Hamiltonian paths and cycles in hypertournaments *

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Abstract

Given two integers n and k, $n \geq k > 1$, a k-hypertournament T on n vertices is a pair (V, A), where V is a set of vertices, $|V| = n$ and A is a set of k-tuples of vertices, called arcs, so that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. A 2-hypertournament is merely an (ordinary) tournament. A path is a sequence $v_1a_1v_2a_2v_3...v_{t-1}a_{t-1}v_t$ of distinct vertices $v_1, v_2, ..., v_t$ and distinct arcs $a_1, ..., a_{t-1}$ such that v_i precedes v_{i+1} in a_i, $1 \leq i \leq t-1$. A cycle can be defined analogously. A path or cycle containing all vertices of T (as v_i's) is Hamiltonian. T is strong if T has a path from x to y for every choice of distinct $x, y \in V$. We prove that every k-hypertournament on n $(> k)$ vertices has a Hamiltonian path (an extension of Redei's theorem on tournaments) and every strong k-hypertournament with n $(> k+1)$ vertices has a Hamiltonian cycle (an extension of Camion's theorem on tournaments). Despite the last result, it is shown that the Hamiltonian cycle problem remains polynomial time solvable only for $k \leq 3$ and becomes NP-complete for every fixed integer $k \geq 4$.

1 Introduction, terminology and notation

Hypertournaments have been studied by a number of authors (cf. Assous [1], Barbut and Bialostocki [2, 3], Bialostocki [5], Frankl [6] and Marshall [9, 10]). Reid [12] (Section 8) describes several results on hypertournaments obtained by the authors above and poses some interesting problems on the topic. In particular, he raises the problem of extending the most important results on tournaments to hypertournaments.

*This paper is dedicated to the memory of Paul Erdős.
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In this paper, we obtain extensions of two of the most basic theorems on tournaments: every tournament has a Hamiltonian path (Redei's theorem), and every strong tournament has a Hamiltonian cycle (Camion's theorem) [11]. We prove that every k-hypertournament on n ($> k$) vertices has a Hamiltonian path and every strong k-hypertournament on $n \geq k+2 \geq 5$ vertices contains a Hamiltonian cycle. We also describe an infinite family of strong k-hypertournaments on $k+1 \geq 4$ vertices which have no Hamiltonian cycles. We consider the complexity of the Hamiltonian cycle problem for k-hypertournaments and prove that the problem remains polynomial time solvable when $k=3$ and becomes NP-complete for every fixed integer $k \geq 4$.

Given two integers n and k, $n \geq k > 1$, a *k-hypertournament* T on n vertices is a pair (V, A), where V is a set of *vertices*, $|V| = n$ and A is a set of k-tuples of vertices, called *arcs*, so that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. That is, T may be thought of as arising from an orientation of the hyperedges of the complete k-uniform hypergraph. Clearly, a 2-hypertournament is merely a tournament. For an arc a of T, $\bar{a}$ denotes the set of vertices contained in a.

Let $T = (V, A)$ denote a k-hypertournament T on n vertices. A *path* in T is a sequence $v_1 a_1 v_2 a_2 v_3 ... v_{t-1} a_{t-1} v_t$ of distinct vertices $v_1, v_2, ..., v_t$, $t \geq 1$, and distinct arcs $a_1, ..., a_{t-1}$ such that v_i precedes v_{i+1} in a_i, $1 \leq i \leq t-1$. A *cycle* in T is a sequence $v_1 a_1 v_2 a_2 v_3 ... v_{t-1} a_{t-1} v_t a_t v_1$ of distinct vertices $v_1, v_2, ..., v_t$ and distinct arcs $a_1, ..., a_t$, $t \geq 1$, such that v_i precedes v_{i+1} in a_i, $1 \leq i \leq t$ ($a_{t+1} = a_1$). The above definitions of a path and cycle in a hypertournament are oriented analogs of the corresponding definitions of a path and cycle in a hypergraph (cf. [4, 10]).

For a path or cycle Q, $V(Q)$ and $A(Q)$ denote the set of vertices (v_i's above) and the set of arcs (a_j's above), respectively. For a pair of vertices v_i and v_j of a path or cycle Q, $Q[v_i, v_j]$ denotes the subpath of Q from v_i to v_j (which can be empty). A path or cycle Q in T is *Hamiltonian* if $V(Q) = V(T)$. T is *Hamiltonian* if it has a Hamiltonian cycle. A path from x to y is an *(x, y)-path.* T is called *strong* if T has an (x, y)-path for every (ordered) pair x, y of distinct vertices in T.

We also consider paths and cycles in digraphs which will be denoted as sequences of the corresponding vertices.

For a pair of distinct vertices x and y in T, $A_T(x, y)$ denotes the set of all arcs of T in which x precedes y. Clearly, for all distinct $x, y \in V(T)$,

$$|A_T(x, y)| + |A_T(y, x)| = \binom{n-2}{k-2}. \tag{1}$$

2 Hamiltonian paths

Clearly, no k-hypertournament with $k \geq 3$ vertices has a Hamiltonian path. However, all other hypertournaments have Hamiltonian paths:

Theorem 2.1 *Every k-hypertournament with n $(> k)$ vertices contains a Hamiltonian path.*

Proof: Let $T = (V, A)$ be a k-hypertournament T on n vertices 1,2,...,n. We consider the cases $k = n - 1$ and $k < n - 1$ separately.

