Hamiltonian Field Description of the One-Dimensional Poisson-Vlasov Equations

by

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Abstract

The one-dimensional Poisson-Vlasov equations are cast into Hamiltonian form. A Poisson Bracket in terms of the phase space density, as sole dynamical variable, is presented. This Poisson bracket is not of the usual form, but possesses the commutator properties of antisymmetry, bilinearity, and nonassociativity by virtue of the Jacobi requirement. Clebsch potentials are seen to yield a conventional (canonical) formulation. This formulation is discretized by expansion in terms of an arbitrary complete set of basis functions. In particular, a wave field representation is obtained.

The Vlasov equations, in Lagrangian variables, were shown by Low[1] to possess an underlying Hamilton's principle. Here we discuss the Hamiltonian formulation of the one-dimensional version of these equations in terms of the usually encountered Eulerian description.

In connection with the development of the inverse scattering method,[2] Gardner[3] showed that the Korteweg-deVries equation is an infinite dimensional Hamiltonian system. In so doing he obtained a Poisson bracket in terms of the noncanonical variable for this system. In a similar vein we have obtained Poisson brackets for the equations that describe a perfect fluid and ideal magnetohydrodynamics,[4] the Maxwell-Vlasov and Poisson-Vlasov equations,[5] and the equations which describe two-dimensional vortex fluids or guiding center plasmas.[6]

In the following we discuss the one-dimensional version of the noncanonical Poisson bracket developed in Ref. 5. It is shown that the introduction of Clebsch[7] potentials transforms the noncanonical formulation into a canonical formulation. This formulation is discretized by expanding in terms of an arbitrary complete orthonormal set of basis functions. The exact Hamiltonian has quadratic plus quartic terms. Following this, the particular basis of Fourier decomposition is developed.

The one-dimensional Poisson-Vlasov equations are the following:

$$\frac{\partial f_\alpha}{\partial t}(z,t) = -v\frac{\partial f_\alpha}{\partial x}(z,t) + \frac{e_\alpha}{m_\alpha}\frac{\partial\phi}{\partial x}(x,t)\frac{\partial f_\alpha}{\partial v}(z,t) \tag{1}$$

$$\frac{\partial^2\phi}{\partial x^2}(x,t) = -\sum_\alpha e_\alpha \int f_\alpha(z,t)\, dv \tag{2}$$

where f_α is the phase space probability density, ϕ is the electrostatic potential, $z \equiv (x,v)$, α designates the species, and e_α and m_α are the signed particle charge and mass respectively. The potential ϕ is defined on R and f_α on $R \times R$. [R is the interval $(-\infty,\infty)$]. Integrating Eq. (2) twice, seeking solutions with asymptotic charge neutrality and vanishing electric field, we obtain

$$\phi(x) = -\sum_\alpha e_\alpha \int K(x|x')\, f_\alpha(z')\, dz' \quad ,$$

where $K(x|x') = 1/2\ |x - x'|$. (For convenience, we display only the arguments necessary to avoid confusion.) Equations (1) and (2) can be written compactly in the following form which is suggestive of the equations for two-dimensional vortex motion:[8]

$$\frac{\partial f_\alpha}{\partial t} = -\underline{w}_\alpha \cdot \nabla_p f_\alpha \quad , \tag{3}$$

where $\nabla_p \equiv (\partial/x,\ \partial/v)$ and

$$\underline{w}_\alpha = \left(v\ ,\ \frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial x}\sum_\beta e_\beta \int K(x|x')\, f_\beta(z')\, dz'\right) \quad .$$

Observe $\nabla_p \cdot \underline{w}_\alpha = 0$.

