points, namely the solutions of (33) for $\lambda = 0$, the solutions of (33) for $\lambda = 1$, and broken trajectories.

1. Since $[\omega]$ vanishes on $\pi_2(M)$, the only solution of (33) for $\lambda = 0$ is the constant map $u \equiv p$.

2. The solutions of (33) for $\lambda = 1$ are the solutions of (32) which we want to count.

3. Solutions of (33) are in bijection with solutions as in Figure 2 consisting of half a Morse flow line followed by a Floer disc. For generic choice of the family J_s^λ, these solutions break off only once, either along the Morse flow line or along the Floer disc. More precisely, for generic choice of J_s^λ, there are finitely many values $0 < \lambda_1 < \ldots < \lambda_n < 1$ for which there are broken trajectories consisting either of pairs $u_1 \in C^\infty(\mathbb{R}, M)$, $u_2 \in C^\infty(\mathbb{R}\times S^1, M)$ which satisfy, for some $i \in \{1,\ldots,n\}$,

$$\left\{\begin{array}{rcl} \partial_s u_1 &=& -\nabla F(u_1), \\ \partial_s u_2 + J_{s,t}^{\lambda_i}\big(\partial_t u_2 - \lambda_i X_{H_{s,t}}(u_2)\big) &=& 0, \\ \lim\limits_{s\to-\infty} u_1(s,t) &=& p, \\ \lim\limits_{s\to\infty} u_1(s,t) &=:& q \in \mathrm{Crit}_{2n-1}(F), \\ \lim\limits_{s\to-\infty} u_2(s,t) &\in& W_F^u(q), \\ \lim\limits_{s\to\infty} u_2(s) &=& p, \\ c_1(u_2) &=& 0, \end{array}\right. \tag{35}$$

or pairs $u_1, u_2 \in C^\infty(\mathbb{R}\times S^1, M)$ which satisfy, for some $i \in \{1,\ldots,n\}$,

$$\left\{\begin{array}{rcl} \partial_s u_1 + J_{s,t}^{\lambda_i}\big(\partial_t u_1 - \lambda_i X_{H_{s,t}}(u_1)\big) &=& 0, \\ \partial_s u_2 + J_t^{\lambda_i,+}\left(\partial_t u_2 - \lambda_i X_{H_t}(u_2)\right) &=& 0, \\ \lim\limits_{s\to-\infty} u_1(s,t) &\in& W_F^u(p), \\ \lim\limits_{s\to\infty} u_1(s,t) = \lim\limits_{s\to-\infty} u_2(s,t) &\in& \mathrm{Crit}\left(\mathcal{A}_{\lambda_i H}\right)\setminus\{p\}, \\ \lim\limits_{s\to\infty} u_2(s,t) &=& p, \\ c_1(u_1\# u_2) &=& 0, \end{array}\right. \tag{36}$$

where $u_1\# u_2$ is the connected sum of the oriented spheres u_1 and u_2. Since p is the only maximum of F, for each critical point q of F of index $2n-1$ there is an even number of Morse flow lines u_1 emanating from p and ending in q. This shows that there is an even number of solutions of (35). Moreover, it follows from formula (20) and from assumption (31) that solutions u_2 of problem (36) have non-positive energy and hence cannot exist. We conclude that there is an even number of broken trajectories.

In view of (34) and 1. and 3. we conclude that for generic choice of J_s the number of solutions of (32) is odd. This proves Lemma 5.4, and so Theorem 5.3 is also proved. □

6. The action spectrum

Recall that the action spectrum Σ_H of $H \in \mathcal{H}_c(M)$ is the set

$$\Sigma_H \;=\; \{\mathcal{A}_H(x) \mid x \in \mathcal{P}_H\}\,.$$

Difficult work of Seidel implies that for a *closed* weakly exact symplectic manifold, $\Sigma_H = \Sigma_K$ whenever $H, K \in \mathcal{H}_c(M)$ generate the same Hamiltonian diffeomorphisms $\varphi_H = \varphi_K$, see [70] and [51, Corollary 8.6.10]. As we shall show in this section by an elementary argument, the same holds true for an open (i.e., not closed) weakly exact symplectic manifold.

Let (M, ω) be an open weakly exact symplectic manifold, and let $G \in \mathcal{H}_c(M)$ be such that $\varphi_G = \mathrm{id}$. To $q \in M$ we associate the loop

$$x_q(t) := \varphi_G^t(q), \quad t \in [0,1].$$

If $q \in M \setminus \operatorname{supp} G$, then x_q is the constant loop. This and the continuity of the map $q \mapsto x_q$ from M to the free loop space of M show that $x_q \in \mathcal{P}_G$ for all $q \in M$. We define the function $I_G \colon M \to \mathbb{R}$ by

$$I_G(q) \;\equiv\; \mathcal{A}_G(x_q) \;=\; -\int_{D^2} \bar{x}_q^* \omega - \int_0^1 G(t, x_q(t))\, dt$$

where $\bar{x}_q$ is a smooth extension of x_q to the unit disc D^2.

Proposition 6.1. *The function I_G vanishes identically.*

Proof. If $q \in M \setminus \operatorname{supp} G$, then $I_G(q) = 0$. It remains to show that I_G is constant. To this end we choose a path $r \mapsto q(r)$ and compute

$$\begin{aligned} \frac{d}{dr} I_G(q(r)) \;&=\; -\int_0^1 \omega\left(d\varphi_G^t(q) q'(r), X_{G_t}\left(\varphi_G^t(q)\right)\right) dt \\ &\quad - \int_0^1 dG_t(\varphi_G^t(q))\left(d\varphi_G^t(q) q'(r)\right) dt \;=\; 0, \end{aligned}$$

as desired. □

Consider $H, K \in \mathcal{H}_c(M)$ such that $\varphi_H = \varphi_K$. We choose a smooth function $\alpha \colon [0,1] \to [0,1]$ such that

$$\alpha(t) \;=\; \begin{cases} 0, & t \le 1/6, \\ 1, & t \ge 1/3. \end{cases} \tag{37}$$

The Hamiltonian $G \in \mathcal{H}_c(M)$ defined by

$$G(t,x) \;=\; \begin{cases} \alpha'(t)\, H(\alpha(t),x), & 0 \le t \le 1/2, \\ -\alpha'(1-t)\, K(\alpha(1-t),x), & 1/2 \le t \le 1, \end{cases} \tag{38}$$

then generates the loop

$$\varphi_G^t \;=\; \begin{cases} \varphi_H^{\alpha(t)}, & 0 \le t \le 1/2, \\ \varphi_K^{\alpha(1-t)}, & 1/2 \le t \le 1, \end{cases}$$

in $\mathrm{Ham}_c(M,\omega)$. Since all loops $x_q(t) = \varphi_G^t(q)$, $q \in M$, $t \in [0,1]$, are contractible, the sets $\mathcal{P}_H$ and $\mathcal{P}_K$ can be canonically identified, and the set

$$\mathrm{Fix}^\circ(\varphi_H) \;=\; \{x(0) \mid x \in \mathcal{P}_H\}$$

of "contractible fixed points" of φ_H does not depend on H. The action of a fixed point $x \in \mathrm{Fix}^\circ(\varphi_H)$ is defined as the action of the loop $\varphi_H^t(x)$,

$$\mathcal{A}_H(x) \;:=\; \mathcal{A}_H\left(\varphi_H^t(x)\right).$$

Corollary 6.2. *Assume that $H, K \in \mathcal{H}_c(M)$ are such that $\varphi_H = \varphi_K$. Then $\mathcal{A}_H(x) = \mathcal{A}_K(x)$ for all $x \in \mathrm{Fix}^\circ(\varphi_H)$. In particular, $\Sigma_H = \Sigma_K$.*

Proof. Define $G \in \mathcal{H}_c(M)$ as in (38). Then

$$\mathcal{A}_G\left(\varphi_G^t(x)\right) \;=\; \mathcal{A}_H\left(\varphi_H^t(x)\right) - \mathcal{A}_K\left(\varphi_K^t(x)\right)$$

for all $x \in \mathrm{Fix}^\circ(\varphi_H) = \mathrm{Fix}^\circ(\varphi_K)$, and so Corollary 6.2 follows from Proposition 6.1. □

Recall that the inverse of $H \in \mathcal{H}_c(M)$ is defined as

$$H_t^-(x) := -H_t\left(\varphi_H^t(x)\right).$$

To $x \in \mathcal{P}_H$ we associate the loop x^- defined as

$$x^-(t) := \varphi_{H^-}^t(x(0)).$$

Corollary 6.3. *If $x \in \mathcal{P}_H$, then $x^- \in \mathcal{P}_{H^-}$ and $\mathcal{A}_{H^-}(x^-) = -\mathcal{A}_H(x)$. In particular, $\Sigma_{H^-} = -\Sigma_H$.*