Case 1: $k = n - 1$. We proceed by induction on $k \geq 2$. By Redei's theorem, this theorem holds for $k = 2$. Hence, suppose that $k \geq 3$. Assume w.l.o.g. that T contains the arc $a = (23...n)$. Let b be the arc of T that has the vertices $1, 2, ..., n - 1$. Consider the $k - 1$-hypertournament $T' = (V', A')$ obtained from T by deleting the arc a, deleting n from the arcs in $A - \{a, b\}$, and finally deleting 1 from b. So, $V' = \{1, 2, ..., n - 1\}$, $A' = \{e' : \ e' \text{ is } e \text{ without } n, \ e \in A - \{a, b\}\} \cup \{b'\}$, where b' is b without the vertex 1. By the induction hypothesis, T' has a Hamiltonian path $x_1a'_1x_2a'_2...a'_{n-2}x_{n-1}$. This path corresponds to the path $Q = x_1a_1x_2a_2...a_{n-2}x_{n-1}$ in T. Clearly, $\{x_1, ..., x_{n-1}\} = \{1, ..., n - 1\}$ and $A - \{a_1, ..., a_{n-2}\}$ consists of the arc a and another arc c.

If $x_{n-1} \neq 1$, then Qan is a Hamiltonian path in T. Hence from now on assume that $x_{n-1} = 1$. Consider two subcases.

Subcase 1.1: $c \neq b$. If the last vertex of c is n, then Qcn is a Hamiltonian path in T. Otherwise, x_j is the last vertex of c for some $j \leq n - 1$. If $j > 1$ we replace a_{j-1} by anc in Q in order to obtain a Hamiltonian path in T. If $j = 1$, then ncQ is a Hamiltonian path in T.

Subcase 1.2: $c = b$. If $c \neq (x_{n-1}x_{n-2}...x_1)$ so that x_i precedes x_{i+1}, for some i, $1 \leq i \leq n - 2$, in c, then $P = Q[x_1, x_i]cQ[x_{i+1}, x_{n-1}]$ is a path in T. Since $a_i \neq b$, one can construct a Hamiltonian path in T from P as in Subcase 1.1. If $c = (x_{n-1}x_{n-2}...x_1)$, then $Q[x_2, x_{n-1}]cx_1an$ is a Hamiltonian path in T.

Case 2: $k < n - 1$. We proceed by induction on $n \geq 4$. The case $n = 4$ (and, hence, $k = 2$) follows from Redei's theorem. Therefore, suppose that $n \geq 5$. Consider the new k-hypertournament T'' obtained from T by deleting the vertex n along with all arcs in A containing n. T'' has a Hamiltonian path because of either Case 1 if $n = k - 2$ or the induction hypothesis, otherwise.

Let $P = x_1a_1x_2a_2...a_{n-2}x_{n-1}$ be a Hamiltonian path in T''. If T has an arc $a \in A_T(x_{n-1}, n)$, then Pan is a Hamiltonian path in T. Suppose that $A_T(x_{n-1}, n) = \emptyset$.

Then either $\cup_{i=1}^{n-1} A_T(x_i, n) = \emptyset$, or there is an i so that T has no arc where any x_j, $j > i$ precedes n and T contains an arc b where x_i precedes n. In the first case, ncP is a Hamiltonian path in T, where c is an arc of T containing both x_1 and n. In the second case, $P[x_1, x_i]bndP[x_{i+1}, x_{n-1}]$ is a Hamiltonian path in T, where d is an arc of T containing both x_{i+1} and n and distinct from b.

$\square$.

3 Hamiltonian cycles

Clearly, every Hamiltonian hypertournament is strong. In this section, we prove that every strong k-hypertournament with n vertices, where $3 \leq k \leq n-2$, is Hamiltonian.

However, for every $k \geq 3$, there exists a strong k-hypertournament with $n = k + 1$ vertices which is not Hamiltonian. Indeed, let the $n-1$-hypertournament H_n have vertex set $\{x_1, ..., x_n\}$ and arc set $\{a_1, a_2, ..., a_n\}$, where $a_1 = (x_2x_3...x_{n-2}x_nx_{n-1})$, $a_2 = (x_1x_3x_4...x_n)$, $a_3 = (x_1x_2x_4x_5...x_n)$, $a_4 = (x_2x_3x_1x_5x_6...x_n)$, and

$$a_i = (x_1x_2...x_{i-4}x_{i-3}x_{i-1}x_{i-2}x_{i+1}x_{i+2}...x_n) \text{ for } 5 \leq i \leq n.$$

The hypertournament H_n is strong because of the following paths: $x_ia_2x_j$ for all $i < j$, $2 \notin \{i, j\}$, $x_1a_3x_2$, $x_2a_1x_j$ for all $j > 2$; $x_ja_{j+1}x_{j-1}a_jx_{j-2}...x_{i+1}a_{i+2}x_i$ for all $3 \leq i < j \leq n$, where $a_{n+1} = a_1$, $x_ja_{j+1}x_{j-1}...x_4a_5x_3a_4x_1a_3x_2$ for all $3 \leq j \leq n$, $x_2a_4x_1$, and $x_ja_{j+1}x_{j-1}...x_3a_4x_1$ for all $3 \leq j \leq n$.

