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Hamiltonian time-dependent mechanics

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The usual formulation of time-dependent mechanics implies a given splitting $Y = \mathbf{R}\times M$ of an event space Y. This splitting, however, is broken by any time-dependent transformation, including transformations between inertial frames. The goal is the frame-covariant formulation of time-dependent mechanics on a bundle $Y\to\mathbf{R}$, whose fibration $Y\to M$ is not fixed. Its phase space is the vertical cotangent bundle V^*Y, provided with the canonical 3-form and the corresponding canonical Poisson structure. An event space of relativistic mechanics is a manifold Z whose fibration $Z\to\mathbf{R}$ is not fixed.
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I. INTRODUCTION

The symplectic technique is well known to provide an adequate mathematical formulation of autonomous mechanics, when Hamiltonians are independent of time.[1–4] Its canonical example is a mechanical system whose phase space is the cotangent bundle T^*M of an event manifold M. This phase space is provided with the canonical symplectic form

$$\Omega = dp_i \wedge dy^i,$$

written with respect to the holonomic coordinates $(y^i,\ p_i = \dot{q}_i)$ on T^*M. A Hamiltonian $\mathcal{H}$ is defined as a real function on T^*M. The motion trajectories are integral curves of the Hamiltonian vector field $\vartheta = \vartheta_i \partial^i + \vartheta^i \partial_i$ on T^*M, which obeys the Hamilton equations

$$\begin{gathered}\vartheta \rfloor \Omega = d\mathcal{H}, \\ \vartheta^i = \partial^i \mathcal{H}, \quad \vartheta_i = -\partial_i \mathcal{H}.\end{gathered} \tag{1}$$

The usual formulation of time-dependent mechanics implies a splitting $Y = \mathbf{R}\times M$ of the event manifold Y and the corresponding splitting $\mathbf{R}\times T^*M$ of the phase space.[5–10] The phase space is provided with the pull-back $\mathrm{pr}_2^*\Omega$ of the symplectic form on T^*M. By a time-dependent Hamiltonian is meant a real function on $\mathbf{R}\times T^*M$, while motion trajectories are integral curves of the time-dependent Hamiltonian vector field,

$$\vartheta : \mathbf{R}\times T^*M \to TT^*M,$$

which obeys the Hamilton equations (1). The problem is that the above-mentioned splittings are broken by any time-dependent transformation, including the inertial frame transformations. Therefore, the form $\mathrm{pr}_2^*\Omega$ on the phase space of time-dependent mechanics fails to be canonical.[11]

We will formulate first-order time-dependent mechanics as a particular field theory, when an event space is a fibered manifold $Y\to\mathbf{R}$, coordinated by (t,y^i).[5,12–14] Unless otherwise stated, the base $\mathbf{R}$ is parametrized by the coordinates t with transition functions $t' = t + \mathrm{const}$. Relative to these coordinates, $\mathbf{R}$ is equipped with the standard vector field ∂_t and the standard 1-form dt, which is also the volume element on $\mathbf{R}$. This is not the case for relativistic mechanics.

The configuration space of time-dependent mechanics is the first-order jet manifold J^1Y of sections of $Y\to\mathbf{R}$, which is provided with the adapted coordinates (t,y^i,y^i_t). There is the canonical monomorphism,

$$\lambda : J^1Y \hookrightarrow TY, \quad \lambda = \partial_t + y^i_t \partial_i,$$

over Y. It is easy to see that $\pi^1_0 : J^1Y \to Y$ is an affine bundle modeled over the vertical tangent bundle $VY \to Y$. For the sake of simplicity, we will identify J^1Y with the corresponding subbundle of TY. In particular, every connection on a bundle $Y \to \mathbf{R}$ can be identified with the horizontal lift,

$$\Gamma : Y \to J^1Y \subset TY, \quad \Gamma = \partial_t + \Gamma^i \partial_i, \tag{2}$$

of the standard vector field ∂_t by this connection. We will call (2) a connection in order to refer to the standard properties of connections without an additional explanation.

The one-dimensional reduction of the polysymplectic Hamiltonian formalism[13–16] provides the adequate mathematical formulation of time-dependent Hamiltonian mechanics on the Legendre bundle $\pi_\Pi : \Pi = V^*Y \to Y$. The phase space V^*Y is endowed with the canonical 3-form,

$$\mathbf{\Omega} = dp_i \wedge dy^i \wedge dt, \tag{3}$$

written with respect to the holonomic coordinates $(t,\ y^i,\ p_i = \dot{y}_i)$ on V^*Y.

The following peculiarities of time-dependent Hamiltonian mechanics should be emphasized.

(i) The form $\mathbf{\Omega}$ (3) defines the canonical degenerate Poisson structure on the phase space V^*Y.

(ii) A Hamiltonian is not a function on a phase space. As a consequence, the evolution equation is not reduced to a Poisson bracket, and integrals of motion cannot be defined as functions in involution with a Hamiltonian.

(iii) Hamiltonian and Lagrangian formulations of time-dependent mechanics are equivalent in the case of hyperregular Lagrangians. A degenerate Lagrangian requires a set of associated Hamiltonians and Hamilton equations in order to exhaust all solutions of the Lagrange ones.

(iv) A complete connection Γ on the event space $Y \to \mathbf{R}$ defines a reference frame so that one can think of the difference $y^i_t - \Gamma^i(y)$ as being the velocity relative to the reference frame Γ. There is one-to-one correspondence between the complete connections Γ and the trivializations of $Y \to \mathbf{R}$.

For the sake of simplicity, $Y \to \mathbf{R}$ is assumed to be a bundle with a typical fiber M. It is trivial. Different trivializations,

$$Y \cong \mathbf{R} \times M, \tag{4}$$

differ from each other in fibrations $Y \to M$, while the fibration $\pi : Y \to \mathbf{R}$ is once for all. Given a trivialization (4), there are the corresponding splittings of the configuration and phase spaces,

$$J^1Y \cong \mathbf{R} \times TM, \quad V^*Y \cong \mathbf{R} \times T^*M.$$

If a fibration $Z \to \mathbf{R}$ of an event space Z is not fixed, we obtain the general formulation of relativistic mechanics, including Special Relativity on the Minkowski space $Z = \mathbf{R}^4$. Its configuration space is the first-order jet manifold J^1_1Z of one-dimensional submanifolds of Z. The bundle $J^1_1Z \to Z$ is projective, and one can think on its fibers as being spaces of nonrelativistic velocities. Relativistic velocities are represented by elements of the tangent bundle TZ, while the cotangent bundle T^*Z plays the role of the phase space of relativistic mechanics.

All manifolds throughout are assumed to be paracompact and connected.

II. CANONICAL POISSON STRUCTURE

The Legendre bundle V^*Y of time-dependent mechanics is provided with the canonical Poisson structure as follows. Let us consider the cotangent bundle T^*Y with the holonomic coordinates (t, y^i, p, p_i), which is the homogeneous Legendre bundle of time-dependent mechanics. It admits the canonical Liouville form

$$\Xi = p\,dt + p_i\,dy^i, \tag{5}$$

and the canonical symplectic form

$$d\Xi = dp \wedge dt + dp_i \wedge dy^i.$$

The corresponding Poisson bracket on the space $C^\infty(T^*Y)$ of functions on T^*Y reads as

$$\{f,g\}=\partial^p f\partial_t g-\partial^p g\partial_t f+\partial^i f\partial_i g-\partial^i g\partial_i f. \tag{6}$$

Let us consider the subspace of $C^\infty(T^*Y)$ that comprises the pull-backs of functions on V^*Y by the projection $T^*Y\to V^*Y$. This subspace is closed under the Poisson bracket (6). Then there exists the canonical Poisson structure,

$$\{f,g\}_V=\partial^i f\partial_i g-\partial^i g\partial_i f, \tag{7}$$

on V^*Y induced by (6).[3] The corresponding Poisson bivector,

$$w(df,dg)=\{f,g\}_V,$$

on V^*Y is vertical with respect to the fibration $V^*Y\to\mathbf{R}$, and reads as

$$w^{ij}=0,\quad w_{ij}=0,\quad w^i_j=1.$$

A glance at this expression shows that the holonomic coordinates on V^*Y are canonical for the Poisson structure (7), which is regular and degenerate.

