HAMILTONIAN FORMALISM FOR SOLVING THE VLASOV-POISSON EQUATIONS AND ITS APPLICATION TO THE COHERENT BEAM-BEAM INTERACTION

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Abstract

A Hamiltonian approach to the solution of the Vlasov-Poisson equations has been developed. Based on a nonlinear canonical transformation, the rapidly oscillating terms in the original Hamiltonian are transformed away, yielding a new Hamiltonian that contains slowly varying terms only. The formalism has been applied to the coherent beam-beam interaction, and a stationary solution to the transformed Vlasov equation has been obtained.

1 INTRODUCTION

The evolution of charged particle beams in accelerators and storage rings can often be described by the Vlasov-Maxwell equations. At high energies the discrete-particle collision term [1] comprises a small correction to the dynamics and can be neglected. Radiation effects at sufficiently high energies for leptons can be a significant feature of the dynamics, and should be included in the model under consideration.

The Vlasov-Maxwell equations constitute a considerable simplification in the description of charged particle beam propagation. Nonetheless there are only a few cases that are tractable analytically. Therefore, it is of utmost the importance to develop a systematic perturbation approach, able to provide satisfactory results in a wide variety of cases of physical interest.

Particle beams are subject to external forces that are often rapidly oscillating, such as quadrupolar focusing forces, RF fields, etc. In addition, the collective self-field excitations can be rapidly oscillating as well. A typical example is a colliding-beam storage ring device, where the evolution of each beam is strongly affected by the electromagnetic force produced by the counter-propagating beam. The beam-beam kick each beam experiences is localized only in a small region around the interaction point, and is periodic with a period of one turn.

In this and other important cases one is primarily interested in the long-time behavior of the beam, thus discarding the fast processes on time scales of order the period of the rapid oscillations. To extract the relevant information, an efficient method of averaging is developed in the next section. Unlike the standard canonical perturbation technique [2, 3], the approach used here is carried out in a "mixed" phase space (old coordinates and new canonical momenta), which is simpler and more efficient in a computational sense. The canonical perturbation method developed here is further applied to the coherent beam-beam interaction, and a coupled set of nonlinear integral equations for the equilibrium beam densities has been derived.

2 THE HAMILTONIAN FORMALISM

We consider a N-dimensional dynamical system, described by the canonical conjugate pair of vector variables $(\mathbf{q}, \mathbf{p})$ with components

$$\mathbf{q} = (q_1, q_2, \ldots, q_N), \quad \mathbf{p} = (p_1, p_2, \ldots, p_N). \tag{2.1}$$

The Vlasov equation for the distribution function $f(\mathbf{q}, \mathbf{p}; t)$ can be expressed as

$$\frac{\partial f}{\partial t} + [f, H] = 0, \tag{2.2}$$

where

$$[F, G] = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \tag{2.3}$$

is the Poisson bracket, $H(\mathbf{q}, \mathbf{p}; t)$ is the Hamiltonian of the system, and summation over repeated indices is implied. Next we define a canonical transformation via the generating function of the second type according to

$$S = S(\mathbf{q}, \mathbf{P}; t), \tag{2.4}$$

and assume that the Jacobian matrix

$$\mathcal{J}_{ij}(\mathbf{q}, \mathbf{P}; t) = \frac{\partial^2 S}{\partial q_i \partial P_j} \tag{2.5}$$

is non-degenerate with

$$\det(\mathcal{J}_{ij}) \neq 0, \tag{2.6}$$

so that the inverse $\mathcal{J}_{ij}^{-1}$ exists. Let us also define the distribution function in terms of the new coordinates $(\mathbf{Q}, \mathbf{P})$ and the mixed pair $(\mathbf{q}, \mathbf{P})$ as

$$f(\mathbf{q}, \mathbf{p}; t) = f_0(\mathbf{Q}, \mathbf{P}; t) = F_0(\mathbf{q}, \mathbf{P}; t). \tag{2.7}$$

The new canonical variables $(\mathbf{Q}, \mathbf{P})$ are defined by the canonical transformation as

$$p_i = \frac{\partial S}{\partial q_i}, \qquad Q_i = \frac{\partial S}{\partial P_i}. \tag{2.8}$$