However, H_n is not Hamiltonian. To prove that, assume that H_n has a Hamiltonian cycle C. We will try to construct C starting from the vertex x_n. Since a_1 is the only arc which has a vertex that succeeds x_n, C has the form $x_na_1x_{n-1}....$ Since a_n is the only arc which has a vertex different from x_n that succeeds x_{n-1}, $C = x_na_1x_{n-1}a_nx_{n-2}....$ Continuing this process, we obtain that $C = x_na_1x_{n-1}...x_4a_5x_3....$ The only arc where x_3 precedes x_1 or x_2 is a_4. Hence, $C = x_na_1x_{n-1}...x_4a_5x_3a_4x_1....$ Now we need to include x_2, a_3 and a_2 into C. However, this is impossible because only one of the arcs a_3, a_2 contains x_2.

The *majority digraph* $M(H)$ of a k-hypertournament with n vertices H has the same vertex set V as H and, for every pair x, y of distinct vertices in V, the arc xy is in $M(H)$ iff $A_H(x, y) \geq A_H(y, x)$ (or, by (1), $|A_T(x, y)| \geq \frac{1}{2}\binom{n-2}{k-2}$). Obviously, $M(H)$ is a *semicomplete digraph*, i.e. every pair of vertices in $M(H)$ is adjacent.

Let $C_1, C_2, \ldots, C_t$ be the strong components of $M(H)$ such that there is no arc from $V(C_j)$ to $V(C_i)$ if $1 \leq i < j \leq t$ (if $M(H)$ is strong, then $t = 1$). Define the function cn (*component number*) such that $cn(x) = r$ if $x \in V(C_r)$. We say that (P, Q) is a

Hamiltonian pair of paths, if P is a (x,y)-path in H and Q is a (y,x)-path in $M(H)$ such that $V(P) \cup V(Q) = V(H)$, $V(P) \cap V(Q) = \{x,y\}$ and if $M(H)$ is not strong, then $cn(y) < cn(x)$.

The main result of this section is the following:

Theorem 3.1 *Every strong k-hypertournament with n vertices, where $3 \leq k \leq n-2$, contains a Hamiltonian cycle.*

Theorem 3.1 follows immediately from Lemma 3.3 (the case $k = 3$), Lemma 3.4 (the case $k \geq 4$ and $n \geq 7$) and Lemma 3.6 (the remaining case $k = 4$ and $n = 6$). Proofs of these lemmas are given in the rest of this section and based on the following:

Lemma 3.2 *For every strong k-hypertournament with n vertices H, there exists a Hamiltonian pair of paths.*

Proof: Suppose first that $M(H)$ is not strong. Let C_1 (C_t, resp.) be the first (terminal, resp.) strong component of $M(H)$. Since H is strong there exists a path $P = x_1a_1x_2a_2\ldots a_{m-1}x_m$ from C_t to C_1 in H. Suppose that P is a shortest such path. Then, $x_1 \in V(C_t)$, $x_m \in V(C_1)$ and $\{x_2, x_3, ..., x_{m-1}\} \cap (V(C_1) \cup V(C_t)) = \emptyset$. Since $M' = M(H) - \{x_2, x_3, \ldots, x_{m-1}\}$ is semicomplete and x_m (x_1) is in the first (terminal) strong component of M', there exists a Hamiltonian path in M' from x_m to x_1. Let $Q = y_1y_2\ldots y_l$ ($x_1 = y_l$ and $x_m = y_1$), be such a path. Clearly (P,Q) is a Hamiltonian pair of paths.

Suppose that $M(H)$ is strong. Then, there is a Hamiltonian cycle $R = x_1x_2\ldots x_nx_1$ in $M(H)$. Clearly $P = x_1ax_2$, where $a \in A_H(x_1,x_2)$, and $Q = R[x_2,x_1]$ form a Hamiltonian pair of paths. $\square$.

Lemma 3.3 *Every strong 3-hypertournament with n vertices, where $n \geq 5$, contains a Hamiltonian cycle.*

Proof: Let H be a 3-hypertournament with $n \geq 5$ vertices and let M be the majority digraph of H.

By Lemma 3.2, there exists a Hamiltonian pair of paths (P_i, Q), where

$P_i = x_1a_1x_2a_2\ldots a_{i-1}x_i$ is a path in H and $Q = x_ix_{i+1}\ldots x_nx_1$ is a path in M. Observe that if uv is an arc of M, then $|A_H(u,v)| \geq 2$. Since $|A_H(x_i,x_{i+1})| \geq 2$, there is an arc $a_i \in A_H(x_i,x_{i+1}) - a_{i-1}$. The arc $a_i \notin \{a_1,a_2,\ldots,a_{i-1}\}$, since if $a_i = a_j$ then $j < i-1$ and the arc a_i includes the vertices x_j, x_{j+1}, x_i and x_{i+1}, which is impossible as H is a 3-hypertournament. Thus, we can extend P_i to $P_{i+1} = P_ia_ix_{i+1}$. Continuing this process we obtain a (x_1, x_n)-path, $P_n = x_1a_1x_2 \dots a_{n-1}x_n$, which is Hamiltonian, and $|A_H(x_n, x_1)| \geq 2$.

If $A_H(x_n, x_1) = \{a_{n-1}, a_1\}$, then there is an arc $b \in A_H(x_1, x_n)$, and $a_1 = (x_nx_1x_2)$ and $a_{n-1} = (x_{n-1}x_nx_1)$. We now obtain the Hamiltonian cycle $P_n[x_2, x_{n-1}]a_{n-1}x_1bx_na_1x_2$.

If $A_H(x_n, x_1) \neq \{a_{n-1}, a_1\}$, then there is an arc $b \in A_H(x_n, x_1) - \{a_{n-1}, a_1\}$. As before we see that $b \notin \{a_1, a_2, \dots, a_{n-1}\}$, and therefore we get the Hamiltonian cycle P_nbx_1.