Given the Poisson bracket (7), the Hamiltonian vector field ϑ_f of a function f on V^*Y, defined by the relation $\{f,g\}_V=\vartheta_f\rfloor dg$, $\forall g\in C^\infty(V^*Y)$, is the vertical vector field

$$\vartheta_f=\partial^i f\partial_i-\partial_i f\partial^i, \tag{8}$$

on $V^*Y\to\mathbf{R}$. Hence, the characteristic distribution of the Poisson structure (7), generated by Hamiltonian vector fields, is precisely the vertical tangent bundle VV^*Y of $V^*Y\to\mathbf{R}$.

By virtue of the well-known theorem,[4] the Poisson structure (7) defines the symplectic foliation on V^*Y that coincides with the fibration $V^*Y\to\mathbf{R}$. The symplectic forms on the fibers of $V^*Y\to\mathbf{R}$ are the pull-backs,

$$\Omega_t=dp_i\wedge dy^i,$$

of the canonical symplectic form on the typical fiber T^*M of $V^*Y\to\mathbf{R}$ with respect to trivialization morphisms.[11]

The Poisson structure (7) can be introduced in a different way. The Legendre bundle V^*Y admits the canonical closed 3-form (3), which is the particular case of the polysymplectic form.[14–16] Then every function f on V^*Y defines the corresponding Hamiltonian vector field ϑ_f (8) by the relation

$$\vartheta_f\rfloor\mathbf{\Omega}=-df\wedge dt,$$

while the Poisson bracket (7) is recovered by the condition

$$\{f,g\}_V dt=\vartheta_g\rfloor\vartheta_f\rfloor\mathbf{\Omega}.$$

III. HAMILTONIAN FORMS

In comparison with autonomous mechanics, Hamiltonian time-dependent mechanics is formulated in the terms of Hamiltonian connections and Hamiltonian forms.

We will say that a connection,

$$\gamma=\partial_t+\gamma^i\partial_i+\gamma_i\partial^i, \tag{9}$$

on the phase space $V^*Y\to\mathbf{R}$ is locally Hamiltonian if the exterior form $\gamma\rfloor\mathbf{\Omega}$ is closed, i.e.,

$$\mathbf{L}_\gamma\mathbf{\Omega}=d(\gamma\rfloor\mathbf{\Omega})=0, \tag{10}$$

where $\mathbf{L}$ denotes the Lie derivative.

For instance, every connection Γ on the event space $Y\to\mathbf{R}$ gives rise to the locally Hamiltonian connection,

$$\widetilde{\Gamma}=\partial_t+\Gamma^i\partial_i-p_i\partial_j\Gamma^i\partial^j,$$

on the phase space, such that

$$\widetilde{\Gamma}\rfloor\mathbf{\Omega}=dH_\Gamma, \quad H_\Gamma=p_idy^i-p_i\Gamma^idt. \tag{11}$$

Locally Hamiltonian connections constitute an affine space modeled over the linear space of vertical vector fields ϑ on $V^*Y\to\mathbf{R}$ that obey the same condition (10), and are locally Hamiltonian vector fields as follows.

Lemma 1: Every closed form $\gamma\rfloor\mathbf{\Omega}$ on $V^*Y\to\mathbf{R}$ is exact.

Proof: Let us consider the decomposition

$$\gamma=\widetilde{\Gamma}+\vartheta, \tag{12}$$

where Γ is a connection on $Y\to\mathbf{R}$, while ϑ satisfies the relation $d(\partial\rfloor\mathbf{\Omega})=0$. It is easily seen that $\vartheta\rfloor\mathbf{\Omega}=\sigma\wedge dt$, where σ is a 1-form. Using properties of the De Rham cohomology groups of a manifold product, one can show that every closed 2-form $\sigma\wedge dt$ on V^*Y is exact, and so is $\gamma\rfloor\mathbf{\Omega}$. Moreover, in accordance with the relative Poincaré lemma, we can write locally $\vartheta\rfloor\mathbf{\Omega}=df\wedge dt$.

Definition 2: A 1-form H on the Legendre bundle V^*Y is called locally Hamiltonian if

$$\gamma\rfloor\mathbf{\Omega}=dH,$$

for a connection γ on $V^*Y\to\mathbf{R}$.

By virtue of Proposition 1, there is one-to-one correspondence between the locally Hamiltonian connections and the locally Hamiltonian forms considered throughout modulo closed forms.

Definition 3: By a Hamiltonian form H on the phase space V^*Y is meant the pull-back,

$$H=h^*\Xi=p_idy^i-\mathscr{H}dt, \tag{13}$$

of the Liouville form Ξ (5) on T^*Y by a section h of the bundle

$$\zeta:T^*Y\to V^*Y. \tag{14}$$

Given a trivialization (4) of $Y\to\mathbf{R}$ the Hamiltonian form (13) looks like the well-known Poincaré–Cartan integral.[2] However, the Hamiltonian $\mathscr{H}$ in the expression (13) is not a function as follows. Any connection Γ on $Y\to\mathbf{R}$ defines the Hamiltonian form H_Γ (11) on V^*Y, and every Hamiltonian form on V^*Y admits the splitting

$$H=p_idy^i-(p_i\Gamma^i+\widetilde{\mathscr{H}}_\Gamma)dt, \tag{15}$$

where Γ is a connection on $Y\to\mathbf{R}$ and $\widetilde{\mathscr{H}}_\Gamma$ is a real function on V^*Y, called the Hamiltonian function.

Proposition 4: Locally Hamiltonian forms are Hamiltonian forms locally.

Proof: Given locally Hamiltonian forms H_γ and $H_{\gamma'}$, their difference,

$$\sigma=H_\gamma-H_{\gamma'}, \quad d\sigma=(\gamma-\gamma')\rfloor\mathbf{\Omega},$$

is a 1-form on V^*Y such that the 2-form $\sigma\wedge dt$ is closed and, consequently, exact. In accordance with the relative Poincaré lemma, this condition implies that $\sigma=fdt+dg$, where f and g are local functions on V^*Y. Then it follows from the splitting (12) that, in a neighborhood of every point $p\in V^*Y$, a locally Hamiltonian form H_γ coincides with the pull-back of the Liouville form Ξ on T^*Y by the local section

$$(t,y^i,p_i)\mapsto(t,y^i,p_i,p=-p_i\Gamma^i+f\ )$$

of $T^*Y \to V^*Y$, where f is a local function on V^*Y.

Proposition 5: Conversely, let H be a Hamiltonian form H on the Legendre bundle V^*Y. There exists a unique connection γ_H on $V^*Y \to \mathbf{R}$, called the Hamiltonian connection, such that $\gamma_H \rfloor \mathbf{\Omega} = dH$.

Proof: Let us introduce the Hamilton operator,

$$\mathcal{E}_H = dH - \lambda \rfloor \mathbf{\Omega} = [(y_t^i - \partial^i \mathcal{H}) dp_i - (p_{ti} + \partial_i \mathcal{H}) dy^i] \wedge dt,$$

on the phase space V^*Y. It is readily observed that this operator is an image of the global section,

$$\gamma_H = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i, \tag{16}$$

of the jet bundle $J^1 V^*Y \to V^*Y$. This is the unique solution of the first-order differential Hamilton equations,

$$y_t^i = \partial^i \mathcal{H}, \quad p_{ti} = -\partial_i \mathcal{H}, \tag{17}$$

on V^*Y, and is a Hamiltonian connection for the Hamiltonian form H.