Proof. Choose $\alpha\colon [0,1] \to [0,1]$ as in (37) and define $G \in \mathcal{H}_c(M)$ by

$$G(t,x) \;=\; \begin{cases} \alpha'(t)\, H(\alpha(t),x), & 0 \le t \le 1/2, \\ \alpha'(t-1/2)\, H^-(\alpha(t-1/2),x), & 1/2 \le t \le 1. \end{cases}$$

Then $\varphi_G = \mathrm{id}$. For $x \in \mathcal{P}_H$ the loop x^- therefore belongs to $\mathcal{P}_{H^-}$. Moreover,

$$I_G(x(0)) \;=\; \mathcal{A}_H(x) + \mathcal{A}_{H^-}(x^-),$$

and so Proposition 6.1 yields $\mathcal{A}_{H^-}(x^-) = -\mathcal{A}_H(x)$. Since the map $x \mapsto x^-$ is a bijection between $\mathcal{P}_H$ and $\mathcal{P}_{H^-}$, we conclude $\Sigma_{H^-} = -\Sigma_H$. □

7. The spectral metric

We consider a weakly exact compact split-convex symplectic manifold (M, ω). For $H \in \mathcal{H}_c(M)$ let $c(H) \in \Sigma_H$ be the special critical value of $\mathcal{A}_H$ defined in Section 5.

Proposition 7.1. *Assume that $H, K \in \mathcal{H}_c(M)$ satisfy $\varphi_H = \varphi_K$. Then*

$$c(H) = c(K).$$

Proof. Since $\varphi_{H \diamond K^-} = \varphi_0 = \mathrm{id}$, Corollary 6.2 shows that $\Sigma_{H \diamond K^-} = \Sigma_0 = \{0\}$. This and Proposition 5.1 yield $c\left(H \diamond K^-\right) = 0$. Together with the triangle inequality (27) we conclude

$$c(H) = c\left(H \diamond K^- \diamond K\right) \leq c\left(H \diamond K^-\right) + c(K) = c(K).$$

Interchanging the roles of H and K we obtain $c(K) \leq c(H)$. Proposition 7.1 follows. □

In view of Proposition 7.1 we can define $c\colon \mathrm{Ham}_c(M) \to \mathbb{R}$ by

$$c(\varphi) = c(H) \quad \text{if } \varphi = \varphi_H.$$

Following Viterbo, [71], and Oh, [54, 55], we define the *spectral norm* $\gamma\colon \mathrm{Ham}_c(M) \to \mathbb{R}$ by

$$\gamma(\varphi) = c(\varphi) + c\left(\varphi^{-1}\right). \tag{39}$$

We shall often write $\gamma(H)$ instead of $\gamma(\varphi_H)$. By Proposition 5.1 and Corollary 6.3, $c(H) \in \Sigma_H$ and $-c(H^-) \in \Sigma_H$, and so $\gamma(\varphi_H) = \gamma(H) = c(H) + c(H^-)$ is the difference of two special actions of φ_H.

Proposition 7.2. *For every C^2-small time-independent $H \in \mathcal{H}_c(M)$ we have $\gamma(H) = \|H\|$.*

Proof. According to Theorem 5.3 we have $c(H) = -\min H$ and $c(H^-) = c(-H) = \max H$, and so $\gamma(H) = c(H) + c(H^-) = \|H\|$. □

We recall that $\mathrm{Symp_c}(M)$ denotes the group of symplectomorphisms of (M, ω) whose support is contained in $M \setminus \partial M$. The following theorem, which implies Theorem 6 of the introduction, justifies that γ is called a norm.

Theorem 7.3. *The spectral norm γ on $\mathrm{Ham}_c(M)$ has the following properties.*

- (S1) $\gamma(\mathrm{id}) = 0$ *and* $\gamma(\varphi) > 0$ *if* $\varphi \neq \mathrm{id}$;
- (S2) $\gamma(\varphi\psi) \leq \gamma(\varphi) + \gamma(\psi)$;
- (S3) $\gamma(\vartheta\varphi\vartheta^{-1}) = \gamma(\varphi)$ *for all* $\vartheta \in \mathrm{Symp_c}(M)$;
- (S4) $\gamma(\varphi) = \gamma\left(\varphi^{-1}\right)$;
- (S5) $\gamma(\varphi) \leq d_H(\varphi, \mathrm{id})$.

Proof. The triangle inequality (S2) follows from the triangle inequality (27) for c. For $\varphi_H \in \mathrm{Ham}_c(M)$ and $\vartheta \in \mathrm{Symp_c}(M)$ we have

$$\vartheta \circ \varphi_H^t \circ \vartheta^{-1} = \varphi_{H_\vartheta}^t \quad \text{for all } t$$

where $H_\vartheta(t,x) = H\left(t,\vartheta^{-1}(x)\right)$. This and the invariance of the Floer equation imply the invariance property (S3). The symmetry property (S4) follows from definition (39). In order to prove the estimate (S5) we need to show that $c\,(H) + c\,(H^-) \le \|H\|$ for all $H \in \mathcal{H}_c(M)$. In view of the continuity of c, it suffices to show this for $H \in \mathcal{H}_{\mathrm{reg}}$. According to Proposition 5.2 we have

$$c\,(H) \;\le\; -\int_0^1 \inf_{x\in M} H_t(x)\,dt, \tag{40}$$

and combining Proposition 5.2 with

$$\inf_{x\in M} H_t^-(x) \;=\; \inf_{x\in M}\left(-H_t\left(\varphi_H^t(x)\right)\right) \;=\; \inf_{x\in M}\left(-H_t(x)\right) \;=\; -\sup_{x\in M} H_t(x)$$

we find

$$c\left(H^-\right) \;\le\; -\int_0^1 \inf_{x\in M} H_t^-(x)\,dt \;=\; \int_0^1 \sup_{x\in M} H_t(x)\,dt. \tag{41}$$

Adding (40) and (41) we obtain $c\,(H) + c\,(H^-) \le \|H\|$, as desired. □

We are left with proving (S1). If $\varphi = \varphi_0 = \mathrm{id}$, then $c\,(\varphi) = c\,(\varphi^{-1}) = c\,(0) = 0$ and so $\gamma\,(\varphi) = 0$. In order to verify that γ is non-degenerate, we shall need the following proposition, which will be crucial for most of our applications. Recall that $\operatorname{supp}\varphi_H = \bigcup_{t\in[0,1]} \operatorname{supp}\varphi_H^t = \bigcup_{t\in[0,1]} \operatorname{supp} X_{H_t}$.

Proposition 7.4. *Assume that $\varphi_H, \psi \in \mathrm{Ham}_c(M)$ are such that ψ displaces $\operatorname{supp}\varphi_H$. Then $\gamma\,(\varphi_H^n) \le 2\,\gamma\,(\psi)$ for all $n \in \mathbb{N}$.*

Proof. We closely follow the proof of Proposition 5.1 in [70]. Note that $\varphi_H^n = \varphi_{H^{(n)}}$ with $H^{(n)}(t,x) := nH(nt,x)$ for all $n \in \mathbb{N}$. Since $\operatorname{supp}\varphi_{H^{(n)}} = \operatorname{supp}\varphi_H$, it is enough to prove the claim for $n = 1$. Assume that $\psi = \varphi_K$. After reparametrizing in t we can assume that $H_t = 0$ for $t \in [0,1/2]$ and $K_t = 0$ for $t \in [1/2,1]$. With this choice of H and K and since φ_K displaces $\operatorname{supp}\varphi_{\epsilon H} = \operatorname{supp}\varphi_H$ for each $\epsilon \in\,]0,1]$ it is clear that

$$\mathrm{Fix}^\circ\left(\varphi_{\epsilon H\diamond K}\right) \;=\; \mathrm{Fix}^\circ\left(\varphi_K\right) \;\subset\; M \setminus \operatorname{supp}\varphi_{\epsilon H},$$

and so $\mathcal{P}_{\epsilon H\diamond K} = \mathcal{P}_K$ and $\Sigma_{\epsilon H\diamond K} = \Sigma_K$ for each $\epsilon \in [0,1]$. The set $\Sigma_K = \Sigma_{\epsilon H\diamond K}$ is nowhere dense, see [70, Proposition 3.7]. This and the continuity of c imply that the map

$$[0,1] \to \Sigma_K, \qquad \epsilon \mapsto c\,(\epsilon H \diamond K)\,,$$

is constant. In particular, $c\,(H\diamond K) = c\,(K)$. Since φ_K displaces $\operatorname{supp}\varphi_H$, its inverse φ_{K^-} displaces $\operatorname{supp}\varphi_{H^-} = \operatorname{supp}\varphi_H$. An argument analogous to the above then yields $c\left((H\diamond K)^-\right) = c\,(K^-\diamond H^-) = c\,(K^-)$. Summarizing we find $\gamma\,(H\diamond K) = \gamma\,(K)$. Together with (S2) and (S4) we can thus conclude

$$\gamma\,(H) \;=\; \gamma\left(H\diamond K\diamond K^-\right) \;\le\; \gamma\,(H\diamond K) + \gamma\left(K^-\right) \;=\; 2\,\gamma\,(K)\,,$$

as desired. □

Assume now that $\varphi \neq \mathrm{id}$. We then find a non-empty open subset $U \subset M$ such that φ displaces U. According to Proposition 7.2 we can choose $H \in \mathcal{H}_c(M)$ such that $\gamma(H) > 0$. Applying Proposition 7.4 with $\psi = \varphi$ we get $0 < \gamma(H) \leq 2\gamma(\varphi)$. The proof of Theorem 7.3 is complete. □

