# Hamiltonians separable in cartesian coordinates and third-order integrals of motion 

Simon Gravel<br>Département de physique et Centre de recherche mathématiques Université de Montréal, C.P.6128, Succursale Centre-Ville<br>Montréal, Québec<br>H3C 3J7<br>Canada<br>Simon.Gravel@UMontreal.CA

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#### Abstract

We present in this article all Hamiltonian systems in $E(2)$ that are separable in cartesian coordinates and that admit a third-order integral, both in quantum and in classical mechanics. Many of these superintegrable systems are new, and it is seen that there exists a relation between quantum superintegrable potentials, invariant solutions of the Korteweg-De Vries equation and the Painlevé transcendents.


## I Introduction

In classical mechanics, an n-dimensional Hamiltonian system is called Liouville integrable if it allows $n$ functionally independent integrals of motion in involution (including the Hamiltonian), that is

$$
\begin{aligned}
\left\{H, X_{i}\right\} & =0 \\
\left\{X_{i}, X_{j}\right\} & =0, \forall i, j
\end{aligned}
$$

The Hamiltonian $H=H\left(x_{1}, \ldots x_{n}, p_{1}, \ldots, p_{n}\right)$ and the integrals of motion $X_{i}=X_{i}\left(x_{1}, \ldots x_{n}, p_{1}, \ldots, p_{n}\right)$ must be well defined functions on phase space (1), [10]). The system is superintegrable if it allows more than $n$ functionally independent integrals, $n$ of them in involution. It is called maximally superintegrable if it allows $2 n-1$ integrals of motion. The best known superintegrable systems in $n$ dimensions are the harmonic oscillator $V=\omega r^{2}$ and the Coulomb potential $V=\frac{\alpha}{r}$, and they are indeed maximally superintegrable. This may be closely related to Bertrand's theorem (1, 2]) which states that these are the only rotationally invariant systems for which all finite trajectories are closed.

In quantum mechanics, a Hamiltonian system is said to be integrable if there exists a set $\left\{X_{i}\right\}$ of $n$ well defined, algebraically independent operators (including the Hamiltonian) that commute pairwise. It is superintegrable if it possesses further independent operators, $\left\{Y_{j}\right\}$ that commute with the Hamiltonian. The $Y_{j}$ do not necessarily commute with each other, nor with the $X_{i}$.

The independence of operators in quantum mechanics remains to be defined rigorously [12, 13, 16, 28]. Since we are dealing here only with polynomial differential operators, we can proceed by analogy with the classical case, keeping in mind that a rigorous definition will be needed as soon as we will want to make some more general statements. This choice of a definition will be used only for discussion purposes, since we will find all potentials that admit third-order integrals and all their integrals. The results obtained will therefore hold for any definition of the independence of operators.

Integrable and superintegrable systems, both in quantum and in classical mechanics, attracted considerable interest in the last years. Extensive literature exists about systems with second-order integrals of motion, either in euclidian space [6, 7, 8, 9, 21, 29, or in spaces with nonzero constant [18, 24] or nonconstant curvature [19]. As long as there was no magnetic field in the Hamiltonian, the quantum and classical integrals of motion obeyed the same determining equations, and therefore quantum and classical integrability were very similar. Both properties were related to separation of the Hamilton-Jacobi or Schrödinger equations, and also to exact solvability [26] and generalized symmetries [25].

Systems with higher-order integrals have been studied and classified as early as 1935 in a well-known paper by Drach. This paper considered classical Hamiltonians in complex Euclidian space. Efforts were made recently to understand and classify more completely systems with higher-order integrals in classical
[12 27] and quantum mechanics 12. In spite of these efforts still relatively few such systems are known. This paper is the logical sequel of a systematic search for superintegrable systems with higher order integrals started in 12. Here we consider two-dimensional real Euclidian space with a one-particle Hamiltonian;

$$
H=\frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+V(x, y)
$$

We request the existence of two additional integrals of motion, one of second order in the momenta and the other of third order.

The condition of existence for second-order integral implies, both in classical and quantum mechanics, that the Hamiltonian be separable in cartesian, polar, parabolic or elliptic coordinates. In this paper we consider potentials that are separable in cartesian coordinates;

$$
H=\frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+V_{1}(x)+V_{2}(y)
$$

We found all such systems which admit third-order integrals. Quantum and classical mechanics will be treated simultaneously, for the conditions of existence of integrals of motion, even though not equivalent, are quite similar in both cases.