The integral curves of the Hamiltonian connection (16) are classical solutions of the Hamilton equations (17). Conversely, since $r(\mathbf{R}) \subset V^*Y$ is closed, every classical solution $r:\mathbf{R} \to V^*Y$ of the Hamilton equations (17) can be extended to a unique Hamiltonian connection for H.

Hamiltonian connections γ_H (16) form an affine space modeled over the linear space of Hamiltonian vector fields (8).

Remark: The Hamilton equations (17) can be introduced without appealing to the Hamilton operator. They are equivalent to the relation

$$r^*(u \rfloor dH) = 0,$$

which is assumed to hold for any vertical vector field u on $V^*Y \to \mathbf{R}$.

With a Hamiltonian form H (15) and the corresponding Hamiltonian connection γ_H (16), we have the Hamilton evolution equation,

$$d_{Ht} f = \gamma_H \rfloor df = (\partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i) f, \tag{18}$$

on functions on the Legendre bundle V^*Y. Substituting a classical solution of the Hamilton equations (17) in (18), we obtain the time evolution of the function f. Given the splitting (15) of a Hamiltonian form H, the Hamilton evolution equation (18) is brought into the form

$$d_{Ht} f = \partial_t f + (\Gamma^i \partial_i - \partial_i \Gamma^j p_j \partial^i) f + \{\widetilde{\mathcal{H}}_\Gamma, f\}_V. \tag{19}$$

A glance at this expression shows that the Hamilton evolution equation in time-dependent mechanics does not reduce to the Poisson bracket. This fact may be relevant to the quantization problem. Under quantization, the Poisson bracket on the right-hand side of the equation (19) becomes the operator one, while the second term remains classical and depends on the choice of a reference frame.

IV. CANONICAL TRANSFORMATIONS

Canonical transformations in time-dependent mechanics are not compatible with the fibration $V^*Y \to Y$, in general.

Definition 6: By a canonical automorphism is meant an automorphism ρ over $\mathbf{R}$ of the bundle $V^*Y \to \mathbf{R}$ that preserves the canonical Poisson structure (7) on V^*Y, i.e.,

$$\{f \circ \rho, g \circ \rho\}_V = \{f, g\}_V \circ \rho,$$

and, equivalently, the canonical form $\mathbf{\Omega}$ (3) on V^*Y, i.e., $\mathbf{\Omega} = \rho^* \mathbf{\Omega}$.

The bundle coordinates on $V^*Y \to \mathbf{R}$ are called canonical if they are canonical for the Poisson structure (7). Canonical coordinate transformations satisfy the relations

$$\frac{\partial p'_i}{\partial p_j}\frac{\partial y'^i}{\partial p_k}-\frac{\partial p'_i}{\partial p_k}\frac{\partial y'^i}{\partial p_j}=0,\quad \frac{\partial p'_i}{\partial y^j}\frac{\partial y'^i}{\partial y^k}-\frac{\partial p'_i}{\partial y^k}\frac{\partial y'^i}{\partial y^j}=0,\quad \frac{\partial p'_i}{\partial p_k}\frac{\partial y'^i}{\partial y^j}-\frac{\partial p'_i}{\partial y^j}\frac{\partial y'^i}{\partial p_k}=\delta^k_j.$$

By definition, the holonomic coordinates on V^*Y are the canonical ones. Accordingly, holonomic automorphisms,

$$(y^i,p_i)\mapsto\left(y'^i,p'_i=\frac{\partial y^j}{\partial y'^i}\,p_j\right)$$

of the phase space $V^*Y\to Y$ induced by the vertical automorphisms of $Y\to\mathbf{R}$ are also canonical.

Proposition 7: Canonical automorphisms send locally Hamiltonian connections onto the locally Hamiltonian ones (and, consequently, locally Hamiltonian forms onto each other).

Proof: If γ is a locally Hamiltonian connection for H, we have

$$T\rho(\gamma)\rfloor\mathbf{\Omega}=(\rho^{-1})^*(\gamma\rfloor\mathbf{\Omega})=d((\rho^{-1})^*H).$$

It should be emphasized that, in general, canonical automorphisms do not send Hamiltonian forms onto Hamiltonian forms, but only locally.

Proposition 8: Let γ be a complete locally Hamiltonian connection on $V^*Y\to\mathbf{R}$, i.e., the vector field (9) is complete. There exist canonical coordinate transformations that bring all components of γ to zero, i.e., $\gamma=\partial_t$.

Proof: A glance at the relation (10) shows that each locally Hamiltonian connection γ is the generator of a local one-parameter group G_γ of canonical automorphisms of the phase space $V^*Y\to\mathbf{R}$. Let V^*_0Y be the fiber of $V^*Y\to\mathbf{R}$ at $0\in\mathbf{R}$. Then canonical coordinates of V^*_0Y dragged along integral curves of the complete vector field γ satisfy the statement of the proposition.

In particular, let H be a Hamiltonian form (15) such that the corresponding Hamiltonian connection γ_H (16) is complete. By virtue of Proposition 8, there exist canonical coordinate transformations that bring the Hamiltonian $\mathcal{H}$ into zero. Then the corresponding Hamilton equations reduce to the equilibrium ones. Accordingly, any Hamiltonian for H can be locally brought into the form where $\mathcal{H}=0$ by local canonical coordinate transformations.

Every canonical transformation admits a local generating function as follows. Let H be a Hamiltonian form (13) on V^*Y. Given a canonical automorphism ρ, we have

$$d(\rho^*H-H)=0,$$

$$\rho^*H-H=dS,$$

where S is a local function on V^*Y. We can write locally

$$\rho^*H=\rho_i d\rho^i-\mathcal{H}\circ\rho dt.$$

Then the corresponding coordinate relations read as

$$\partial_i S=\rho_j\partial_i\rho^j-p_i,\quad \partial^i S=\rho_j\partial^i\rho^j,\quad \mathcal{H}'-\mathcal{H}=\rho_i\partial_t\rho^i-\partial_t S.$$

Taken on the graph,

$$\Delta_\rho=\{(p,\rho(p))\in V^*Y\times V^*Y\},$$

of the canonical automorphism, the function S plays the role of a local generating function. For instance, if the graph Δ_ρ is coordinated by (t,y^i,y'^i), we obtain the familiar expression $\mathcal{H}'-\mathcal{H}=\partial_t S(t,y^i,y'^i)$.

V. REFERENCE FRAMES

Every connection Γ on the bundle $\pi:Y\to\mathbf{R}$ defines a horizontal foliation on $Y\to\mathbf{R}$ whose leaves are the integral curves of the nowhere vanishing vector field (2). Conversely, let Y admit a horizontal foliation such that, for each point $y\in Y$, the leaf of this foliation through y is locally determined by a section s_y of $V^*Y\to\mathbf{R}$ through y. Then, the map

$$\Gamma: Y \to J^1Y, \quad \Gamma(y) = j^1_t s_y, \quad \pi(y) = t,$$

is well defined. This is a connection on $Y \to \mathbf{R}$.

Given a horizontal foliation on Y, there exists the associated atlas of constant local trivializations of Y such that every leaf of this foliation is locally generated by the equations $y^i = \text{const}$, and the transition functions $y^i \to y'^i(y^j)$ are independent of the coordinate t.[16,17] Two such atlases are said to be equivalent if their union is also an atlas of constant local trivializations. They are associated with the same horizontal foliation. Thus, we have proved the following assertion.