In Refs. 4, 5, and 6 we discuss the properties of Poisson brackets. The brackets presented are not of the usual form, but do possess the Lie algebraic properties: bilinearity, antisymmetry, and nonassociativity as determined by the Jacobi requirement. In Ref. 5 the three-dimensional version of Eq. (3) is expressed in Heisenberg form; i.e.,

$$\frac{\partial f_\alpha}{\partial t} = [f_\alpha, H] \quad , \tag{4}$$

where the bracket on the right hand side is the field Poisson bracket. With slight modification we obtain the one-dimensional Poisson bracket

$$[A,B] = \sum_\alpha \int \frac{f_\alpha(z)}{m_\alpha} \left\{ \frac{\delta A}{\delta f_\alpha} , \frac{\delta B}{\delta f_\alpha} \right\} dz \quad , \tag{5}$$

where the quantities A and B, on which the bracket acts, are functionals such as the integral of the Hamiltonian density:

$$H[f_\alpha] = \sum_\alpha \frac{1}{2} m_\alpha \int v^2 f_\alpha(z)\, dz$$

$$- \frac{1}{2} \sum_{\alpha,\beta} e_\alpha e_\beta \int K(x|x')\, f_\alpha(z) f_\beta(z')\, dz\, dz' \quad . \tag{6}$$

The braces of the integrand of Eq. (5) signify the usual particle Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial x} \quad . \tag{7}$$

The functional derivative, $\delta A/\delta f_\alpha$, is defined by

$$\frac{d}{d\epsilon} A[f + \epsilon h] \Big|_{\epsilon=0} = \int \frac{\delta A}{\delta f}\, h\, dz \quad ,$$

where h is an arbitrary, sufficiently differentiable, function of z. Observe

$$\frac{\delta H}{\delta f_\alpha} = \frac{1}{2} m_\alpha v^2 - \sum_\beta e_\alpha e_\beta \int K(x|x')\, f_\beta(z')\, dz' \equiv H_p \quad ,$$

where H_p is the Hamiltonian (energy) of a particle in the potential

$$\phi(x) = - \sum_\beta e_\beta \int K(x|x') f_\beta (z') dz' \quad .$$

Since $\delta f(z)/\delta f(z') = \delta(z-z')$, where $\delta(z)$ is the Dirac delta function, Eq. (4) with Eq. (6) is equivalent to Eq. (3).[9]

Practical exploitation of the Hamiltonian structure presented here, either numerical or analytical, requires discretization and truncation. For noncanonical Poisson brackets, which are linear in the dynamical variables of the system, we have observed that truncation destroys the Jacobi requirement. In order to rectify this situation we seek canonical variables. To this end we represent the distribution function in terms of Clebsch potentials[7]

$$f_\alpha(z) = \frac{1}{m_\alpha} \left(\frac{\partial\Phi^{(\alpha)}}{\partial x} \frac{\partial T^{(\alpha)}}{\partial v} - \frac{\partial T^{(\alpha)}}{\partial x} \frac{\partial\Phi^{(\alpha)}}{\partial v} \right) \quad . \tag{8}$$

This representation is not unique; e.g., one has the gauge condition that any function $\bar{T}^{(\alpha)}$ can be added to $T^{(\alpha)}$ provided

$$\frac{\partial\Phi^{(\alpha)}}{\partial x} \frac{\partial\bar{T}^{(\alpha)}}{\partial v} - \frac{\partial\bar{T}^{(\alpha)}}{\partial x} \frac{\partial\Phi^{(\alpha)}}{\partial v} = 0 \quad .$$

Substituting Eq. (8) into the equation of motion [Eq. (3)] yields

$$\left[\Phi^{(\alpha)}, \frac{\partial T^{(\alpha)}}{\partial t} + \underline{w}_\alpha \cdot \nabla_p T^{(\alpha)}\right] + \left[\frac{\partial\Phi^{(\alpha)}}{\partial t} + \underline{w}_\alpha \cdot \nabla_p \Phi^{(\alpha)}, T^{(\alpha)}\right] = 0 \quad ,$$

where recall the braces are defined by Eq. (7). Clearly the above is statisfied (although not in the most general manner) if $\Phi^{(\alpha)}$ and $T^{(\alpha)}$ satisfy the following:[10]

$$\frac{\partial \phi^{(\alpha)}}{\partial t} \approx - \underline{w}_\alpha \cdot \nabla_p \phi^{(\alpha)}$$

$$\frac{\partial T^{(\alpha)}}{\partial t} \approx - \underline{w}_\alpha \cdot \nabla_p T^{(\alpha)} \quad . \tag{9}$$

Upon use of the chain rule for functional derivatives, the substitution [Eq. (8)] renders the bracket [Eq. (5)] canonical:[11] Eqs. (9) have the form

$$\frac{\partial \phi^{(\alpha)}}{\partial t} = \frac{\delta H}{\delta T^{(\alpha)}}$$

$$\frac{\partial T^{(\alpha)}}{\partial t} = - \frac{\delta H}{\delta \phi^{(\alpha)}} \quad , \tag{10}$$

where H is now regarded as a functional of $\phi^{(\alpha)}$ and $T^{(\alpha)}$.