## II Existence of a third-order integral

In quantum and classical mechanics, the general third-order commuting operator

$$
X=\sum_{i+j=0}^{3} P_{i j}(x, y) p_{x}^{i} p_{y}^{j}
$$

can be reduced to a much simpler form,

$$
\begin{align*}
X & =\sum_{\substack{i, j, k \\
i+j+k=3}} A_{i j k}\left\{L_{3}^{i}, p_{x}^{j} p_{y}^{k}\right\}+\left\{g_{1}(x, y), p_{x}\right\}+\left\{g_{2}(x, y), p_{y}\right\}  \tag{1}\\
L_{3} & =x p_{y}-y p_{x}
\end{align*}
$$

where the $A_{i j k}$ are real constants, and the $g_{i}$ real functions. The bracket is the anticommutator. It is not needed in classical mechanics, but in quantum mechanics it allows us to get rid of terms with even powers of the $p_{i}$ and to make sure the operator is self-adjoint. Furthermore, its use allows us to see clearly the relations between the quantum and the classical case. Indeed, it was found in [12] that the requirement that the operator commutes (or Poisson-commutes) with the Hamiltonian

$$
H=\frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+V(x, y)
$$

implies equations that behave well in the classical limit. Namely, commutativity implies

$$
\begin{align*}
0= & g_{1} V_{x}+g_{2} V_{y}-\frac{\hbar^{2}}{4}\left(f_{1} V_{x x x}+f_{2} V_{x x y}+f_{3} V_{x y y}+f_{4} V_{y y y}\right.  \tag{2}\\
& \left.+8 A_{300}\left(x V_{y}-y V_{x}\right)+2\left(A_{210} V_{x}+A_{201} V_{y}\right)\right), \\
\left(g_{1}\right)_{x}= & 3 f_{1}(y) V_{x}+f_{2}(x, y) V_{y},  \tag{3}\\
\left(g_{2}\right)_{y}= & f_{3}(x, y) V_{x}+3 f_{4}(x) V_{y},  \tag{4}\\
\left(g_{1}\right)_{y}+\left(g_{2}\right)_{x}= & 2\left(f_{2}(x, y) V_{x}+f_{3}(x, y) V_{y}\right), \tag{5}
\end{align*}
$$

in quantum mechanics, where

$$
\begin{aligned}
f_{1}(y) & =-A_{300} y^{3}+A_{210} y^{2}-A_{120} y+A_{030} \\
f_{2}(x, y) & =3 A_{300} x y^{2}-2 A_{210} x y+A_{201} y^{2}+A_{120} x-A_{111} y+A_{021} \\
f_{3}(x, y) & =-3 A_{300} x^{2} y+A_{210} x^{2}-2 A_{201} x y+A_{111} x-A_{102} y+A_{012} \\
f_{4}(x) & =A_{300} x^{3}+A_{201} x^{2}+A_{102} x+A_{003}
\end{aligned}
$$

The equations in classical mechanics are obtained by setting $\hbar=0$. We may notice from the quantum equations, or directly from the condition $[H, X]=0$, that we can express all quantum integrable potentials as $\hbar^{2} \tilde{V}(x, y)$ where $\tilde{V}$ does not depend on $\hbar$. It is often more natural and interesting though to choose arbitrary parameters contained in $\tilde{V}$ to be depending on $\hbar$, so that the potential $V$ does not vanish in the classical limit. One may always verify though that through an appropriate transformation of the arbitrary parameters one can write the potential as being proportional to $\hbar^{2}$

The three last equations, identical in the quantum and classical cases, yield a linear compatibility condition for $V$, which reads

$$
\begin{align*}
0= & -f_{3} V_{x x x}+\left(2 f_{2}-3 f_{4}\right) V_{x x y}+\left(-3 f_{1}+2 f_{3}\right) V_{x y y}-f_{2} V_{y y y} \\
& +2\left(f_{2 y}-f_{3 x}\right) V_{x x}+2\left(-3 f_{1 y}+f_{2 x}+f_{3 y}-3 f_{4 x}\right) V_{x y}+2\left(-f_{2 y}+f_{3 x}\right) V_{y y} \\
& +\left(-3 f_{1 y y}+2 f_{2 x y}-f_{3 x x}\right) V_{x}+\left(-f_{2 y y}+2 f_{3 x y}-3 f_{4 x x}\right) V_{y} . \tag{6}
\end{align*}
$$

Further nonlinear compatibility conditions can be obtained from (2) to (5) for the potential, and these are listed in [12. They are quite complicated though and were not used for the results stated in this paper.