Proposition 9: There is one-to-one correspondence between the connections Γ on $Y \to \mathbf{R}$ and the equivalence classes of atlases of constant local trivializations of Y such that $\Gamma^i = 0$ relative to the associated coordinates, called adapted to Γ.

Proposition 10: Every trivialization of $Y \to \mathbf{R}$ yields a complete connection on this bundle. Conversely, every complete connection on $Y \to \mathbf{R}$ defines a trivialization $Y \cong \mathbf{R} \times M$ such that the associated coordinates are adapted to Γ.

Proof: Every trivialization of $Y \to \mathbf{R}$ defines the horizontal lift $\Gamma = \partial_t$ onto Y of the standard field ∂_t on $\mathbf{R}$, which is obviously a complete connection on $Y \to \mathbf{R}$. Conversely, let Γ be a complete connection on $Y \to \mathbf{R}$. This is the generator of the one-parameter group G_Γ that acts freely on Y. The orbits of this action are, of course, the integral curves of Γ. Hence, we obtain a projection,

$$\pi_\Gamma: Y \to Y/G_\Gamma = M.$$

This projection together with $\pi: Y \to \mathbf{R}$ defines a trivialization of Y.

One can say that a complete connection Γ on an event space $Y \to \mathbf{R}$ describes a reference frame in time-dependent mechanics. Given a reference frame Γ, we have the corresponding covariant differential,

$$D_\Gamma: \; J^1Y \to VY, \quad (t, y^i, y^i_t) \mapsto (t, y^i, \dot{y}^i = y^i_t - \Gamma^i).$$

Let s be a (local) section of $Y \to \mathbf{R}$. One can think of $D_\Gamma \circ J^1 s$ as being the relative velocity of the motion s with respect to the reference frame Γ. Indeed, $D_\Gamma \circ J^1 s$ vanishes identically iff s is an integral curve of Γ.

Let us consider the Hamilton evolution equation (19). For any connection Γ in the splitting (19), there exist holonomic canonical transformations of V^*Y to the coordinates adapted to Γ that bring (19) into the familiar Poisson bracket form,

$$d_{Ht} f = \partial_t f + \{\widetilde{\mathcal{H}}, f\}_V.$$

VI. LAGRANGIAN POISSON STRUCTURE

In contrast with the Legendre bundle V^*Y, the configuration space J^1Y of time-dependent mechanics does not possess any canonical Poisson structure, in general. A Poisson structure on J^1Y depends on the choice of a Lagrangian,

$$L = \mathcal{L} dt, \quad \mathcal{L}: \; J^1Y \to \mathbf{R}. \tag{20}$$

We will use the notation $\pi_i = \partial^t_i \mathcal{L}$, $\pi_{ij} = \partial^t_i \partial^t_j \mathcal{L}$.

Every Lagrangian L (20) defines the Legendre map,

$$\hat{L}: J^1Y \to V^*Y, \quad p_i \circ \hat{L} = \pi_i. \tag{21}$$

The pull-back on J^1Y of the canonical 3-form $\mathbf{\Omega}$ (3) by the Legendre map $\hat{L}$ (21) reads as

$$\mathbf{\Omega}_L = \hat{L}^* \mathbf{\Omega} = d\pi_i \wedge dy^i \wedge dt.$$

By means of $\mathbf{\Omega}_L$, every vertical vector field $\vartheta = \vartheta^i \partial_i + \dot{\vartheta}^i \partial^t_i$ on $J^1Y \to \mathbf{R}$ yields the 2-form

$$\vartheta \rfloor \mathbf{\Omega}_L = \{[\dot{\vartheta}^j \pi_{ji} + \vartheta^j(\partial_j \pi_i - \partial_i \pi_j)] dy^i - \vartheta^i \pi_{ji} dy^j_t\} \wedge dt.$$

This is one-to-one correspondence, if the Lagrangian L is regular. Indeed, given any 2-form $\phi = (\phi_i dy^i + \dot{\phi}_i dy_t^i)\wedge dt$ on J^1Y, the algebraic equations,

$$\dot{\vartheta}^j \pi_{ji} + \vartheta^j(\partial_j \pi_i - \partial_i \pi_j) = \phi_i, \quad -\vartheta^i \pi_{ji} = \dot{\phi}_j,$$

have a unique solution,

$$\vartheta^i = -(\pi^{-1})^{ij}\dot{\phi}_j, \quad \dot{\vartheta}^j = (\pi^{-1})^{ji}[\phi_i + (\pi^{-1})^{kn}\dot{\phi}_n(\partial_k \pi_i - \partial_i \pi_k)].$$

In particular, every function f on J^1Y determines a vertical vector field,

$$\vartheta_f = (\pi^{-1})^{ij}\partial_j^t f \partial_i - (\pi^{-1})^{ji}[\partial_i f + (\pi^{-1})^{kn}\partial_n^t f(\partial_k \pi_i - \partial_i \pi_k)]\partial_j^t, \tag{22}$$

on $J^1Y \to \mathbf{R}$ in accordance with the relation

$$\vartheta_f \rfloor \mathbf{\Omega}_L = -df \wedge dt.$$

Then the Poisson bracket,

$$\{f,g\}_L dt = \vartheta_g \rfloor \vartheta_f \rfloor \mathbf{\Omega}_L, \quad f,g \in C^\infty(J^1Y), \tag{23}$$

can be defined on functions on J^1Y, and reads as

$$\{f,g\}_L = (\pi^{-1})^{ij}(\partial_i^t f \partial_j g - \partial_i^t g \partial_j f\ ) + (\partial_n \pi_k - \partial_k \pi_n)(\pi^{-1})^{ki}(\pi^{-1})^{nj}\partial_i^t f \partial_j^t g.$$

The vertical vector field ϑ_f (22) is the Hamiltonian vector field of the function f with respect to the Poisson structure (23).

In particular, if the Lagrangian L is hyperregular, that is, the Legendre map $\hat{L}$ is a diffeomorphism, the Poisson structure (23) is obviously isomorphic to the Poisson structure (7) on the phase space V^*Y.

The Poisson structure (23) defines the corresponding symplectic foliation on J^1Y that coincides with the fibration $J^1Y \to \mathbf{R}$. The symplectic form on the leaf J_t^1Y of this foliation is $\Omega_t = d\pi_i \wedge dy^i$.[18]

We will see below that the Lagrangian counterpart of Hamiltonian forms is the Poincaré–Cartan form,

$$H_L = \pi_i dy^i - (\pi_i y_t^i - \mathscr{L})dt.$$

This is the unique Lepagian equivalent of a Lagrangian L that participates in the first variational formula. The Poincaré–Cartan form defines the morphism

$$\hat{H}_L : J^1Y \to T^*Y, \quad p \circ \hat{H}_L = \mathscr{L} - \pi_i y_t^i, \quad p_i \circ \hat{H}_L = \pi_i. \tag{24}$$

There is the relation $\hat{L} = \zeta \circ \hat{H}_L$, where ζ is the canonical projection (14).