Corollary 7.5. *If $\varphi_H \in \mathrm{Ham}_c(M) \setminus \{\mathrm{id}\}$, then the spectrum Σ_H contains not only* 0.

Proof. Recall that $\gamma(H)$ is the difference of two elements of Σ_H. The corollary thus follows from (S1) of Theorem 7.3. □

The spectral metric d_γ on $\mathrm{Ham}_c(M)$ is defined as

$$d_\gamma(\varphi, \psi) := \gamma\left(\varphi \circ \psi^{-1}\right) \qquad \varphi, \psi \in \mathrm{Ham}_c(M).$$

Theorem 7.3 says that d_γ is a biinvariant metric on $\mathrm{Ham}_c(M)$ such that

$$d_\gamma(\varphi, \psi) \leq d_H(\varphi, \psi) \qquad \text{for all } \varphi, \psi \in \mathrm{Ham}_c(M).$$

8. An energy-capacity inequality

In this section we shall compare the π_1-sensitive Hofer–Zehnder capacity $c_{\mathrm{HZ}}^\circ(A)$ of a subset $A \subset (M, \omega)$ with the d_γ-diameter of $\mathrm{Ham}_c(\mathrm{Int}\, A, \omega)$. This will lead to an energy-capacity inequality for c_{HZ}°, which will be a crucial tool in the proofs of Theorems 4.A and 4.B (ii).

8.1. **The π_1-sensitive Hofer–Zehnder capacity.** Let (M, ω) be an arbitrary symplectic manifold. Given a subset $A \subset M$ we consider the function space

$$\mathcal{F}(A) = \{H \in C_c^\infty(\mathrm{Int}\, A) \mid H \geq 0,\ H|_U = \max H \text{ for some open } U \subset A\}.$$

A function $H \in \mathcal{F}(A)$ is HZ-*admissible* if the flow φ_H^t has no non-constant T-periodic orbit with period $T \leq 1$, and $H \in \mathcal{F}(A)$ is HZ$^\circ$-*admissible* if the flow φ_H^t has no non-constant T-periodic orbit with period $T \leq 1$ which is contractible in M. Set

$$\begin{aligned} \mathcal{F}_{\mathrm{HZ}}(A, M, \omega) &= \{H \in \mathcal{F}(A) \mid H \text{ is HZ-admissible}\}, \\ \mathcal{F}_{\mathrm{HZ}}^\circ(A, M, \omega) &= \{H \in \mathcal{F}(A) \mid H \text{ is HZ}^\circ\text{-admissible}\}. \end{aligned}$$

As in [40, 41] and [47, 70] the Hofer–Zehnder capacity and the π_1-sensitive Hofer–Zehnder capacity of $A \subset (M, \omega)$ are defined as

$$\begin{aligned} c_{\mathrm{HZ}}(A, M, \omega) &= \sup\{\|H\| \mid H \in \mathcal{F}_{\mathrm{HZ}}(A, M, \omega)\}, \\ c_{\mathrm{HZ}}^\circ(A, M, \omega) &= \sup\{\|H\| \mid H \in \mathcal{F}_{\mathrm{HZ}}^\circ(A, M, \omega)\}. \end{aligned}$$

From now on we suppress ω from the notation. Of course, $c_{\mathrm{HZ}}(A, M) \leq c_{\mathrm{HZ}}^\circ(A, M)$. Example 8.1 below shows that this inequality can be strict. It also shows that in contrast to c_{HZ}, the π_1-sensitive Hofer–Zehnder capacity c_{HZ}° is not an intrinsic symplectic capacity as defined in [41]; it is, however, a relative symplectic capacity and in particular satisfies the relative monotonicity axiom

$$c_{\mathrm{HZ}}^\circ(A, M) \leq c_{\mathrm{HZ}}^\circ(B, M) \qquad \text{whenever } A \subset B \subset M. \tag{42}$$

Example 8.1. Consider the annulus $A = \{z \in \mathbb{R}^2 \mid 0 < |z| < 1\}$ in $(\mathbb{R}^2, \omega_0)$. Then $c_{\mathrm{HZ}}(A, A) = c^\circ_{\mathrm{HZ}}(A, \mathbb{R}^2) = \pi$ and $c^\circ_{\mathrm{HZ}}(A, A) = \infty$.

Corollary 8.2. *For any subset A of a weakly exact compact split-convex symplectic manifold (M, ω),*

$$c^\circ_{\mathrm{HZ}}(A, M) \;=\; \sup \{\gamma_M(\varphi_H) \mid H \in \mathcal{F}^\circ_{\mathrm{HZ}}(A, M)\}\,.$$

Proof. Fix $H \in \mathcal{F}^\circ_{\mathrm{HZ}}(A, M)$. Then both H and $H^- = -H$ meet the assumptions of Theorem 5.3, and so $c(H) = 0$ and $c(H^-) = \|H\|$. Therefore, $\gamma_M(H) = c(H) + c(H^-) = \|H\|$. □

8.2. **An energy-capacity inequality for c°_{HZ}.** Following [70] we define for any subset A of a weakly exact compact split-convex symplectic manifold (M, ω) the relative capacity $c_\gamma(A, M) = c_\gamma(A, M, \omega) \in [0, \infty]$ as

$$c_\gamma(A, M) \;=\; \sup \{\gamma_M(\varphi) \mid \varphi \in \mathrm{Ham}_c(\mathrm{Int}\, A, \omega)\}\,.$$

Notice that $c_\gamma(A, M)$ is the diameter of $\mathrm{Ham}_c\,(\mathrm{Int}\, A, \omega\}$. We recall that for any symplectic manifold (M, ω) the displacement energy $e(A, M) = e(A, M, \omega)$ of a subset A of M is defined as

$$e(A, M) \;=\; \inf \{d_H(\varphi, \mathrm{id}) \mid \varphi \in \mathrm{Ham}_c(M, \omega),\, \varphi(A) \cap A = \emptyset\}\,.$$

Corollary 8.2, Proposition 7.4 and (S5) of Theorem 7.3 yield

Corollary 8.3. *For any subset A of a weakly exact compact split-convex symplectic manifold (M, ω),*

$$c_{\mathrm{HZ}}(A, M) \;\le\; c^\circ_{\mathrm{HZ}}(A, M) \;\le\; c_\gamma(A, M) \;\le\; 2\, e(A, M).$$

Remark 8.4. It was noticed in [30] that working directly with the action selector c instead of the metric γ, one finds that the factor 2 in the inequality $c^\circ_{\mathrm{HZ}}(A, M) \le 2\, e(A, M)$ can be omitted, see also [23]. ◇

This concludes the construction of our tools. In the next five sections we shall use them to study Hamiltonian diffeomorphisms on weakly exact symplectic manifolds which away from a compact subset look like a product of convex symplectic manifolds. To be precise, we recall from Definition 3.1 that a compact symplectic manifold with corners (M, ω) is split-convex if there exist compact convex symplectic manifolds (M_j, ω_j), $j = 1, \dots, k$, and compact subsets $K \subset M \setminus \partial M$ and $K' \subset M_1 \times \cdots \times M_k$ such that

$$(M \setminus K, \omega) \;=\; ((M_1 \times \cdots \times M_k) \setminus K', \omega_1 \oplus \cdots \oplus \omega_k)\,.$$

We say that a non-compact symplectic manifold (M, ω) is *split-convex* if there exists an increasing sequence of compact split-convex submanifolds $M_i \subset M$ exhausting M, that is,

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots \subset M \quad \text{and} \quad \bigcup_i M_i = M.$$

Extension 8.5. The construction of our tools for weakly exact split-convex symplectic manifolds is readily extended to weakly exact symplectic manifolds which away from a compact subset look like a product $(M \times P, \omega_M \oplus \omega_P)$, where (M, ω_M) is split-convex and P is closed. The subsequent applications stated for weakly exact split-convex symplectic manifolds will thus hold for the wider class of weakly exact symplectic manifolds of this form. We shall make use of this extension only in the proof of Theorem 12.5 (ii).