$\square$.

Lemma 3.4 *Every strong k-hypertournament with n vertices, where $4 \leq k \leq n-2$ and $n \geq 7$, contains a Hamiltonian cycle.*

Proof: It is easy to check that, for $4 \leq k \leq n-2$, $\binom{n-2}{k-2} \geq 2n-4$ if and only if $n \geq 7$. Let H be a k-hypertournament with n vertices, such that $4 \leq k \leq n-2$ and $\binom{n-2}{k-2} \geq 2n-4$, and let $M = M(H)$ be the majority digraph of H. Now consider the following two cases.

Case 1: M is not strong. Let C_1 (C_t) be the first (terminal) strong component of M. We first prove that H has a pair of distinct vertices x, y such that

$$\text{there exists a Hamiltonian } (x,y)\text{-path in } H \text{ and } |A_H(y,x)| \geq n-1. \tag{2}$$

By Lemma 3.2, there exists a Hamiltonian pair of paths (P, Q), where

$P = x_1a_1x_2a_2 \dots a_{m-1}x_m$ is a path in H, $x_1 \in V(C_t)$, $x_m \in V(C_1)$, and $Q = y_1y_2...y_l$ is a path in M. Recall that $y_1 = x_m$ and $y_l = x_1$.

We may assume w.l.o.g that, for some $i > 1$, $cn(y_i) < cn(y_{i+1})$ (the case $i = 1$ can be considered analogously). It follows from the definition of M that

$$|A_H(y_j, y_{j+1})| \geq n-2 \text{ for } j = 1, 2, \dots, l-1, \tag{3}$$

$$|A_H(y_p, y_q)| \geq n-1 \text{ for } 1 \leq p \leq i < q \leq l. \tag{4}$$

If $l = 2$, then P is a path satisfying (2). Hence we may assume that $l > 2$. By (3), we can extend the path P to a path $R = r_1b_1r_2b_2 \dots b_{n-2}r_{n-1}$ in H with $r_1 = y_{i+1}$, $r_{n-1} = y_{i-1}$, $V(R) = V(H) - y_i$. If there is an arc in H in which y_{i-1} precedes y_i and which is not already used in R, then we can find a Hamiltonian (y_{i+1}, y_i)-path in H and $|A_H(y_i, y_{i+1})| \geq n-1$, therefore we may assume that $A_H(y_{i-1}, y_i) = A(R)$.

Since $A(y_{i-1}, y_i) = A(R)$, we observe that b_{n-2} contains the vertices r_{n-2}, y_{i-1}, y_i (in that order), thus, $b_{n-2} \in A_H(r_{n-2}, y_i)$. Let c be an arbitrary arc in $A_H(y_i, y_{i-1})$ $(A_H(y_i, y_{i-1}) \neq \emptyset$ since $|A_H(y_i, y_{i-1})| \geq n-2)$. We now obtain a path

$R' = R[r_1, r_{n-2}]b_{n-2}y_icy_{i-1}$ which satisfies (2) because of (4). Thus the claim (2) is completely proved.

Let $S = s_1d_1s_2d_2 \dots d_{n-1}s_n$ be a Hamiltonian path in H such that $|A_H(s_n, s_1)| \geq n-1$.

If $A_H(s_n, s_1) \neq A(S)$ then there is an arc $e \in A_H(s_n, s_1) - A(S)$, since $|A_H(s_n, s_1)| \geq n-1$ and $|A(S)| = n-1$. Now Ses_1 is a Hamiltonian cycle in H.

If $A_H(s_n, s_1) = A(S)$, then $|A_H(s_n, s_1)| = n-1$. Let f be an arbitrary arc in $A_H(s_1, s_n)$ ($A_H(s_1, s_n)$ is not empty since $|A_H(s_1, s_n)| \geq n-3 \geq 3$). Since $A_H(s_n, s_1) = A(S)$, it follows that $d_{n-1} \in A_H(s_{n-1}, s_1)$ and $d_1 \in A_H(s_n, s_2)$. This implies that $S[s_2, s_{n-1}]d_{n-1}s_1fs_nd_1s_2$ is a Hamiltonian cycle of H.

Case 2: M is strong. There is a Hamiltonian cycle, $C = x_1x_2 \dots x_nx_1$, in M. Since $k \geq 4$, there exist distinct arcs a_1 and a_2, such that $\{a_1, a_2\} \subseteq A_H(x_1, x_2)$. Since $k \geq 4$ and a_1 and a_2 cannot include exactly the same vertices, either a_1 or a_2 does not contain at least one vertex from the set $\{x_4, x_5, \dots, x_{n-1}\}$. Assume w.l.o.g. that $x_i \notin a_1$, where $i \in \{4, 5, \dots, n-1\}$. Since $|A_H(x_j, x_{j+1})| \geq n-2$ for all $j = 1, 2, \dots, n-1$ we can find distinct arcs in H, $b_1, b_2, \dots, b_{n-3}$, such that the following sequence is a path in H:

$$P = x_{i+1}b_1x_{i+2}b_2 \dots x_nb_{n-i}x_1a_1x_2b_{n-i+1}x_3 \dots b_{n-3}x_{i-1}.$$

Since $a_1 \notin A_H(x_{i-1}, x_i)$ and $|A_H(x_{i-1}, x_i)| \geq n-2$, there is an arc $b_{n-2} \in A_H(x_{i-1}, x_i) - A(P)$.