Let

$$u = u^t \partial_t + u^i \partial_i, \quad u^t = 0,1, \tag{25}$$

be a vector field on $Y \to \mathbf{R}$. The first variational formula provides the canonical decomposition of the Lie derivative,

$$\mathbf{L}_{J^1u}L = (J^1u \rfloor d\mathscr{L})dt = (u^t\partial_t + u^i\partial_i + d_t u^i \partial_i^t)\mathscr{L}dt,$$
$$d_t = \partial_t + y_t^i \partial_i + \cdots, \tag{26}$$

in accordance with the variational task.[16,19] We have

$$J^1u \rfloor d\mathscr{L} = (u^i - u^t y_t^i)\mathscr{E}_i + d_t(u \rfloor H_L), \tag{27}$$

where

$$\mathcal{E}_L=(\partial_i-d_t\partial_i^t)\mathcal{L}dy^i\wedge dt \tag{28}$$

is the Euler–Lagrange operator for L. The kernel Ker $\mathcal{E}_L\subset J^2Y$ of this operator defines the second–order Lagrange equations on Y,

$$(\partial_i-d_t\partial_i^t)\mathcal{L}=0. \tag{29}$$

Definition 11: A connection,

$$\xi=\partial_t+\xi^i\frac{\partial}{\partial^i}+\xi_t^i\frac{\partial}{\partial\hat{y}_t^i},$$

on the configuration space $J^1Y\to\mathbf{R}$ is said to be a Lagrangian connection for the Lagrangian L if it obeys the condition

$$\xi\rfloor\Omega_L=dH_L, \tag{30}$$

which takes the coordinate form

$$(\xi^i-y_t^i)\pi_{ij}=0,$$

$$\partial_i\mathcal{L}-\partial_t\pi_i-\xi^j\partial_j\pi_i-\xi_t^j\pi_{ij}+(\xi^j-y_t^j)\partial_i\pi_j=0,$$

relative to the adapted coordinates $(t,y^i,y_t^i,\hat{y}_t^i,y_{tt})$ on J^1J^1Y.

In order to clarify the meaning of (30), let us consider the Lagrangian,

$$\overline{L}=L+(\hat{y}_t^i-y_t^i)\pi_i dt,$$

on the repeated jet manifold J^1J^1Y. The corresponding Euler–Lagrange operator, called the Euler–Lagrange–Cartan one, reads as

$$\begin{gathered}\mathcal{E}_{\overline{L}}=[(\partial_iL-\hat{d}_t\pi_i+\partial_i\pi_j(\hat{y}_t^j-y_t^j))dy^i+\partial_i^t\pi_j(\hat{y}_t^j-y_t^j)dy_t^i]\wedge dt,\\ \hat{d}_t=\partial_t+\hat{y}_t^i\partial_i+y_{tt}^i\partial_i^t.\end{gathered} \tag{31}$$

Then the condition (30) is equivalent to the one Im $\xi\subset$Ker $\mathcal{E}_{\overline{L}}$, and leads to the first-order differential equations on the jet manifold J^1Y, called the Cartan equations,

$$\partial_i^t\pi_j(\hat{y}_t^j-y_t^j)=0,\quad \partial_iL-\hat{d}_t\pi_i+(\hat{y}_t^j-y_t^j)\partial_i\pi_j=0. \tag{32}$$

Integral curves of Lagrangian connections ξ for L provide classical solutions $\overline{s}:\mathbf{R}\to J^1Y$ of these equations.

The restriction of $\mathcal{E}_{\overline{L}}$ to the holonomic jet manifold J^2Y defines the first-order Euler–Lagrange operator whose kernel is the system of the first-order Lagrange equations,

$$\hat{y}_\lambda^i-y_\lambda^i=0,\quad (\partial_i-d_\lambda\partial_i^\lambda)\mathcal{L}=0. \tag{33}$$

These are equivalent to the second-order Lagrange equations (29), and represent their familiar first-order reduction.

It is easily seen that the first-order Lagrange equations (33) [and consequently the second-order ones (29)] are equivalent to the Cartan equations (32) on the integrable sections $\overline{s}=J^1s$ of $J^1Y\to\mathbf{R}$. They are completely equivalent to the Cartan equations in the case of regular Lagrangians.

VII. DEGENERATE SYSTEMS

Let us consider the relations between the solutions of Hamilton and Lagrange equations. Every Hamiltonian form H on the phase space V^*Y defines the Hamiltonian map

$$\hat{H}: V^*Y \to J^1Y, \quad y_t^i \circ \hat{H} = \partial^i \mathcal{H}.$$

Its jet prolongation reads as

$$J^1\hat{H}: J^1V^*Y \to J^1J^1Y, \quad (y_t^i, \hat{y}_t^i, y_{tt}^i) \circ J^1\hat{H} = (\partial^i \mathcal{H}, y_t^i, d_t \partial^i \mathcal{H}).$$

Following the general polysymplectic scheme, we say that a Hamiltonian form H on V^*Y is associated with a Lagrangian L if H obeys the conditions[14–16]

$$\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L}, \tag{34a}$$

$$p_i \partial^i \mathcal{H} - \mathcal{H} = \mathcal{L} \circ \hat{H}. \tag{34b}$$

It follows from the condition (34a) that $\hat{L} \circ \hat{H}$ is the projection operator to $Q = \hat{L}(J^1Y) \subset V^*Y$, called the Lagrangian constraint space, and $\hat{H} \circ \hat{L}$ is the projection operator to $\hat{H}(Q) \subset J^1Y$. Given a Hamiltonian form H associated with L, the Lagrangian constraint space is defined by the coordinate relation,

$$p_i = \partial_i^t \mathcal{L}(t, y^j, \partial^j \mathcal{H}).$$

If a Lagrangian L is hyperregular, there exists a unique Hamiltonian form associated with L.

Let a Lagrangian L be semiregular, i.e., the preimage $\hat{L}^{-1}(p)$ of any point $p \in Q$ is a connected submanifold of J^1Y. The following assertions issue from the corresponding theorems of a polysymplectic formalism.[14,16,20]

Proposition 12: All Hamiltonian forms H associated with a semiregular Lagrangian L coincide on the Lagrangian constraint space Q, and the Poincaré–Cartan form H_L is the pull-back of any such a Hamiltonian form H by the Legendre map $\hat{L}$, i.e.,

$$\pi_i y_t^i - \mathcal{L} = \mathcal{H}(t, y^i, \pi_i). \tag{35}$$

Proposition 13: Let a section r of the bundle $V^*Y \to \mathbf{R}$ be a solution of the Hamilton equations (17) for a Hamiltonian form H associated with a semiregular Lagrangian density L. If r lives in the Lagrangian constraint space Q, the section $s = \pi_\Pi \circ r$ of the bundle $Y \to \mathbf{R}$ satisfies the Lagrange equations (29), while its jet prolongation $\overline{s} = \hat{H} \circ r = J^1 s$ obeys the Cartan equations (32). Conversely, let a section $\overline{s}$ of $J^1Y \to \mathbf{R}$ be a solution of the Cartan equations (32) for a semiregular Lagrangian L. Let H be a Hamiltonian form associated with L so that the corresponding Hamiltonian map satisfies the condition

$$\hat{H} \circ \hat{L} \circ \overline{s} = J^1(\pi_0^1 \circ \overline{s}). \tag{36}$$

Then the section $r = \hat{L} \circ \overline{s}$ of $V^*Y \to \mathbf{R}$ is a solution of the Hamilton equations for H.

Proof: One can show that, in the case of a semiregular Lagrangian L, the Euler–Lagrange–Cartan operator (31) is the pull-back,

$$\mathcal{E}_{\overline{L}} = (J^1\hat{L})^* \mathcal{E}_H, \tag{37}$$

of the Hamilton operator for a Hamiltonian form H associated with L. In accordance with the relation (37), if γ_H is a Hamiltonian connection for H, the composition,

$$J^1\hat{H} \circ \gamma_H : V^*Y \to J^2Y, \quad (y_t^i, \hat{y}_t^i, y_{tt}^i) \circ J^1\hat{H} \circ \gamma_H = (\partial^i \mathcal{H}, \partial^i \mathcal{H}, d_{Ht} \partial^i \mathcal{H}),$$

takes its values into the kernel of the Euler–Lagrange operator $\mathcal{E}_L$. Then the morphism $J^1\hat{H}\circ\gamma_H\circ\hat{L}$ restricted to $\hat{H}(Q)$ is a local Lagrangian connection on $\hat{H}(Q)\subset J^1Y$. Conversely, the first of the Hamilton equations (17) is satisfied due to the condition (36), while the second one reduces to the second of the Cartan equations (32) because of the relation (35).