9. Existence of a closed orbit with non-zero action

The following result is a generalization of Theorem 1.

Theorem 9.1. *Assume that (M, ω) is a weakly exact split-convex symplectic manifold. Then for every Hamiltonian function $H \in \mathcal{H}_c(M)$ generating $\varphi_H \in \mathrm{Ham}_c(M, \omega) \setminus \{\mathrm{id}\}$ there exists $x \in \mathcal{P}_H$ such that $\mathcal{A}_H(x) \neq 0$.*

Proof. Assume that $\bigcup_{i \geq 1} M_i$ is an exhaustion of M by compact split-convex submanifolds. Given $H \in \mathcal{H}_c(M)$ generating $\varphi_H \neq \mathrm{id}$ we choose i so large that $\mathrm{supp}\, \varphi_H \subset M_i$. Since $\varphi_H \in \mathrm{Ham}_c(M_i) \setminus \{\mathrm{id}\}$, Corollary 7.5 guarantees the existence of $x \in \mathcal{P}_H$ with $\mathcal{A}_H(x) \neq 0$, and so Theorem 9.1 follows. □

10. Infinitely many periodic points of Hamiltonian diffeomorphisms

We first consider a weakly exact compact split-convex symplectic manifold (M, ω), and we let γ be the spectral norm on $\mathrm{Ham}_c(M, \omega)$ constructed in Section 7.

Theorem 10.1. *Assume that $\varphi_H \in \mathrm{Ham}_c(M, \omega) \setminus \{\mathrm{id}\}$ is such that*

$$\gamma(\varphi_H^n) \leq C \quad \textit{for all } n \in \mathbb{N} \textit{ and some } C < \infty.$$

Then φ_H has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits.

Proof. We closely follow [70].

Case 1. $\varphi_H^n = \mathrm{id}$ *for some* $n \in \mathbb{N}$. Then every $x \in M$ is a periodic point of φ_H, and since the support of φ_H is not all of M and since M is connected, every $x \in M$ is a periodic point of φ_H corresponding to a contractible periodic orbit. Since $\varphi_H \neq \mathrm{id}$, infinitely many among these periodic points are non-trivial.

Case 2. $\varphi_H^n \neq \mathrm{id}$ *for all* $n \in \mathbb{N}$. According to Corollary 7.5, φ_H has at least 1 nontrivial periodic point corresponding to a contractible periodic orbit. Arguing by contradiction, we assume that φ_H has only finitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits, say $x_1, \dots, x_N$. The period of x_i is defined as the minimal $k_i \in \mathbb{N}$ such that $\varphi_H^{k_i}(x_i) = x_i$. Set $k = k_1 k_2 \cdots k_N$ and $G(t,x) = kH(kt,x)$. Then $\varphi_G = \varphi_H^k$, and $x_1, \dots, x_N$ are the nontrivial periodic points of φ_G corresponding to contractible periodic orbits. Their period is 1. By assumption,

$$\gamma\left(\varphi_G^n\right) = \gamma\left(\varphi_H^{nk}\right) \leq C \quad \text{for all } n \in \mathbb{N}. \tag{43}$$

The spectrum Σ_G consists of 0 (coming from trivial periodic points) and $\mathcal{A}_G(x_i)$, $i = 1, \dots, N$. Set $G^{(n)}(t,x) = nG(nt,x)$. Since φ_G has no other nontrivial periodic points corresponding to contractible periodic orbits than $x_1, \dots, x_N$,

$$\Sigma_{G^{(n)}} = n\Sigma_G = \{0, n\mathcal{A}_G(x_1), \dots, n\mathcal{A}_G(x_N)\}. \tag{44}$$

By assumption, $\varphi_G^n = \varphi_H^{nk} \neq \mathrm{id}$ for all n, and so

$$\gamma(\varphi_G^n) = \gamma\left(G^{(n)}\right) = c\left(G^{(n)}\right) + c\left(\left(G^{(n)}\right)^-\right) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Recall now that $c\left(G^{(n)}\right) + c\left(\left(G^{(n)}\right)^-\right)$ is the difference of two action values in $\Sigma_{G^{(n)}}$. We thus infer from (44) that $\gamma(\varphi_G^n) \to \infty$ as $n \to \infty$, contradicting (43). □

Theorem 2 is a special case of

Corollary 10.2. *Assume that (M,ω) is a weakly exact split-convex symplectic manifold. If the support of $\varphi_H \in \mathrm{Ham}_c(M,\omega) \setminus \{\mathrm{id}\}$ is displaceable, then φ_H has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits.*

Proof. Choose $\psi \in \mathrm{Ham}_c(M,\omega)$ which displaces $\mathrm{supp}\,\varphi_H$, and choose i so large that $\mathrm{supp}\,\psi \subset M_i$. According to Proposition 7.4, $\gamma_{M_i}(\varphi_H^n) \leq 2\,\gamma_{M_i}(\psi)$ for all $n \in \mathbb{N}$, and so the corollary follows from Theorem 10.1. □

Proof of Corollary 2: Consider a subcritical Stein manifold (V,J,f) and $\varphi_H \in \mathrm{Ham}_c(V,\omega_f) \setminus \{\mathrm{id}\}$. Since f is proper, we find a regular value R such that $S = \mathrm{supp}\,\varphi_H$ is contained in $V_R = \{x \in V \mid f(x) \leq R\}$. After composing f with an appropriate smooth function $h \colon \mathbb{R} \to \mathbb{R}$ such that $h(r) = r$ for $r \leq R$ we obtain a subcritical Stein manifold $(V,J,h \circ f)$ such that the gradient vector field $\nabla(h \circ f)$ of $h \circ f$ with respect to the Riemannian metric $g_{h\circ f}$ is complete, see [1, Lemma 3.1]. Since $S \subset V_R$ and $\omega_f|_{V_R} = \omega_{h\circ f}|_{V_R}$, we have $\varphi_H \in \mathrm{Ham}_c(V,\omega_{h\circ f}) \setminus \{\mathrm{id}\}$. Let $\mathrm{Crit}_R(h \circ f)$ be the set of critical points of $h \circ f$ in V_R, and consider the union

$$\Delta_R = \bigcup_{x \in \mathrm{Crit}_R(h\circ f)} W_x^s(\nabla(h \circ f))$$

of those stable manifolds of $\nabla(h \circ f)$ which are contained in V_R. Applying the proof of Lemma 3.2 in [1] to S and Δ_R we find a compactly supported Hamiltonian isotopy of $(V,\omega_{h\circ f})$ displacing S from itself. Theorem 2 now shows that φ_H has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic orbits. □

11. The Weinstein conjecture

Consider a weakly exact split-convex symplectic manifold (M,ω). A *hypersurface* S in M is a C^2-smooth compact connected orientable codimension 1 submanifold of M without boundary. We recall that a closed characteristic on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x,\xi) \in TS \mid \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_xS\}.$$

We denote by $\mathcal{P}^\circ(S)$ the set of closed characteristics on S which are contractible in M. The *reduced action* of $x \in \mathcal{P}^\circ(S)$ is defined as

$$\mathcal{A}(x) = \left| \int_{D^2} \overline{x}^* \omega \right|$$

where $\overline{x}\colon D^2 \to M$ is a smooth disc in M bounding x. The *action spectrum* of S is the subset $\sigma(S) = \{\mathcal{A}(x) \mid x \in \mathcal{P}^\circ(S)\}$ of $\mathbb{R}$. If $\sigma(S)$ is non-empty, we define $\lambda_1(S) \in [0,\infty[$ as

$$\lambda_1(S) = \inf\{\lambda \in \sigma(S)\}.$$

Examples show that $\sigma(S)$ can be empty, see [29, 31]. We therefore follow [39] and consider parametrized neighbourhoods of S. Since S is orientable, there exists (after adding a collar $\partial M_j \times]0,\epsilon]$ to each M_j, $j = 1,\dots,k$, in case S touches ∂M) an open neighbourhood I of 0 and a C^2-smooth diffeomorphism

$$\psi\colon S \times I \to U \subset M$$

such that $\psi(x,0) = x$ for $x \in S$. We call ψ a *thickening of* S, and we abbreviate $S_\epsilon = \psi(S \times \{\epsilon\})$ and shall often write (S_ϵ) instead of $\psi\colon S \times I \to U$.