If $A_H(x_i, x_{i+1}) \neq \{b_1, b_2, \dots, b_{n-2}\}$, then let $b_{n-1} \in A_H(x_i, x_{i+1}) - \{b_1, b_2, \dots, b_{n-2}\}$ be arbitrary. Now $Pb_{n-2}x_ib_{n-1}x_{i+1}$ is a Hamiltonian cycle in H.

If $A_H(x_i, x_{i+1}) = \{b_1, b_2, \dots, b_{n-2}\}$, then let $c \in A_H(x_{i+1}, x_i)$ be arbitrary. Observe that $b_1 \in A_H(x_i, x_{i+1}) \cup A_H(x_{i+1}, x_{i+2})$, thus, $b_1 \in A_H(x_i, x_{i+2})$, and $b_{n-2} \in A_H(x_{i-1}, x_i) \cup A_H(x_i, x_{i+1})$, thus, $b_{n-2} \in A_H(x_{i-1}, x_{i+1})$. We now obtain the Hamiltonian cycle

$$P[x_{i+2}, x_{i-1}]b_{n-2}x_{i+1}cx_ib_1x_{i+2},$$

where we define $x_{n+1} = x_1$ (when $i = n-1$). $\square$.

In the rest of this section we adopt the following: H is a strong 4-hypertournament with 6 vertices and $M = M(H)$ is the majority digraph of H.

To prove Lemma 3.6, we need one more lemma.

Lemma 3.5 *If M contains a Hamiltonian path $P = x_1x_2x_3x_4x_5x_6$ such that $A_H(x_6, x_1) \neq \emptyset$, then H is Hamiltonian.*

Proof: To show that H is Hamiltonian, it is sufficient to prove that the family of sets $A_1, A_2, ..., A_6$, where $A_i = A_H(x_i, x_{i+1})$ for $1 \le i \le 5$ and $A_6 = A_H(x_6, x_1)$, has a system of distinct representatives (arcs of H). By P. Hall's matching theorem, such a system exists iff

$$| \cup_{r \in R} A_r| \ge |R| \text{ for all subsets } R \text{ of } \{1,2,3,4,5,6\}. \quad (5)$$

If $|R| \le 3$, then (5) holds by the definition of M ($|A_i| \ge 3$ for all $1 \le i \le 5$). If $4 \le |R| \le 5$, then R contains two integers i, j such that $1 < i+1 < j \le 5$. Obviously, $|A_i \cap A_j| \le 1$, $|A_i|$, $|A_j| \ge 3$. Hence, $| \cup_{r \in R} A_r| \ge |A_i \cup A_j| \ge 5$.

If $|R| = 6$, then

$$| \cup_{r \in R} A_r| \ge |A_1 \cup A_3 \cup A_5| \ge 6.$$

□.

Lemma 3.6 *Every strong* 4*-hypertournament with* 6 *vertices contains a Hamiltonian cycle.*

Proof: Assume that H is not Hamiltonian.

Assume that M is strong. Since M is semicomplete, M has a Hamiltonian cycle. Hence, H is Hamiltonian by Lemma 3.5. Therefore, we may and will assume that M is not strong.

Let (P, Q) be a Hamiltonian pair of paths such that P has maximum possible length and let $P = x_1 a_1 x_2 a_2 \dots a_{m-1} x_m$, $Q = y_1 y_2 \dots y_l$ ($l = 8 - m$, $x_1 = y_l$ and $x_m = y_1$). If $l \ge 3$, then assume w.l.o.g. that $cn(y_2) < cn(y_l)$ (otherwise $cn(y_1) < cn(y_{l-1})$, so we may reverse all arcs). Since $y_1 y_2$ and $y_{l-1} y_l$ are arc in M, we have $|A_H(y_1, y_2)| \ge 3$ and $|A_H(y_{l-1}, y_l)| \ge 3$. By the maximality of P and the fact that H is not Hamiltonian, we conclude that

$$A_H(y_1, y_2) \subseteq A(P), \ A_H(y_{l-1}, y_l) \subseteq A(P). \quad (6)$$

Since $m - 1 = |A(P)| \ge |A_H(y_1, y_2)| \ge 3$, we obtain $4 \le m \le 6$. Consider the following three cases depending on the value of m.

Case 1 ($m = 4$)**:** By $|A_H(y_1, y_2)| \ge 3$, $|A_H(y_{l-1}, y_l)| \ge 3$ and (6), we conclude that $A_H(y_1, y_2) = A_H(y_{l-1}, y_l) = \{a_1, a_2, a_3\}$. The last formula and $|\{y_1, y_2, y_3, y_4\}| = 4 = k$ imply that a_1, a_2, a_3 consist of the same vertices, which is impossible.

Case 2 **(**$m = 5$**):** By (6) and since $|A(P)| < 6$, there exists an arc $d \in (A_H(y_1, y_2) \cup A_H(y_2, y_1)) - A(P) = A_H(y_2, y_1) - A(P)$. If $a_4 \in A_H(y_1, y_2)$ then $(x_1a_1x_2a_2x_3a_3x_4a_4y_2dy_1, y_1y_3)$ is a Hamiltonian pair of paths (since $cn(y_1) < cn(y_3)$). This contradicts the maximality of m. Therefore $a_4 \notin A_H(y_1, y_2)$ which together with (6) and $|A_H(y_1, y_2)| \geq 3$ implies that $A_H(y_1, y_2) = \{a_1, a_2, a_3\}$. Now, the last formula, (6) and $|A_H(y_2, y_3)| \geq 3$ imply that at least two of a_1, a_2, a_3 contain all of y_1, y_2, y_3. However, a_2 and a_3 contain at least two vertices which are not in $\{y_1, y_2, y_3\}$, a contradiction (as $k = 4$).