Since, by Proposition 13, solutions of the Lagrange equations for a degenerate Lagrangian may correspond to solutions of different Hamilton equations, we can conclude that, roughly speaking, the Hamilton equations involve some additional conditions in comparison with the Lagrange ones. Therefore, let us separate a part of the Hamilton equations which are defined on the Lagrangian constraint space Q in the case of almost regular Lagrangians.

Definition 14: A semiregular Lagrangian L is said to be almost regular if (i) the Lagrangian constraint space $Q\to Y$ is a closed imbedded subbundle $i_Q:Q\hookrightarrow V^*Y$ of the Legendre bundle $V^*Y\to Y$; and (ii) the Legendre map $\hat{L}:J^1Y\to Q$ is a bundle.

Let $H_Q=i_Q^*H$ be the restriction of a Hamiltonian form H associated with L to the constraint space Q. By virtue of Proposition 12, this restriction, called the constrained Hamiltonian form, is uniquely defined, and $H_L=\hat{L}^*H_Q$. For sections r of the bundle $Q\to\mathbf{R}$, we can write the constrained Hamilton equations,

$$r^*(u_Q\rfloor dH_Q)=0, \tag{38}$$

where u_Q is an arbitrary vertical vector field on $Q\to\mathbf{R}$.[14,16,21] In brief, we can identify a vertical vector field u_Q on $Q\to\mathbf{R}$ with its image $Ti_Q(u_Q)$ and can bring the constrained Hamilton equations (38) into the form

$$r^*(u_Q\rfloor dH)=0, \tag{39}$$

where r is a section of $Q\to\mathbf{R}$ and u_Q is an arbitrary vertical vector field on $Q\to\mathbf{R}$. These equations fail to be equivalent to the Hamilton equations restricted to the constraint space Q.

The following two assertions together with Proposition 13 give the relations between Cartan, Hamilton, and constrained Hamilton equations when a Lagrangian is almost regular.[16]

Proposition 15: For any Hamiltonian form H associated with an almost regular Lagrangian L, every solution r of the Hamilton equations that lives in the Lagrangian constraint space is a solution of the constrained Hamilton equations (39).

Proposition 16: A section $\overline{s}$ of $J^1Y\to\mathbf{R}$ is a solution of the Cartan equations (32) iff $\hat{L}\circ\overline{s}$ is a solution of the constrained Hamilton equations (39).

In the spirit of a well-known Dirac–Bergmann algorithm for analyzing constrained systems in symplectic mechanics, the Lagrangian constraint space Q plays the role of the primary constraint one. However, we cannot extend this algorithm to time-dependent mechanics because the Poisson bracket of a Hamiltonian and constraint functions fails to be defined.

At the same time, one can hope to find a complete family of Hamiltonian forms associated with a degenerate Lagrangian L so that, for each solution of the Lagrange equations, there exists the corresponding solution of the Hamilton ones for some Hamiltonian form from this family. By Proposition 13, if L is semiregular, such a complete family of associated Hamiltonian forms exists iff, for every solution s of the Lagrange equations, there is a Hamiltonian form H from this family such that

$$\hat{H}\circ\hat{L}\circ J^1s=J^1s.$$

In particular, a complete family of Hamiltonian forms associated with an almost regular quadratic Lagrangian always exists.[14–16]

Note that, since Hamiltonians in time-dependent mechanics are not functions on a phase space, we cannot apply to them the well-known analysis of the normal forms[22] (e.g., quadratic Hamiltonians[2]) in symplectic mechanics.

VIII. CONSERVATION LAWS

In autonomous mechanics, an integral of motion, by definition, is a function on the phase space whose Poisson bracket with a Hamiltonian is equal to zero. This notion cannot be extended to time-dependent mechanics because the Hamiltonian evolution equation (19) is not reduced to the Poisson bracket.

Let us start from conservation laws in Lagrangian mechanics. To obtain differential conservation laws, we use the first variational formula (27).[16,19] On shell, this leads to the weak identity

$$J^1u\rfloor d\mathcal{L}\approx -d_t\mathcal{T}, \tag{40}$$

where

$$\mathcal{T}=\pi_i(u^ty_t^i-u^i)-u^t\mathcal{L} \tag{41}$$

is the current along the vector field u (25). If the Lie derivative $\mathbf{L}_{J^1u}L$ (26) vanishes, we have the conservation law

$$0\approx -d_t[\pi_i(u^ty_t^i-u^i)-u^t\mathcal{L}].$$

This is brought into the differential conservation law,

$$0\approx -\frac{d}{dt}\,(\pi_i\circ s(u^t\partial_ts^i-u^i\circ s)-u^t\mathcal{L}\circ s),$$

on solutions s of the Lagrange equations. A glance at this expression shows that, in time-dependent mechanics, the conserved current (41) plays the role of an integral of motion.

Every symmetry current (41) along a vector field u (25) on Y can be represented as a superposition of the Nöther current along a vertical vector field ϑ and of the energy current along some connection Γ (2) on $Y\to\mathbf{R}$, where $u=\vartheta+\Gamma$.[19,23]

If ϑ is a vertical vector field, the weak identity (40) reads as

$$(\vartheta^i\partial_i+d_t\vartheta^i\partial_i^t)\mathcal{L}\approx d_t(\pi_i\vartheta^i).$$

If the Lie derivative of L along ϑ equals zero, we have the integral of motion $\mathcal{T}=\pi_i\vartheta^i$.

In the case of a connection Γ (2), the weak identity (40) takes the form

$$(\partial_t+\Gamma^i\partial_i+d_t\Gamma^i\partial_i^t)\mathcal{L}\approx -d_t(\pi_i(y_t^i-\Gamma^i)-\mathcal{L}), \tag{42}$$

where one can think of

$$\mathcal{T}_\Gamma=\pi_i(y_t^i-\Gamma^i)-\mathcal{L} \tag{43}$$

as being the energy function with respect to the reference frame Γ. In particular, the energy conservation law (42) written relative to the coordinates adapted to Γ takes the familiar form

$$\partial_t\mathcal{L}=-d_t(\pi_iy_t^i-\mathcal{L}). \tag{44}$$

To discover conservation laws within the framework of Hamiltonian formalism, we will do the following trick. Given a Hamiltonian form H (13) on V^*Y, let us consider the Lagrangian,

$$L_H=(p_iy_t^i-\mathcal{H})dt, \tag{45}$$

on the jet manifold J^1V^*Y. It is readily observed that the Poincaré–Cartan form of the Lagrangian L_H coincides with the Hamiltonian form H, and the Euler–Lagrange operator for L_H is precisely the Hamilton operator for H. As a consequence, the Lagrange equations for L_H are equivalent to the Hamilton equations for H. Therefore, let us apply the first variational formula (27) to the Lagrangian (45).[19]

Given a vector field (25) on the event bundle Y, its canonical lift onto V^*Y reads as

$$\widetilde{u}=u^t\partial_t+u^i\partial_i-\partial_i u^j p_j\partial^i,\quad u^t=0,1.$$

Substituting this vector field into the weak identity (40), we obtain

$$-u^i\partial_i\mathcal{H}-u^t\partial_t\mathcal{H}+p_i d_t u^i\approx -d_t(-p_i u^i+u^t\mathcal{H}), \tag{46}$$

for the current

$$\widetilde{\mathcal{T}}=-p_i u^i+u^t\mathcal{H}. \tag{47}$$

In the case of $u=\Gamma$, the weak identity (46) takes the form

$$-\partial_t\mathcal{H}-\Gamma^i\partial_i\mathcal{H}+p_i d_t\Gamma^i\approx -d_t\widetilde{\mathcal{H}}_\Gamma,$$

where $\widetilde{\mathcal{H}}_\Gamma=\mathcal{H}-p_i\Gamma^i$ is the Hamiltonian function in the splitting (15).