Because

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial^2 S}{\partial q_i \partial P_j} = \mathcal{J}_{ij} \qquad \Longrightarrow \qquad \frac{\partial P_i}{\partial p_j} = \mathcal{J}_{ij}^{-1}, \tag{2.9}$$

we can express the Poisson bracket in terms of the mixed variables in the form

$$[F,G] = \mathcal{J}_{ji}^{-1}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_j}\frac{\partial G}{\partial q_i}\right). \tag{2.10}$$

Differentiation of Eq. (2.8) with respect to time t, keeping the old variables $(\mathbf{q},\mathbf{p})$ fixed, yields

$$\frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial^2 S}{\partial q_i \partial P_j}\left(\frac{\partial P_j}{\partial t}\right)_{qp} = 0, \tag{2.11}$$

$$\left(\frac{\partial Q_i}{\partial t}\right)_{qp} = \frac{\partial^2 S}{\partial P_i \partial t} + \frac{\partial^2 S}{\partial P_i \partial P_j}\left(\frac{\partial P_j}{\partial t}\right)_{qp}, \tag{2.12}$$

or

$$\left(\frac{\partial P_j}{\partial t}\right)_{qp} = -\mathcal{J}_{ji}^{-1}\frac{\partial^2 S}{\partial q_i \partial t}. \tag{2.13}$$

Our goal is to express the Vlasov equation (2.2) in terms of the mixed variables $(\mathbf{q},\mathbf{P})$. Taking into account the identities

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial^2 S}{\partial q_j \partial P_i} = \mathcal{J}_{ji} \quad \Longrightarrow \quad \frac{\partial q_i}{\partial Q_j} = \mathcal{J}_{ji}^{-1}, \tag{2.14}$$

$$\frac{\partial f_0}{\partial Q_i} = \mathcal{J}_{ij}^{-1}\frac{\partial F_0}{\partial q_j}, \tag{2.15}$$

$$\frac{\partial f_0}{\partial P_i} = \frac{\partial F_0}{\partial P_i} - \frac{\partial f_0}{\partial Q_j}\frac{\partial^2 S}{\partial P_i \partial P_j}, \tag{2.16}$$

we obtain

$$\left(\frac{\partial f}{\partial t}\right)_{qp} = \frac{\partial f_0}{\partial t} + \frac{\partial f_0}{\partial Q_i}\left(\frac{\partial Q_i}{\partial t}\right)_{qp} + \frac{\partial f_0}{\partial P_i}\left(\frac{\partial P_i}{\partial t}\right)_{qp}$$

$$= \frac{\partial F_0}{\partial t} + \mathcal{J}_{ji}^{-1}\left(\frac{\partial F_0}{\partial q_i}\frac{\partial^2 S}{\partial t \partial P_j} - \frac{\partial F_0}{\partial P_j}\frac{\partial^2 S}{\partial t \partial q_i}\right)$$

$$= \frac{\partial F_0}{\partial t} + \left[F_0, \frac{\partial S}{\partial t}\right]. \tag{2.17}$$

Furthermore, using the relation

$$[f,H] = [F_0,\mathcal{H}], \tag{2.18}$$

where

$$\mathcal{H}(\mathbf{q},\mathbf{P};t) = H(\mathbf{q},\nabla_q S;t), \tag{2.19}$$

we express the Vlasov equation in terms of the mixed variables according to

$$\frac{\partial F_0}{\partial t} + [F_0,\mathcal{K}] = 0, \tag{2.20}$$

where

$$\mathcal{K}(\mathbf{q},\mathbf{P};t) = \frac{\partial S}{\partial t} + H(\mathbf{q},\nabla_q S;t) \tag{2.21}$$

is the new Hamiltonian. For the distribution function $f_0(\mathbf{Q},\mathbf{P};t)$ depending on the new canonical variables, we clearly obtain

$$\frac{\partial f_0}{\partial t} + [f_0,\mathcal{K}] = 0, \tag{2.22}$$

where the new Hamiltonian $\mathcal{K}$ is a function of the new canonical pair $(\mathbf{Q},\mathbf{P})$, such that

$$\mathcal{K}(\nabla_P S,\mathbf{P};t) = \frac{\partial S}{\partial t} + H(\mathbf{q},\nabla_q S;t), \tag{2.23}$$

and the Poisson bracket entering Eq. (2.22) has the same form as Eq. (2.3), expressed in the new canonical variables.