Theorem 11.1. *Assume that S is a displaceable hypersurface of a weakly exact split-convex symplectic manifold (M,ω), and let (S_ϵ) be a thickening of S. For every $\delta > 0$ there exists $\epsilon \in [-\delta,\delta]$ such that*

$$\mathcal{P}^\circ(S_\epsilon) \neq \emptyset \quad \textit{and} \quad \lambda_1(S_\epsilon) \leq 2\,e(S,M) + \delta.$$

Proof. Fix $\delta > 0$. We choose $K \in \mathcal{H}_c(M)$ such that φ_K displaces S and $\|K\| < e(S,M) + \delta/2$. Let $\rho \in\,]0,\delta]$ be so small that φ_K displaces the whole neighbourhood $\mathcal{N}_\rho := \psi(S \times [-\rho,\rho])$ of S. We abbreviate $E = 2e(S,M) + \delta$ and choose a C^∞-function $f\colon \mathbb{R} \to [0,E]$ such that

$$f(t) = 0 \text{ if } t \notin [-\rho,\rho], \quad f(0) = E, \quad f'(t) \neq 0 \text{ if } t \in\,]-\rho,\rho[\setminus\{0\},$$

see Figure 4.

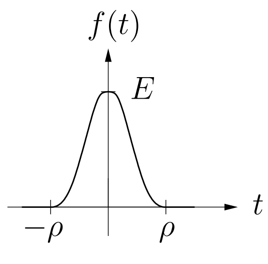

FIGURE 4. The function f.

We define the time-independent Hamiltonian $H \in \mathcal{H}_c(M)$ by

$$H(x) \;=\; \begin{cases} f(t) & \text{if } x \in S_t, \\ 0 & \text{otherwise.} \end{cases}$$

If $\bigcup_{i\geq 1} M_i$ is an exhaustion of M, we choose i so large that $\operatorname{supp}\varphi_K \subset M_i$. Since $\varphi_H \neq \mathrm{id}$ and since φ_H is supported in $\mathcal{N}_\rho \subset \operatorname{supp}\varphi_K \subset M_i$, we read off from (S1) of Theorem 7.3 and from Corollary 8.3 that

$$0 \;<\; \gamma_{M_i}(H) \;\leq\; 2\,\|K\| \;<\; E. \tag{45}$$

Let $x^+ \in \mathcal{P}_H$ and $x^- \in \mathcal{P}_{H^-}$ be closed orbits for which

$$c(H) = \mathcal{A}_H\left(x^+\right) \qquad \text{and} \qquad c\left(H^-\right) = \mathcal{A}_{H^-}\left(x^-\right).$$

Proposition 5.2 applied to H and $H^- = -H$ yields

$$c(H) \;=\; \mathcal{A}_H\left(x^+\right) \;=\; -\int_{D^2} \left(\overline{x^+}\right)^* \omega - \int_0^1 H\left(x^+(t)\right) dt \;\leq\; 0, \tag{46}$$

$$c\left(H^-\right) \;=\; \mathcal{A}_{H^-}\left(x^-\right) \;=\; -\int_{D^2} \left(\overline{x^-}\right)^* \omega + \int_0^1 H\left(x^-(t)\right) dt \;\leq\; E. \tag{47}$$

Notice that not both x^+ and x^- are constant orbits. Indeed, if they were, our choice of H would yield $c(H) \in \{0, -E\}$ and $c\left(H^-\right) \in \{0, E\}$, and so $\gamma(H) = c(H) + c\left(H^-\right) \in \{-E, 0, E\}$, contradicting (45).

Case 1. The orbit x^+ is not constant. By construction of H there exists $\epsilon \in [-\rho, \rho] \subset [-\delta, \delta]$ such that $x^+ \in \mathcal{P}^\circ\left(S_\epsilon\right)$. The choice of H and (46) yield $-\int_{D^2} \left(\overline{x^+}\right)^* \omega \leq E$. Assume that $-\int_{D^2} \left(\overline{x^+}\right)^* \omega < -E$. Then (46) yields $c(H) < -E$, and so, together with (47), $\gamma(H) = c(H) + c\left(H^-\right) < 0$, contradicting (45). We conclude that $\mathcal{A}\left(x^+\right) = \left|\int_{D^2} \left(\overline{x^+}\right)^* \omega\right| \leq E$.

Case 2. The orbit x^- is not constant. Again we find $\epsilon \in [-\delta, \delta]$ such that $x^- \in \mathcal{P}^\circ\left(S_\epsilon\right)$, and arguing similarly as in Case 1 we find that $\mathcal{A}\left(x^-\right) \leq E$. The proof of Theorem 11.1 is complete. □

A hypersurface S is *stable* if there exists a thickening (S_ϵ) of S such that the local flow ψ_t around S induced by $\psi\colon S \times I \to U$ induces bundle isomorphisms

$$T\psi_\epsilon\colon \mathcal{L}_S \;\to\; \mathcal{L}_{S_\epsilon}$$

for every $\epsilon \in I$. It then follows that $\psi_{-\epsilon}(x) \in \mathcal{P}^\circ(S)$ for every $x \in \mathcal{P}^\circ\left(S_\epsilon\right)$. Since $\psi_\epsilon \to \mathrm{id}$ in the C^1-topology as $\epsilon \to 0$, we conclude from Theorem 11.1 the

Corollary 11.2. *Assume that S is a displaceable stable hypersurface of a weakly exact split-convex symplectic manifold (M, ω). Then $\mathcal{P}^\circ(S) \neq \emptyset$ and $\lambda_1(S) \leq 2\,e(S)$.*

It is well known that every hypersurface of contact type is stable, see [41], and so Theorem 3 follows from Corollary 11.2. Corollary 3 follows from Theorem 3 by using

Cieliebak's result in [3] or by arguing as in the proof of Corollary 2 given in the previous section.

Remark 11.3. Theorem 11.1 implies that for any displaceable thickening (S_ϵ) in a weakly exact split-convex symplectic manifold it holds that $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for a dense set of $\epsilon \in I$. This result and hence the first statement of Corollary 11.2 and Theorem 3 are proved in [67] for all geometrically bounded symplectic manifolds. The proof there uses results from Hofer geometry. For a list of references with further results on the Weinstein conjecture we also refer to [67].

Example 11.4. We consider a stable hypersurface S in $(\mathbb{R}^{2n}, \omega_0)$. If S has diameter $\operatorname{diam}(S)$, then S is contained in a ball of radius $\operatorname{diam}(S)$. Since $e\left(B^{2n}(r)\right) = \pi r^2$, we find $e(S) \leq \pi \operatorname{diam}(S)^2$, and so

$$\lambda_1(S) \;\leq\; 2\pi \operatorname{diam}(S)^2,$$

improving the estimate in [39].

Remarks 11.5. **1.** Working directly with c instead of γ, one finds that the factor 2 in the estimate in Theorem 11.1 and hence in the estimates in Corollary 11.2 and Example 11.4 can be omitted, see [23]. The estimate $\lambda_1(S) \leq e(S)$ was known before for hypersurfaces S of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$, see [36]. If S bounds a convex domain $U \subset \mathbb{R}^{2n}$, then $\lambda_1(S) = c_{\mathrm{HZ}}(U) \leq e(U) = e(S)$ where c_{HZ} is the Hofer–Zehnder capacity, [40].
2. Assume that $S \subset (M, \omega)$ is a hypersurface of contact type and that one of the following conditions is met.

- S is simply connected.
- $\omega = d\lambda$ is exact and $H^1(S; \mathbb{R}) = 0$.

Then $0 \notin \sigma(S)$ and $\sigma(S)$ is closed, cf. [39]. Therefore, $\lambda_1(S) > 0$. ◇

Recall that Theorem 11.1 implies that for any displaceable thickening $\psi\colon S \times I \to U$ it holds that $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for a *dense* set of $\epsilon \in I$. Following Hofer and Zehnder, [41, Section 4.2], we shall use the finiteness of $c^\circ_{\mathrm{HZ}}(U)$ to improve this result considerably. Let μ denote the Lebesgue measure on $\mathbb{R}$.