The following assertion shows that the Hamiltonian function $\widetilde{\mathcal{H}}_\Gamma$ is the Hamiltonian counterpart of the Lagrangian energy function $\mathcal{T}_\Gamma$ (43) in the case of semiregular Lagrangians.[16]

Proposition 17: Let a Hamiltonian form H on the Legendre bundle V^*Y be associated with a semiregular Lagrangian $\mathcal{L}$ on J^1Y. Let r be a solution of the Hamilton equations (17) for H that lives in the Lagrangian constraint space Q and s the corresponding solution of the Lagrange equations for L. Then, we have

$$\widetilde{\mathcal{T}}(r)=\mathcal{T}(\hat{H}\circ r),\quad \widetilde{\mathcal{T}}(\hat{L}\circ J^1 s)=\mathcal{T}(s),$$

where $\mathcal{T}$ is the current (41) on J^1Y and $\widetilde{\mathcal{T}}$ is the current (47) on V^*Y.

Therefore, we can treat $\widetilde{\mathcal{H}}_\Gamma$ as the energy function with respect to the reference frame Γ. In particular, if $\Gamma^i=0$, we obtain the well-known energy conservation law

$$\partial_t\mathcal{H}\approx d_t\mathcal{H},$$

which is the Hamiltonian variant of the Lagrangian conservation law (44).

IX. RELATIVISTIC MECHANICS

Let us consider a mechanic system whose event space Z has no fibration $Z\to\mathbf{R}$ or admits different such fibrations. We come to relativistic mechanics where a configuration space is the jet manifold of one-dimensional submanifolds of Y that generalizes the notion of jets of sections of a bundle.[16,24,25]

Let Z be a manifold of dimension $m+n$. The first-order jet manifold J^1_nZ of n-dimensional submanifolds of Z comprises the equivalence classes $[S]^1_z$ of n-dimensional imbedded submanifolds of Z that pass through $z\in Z$ and that are tangent to each other at z. It is provided with a manifold structure as follows.

Let $Y\to X$ be an $(m+n)$-dimensional bundle over an n-dimensional base X and Φ an imbedding of Y into Z. Then there is the natural injection,

$$J^1\Phi\colon\ J^1Y\to J^1_nZ,$$

$$j^{\prime}_x s\mapsto [S]^1_{\Phi(s(x))},\quad S=\mathrm{Im}(\Phi\circ s),$$

where s are sections of $Y\to X$. This injection defines a chart on J^1_nZ. Such charts cover the set J^1_nZ, and transition functions between these charts are differentiable. They provide J^1_nZ with the structure of a finite-dimensional manifold.

Hereafter, we will use the following coordinate atlases on the jet manifold J^1_nZ of submanifolds of Z. Let Z be endowed with a manifold atlas with coordinate charts,

$$(\mathrm{z}^A),\quad A=1,...,n+m. \tag{48}$$

Though $J^0_n Z$, by definition, is diffeomorphic to Z, let us provide $J^0_n Z$ with the atlas obtained by replacing every chart (z^A) on a domain $U \subset Z$ with the charts on the same domain U that correspond to the different partitions of the collection (z^A) in collections of n and m coordinates. We denote these coordinates by

$$(x^\lambda, y^i), \quad \lambda = 1,...,n, \quad i = 1,...,m. \tag{49}$$

The transition functions between the coordinate charts (49) of $J^0_n Z$ associated with the coordinate chart (48) of Z are reduced simply to exchange between coordinates x^λ and y_i. Transition functions between arbitrary coordinate charts of the manifold $J^0_n Z$ take the form

$$\widetilde{x}^\lambda = \widetilde{g}^\lambda (x^\mu, y^j), \quad \widetilde{y}^i = \widetilde{f}^i (x^\mu, y^j). \tag{50}$$

Given the coordinate atlas (49) of the manifold $J^0_n Z$, the jet manifold $J^1_n Z$ of Z is endowed with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$. Using the formal total derivatives $d_\lambda = \partial_\lambda + y^i_\lambda \partial^i$, one can write the transformation rules for these coordinates in the following form. Given the coordinate transformations (50), it is easy to find that

$$d_{\widetilde{x}^\lambda} = [d_{\widetilde{x}^\lambda} g^\alpha (\widetilde{x}^\lambda, \widetilde{y}^i)] d_{x^\alpha}. \tag{51}$$

Then we have

$$\widetilde{y}^i_\lambda = \left[\left(\frac{\partial}{\partial \widetilde{x}^\lambda} + \widetilde{y}^p_\lambda \frac{\partial}{\partial \widetilde{y}^p} \right) g^\alpha (\widetilde{x}^\lambda, \widetilde{y}^i) \right] \left(\frac{\partial}{\partial x^\alpha} + y^j_\alpha \frac{\partial}{\partial y^j} \right) \widetilde{f}^i (x^\mu, y^j). \tag{52}$$

Remark: Given a manifold Z, there is one-to-one correspondence between the jets $[S]^1_z$ at a point $z \in Z$ and the n-dimensional vector subspaces of the tangent space $T_z Z$:

$$[S]^1_z \mapsto \dot{x}^\lambda (\partial_\lambda + y^i_\lambda ([S]^1_z) \partial_i).$$

The bundle $J^1_n Z \to Z$ possesses the structure group $\mathrm{GL}(n,m;\mathbf{R})$ of linear transformations of the vector space $\mathbf{R}^{m+n}$ that preserve the subspace $\mathbf{R}^n$. Its typical fiber is the Grassmann manifold $\mathrm{GL}(n+m;\mathbf{R})/\mathrm{GL}(n,m;\mathbf{R})$ of n-dimensional vector subspaces of the vector space $\mathbf{R}^{m+n}$. In particular, if $n=1$, the fiber coordinates y^i_0 of $J^1_1 Z \to Z$ with the transition functions (52) are exactly the standard coordinates of the projective space $\mathbf{RP}^m$.

When $n=1$, the formalism of jets of submanifolds provides the adequate mathematical description of relativistic mechanics as follows.

Let Z be a $(m+1)$-dimensional manifold equipped with an atlas of coordinates (z^0, z^i), $i = 1,...,m$, (49) with the transition functions (50), which take the form

$$z^0 \to \widetilde{z}^0 (z^0, z^j), \quad z^i \to \widetilde{z}^i (z^0, z^j). \tag{53}$$

The coordinates z^0 in different charts of Z play the role of the temporal ones.

Let $J^1_1 Z$ be the jet manifold of one-dimensional submanifolds of Z. This is provided with the adapted coordinates (z^0, z^i, z^i_0). Then one can think of z^i_0 as being the coordinates of nonrelativistic velocities. Their transition functions are obtained as follows.

Given the coordinate transformations (53), the total derivative (51) reads as

$$d_{\widetilde{z}^0} = d_{\widetilde{z}^0}(z^0) d_{z^0} = \left(\frac{\partial z^0}{\partial \widetilde{z}^0} + \widetilde{z}^k_0 \frac{\partial z^0}{\partial \widetilde{z}^k} \right) d_{z^0}.$$

In accordance with the relation (52), we have

$$\widetilde{z}^i_0 = d_{\widetilde{z}^0}(z^0) d_{z^0}(\widetilde{z}^i) = \left(\frac{\partial z^0}{\partial \widetilde{z}^0} + \widetilde{z}^k_0 \frac{\partial z^0}{\partial \widetilde{z}^k} \right) \left(\frac{\partial \widetilde{z}^i}{\partial z^0} + z^j_0 \frac{\partial \widetilde{z}^i}{\partial z^j} \right).$$

The solution of this equation is

$$\widetilde{z}^i_0 = \left(\frac{\partial \widetilde{z}^i}{\partial z^0} + z^j_0 \, \frac{2\,\partial \widetilde{z}^i}{\partial z^j} \right) \Bigg/ \left(\frac{\partial \widetilde{z}^0}{\partial z^0} + z^k_0 \, \frac{\partial \widetilde{z}^0}{\partial z^k} \right).$$

This is the transformation law of nonrelativistic velocities, which illustrates that the jet bundle $J^1_1 Z \to Z$ is not affine, but projective.