3 COHERENT BEAM-BEAM INTERACTION

As an application of the formalism developed in the previous section we study here the evolution of two counter-propagating beams, nonlinearly coupled by the electromagnetic interaction between the beams at collision. For simplicity, we consider one-dimensional motion in the vertical (q) direction, described by the nonlinear Vlasov-Poisson equations

$$\frac{\partial f_k}{\partial \theta} + [f_k, H_k] = 0, \tag{3.1}$$

$$\frac{\partial^2 V_k}{\partial q^2} = 4\pi \int dp f_{3-k}(q,p;\theta), \tag{3.2}$$

where

$$H_k = \frac{\nu_k}{2}\left(p^2+q^2\right) + \lambda_k \delta_p(\theta) V_k(q;\theta) \tag{3.3}$$

is the Hamiltonian. The notation in Eqs. (3.1) - (3.3) is the same as in Ref. [4]. Our goal is to determine a canonical transformation such that the new Hamiltonian is time-independent. As a consequence, the stationary solution of the Vlasov equation (2.20) is expressed as a function of the new Hamiltonian. Following the procedure outlined in the preceding section we transform Eqs. (3.1) - (3.3) according to

$$\left[F_0^{(k)}, \mathcal{K}_k\right] \equiv 0, \tag{3.4}$$

$$\frac{\partial S_k}{\partial \theta} + \epsilon H_k\left(q, \frac{\partial S_k}{\partial q}; \theta\right) = \mathcal{K}_k(q,P), \tag{3.5}$$

$$\frac{\partial^2 V_k}{\partial q^2} = 4\pi \int dP \frac{\partial^2 S_k}{\partial q \partial P} F_0^{(3-k)}(q,P), \tag{3.6}$$

where ϵ is formally a small parameter, which will be set equal to unity at the end of the calculation. The next step is to expand the quantities S_k, $\mathcal{K}_k$ and V_k in a power series in ϵ as

$$S_k = qP + \epsilon G_k^{(1)} + \epsilon^2 G_k^{(2)} + \epsilon^3 G_k^{(3)} + \ldots, \tag{3.7}$$

$$\mathcal{K}_k = \epsilon \mathcal{K}_k^{(1)} + \epsilon^2 \mathcal{K}_k^{(2)} + \epsilon^3 \mathcal{K}_k^{(3)} + \ldots, \tag{3.8}$$

$$V_k = \widetilde{V}_k + \epsilon V_k^{(1)} + \epsilon^2 V_k^{(2)} + \epsilon^3 V_k^{(3)} + \ldots, \tag{3.9}$$

where

$$\frac{\partial^2 \widetilde{V}_k}{\partial q^2} = 4\pi \int dP F_0^{(3-k)}(q,P). \tag{3.10}$$

Substitution of the above expansions (3.7) - (3.9) into Eqs. (3.5) and (3.6) yields perturbation equations that can be solved successively order by order. The results are:

First Order: $O(\epsilon)$

$$\mathcal{K}_k^{(1)}(q,P) = \frac{\nu_k}{2}\left(P^2 + q^2\right) + \frac{\lambda_k}{2\pi}\widetilde{V}_k(q), \tag{3.11}$$

$$G_k^{(1)}(q,P;\theta) = \frac{i\lambda_k}{2\pi}\widetilde{V}_k(q)\sum_{n\neq 0}\frac{e^{in\theta}}{n}, \tag{3.12}$$