Case 3 **(**$m = 6$**):** By the definition of a Hamiltonian pair of paths, we have that $cn(x_6) < cn(x_1)$, which implies that $|A_H(x_6, x_1)| \geq 4$. Suppose that $\{a_1, a_5\} \subseteq A_H(x_6, x_1)$. By (6), there is an arc $c_1 \in A_H(x_1, x_6) - A(P)$. Thus, $x_1c_1x_6a_1x_2a_2x_3a_3x_4a_4x_5a_5x_1$ is a Hamiltonian cycle in H. This implies that $\{a_1, a_5\} \not\subseteq A_H(x_6, x_1)$. Assume w.l.o.g. that $A_H(x_6, x_1) = \{a_1, a_2, a_3, a_4\}$ (otherwise $A_H(x_6, x_1) = \{a_2, a_3, a_4, a_5\}$ and reverse all arcs). Observe that a_1 includes the vertices $\{x_1, x_2, x_6\}$, $\bar{a_2} = \{x_1, x_2, x_3, x_6\}$, $\bar{a_3} = \{x_1, x_3, x_4, x_6\}$, $\bar{a_4} = \{x_1, x_4, x_5, x_6\}$, and a_5 includes the vertices $\{x_5, x_6\}$.

Suppose that $|A_H(x_5, x_1)| \geq 3$. Observe that $|A_H(x_1, x_6)| = 2$ and $A_H(x_1, x_6) \cap \{a_1, a_2, a_3, a_4\} = \emptyset$. Therefore there exists an arc $c_1 \in A_H(x_5, x_1) - \{a_1, a_4\}$ and an arc $c_2 \in A_H(x_1, x_6) - \{a_1, a_2, a_3, a_4, c_1\}$. Note that $c_1 \notin \{a_2, a_3\}$. This implies that $x_1c_2x_6a_1x_2a_2x_3a_3x_4a_4x_5c_1x_1$ is a Hamiltonian cycle in H, a contradiction.

Therefore, $|A_H(x_5, x_1)| < 3$. This implies that x_1x_5 is in M, thus, $cn(x_5) \geq cn(x_1)$. We note that $A_H(x_1, x_2) \subseteq \{a_1, a_2, a_5\}$, since we could otherwise find an arc $c_1 \in A_H(x_1, x_2) - \{a_1, a_2, a_3, a_4, a_5\}$, such that $x_1c_1x_2a_2x_3a_3x_4a_4x_5a_5x_6a_1x_1$ is a Hamiltonian cycle in H. Analogously, we can show that $A_H(x_2, x_3) \subseteq \{a_1, a_2, a_5\}$, $A_H(x_3, x_4) \subseteq \{a_3, a_5\}$ and $A_H(x_4, x_5) \subseteq \{a_4, a_5\}$. This implies that $cn(x_1) \geq cn(x_2) \geq cn(x_3) \geq cn(x_4) \geq cn(x_5) \geq cn(x_1)$, which in turn yields $cn(x_1) = cn(x_2) = cn(x_3) = cn(x_4) = cn(x_5)$. Hence, M contains exactly two strong components: the initial one consists of the vertex x_6 and the second one contains all other vertices. Since the second strong component has a Hamiltonian cycle, M contains a Hamiltonian (x_6, x_1)-path. Moreover, $|A_H(x_1, x_6)| = 2$. Therefore, by Lemma 3.5, H is Hamiltonian, a contradiction.

4 Complexity of the Hamiltonian cycle problem for k-hypertournaments

It is well known (see [8] for an efficient algorithm) that the Hamiltonian cycle problem for 2-hypertournaments, i.e. (ordinary) tournaments is polynomial time solvable. The aim of this section is to show that the problem remains polynomial time solvable for $k = 3$, but becomes NP-complete for every fixed $k \geq 4$.

Let $H = (V, A)$ be a k-hypertournament, $A = \{a_1, ..., a_m\}$. Associate with H the following edge-coloured directed multigraph $D(H)$: the vertex set of $D(H)$ is V; for distinct vertices $x, y \in V$, $D(H)$ has the arc xy of colour i iff $a_i \in A_H(x, y)$. Clearly, H contains a path from a vertex x to another vertex y iff $D(H)$ has a path P from x to y such that no two arcs in P have the same colour.

Proposition 4.1 *The Hamiltonian cycle problem for 3-hypertournaments is polynomial time solvable.*

Proof: Let H be a 3-hypertournament. By Theorem 3.1, it suffices to prove that one can check the existence of a path, in H, from a vertex x to another vertex y in polynomial time. Construct the edge-coloured directed multigraph $D(H)$ as above. We prove that H has a path from x to y iff $D(H)$ has some (x, y)-path. Clearly, if H has a path from x to y, then $D(H)$ contains such a path. Suppose that $D(H)$ has a path $Q = x_1...x_p$ from $x = x_1$ to $y = x_p$. If Q has no arcs of the same colour, then Q corresponds, in the obvious way, to an (x, y)-path of H. Suppose that Q contains arcs of the same colour. This means that there exist a subscript i and an integer j such that the arcs $x_{i-1}x_i$ and x_ix_{i+1} have the same colour j (these two are the only arcs of colour j which can be in Q). We can replace Q by the path $Q[x_1, x_{i-1}]Q[x_{i+1}, x_p]$. Continuing this process, we obtain a new path, in $D(H)$, from x to y without repetition of colours. The new path corresponds to an (x, y)-path in H. $\square$.