Theorem 11.6. *Assume that (S_ϵ) with $\epsilon \in I$ is a displaceable thickening of a C^2-hypersurface S in a weakly exact split-convex symplectic manifold (M, ω). Then*

$$\mu\left\{\epsilon \in I \mid \mathcal{P}^\circ(S_\epsilon) \neq \emptyset\right\} \;=\; \mu(I).$$

Proof. We can assume that M is compact. We can also assume that $I =]-1,1[$, and for each $\epsilon \in]0,1[$ we set $U_\epsilon = \psi\left(S \times]-\epsilon, \epsilon[\right)$. According to Corollary 8.3 and since U_1 is displaceable,

$$c^\circ_{\mathrm{HZ}}(U_1, M) \;\leq\; 2\,e(U_1, M) \;<\; \infty,$$

and in view of the relative monotonicity property (42) of c°_{HZ} the function $\epsilon \mapsto c^\circ_{\mathrm{HZ}}(U_\epsilon, M)$ is monotone increasing. Theorem 11.6 now follows from repeating the proof of Theorem 1 in [49] with C^2-smooth instead of C^∞-smooth Hamiltonians and with c_{HZ} replaced by c°_{HZ}. □

12. Closed trajectories of a charge in a magnetic field

In this section we apply Theorem 11.6 to the existence problem of closed trajectories in a magnetic field. We first prove Theorem 4.A and then discuss the number $d > 0$ appearing in Theorem 4.A. In Section 12.3 we prove a generalization of Theorem 4.B.

12.1. **Proof of Theorem 4.A.** Let (N, g) and (T^*N, ω_σ) be as in Theorem 4.A of the introduction. Since $\sigma = d\alpha$ is exact, the symplectic form $\omega_\sigma = -d(\lambda + \pi^*\alpha)$ is exact. Since σ does not vanish, N is at least 2-dimensional. For each $c > 0$ the energy level $E_c = \{H = c\}$ is therefore a connected C^2-hypersurface, which bounds the sublevel set

$$H^c \;=\; \left\{(q,p) \in T^*N \mid H(q,p) = \tfrac{1}{2}|p|^2 \le c\right\}.$$

We define the norm of σ as

$$\|\sigma\| \;=\; \inf\left\{\|\alpha\| \mid d\alpha = \sigma\right\}$$

where $\|\alpha\| = \max_{x\in N} |\alpha(x)|$. In order to apply Theorem 11.6 we need

Lemma 12.1. *The symplectic manifold (T^*N, ω_σ) is convex. Indeed, H^c is convex whenever $c > \frac{1}{2}\|\sigma\|^2$.*

Proof. We choose a 1-form α on N such that $d\alpha = \sigma$. Under the symplectomorphism

$$\Phi\colon\ (T^*N, \omega_\sigma) \to (T^*N, \omega_0)\,, \qquad (q,p) \mapsto (q, p + \alpha(q))$$

the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2$ on (T^*N, ω_σ) corresponds to the Hamiltonian $H_\alpha(q,p) = \frac{1}{2}|p-\alpha|^2$ on (T^*N, ω_0). If $c > \frac{1}{2}\|\alpha\|^2$, then the sublevel set $H^c_\alpha = \{(q,p) \mid H_\alpha(q,p) \le c\}$ contains N, and so the Liouville vector field $\sum_i p_i \frac{\partial}{\partial p_i}$ for ω_0 intersects the boundary of H^c_α transversally. Therefore, H^c_α is convex. It follows that $H^c = \Phi^{-1}(H^c_\alpha)$ is convex whenever $c > \frac{1}{2}\|\alpha\|^2$. Since this is true for any α with $d\alpha = \sigma$, the lemma follows. □

Let $\chi(N)$ be the Euler characteristic of N.

Case 1. $\chi(N) = 0$. We set

$$d \;=\; d(g,\sigma) \;=\; \sup\left\{c \ge 0 \mid H^c \text{ is displaceable in } (T^*N, \omega_\sigma)\right\}. \tag{48}$$

Notice that

$$d \;=\; \sup\left\{c \ge 0 \mid E_c \text{ is displaceable in } (T^*N, \omega_\sigma)\right\}.$$

Since $\sigma \ne 0$, the zero section N of T^*N is not Lagrangian, and so a remarkable theorem of Polterovich, [61, 44], implies that $d > 0$. We shall see below that $d < \infty$. Theorem 4.A follows from applying Theorem 11.6 to $S = E_{d/2}$ and a thickening

$$\psi\colon\ S \times \left]-d/2, d/2\right[\;\to \bigcup_{0<c<d} E_c$$

such that $\psi(S \times \{\epsilon\}) = E_{\epsilon + d/2}$.

Case 2. $\chi(N) \ne 0$. In this case the zero section N is not displaceable for topological reasons. We use a stabilization trick used before by Macarini in [48]. Let S^1 be the unit circle, and denote canonical coordinates on T^*S^1 by (x,y). We consider the manifold

$T^*(N \times S^1) = T^*N \times T^*S^1$ endowed with the split symplectic form $\omega = \omega_\sigma \oplus \omega_{S^1}$, where $\omega_{S^1} = dx \wedge dy$. In view of Lemma 12.1, $(T^*N \times T^*S^1, \omega)$ is a weakly exact convex symplectic manifold. Moreover, $N \times S^1$ is not Lagrangian, and $\chi(N \times S^1) = 0$. Let

$$H_1(q,p) = \tfrac{1}{2}|p|^2, \quad H_2(x,y) = \tfrac{1}{2}|y|^2, \quad H(q,p,x,y) = \tfrac{1}{2}|p|^2 + \tfrac{1}{2}|y|^2$$

be the metric Hamiltonians on T^*N, T^*S^1 and $T^*N \times T^*S^1$. In order to avoid confusion, we denote their energy levels by $E_c(H_1)$, $E_c(H_2)$ and $E_c(H)$. Repeating the argument given in Case 1 for the Hamiltonian system

$$H\colon \big(T^*N \times T^*S^1, \omega\big) \to \mathbb{R} \tag{49}$$

and

$$d = d(g,\sigma) = \sup\big\{c \geq 0 \mid H^c \text{ is displaceable in } \big(T^*N \times T^*S^1, \omega\big)\big\} \tag{50}$$

we find that

$$\mu\,\{c \in \,]0,d[\, \mid \mathcal{P}^\circ\,(E_c(H)) \neq \emptyset\} = d.$$

Fix $c \in \,]0,d[$ such that $\mathcal{P}^\circ\,(E_c(H)) \neq \emptyset$. Since the Hamiltonian system (49) splits, a contractible closed orbit $x(t)$ on $E_c(H)$ is of the form $(x_1(t), x_2(t))$, where x_1 is a contractible closed orbit on $E_{c_1}(H_1)$ and x_2 is a contractible closed orbit on $E_{c_2}(H_2)$ and $c_1 + c_2 = c$. Since the only contractible orbits of $H_2\colon T^*S^1 \to \mathbb{R}$ are the constant orbits on $E_0(H_2)$, we conclude that $c_2 = 0$ and $c_1 = c$, and so $x_1 \in \mathcal{P}^\circ\,(E_c(H_1))$. It follows that

$$\mu\,\{c \in \,]0,d[\, \mid \mathcal{P}^\circ\,(E_c(H_1)) \neq \emptyset\} = d.$$

The proof of Theorem 4.A is complete. □

Remark 12.2. The above proof makes crucial use of the fact that the closed characteristics guaranteed by Theorem 11.6 and found via Floer homology are contractible. Without this information, we could prove Theorem 4.A only for manifolds with $\chi(N) = 0$. ◇

12.2. **Comparison of $d(g,\sigma)$ and $\frac{1}{2}\|\sigma\|^2$.** It would be important to know a computable lower bound of $d(g,\sigma)$. An upper bound can be described in a variety of ways.

Proposition 12.3. *We have $d(g,\sigma) \leq \frac{1}{2}\|\sigma\|^2$.*

Proof. We assume first that $\chi(N) = 0$. Arguing by contradiction, we assume that $d = d(g,\sigma) > \frac{1}{2}\|\sigma\|^2$. We then find a 1-form α on N such that $d\alpha = \sigma$ and $d > \frac{1}{2}\|\alpha\|^2$. By definition of d, the graph $\Gamma_{-\alpha}$ of $-\alpha$, which is contained in $H^{\frac{1}{2}\|\alpha\|^2}$, is then a displaceable subset of (T^*N, ω_σ), and so the zero section $\Phi(\Gamma_{-\alpha})$ of T^*N is a displaceable subset of (T^*N, ω_0). This contradicts a Lagrangian intersection result of Gromov, [35].