Let us now expand $\phi^{(\alpha)}$ and $T^{(\alpha)}$ in terms of an arbitrary orthonormal complete set of basic functions $\mu_{\underline{k}}(z)$,

$$\phi^{(\alpha)} = \sum_{\underline{k}} \phi_{\underline{k}}^{(\alpha)} \mu_{\underline{k}}(z)$$

$$T^{(\alpha)} = \sum_{\underline{k}} T_{\underline{k}}^{(\alpha)} \mu_{\underline{k}}(z) \quad . \tag{11}$$

Here $\mu_{\underline{k}}$ is doubly indexed by $\underline{k}$ and orthonormality, is with respect to the following inner product:

$$< \mu_{\underline{k}} | \mu_{\underline{\ell}} > = \int \mu_{\underline{k}}^{*} \mu_{\underline{\ell}} \, dz = \delta_{\underline{k},\underline{\ell}} \quad .$$

(* refers to complex conjugation and $\delta_{\underline{k},\ell}$ is the Kronecker delta function.) The reality condition is assumed; e.g., in the case where $\mu_{\underline{k}}$ is complex [e.g., Eq. (15)] in both its arguments, x and v, we require $\phi_{\underline{k}}^{(\alpha)*} = \phi_{-\underline{k}}^{(\alpha)}$ and similarly for $T_{\underline{k}}^{(\alpha)}$. In case $\mu_{\underline{k}}$ is real this condition is not applicable and the inner product is symmetric. Since an arbitrary functional $A[\Phi]$ obtains its $\phi_{\underline{k}}$ dependence through Φ, it is not difficult to verify the following:

$$\frac{\partial A}{\partial \phi_{\underline{k}}} = \int \frac{\delta A}{\delta \Phi} \frac{\partial \Phi}{\partial \phi_{\underline{k}}} dz = \int \frac{\delta A}{\delta \Phi} \mu_{\underline{k}} dz \quad . \tag{12}$$

In this expression the A on the left hand side is to be treated as a function of the $\phi_{\underline{k}}$; in the center and on the right, A retains its role as a functional of Φ. From Eq. (12) we obtain the following expression for the expansion of a functional derivative:

$$\frac{\delta A}{\delta \Phi} = \sum_{\underline{k}} \frac{\partial A}{\partial \phi_{\underline{k}}^*} \mu_{\underline{k}} \quad .$$

With this, the discretized version of Eqs. (10) are

$$\dot{\phi}_{\underline{k}}^{(\alpha)} = \frac{\partial H}{\partial T_{\underline{k}}^{(\alpha)*}}$$

$$\dot{T}_{\underline{k}}^{(\alpha)} = -\frac{\partial H}{\partial \phi_{\underline{k}}^{(\alpha)*}} \quad .$$

These equations can be symmetrized by making the substitution

$$\psi_{\underline{k}}^{(\alpha)} = 2^{-1/2} \left[\phi_{\underline{k}}^{(\alpha)} + i\, T_{\underline{k}}^{(\alpha)}\right] \quad . \tag{13}$$

We obtain

$$\dot{\psi}_{\underline{k}}^{(\alpha)} = - i \frac{\partial H}{\partial \psi_{\underline{k}}^{(\alpha)*}} \quad \text{and} \quad \dot{\psi}_{\underline{k}}^{(\alpha)*} = i \frac{\partial H}{\partial \psi_{\underline{k}}^{(\alpha)}} \quad . \tag{14}$$

The distribution function (density) has an expansion which is quadratic in the $\psi_{\underline{k}}^{(\alpha)}$ and $\psi_{\underline{k}}^{(\alpha)*}$, a form which is reminiscent of quantum mechanics. From Eq. (6) we observe that the Hamiltonian has terms quadratic and quartic in the $\psi_{\underline{k}}^{(\alpha)}$ and $\psi_{\underline{k}}^{(\alpha)*}$.