Theorem 4.2 *Let $k \geq 4$ be an integer. The Hamiltonian cycle problem for k-hypertournaments (k-HCHT) is NP-complete.*

Proof: It is easy to see that k-HCHT is in NP.

To show that our problem is NP-hard, we first transform the well known problem 3-SAT ([7], p. 46) to 4-HCHT. Let $U = \{u_1, ..., u_k\}$ be a set of variables, let $C = \{c_1, ..., c_m\}$ be a set of clauses such that every c_i has three literals, and let v_{il} be the l'th literal in the clause c_i. We may and will assume that $m \geq 3$. We shall construct a 4-hypertournament H which is Hamiltonian iff C is satisfiable. Since it is more convenient, we shall actually construct $D = D(H)$ instead of H.

We first construct a spanning subgraph D' of D. The edge-coloured directed multigraph D' consists of $m + k + 1$ parts: the first m parts X_i^1, $i = 1, ..., m$, correspond to the clauses of C, the next k parts X_i^2, $i = 1, ..., k$, correspond to the variables of U, and the last part $X^3 = X_1^3$ is auxiliary. Every part X_i^1 (X_i^2) consists of vertices $x_{ij} = x_{ij}^1$ ($y_{ij} = x_{ij}^2$, resp.). X_1^3 has two vertices $z_1 = x_{11}^3, z_2 = x_{12}^3$. For a pair of distinct vertices v, w in D', we say that $v < w$ if either $v \in X_i^l$, $w \in X_q^j$ such that $l \leq j$ and if $l = j$ then $i < q$, or $v = x_{ij}^l, w = x_{iq}^l$, where $j < q$.

Note that, in the constructions below, different symbols denote different colours.

Each of the first m parts X_i^1 ($i \in \{1, 2, ..., m\}$) consists of six vertices $x_{i1}, ..., x_{i6}$ and the following arcs: there are two arcs from x_{i1} to x_{i2}, the first of colour a_{i1} and the second of colour b_{i1}; there is an arc from x_{i2} to x_{i3} and from x_{i4} to x_{i5} of colours d_{i1} and d_{i2}, respectively; there are three arcs from x_{i3} to x_{i4} of colours a_{i2}, b_{i1} and b_{i2}; and there are two arcs from x_{i5} to x_{i6} of colours a_{i3} and b_{i2}. Every X_i^1 is connected to X_{i+1}^1 by the arc $x_{i6}x_{i+1,1}$ of colour e_i, for $i = 1, 2, ..., m-1$.

For each $i = 1, 2, ..., k$, let f'_{i1} be the number of appearances of the literal u_i in the clauses of C, and let f'_{i2} be the number of appearances of the literal $\bar{u}_i$ in the clauses of C. Define f_{ij}, $i = 1, 2, ..., k$, $j = 1, 2$, as follows: $f_{ij} = f'_{ij} + 1$ if $f'_{ij} > 0$ and $f_{ij} = 0$, otherwise.

Each of the k parts X_i^2 consists of $f_i = f_{i1} + f_{i2}$ vertices $y_{i1}, ..., y_{if_i}$, and the *basic* arcs $y_{ij}y_{i,j+1}$ for every $j = 1, 2, ..., f_i - 1$ and *additional* arcs $y_{i1}y_{if_{i1}}$, $y_{i,f_{i1}+1}y_{if_i}$ (the first additional arc does not exist if $f_{i1} = 0$ and the second additional arc does not exist if $f_{i2} = 0$). For every $j = 1, ..., f_{i1} - 1$, the basic arc $y_{ij}y_{i,j+1}$ has colour a_{qt}, if the j'th appearance of the variable u_i in C is the t'th literal in the q'th clause c_q. For every $j = f_{i1} + 1, ..., f_i - 1$, the basic arc $y_{ij}y_{i,j+1}$ has colour a_{qt}, if the j'th appearance of the negation of the variable u_i in C is the t'th literal in the q'th clause c_q. If both f_{i1} and f_{i2} are positive, then there exists an arc $y_{if_{i1}}y_{i,f_{i1}+1}$ and its colour is g_i. The colour of the additional arcs $y_{i1}y_{if_{i1}}$ and $y_{i,f_{i1}+1}y_{if_i}$ (possibly, only one of these two arcs does exist) is s_i ($i = 1, ..., k$). Every X_i^2 is connected to X_{i+1}^2 by the arc $y_{if_i}y_{i+1,1}$ of colour p_i.

We say that an arc $y_{i1}y_{if_{i1}}$ corresponds to the literal u_i and an arc $y_{i,f_{i1}+1}y_{if_i}$ corresponds to the literal $\bar{u}_i$. We also say that an arc of D' of colour a_{il} corresponds to the literal v_{il}.

The part X^3 consists of two vertices z_1, z_2 and an arc z_1z_2 of colour c_3. Add two more arcs: $x_{m6}y_{11}$ of colour c_1 and $y_{kf_k}z_1$ of colour c_2.

We have obtained the edge-coloured directed multigraph D'. We shall prove that D' has a path P from x_{11} to z_2 such that no colour in P appears twice iff C is satisfiable.