Assume now that $\chi(N) \neq 0$. We denote by g_{S^1} the Riemannian metric of the unit circle. By definition of $d(g,\sigma)$ and by the already proved case,

$$d(g,\sigma) = d\,(g \oplus g_{S^1}, \sigma \oplus 0) \leq \tfrac{1}{2}\|\sigma \oplus 0\|^2 \leq \tfrac{1}{2}\|\sigma\|^2.$$

The proof of Proposition 12.3 is complete. □

An important number associated with the Hamiltonian system (3) is *Mañé's strict critical value* $c_0(g,\sigma)$ for whose definition and relevance we refer to [58, 8, 59]. Let α be such that $d\alpha = \sigma$. According to Corollary 1 in [8], $c_0(g,\sigma)$ is given by

$$c_0(g,\sigma) \;=\; \inf \max_{x \in N} \tfrac{1}{2} \left|\beta(x) - \alpha(x)\right|^2 \tag{51}$$

where the infimum is taken over all closed 1-forms β on N. It follows that

$$c_0(g,\sigma) \;=\; \tfrac{1}{2} \left\|\sigma\right\|^2 . \tag{52}$$

We denote by $\Lambda_{-\alpha}$ the set of Lagrangian submanifolds in (T^*N, ω_σ) which are Lagrangian isotopic to the graph $\Gamma_{-\alpha}$ of $-\alpha$. Combining (51) with a result in [59], we find

$$c_0(g,\sigma) \;=\; \inf \left\{ c \in \mathbb{R} \mid H^c \text{ contains a Lagrangian submanifold in } \Lambda_{-\alpha} \right\}.$$

This is a purely symplectic characterization of $c_0(g,\sigma) = \frac{1}{2}\left\|\sigma\right\|^2$.

We recall from Theorem 4.A that $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c \in \;]0, d(g,\sigma)]$. It follows from Lemma 12.1 and a theorem of Hofer and Viterbo, [38], that E_c carries a closed orbit whenever $c > \frac{1}{2}\|\sigma\|^2$. More precisely, for every non-trivial homotopy class $h \in \pi_1(N)$ and every $c > c_0(g,\sigma) = \frac{1}{2}\left\|\sigma\right\|^2$ there exists a closed orbit on E_c whose projection to N lies in h, see [9, Theorem 27]. The following example shows that $\mathcal{P}^\circ(E_c)$ can be empty for all $c \geq \frac{1}{2}\left\|\sigma\right\|^2$. It also shows that there can be a gap between $d(g,\sigma)$ and $\frac{1}{2}\left\|\sigma\right\|^2$.

Example 12.4. Let N be a closed orientable surface of genus 2. It has been shown in [58] that there exists a Riemannian metric g and an exact 2-form σ on N such that

(i) $c_0(g,\sigma) > \frac{1}{2}$;
(ii) the restriction of the flow of (3) to E_c is Anosov for all $c \geq \frac{1}{2}$.

Property (ii) implies that $\mathcal{P}^\circ(E_c) = \emptyset$ for all $c \geq \frac{1}{2}$, and so, by Theorem 4.A, Property (i) and (52),

$$d(g,\sigma) \;\leq\; \tfrac{1}{2} \;<\; c_0(g,\sigma) \;=\; \tfrac{1}{2}\left\|\sigma\right\|^2 .$$

12.3. **Proof of Theorem 4.B.** Our most general result about the existence of closed orbits of magnetic flows is

Theorem 12.5. *Assume that* $N = N_1 \times N_2 \times N_3$ *is a closed manifold, where* N_1 *is any closed manifold,* $N_2 = T^m$ *is a torus (or a point), and* $N_3 = \times_i \Sigma_i$ *is a product of closed orientable surfaces of genus at least* 2 *(or a point), and assume that* N *is endowed with a* C^2*-smooth Riemannian metric* g *and a non-vanishing closed* 2*-form* σ *such that*

$$[\sigma] \;=\; 0 \oplus [\sigma_2] \oplus [\sigma_3] \;\in\; H^2(N_1) \oplus H^2(N_2) \oplus H^2(N_3) \;\subset\; H^2(N)$$

and such that σ_3 *is cohomologically split in the sense that*

$$[\sigma_3] \;\in\; \oplus_i \mathbb{R}\,[\Sigma_i] \;=\; \oplus_i H^2(\Sigma_i) \;\subset\; H^2(\times_i \Sigma_i).$$

(i) *If* $[\sigma_2] = 0$*, there exists* $d > 0$ *such that* $\mathcal{P}^\circ(E_c) \neq \emptyset$ *for almost all* $c \in \;]0, d[$.
(ii) *If* $[\sigma_2] \neq 0$*, then* $\mathcal{P}^\circ(E_c) \neq \emptyset$ *for almost all* $c > 0$.

For N_2 and N_3 a point, Theorem 12.5 is Theorem 4.A, and for N_1 a point and N_2 or N_3 a point, Theorem 12.5 is a generalization of Theorem 4.B.

Proof of Theorem 12.5: The proof is divided into four steps.

Step 1. *$(T^*\Sigma, \omega_\sigma)$ is convex if Σ is not a torus.* We consider a closed orientable surface Σ different from the torus, and we endow Σ with a Riemannian metric g of constant curvature k. We fix an orientation of Σ, denote the area form on Σ by τ, and consider the 2-form $\sigma = s\tau$ for some $s \in \mathbb{R}$. Recall that $\omega_\sigma = \omega_0 - \pi^*\sigma$. The following lemma and the subsequent remark were explained to us by Viktor Ginzburg.

Lemma 12.6. *The symplectic manifold $(T^*\Sigma, \omega_\sigma)$ is convex. Indeed, if $\Sigma = S^2$, then H^c is convex for all $c > 0$, and if* $\operatorname{genus}(\Sigma) \geq 2$*, then H^c is convex for all $c > -\frac{s^2}{2k}$.*

Proof. We fix $c > 0$ and consider E_c as an oriented S^1-bundle

$$S^1 \longrightarrow E_c \xrightarrow{\pi_c} N. \tag{53}$$

Let X_c be the geodesic spray on E_c, let Y_c be the vector field on E_c generating the S^1-action, and let α_c be the connection 1-form of the bundle (53). Then

$$\alpha_c(X_c) = 0, \quad \alpha_c(Y_c) = 1, \quad d\alpha_c = -\pi_c^*(k\tau). \tag{54}$$

Varying over $c > 0$ we obtain vector fields X, Y and a 1-form α on $T^*N \setminus N$ such that $\alpha|_{E_c} = \alpha_c$ and $d\alpha = -\pi^*(k\tau)$. Since N is not the torus, $k \neq 0$, and so we can set $\beta = -\frac{s}{k}\alpha$. Then

$$d\beta \;=\; -\tfrac{s}{k} d\alpha \;=\; \pi^*(s\tau) \;=\; \pi^*\sigma \quad \text{on } T^*N \setminus N.$$

Therefore,

$$d(-\lambda - \beta) \;=\; \omega_\sigma. \tag{55}$$

The vector field $X_H = X - sY$ on $T^*N \setminus N$ is the Hamiltonian vector field of $H(q,p) = \frac{1}{2}|p|^2$ with respect to ω_σ. In particular, $X_H|_{E_c}$ is a section of the distinguished line bundle $\mathcal{L}_{E_c}$ for every $c > 0$. Notice that $\lambda(X)|_{E_c} = 2c$ and $\lambda(Y) = 0$. Moreover, $\beta = -\frac{s}{k}\alpha$ and (54) yield $\beta(X) = 0$ and $\beta(Y) = -\frac{s}{k}$. Therefore,

$$(-\lambda - \beta)(X_H) \;=\; -2c - \tfrac{s^2}{k}. \tag{56}$$

Equation (55) and (56) show that if $N = S^2$, then E_c is of contact type for every $c > 0$, and if $\operatorname{genus}(N) \geq 2$, then E_c is of contact type if $c \neq \frac{s^2}{2k}$. If $s = 0$, all these hypersurfaces are convex boundaries of H^c, and so the claim follows. □

Remark 12.7. For homological reasons, (T^*N, ω_σ) is not convex if σ is not exact and N is a 2-torus or $\dim N \geq 3$.