Assuming periodic boundary conditions on a unit phase space box, we obtain the following wave field expression for the phase space density:

$$f_{\alpha}(z) = \sum_{\underline{k}} f_{\underline{k}}^{(\alpha)} \mu_{\underline{k}}(z) \quad ,$$

where the orthonormal functions

$$\mu_{\underline{k}} \equiv \frac{1}{2\pi} e^{ik_1 x + ik_2 v} \quad . \qquad (1')$$

[Here $\underline{k} \equiv (k_1, k_2)$.] The Fourier coefficients of the density are seen by Eqs. (8), (11), and (13), to be related to the "wave function" $\psi_{\underline{k}}^{(\alpha)}$ through

$$f_{\underline{k}}^{(\alpha)} = \sum_{\underline{t}=\underline{k}+\underline{\ell}} a_{\underline{t},\underline{\ell}}^{(\alpha)} \psi_{\underline{t}}^{(\alpha)} \psi_{\underline{\ell}}^{(\alpha)*} \tag{16}$$

where

$$a_{\underline{t},\underline{\ell}}^{(\alpha)} = \frac{i}{2\pi m_{\alpha}} \left(t_1 \ell_2 - \ell_1 t_2\right) \quad .$$

Substituting Eq. (16) into Eq. (6) yields the following expression for the Hamiltonian:

$$H = \sum_{\substack{\alpha \\ \ell_1 = k_1 \\ \ell_2 \neq k_2}} S^{(2)}_{\underline{\ell},\underline{k}} \, \psi_{\underline{\ell}}^{(\alpha)*} \psi_{\underline{k}}^{(\alpha)} + \sum_{\substack{\alpha,\beta \\ s_2 = \ell_2 \\ m_2 = p_2 \\ p_1 - m_1 = \ell_1 - s_1 \neq 0}} \frac{e_\alpha}{m_\alpha} \frac{e_\beta}{m_\beta} S^{(4)}_{\underline{\ell},\underline{s},\underline{m},\underline{p}} \, \psi_{\underline{\ell}}^{(\alpha)*} \psi_{\underline{s}}^{(\alpha)} \psi_{\underline{m}}^{(\beta)*} \psi_{\underline{p}}^{(\beta)}$$

where

$$S^{(2)}_{\underline{\ell},\underline{k}} = \frac{i(-1)^{\ell_2 - k_2}}{(\ell_2 - k_2)^2} (k_1 \ell_2 - \ell_1 k_2)$$

and

$$S^{(4)}_{\underline{\ell},\underline{s},\underline{m},\underline{p}} = \frac{-(\ell_1 s_2 - \ell_2 s_1)(m_1 p_2 - m_2 p_1)}{4\pi (s_1 - \ell_1)(m_1 - p_1)} \quad .$$

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7. The use of potentials to represent vector fields has a long history in hydrodynamics. For a partial historical account see, D. P. Stern, Am. J. Phys. 38, 494 (1970). See also H. Lamb, Hydrodynamics, sixth edition, (Cambridge University Press, Cambridge, 1932) sec. 167 and J. Serrin, in

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8. The Poisson bracket for the Poisson-Vlasov equation is the same as that for the vortex fluid and guiding-center plasma, although the Hamiltonians differ. See Ref. 6.

9. This requires the use of the identity, $\int f\{g,h\}\,dz = -\int g\{f,h\}\,dz$, which is readily verified by integration by parts.

10. It is well-known that, since f_α is constant on characteristics, if $f_\alpha \geq 0$ at $t=0$ for all z then $f_\alpha \geq 0$ for all z and t. If $\phi^{(\alpha)}$ and $\tau^{(\alpha)}$ are chosen at $t=0$ such that f_α, as determined by Eq. (8), is nonnegative, then the dynamical equations for $\tau^{(\alpha)}$ and $\phi^{(\alpha)}$ assure that f_α solves Eqs. (3); hence nonnegativity of f_α is maintained.

11. See Ref. 6 pp. 12 and 13.