Suppose first that D' has a path P from x_{11} to z_2 such that no colour in P appears twice. Hence, for every $i = 1, 2, ..., m$, there is at least one arc of colour a_{i,l_i} which is in $P[x_{11}, x_{m6}]$. Hence, the subpath $P[y_{11}, y_{kf_k}]$ contains the arcs of colours s_j's corresponding to the literals v_{i,l_i}, $i = 1, 2, ..., m$. It follows that if the negation of v_{i,l_i} is also in C, then the arcs of colours a_{jq}'s of D' corresponding to the negation of v_{i,l_i} and belonging to P must be in $P[y_{11}, y_{kf_k}]$ and must not be in $P[x_{11}, x_{m6}]$. This fact allows us to assign "true" to every literal v_{i,l_i}, $i = 1, 2, ..., m$ such that there is an arc of colour a_{il_i} belonging to $P[x_{11}, x_{m6}]$. This assignment is proper and makes C satisfied.

Suppose now that C is satisfiable and consider a truth assignment α for U that satisfies all the clauses in C. Let v_{i,l_i}, $i = 1, 2, ..., m$ be true under α. Then the arcs $x_{i,2l_i-1}x_{i,2l_i}$, $i = 1, 2, ..., m$ and the arcs of colours s_r's corresponding to v_{i,l_i}, $i = 1, 2, ..., m$ can be easily extended to a path P from x_{11} to z_2 such that no colour in P appears twice.

Now we construct D from D'. Choose any four vertices v_1, v_2, v_3, v_4 in D' such that $v_1 < v_2 < v_3 < v_4$. In D, the four vertices together with some arcs between them must form a monochromatic transitive 4-tournament such that the colour of this tournament differs from the colours of all other such transitive 4-tournaments. So, we shall add some arcs to D' in order to meet this condition. The symbol $TT(v_{i_1}v_{i_2}v_{i_3}v_{i_4})$ will denote the transitive 4-tournament with vertex set $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ and arc set $\{v_{i_p}v_{i_q} : 1 \leq q < p \leq 4\}$.

If D' contains arcs v_1v_2 and v_3v_4 of the same colour c (a_{ij} or b_{ij} or s_i), then let v_1, v_2, v_3, v_4 form in D the tournament $TT(v_3v_4v_1v_2)$ of colour c. Otherwise, if $\{v_3, v_4\} \neq \{z_1, z_2\}$, then v_1, v_2, v_3, v_4 form the tournament $TT(v_4v_3v_2v_1)$ of a new colour. If $\{v_3, v_4\} = \{z_1, z_2\}$ and D' has an arc from v_1 to v_2 of colour c that appears in D' only once, then let v_1, v_2, v_3, v_4 form the tournament $TT(z_2z_1v_1v_2)$ of colour c. If $\{v_3, v_4\} = \{z_1, z_2\}$ and either D' has an arc from v_1 to v_2 of colour that appears in D' twice or D' has no arc from v_1 to v_2, but $\{v_1, v_2\} \neq \{x_{11}x_{21}\}$ or $\{x_{11}y_{k,f_k}\}$, then v_1, v_2, v_3, v_4 form the tournament $TT(v_4v_3v_2v_1)$ of a new colour. If $(v_1, v_2, v_3, v_4) = (x_{11}, y_{kf_k}, z_1, z_2)$ $((v_1, v_2, v_3, v_4) = (x_{11}, x_{21}, z_1, z_2)$, resp.), then v_1, v_2, v_3, v_4 form the tournament $TT(z_2y_{kf_k}z_1x_{11})$ ($TT(z_1z_2x_{21}x_{11})$, resp.) of colour c_2 (c_3, resp.).

Observe that D is an edge-coloured directed multigraph of some 4-hypertournament H and all arcs vw of D such that $v < w$ are the arcs of D'. Thus, by the construction of D, D has a path P from x_{11} to z_2 which contains no arcs of the same colour iff D' has such a path. Moreover, if D has such a path P, the path P can be extended to a Hamiltonian cycle W in D which contains no arcs of the same colour. Indeed, only some vertices $y'_r = y_{j_r,q_r} \in \cup_{i=1}^{k} X_i^2$ $(r = 1, 2, ..., p)$ are not in P. If $p = 0$, then we use the arc z_2x_{11} of the tournament $TT(z_2x_{31}x_{21}x_{11})$ to construct W. If $p > 0$, then we use the arcs $z_2y'_r, y'_ry'_{r-1}, ..., y'_2y'_1, y'_1x_{11}$ of the tournaments $TT(z_2y'_rx_{21}x_{11})$, $TT(y'_ry'_{r-1}x_{21}x_{11})$,...,$TT(y'_2y'_1x_{21}x_{11})$, $TT(y'_1x_{31}x_{21}x_{11})$ to construct W.

Therefore, the 4-hypertournament H corresponding to D is Hamiltonian iff C is satisfiable. This completes the proof for 4-hypertournaments. One can easily modify the construction of D such that D will correspond to a q-hypertournament, $q \geq 5$, using $q-2$ vertices, instead of two, in the last part X^3 of D'. □.

5 Remarks and open problems

When all results of this paper except Lemma 3.6 were already proved, Susan Marshall informed us (personal communication) that she independently obtained Theorem 2.1 (unpublished).

We have obtained a characterization of Hamiltonian k-hypertournaments with $n \geq k+2$ vertices. Yet, we were unable to characterize Hamiltonian $n-1$-hypertournaments with n vertices. Note that a non-difficult modification of the construction in the proof of Theorem 4.2 shows that the Hamiltonian cycle problem for $n-1$-hypertournaments with n vertices is NP-complete.

It would also be interesting to characterize pancyclic and vertex pancyclic hypertournaments (extensions of well-known theorems by Moser and Moon, respectively, [11]).

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