Step 2. *Transition to a split form.* Let now N, g and σ be as in Theorem 12.5. We denote the area form τ considered in Lemma 12.6 by τ_Σ. By assumption on the form σ_3 there are real numbers s_i such that $[\sigma_3] = \oplus_i s_i [\tau_{\Sigma_i}] \in H^2(\times_i \Sigma_i)$. Define the closed 2-form σ_0 on $N = N_1 \times N_2 \times N_3$ as

$$\sigma_0 \;=\; 0 \oplus \tau_T \oplus_i s_i \tau_{\Sigma_i}$$

where τ_T is the unique translation-invariant 2-form on T^m cohomologous to σ_2. By assumption on σ there exists a 1-form α on N such that $\sigma = \sigma_0 + d\alpha$. Under the symplectomorphism

$$\Phi\colon\, (T^*N, \omega_\sigma) \to (T^*N, \omega_{\sigma_0}), \quad (q,p) \mapsto (q, p + \alpha(q)) \tag{57}$$

the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2$ on (T^*N, ω_σ) corresponds to the Hamiltonian $H_\alpha(q,p) = \frac{1}{2}|p-\alpha|^2$ on $(T^*N, \omega_{\sigma_0})$. The sublevel set $H_\alpha^c = \Phi(H^c) = \{(q,p) \mid H_\alpha(q,p) \le c\}$ is displaceable in $(T^*N, \omega_{\sigma_0})$ if and only if H^c is displaceable in (T^*N, ω_σ), and Φ maps $\mathcal{P}^\circ(E_c)$ bijectively to the set $\mathcal{P}^\circ(E_c^\alpha)$ of closed characteristics on the boundary $E_c^\alpha = \Phi(E_c)$ of H_α^c which are contractible in T^*N.

Step 3. *Proof of (i).* Since $[\sigma_2] = 0$ we have $\tau_T = 0$. In view of Lemmata 12.1 and 12.6 the symplectic manifold

$$(T^*N, \omega_{\sigma_0}) \;=\; (T^*(N_1 \times N_2), \omega_0) \times_i \left(T^*\Sigma_i, \omega_{s_i \tau_{\Sigma_i}}\right)$$

is a product of weakly exact convex symplectic manifolds. We define $d = d(g,\sigma)$ as in (48) or (50). We can assume without loss of generality that d is finite. Proceeding as in the proof of Theorem 4.A with (T^*N, ω_σ), H, H^c and E_c replaced by $(T^*N, \omega_{\sigma_0})$, H_α, H_α^c and E_c^α we find that $\mathcal{P}^\circ(E_c^\alpha) \neq \emptyset$ for almost all $c \in \,]0,d[$, and so $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c \in \,]0,d[$.

Step 4. *Proof of (ii).* We fix $d > 0$. Since $[\sigma_2] \neq 0$ the manifold $(T^*T^m, \omega_{\tau_T})$ is not convex. However, according to the proof of Theorem 3.1 in [32], there exists a symplectic embedding

$$\varphi\colon\, \left(T^*_{\sqrt{2d}+\|\alpha\|} T^m, \omega_{\tau_T}\right) \hookrightarrow \left(\mathbb{R}^{2k} \times T^{2(m-k)}, \Omega_{\text{can}} \oplus \Omega_T\right) \tag{58}$$

which induces an injection on π_1. Here, $2k > 0$ and Ω_T is a translation-invariant symplectic form on $T^{2(m-k)}$. Notice that

$$(M,\omega) \;=\; (T^*N_1, \omega_0) \times \left(\mathbb{R}^{2k} \times T^{2(m-k)}, \Omega_{\text{can}} \oplus \Omega_T\right) \times_i \left(T^*\Sigma_i, \omega_{s_i\tau_{\Sigma_i}}\right)$$

is the product of a weakly exact split-convex symplectic manifold and a weakly exact closed symplectic manifold, and that every compact subset of (M,ω) is displaceable. Applying Extension 8.5 and Theorem 11.6 to the hypersurfaces

$$E_c' \;=\; (\text{id} \times \varphi \times \text{id})(\Phi(E_c)), \quad c \in \,]0,d[,$$

of (M,ω) we find that $\mathcal{P}^\circ(E_c') \neq \emptyset$ for almost all $c \in \,]0,d[$. Using that φ is injective on π_1, we conclude that $\mathcal{P}^\circ(E_c) \neq \emptyset$ for almost all $c \in \,]0,d[$. Since $d > 0$ was arbitrary, Theorem 12.5 (ii) follows. □

Remarks 12.8. 1. In view of Example 12.4, the number $d > 0$ in Theorem 4.B (i) cannot be chosen arbitrarily large in general. Here is a simpler example illustrating this fact: Let N be a closed oriented surface equipped with a metric of constant curvature -1, and let σ be the area form on N. If $c \geq \frac{1}{2}$, then $\mathcal{P}^\circ(E_c) = \emptyset$, see [28, Example 3.7].

2. In the situation of Theorem 12.5 (ii), $d(g,\sigma)$ defined by (48) is infinite. Indeed, according to the proof of Theorem 3.1 in [32], $(T^*T^m, \omega_{\tau_T})$ is symplectomorphic to the product $\mathbb{R}^{2k} \times W$ with $2k > 0$, where $\mathbb{R}^{2k}$ is equipped with its standard symplectic form and $W = \mathbb{R}^{m-2k} \times T^m$ is given a translation-invariant symplectic form. It follows that $d(g,\sigma_0)$ is infinite, and so $d(g,\sigma)$ is also infinite.

13. Lagrangian intersections

Theorem 5 is a special case of

Theorem 13.1. *Assume that (M,ω) is a weakly exact split-convex symplectic manifold, and let $L \subset M \setminus \partial M$ be a closed Lagrangian submanifold such that*

(i) *the injection $L \subset M$ induces an injection $\pi_1(L) \subset \pi_1(M)$;*

(ii) *L admits a Riemannian metric none of whose closed geodesics is contractible.*

Then L is not displaceable.

Proof. We can assume that M is compact. Arguing by contradiction we assume that $\psi \in \mathrm{Ham}_c(M,\omega)$ displaces L and hence a neighbourhood U of L in M. We choose a Riemannian metric as in (ii) and denote by $|p|$ the length of a covector $(q,p) \in T^*L$. By Weinstein's Theorem we find $\epsilon > 0$ such that a neighbourhood V of L in U can be symplectically identified with $T^*_{3\epsilon}L$. Choose a smooth function $f\colon [0,3\epsilon] \to [0,1]$ such that

$$f(r) = -1 \text{ if } r \leq \epsilon, \quad f(r) = 0 \text{ if } r \geq 2\epsilon, \quad f'(r) > 0 \text{ if } r \in]\epsilon, 2\epsilon[.$$

We choose canonical coordinates (q,p) on $T^*_{3\epsilon}L \equiv V$ and define the autonomous Hamiltonian $H\colon M \to \mathbb{R}$ by

$$H(x) = H(p) = f(|p|) \text{ if } x = (q,p) \in V, \quad H(x) = 0 \text{ otherwise.}$$

Set again $H^{(n)}(t,x) = nH(nt,x)$ so that $\varphi_{H^{(n)}} = \varphi_H^n$. By assumptions (i) and (ii) and by our choice of H, the only contractible periodic orbits of φ_H^t are fixed points, and so $\Sigma_{H^{(n)}} = \{0, n\}$. Since $\varphi_H^n \neq \mathrm{id}$, $\gamma(\varphi_H^n) > 0$, and so we conclude that

$$\gamma(\varphi_H^n) = n \to \infty \quad \text{as } n \to \infty. \tag{59}$$

On the other hand, φ_H^n is supported in V for all n, and $\psi(V) \cap V = \emptyset$. Proposition 7.4 thus yields $\gamma(\varphi_H^n) \leq 2\gamma(\psi)$, which by (59) is a contradiction. □

Remarks 13.2. 1. In [46], Lalonde and Polterovich used the general energy-capacity inequality to prove the conclusion of Theorem 13.1 for *any* symplectic manifold (M,ω) and any closed Lagrangian submanifold $L \subset M \setminus \partial M$ satisfying (i) and

(ii') L admits a Riemannian metric of non-positive curvature.

Of course, (ii') implies (ii). We show by an example that (ii) is a weaker condition than (ii'). Let H be the $(2k+1)$-dimensional Heisenberg group endowed with any left invariant Riemannian metric, and choose a discrete cocompact subgroup $\Gamma \subset H$. The Riemannian exponential map from the Lie algebra of H to H is not injective, but there are no closed geodesics, see e.g. [11]. Therefore, $\Gamma \setminus H$ satisfies condition (ii). On the other hand, $\pi_1(\Gamma \setminus H) = \Gamma$ is nilpotent, and so $\Gamma \setminus H$ cannot satisfy (ii'), see [34, 73].

2. By replacing the spectral norm ingredient in the above proof by results from Hofer geometry, Theorem 13.1 is proved in [67] for all geometrically bounded symplectic manifolds.

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