To obtain the relation between nonrelativistic and relativistic velocities, let us consider the tangent bundle TZ equipped with the induced coordinates $(z^0, z^i, \dot{z}^0, \dot{z}^i)$. There is the morphism

$$\rho: \ TZ \to J^1_1 Z, \quad z^i_0 \circ \rho = \dot{z}^i / \dot{z}^0. \tag{54}$$

It is readily observed that the coordinate transformation laws of z^i_0 and $\dot{z}^i/\dot{z}^0$ are the same. Thus, one can think of the coordinates $(\dot{z}^0, \dot{z}^i)$ as being relativistic velocities.

The morphism (54) is a surjection. Let us assume that the tangent bundle is equipped with a pseudo-Riemannian metric g and $Q_z \subset T_z Z$ is the hyperboloid given by the relation

$$g_{\mu\nu}(z) \dot{z}^\mu \dot{z}^\nu = 1, \quad \mu, \nu = 0,1,...,m. \tag{55}$$

The union of these hyperboloids over Z,

$$Q = \bigcup_{z \in Z} Q_z = Q^+ \cup Q^-$$

is the union of two connected imbedded subbundles of TZ. Then the restriction of the morphism (54) to each of these subbundles is an injection of Q into $J^1_1 Z$.

Let us consider the image of this injection in the fiber of $J^1_1 Z$ over a point $z \in Z$. There are coordinates (z^0, z^i) in a neighborhood around z such that the pseudo-Riemannian metric $g(z)$ at z comes to the Minkowski one,

$$g(z) = \eta = \mathrm{diag}(1, -1, ..., -1).$$

In this coordinates the hyperboloid $Q_z \subset T_z Z$ is given by the relation

$$(\dot{z}^0)^2 - \sum_i \, (\dot{z}^i)^2 = 1.$$

This is the union of the subsets Q^+_z where $z^0 > 0$ and Q^-_z where $z^0 < 0$. The image $\rho(Q^+_z)$ is given by the coordinate relation

$$\sum_i \, (z^i_0) < 1.$$

From the physical viewpoint, this relation means that nonrelativistic velocities are bounded in accordance with Special Relativity.

In general, Lagrangian formalism fails to be appropriate to relativistic mechanics because a Lagrangian $L = \mathcal{L} dz^0$ can be defined only locally. Its Hamiltonian formulation is more promising.

The phase space of relativistic mechanics is the cotangent bundle T^*Z provided with the holonomic coordinates $(z^0, \ z^i, \ p_0 = \dot{z}_0, \ p_i = \dot{z}_i)$. This admits the canonical symplectic form

$$\Omega = dp_\mu \wedge dz^\mu, \quad \mu = 0,1,2,3.$$

Hamiltonian relativistic mechanics is formulated as an autonomous Hamiltonian mechanics on the symplectic manifold T^*Z. Every Hamiltonian $\mathcal{H}$ on the phase space T^*Z defines the Hamiltonian map,

$$\hat{\mathcal{H}}: \ T^*Z \to TZ, \quad \dot{z}^\mu \circ \hat{\mathcal{H}} = \partial^\mu \mathcal{H}.$$

Since relativistic velocities live in the hyperboloid (55), we have the similar constraint,

$$g_{\mu\nu}\frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial\mathcal{H}}{\partial p_\nu}=1, \tag{56}$$

on the phase space T^*Z. Thus, relativistic mechanics is described as a Dirac constraint system on the primary constraint space Q (56). Its solutions are integral curves of the Hamiltonian vector field ϑ on $Q\subset T^*Z$ that obeys the Hamilton equation

$$\vartheta\rfloor i_Q^*\Omega=-i_Q^*d\mathcal{H}. \tag{57}$$

It should be emphasized that, as follows from the relation (24), the coordinate $-p$, but not a Hamiltonian $\mathcal{H}$ plays the role of a relativistic energy function. For example, let us write (i) the Hamiltonian,

$$\mathcal{H}=-\frac{1}{2m}\,\eta^{\mu\nu}p_\mu p_\nu,$$

of a free relativistic mass m in Special Relativity on the Minkowski space $\mathbf{R}^4$; (ii) the Hamiltonian,

$$\mathcal{H}=-\frac{1}{2m}\,\eta^{\mu\nu}(p_\mu-eA_\mu)(p_\nu-eA_\nu),$$

of a relativistic electric charge e in the presence of an electromagnetic field A_μ; and (iii) the Hamiltonian,

$$\mathcal{H}=-\frac{1}{2m}\,\gamma^{\mu\nu}p_\mu p_\nu,$$

of a point mass m in the presence of a metric gravitational field g. Substituting these Hamiltonians into the equations (56) and (57), we obtain the well-known solutions of relativistic mechanics.

[1] R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, London, 1978).
[2] V. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, Berlin, 1978).
[3] P. Libermann and C-M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).
[4] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhäuser Verlag, Basel, 1994).
[5] E. Binz, H. Fischer, and J. Śniatycki, *Geometry of Classical Fields* (North-Holland, Amsterdam, 1988).
[6] M. de León and P. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics* (North-Holland, Amsterdam, 1989).
[7] G. Morandi, C. Ferrario, and G. Lo Vecchio, Phys. Rep. **188**, 147 (1990).
[8] A. Echeverría Enríquez, M. Muñoz Lecanda, and N. Román Roy, Rev. Math. Phys. **3**, 301 (1991).
[9] J. Cariñena and J. Fernández-Núñez, Fortschr. Phys. **41**, 517 (1993).
[10] E. Massa and E. Pagani, Ann. Inst. Henri Poincaré **61**, 17 (1994).
[11] J. Cariñena and M. Rañada, J. Math. Phys. **30**, 2258 (1989).
[12] J. Cariñena, J. Fernández-Núñez, and E. Martínez, Lett. Math. Phys. **23**, 51 (1991).
[13] G. Sardanashvily and O. Zakharov, Diff. Geom. Appl. **3**, 245 (1993).
[14] G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory. Constraint Systems* (World Scientific, Singapore, 1995).
[15] G. Sardanashvily, J. Math. Phys. **35**, 6584 (1994).
[16] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, Singapore, 1997).
[17] D. Canarutto, Bull. U.M.I. Alg. Geom. Ser. VI **V-D**, 18 (1986).
[18] I. Vaisman, Diff. Geom. Appl. **5**, 153 (1995).
[19] G. Sardanashvily, J. Math. Phys. **38**, 847 (1997).
[20] O. Zakharov, J. Math. Phys. **33**, 607 (1992).
[21] G. Giachetta and L. Mangiarotti, Int. J. Theor. Phys. **34**, 2353 (1995).
[22] A. D. Bruno, Usp. Mat. Nauk **43**, 23 (1988) (in Russian).
[23] A. Echeverría Enríquez, M. Muñoz Lecanda, and N. Román Roy, J. Phys. A **19**, 5553 (1995).
[24] I. Krasil'shchik, V. Lychagin, and A. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, Glasgow, 1985).
[25] M. Modugno and A. Vinogradov, Ann. Mat. Pura Appl. **CLXVII**, 33 (1994).