$$V_k^{(1)}(q;\theta) \equiv 0. \tag{3.13}$$

Second Order: $O(\epsilon^2)$

$$\mathcal{K}_k^{(2)}(q,P) \equiv 0, \tag{3.14}$$

$$G_k^{(2)}(q,P;\theta) = -\frac{\lambda_k\nu_k}{2\pi}P\widetilde{V}_k'(q)\sum_{n\neq 0}\frac{e^{in\theta}}{n^2}, \tag{3.15}$$

$$V_k^{(2)}(q;\theta) = -\frac{\lambda_k\nu_k}{2\pi}\widetilde{V}_k^{(2)}(q)\sum_{n\neq 0}\frac{e^{in\theta}}{n^2}, \tag{3.16}$$

where

$$\frac{\partial^2\widetilde{V}_k^{(2)}}{\partial q^2} = 4\pi\widetilde{V}_k''(q)\int dP F_0^{(3-k)}(q,P). \tag{3.17}$$

Third Order: $O(\epsilon^3)$ In third order we are interested in the new Hamiltonian, which is of the form

$$\mathcal{K}_k^{(3)}(q,P) = \frac{\lambda_k^2\nu_k}{4\pi^2}\zeta(2)\left[\widetilde{V}_k'^2(q) - 2\widetilde{V}_k^{(2)}(q)\right], \tag{3.18}$$

where $\zeta(z)$ is Riemann's zeta-function

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}. \tag{3.19}$$

4 THE EQUILIBRIUM DISTRIBUTION FUNCTION

Since the new Hamiltonian $\mathcal{K}_k$ is time-independent (by construction), the equilibrium distribution function $F_0^{(k)}$ [see Eq. (3.4)] is a function of the new Hamiltonian

$$F_0^{(k)}(q,P) = \mathcal{G}_k(\mathcal{K}_k), \tag{4.1}$$

where

$$\mathcal{K}_k(q,P) = \frac{\nu_k}{2}\left(P^2 + q^2\right) + \frac{\lambda_k}{2\pi}\widetilde{V}_k(q) + \frac{\lambda_k^2\nu_k}{4\pi^2}\zeta(2)\left[\widetilde{V}_k'^2(q) - 2\widetilde{V}_k^{(2)}(q)\right]. \tag{4.2}$$

Integrating Eq. (4.1) over P we obtain a nonlinear integral equation of Haissinski type [5] for the equilibrium beam density profile $\varrho_0^{(k)}$

$$\varrho_0^{(k)}(q) = \int dP\mathcal{G}_k(\mathcal{K}_k), \tag{4.3}$$

where

$$\mathcal{K}_k(q,P) = \frac{\nu_k}{2}\left(P^2 + q^2\right) + \lambda_k\int dq'|q-q'|\varrho_0^{(3-k)}(q') + 2\lambda_k^2\nu_k\zeta(2)\mathcal{F}_k(q), \tag{4.4}$$

$$\mathcal{F}_k(q) = \int dq'dq''\mathcal{Z}(q-q',q'-q'') \times\varrho_0^{(3-k)}(q')\varrho_0^{(3-k)}(q''), \tag{4.5}$$

$$\mathcal{Z}(u,v) = \mathrm{sgn}(u)\mathrm{sgn}(v) - 2|u|\delta(v). \tag{4.6}$$

5 CONCLUDING REMARKS

We have developed a systematic canonical perturbation approach that removes rapidly oscillating terms in Hamiltonians of quite general form. The essential feature of this approach is the use of mixed canonical variables. For this purpose the Vlasov-Poisson equations are transformed to mixed canonical variables, and an appropriate perturbation scheme is chosen to obtain the equilibrium phase space density. It is worthwhile to note that the perturbation expansion outlined in the preceding section can be carried out to arbitrary order, although higher-order calculations become very tedious.

The canonical perturbation technique has been applied to study the one-dimensional beam-beam interaction. In particular, rapidly oscillating terms due to the periodic beam-beam kicks have been averaged away, yielding a time-independent new Hamiltonian. Furthermore, the equilibrium distribution functions have been obtained as a general function of the new Hamiltonian, and a coupled set of integral equations for the beam densities has been derived.

6 ACKNOWLEDGMENTS

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7 REFERENCES

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