## III Potentials separable in cartesian coordinates

If we set $V=V_{1}(x)+V_{2}(y)$ in equations (2) to (5), we find

$$
\begin{align*}
0= & g_{1} V_{1 x}+g_{2} V_{2 y}-\frac{\hbar^{2}}{4}\left(f_{1} V_{1 x x x}+f_{4} V_{2 y y y}\right.  \tag{7}\\
& \left.+8 A_{300}\left(x V_{2 y}-y V_{1 x}\right)+2\left(A_{210} V_{1 x}+A_{201} V_{2 y}\right)\right) \\
\left(g_{1}\right)_{x}= & 3 f_{1}(y) V_{1 x}+f_{2}(x, y) V_{2 y}  \tag{8}\\
\left(g_{2}\right)_{y}= & f_{3}(x, y) V_{1 x}+3 f_{4}(x) V_{2 y}  \tag{9}\\
\left(g_{1}\right)_{y}+\left(g_{2}\right)_{x}= & 2\left(f_{2}(x, y) V_{1 x}+f_{3}(x, y) V_{2 y}\right) \tag{10}
\end{align*}
$$

with $\hbar=0$ in the classical case. Equations (8) and (91) are readily integrated, so in the cartesian case two equations remain to be solved.

The compatibility condition (6) allows us to find ODEs for $V_{1}$ and $V_{2}$. If we set alternatively $y=0$ and $x=0$, we find

$$
\begin{align*}
& \left(A_{210} x^{2}+A_{111} x+A_{012}\right) V_{1}^{(3)}(x)+4\left(2 A_{210} x+A_{111}\right) V_{1}^{\prime \prime}(x)+12 A_{210} V_{1}^{\prime}(x)=a x+b  \tag{11}\\
& \left(A_{201} y^{2}-A_{111} y+A_{021}\right) V_{2}^{(3)}(y)+4\left(2 A_{201} y-A_{111}\right) V_{2}^{\prime \prime}(y)+12 A_{201} V_{2}^{\prime}(y)=c y+d \tag{12}
\end{align*}
$$

The solutions to the homogeneous part of these equations are easily found and brought to a simple form by translations in $x$ and $y$. If we take the first one for definiteness, we have four different types of solution. When $A_{210} \neq 0$, we have two possible solutions,

$$
\begin{aligned}
V_{1 \text { hom }} & =\frac{c_{1}}{(x+\alpha)^{2}}+\frac{c_{2}}{(x-\alpha)^{2}} \\
V_{1 \text { hom }} & =\frac{c_{1}}{x^{2}}+\frac{c_{2}}{x^{3}}
\end{aligned}
$$

If $A_{210}=0$ and $A_{111} \neq 0$ we get

$$
V_{1 \text { hom }}=\frac{c_{1}}{x^{2}}+c_{2} x
$$

Finally if only $A_{012} \neq 0$, the solution may be brought to the form

$$
V_{1 \mathrm{hom}}=c_{2} x^{2}+c_{1} x
$$

Special solutions are also simple. If $A_{210} \neq 0$, we have $V_{1 \text { part }}=\alpha x^{2}+\beta x$. Otherwise, when $A_{111} \neq 0, V_{1 \text { part }}=\alpha x^{3}+\beta x^{2}$, finally, if only $A_{012} \neq 0$, $V_{1 \text { part }}=\alpha x^{4}+\beta x^{3}$. Provided that (11) or (12) do not vanish trivially, we can choose $V_{1}$ or $V_{2}$, respectively, amongst the following functions:
(A.1) $f_{1}=\frac{c_{1}}{(x+\alpha)^{2}}+\frac{c_{2}}{(x-\alpha)^{2}}+c_{3} x^{2}+c_{4} x$
(A.2) $f_{2}=\frac{c_{1}}{x^{2}}+\frac{c_{2}}{x^{3}}+c_{3} x^{2}+c_{4} x$

| Superintegrable potentials | Leading-order terms <br> of the integrals |
| :--- | :---: |
| $\star V_{a}=a\left(x^{2}+y^{2}\right)$ | $L^{3} ;\left\{L, p_{x} p_{y}\right\} ;\left\{L, p_{y}^{2}\right\} ;\left\{L, p_{x}^{2}\right\}$ |
| $\star V_{b}=a\left(x^{2}+y^{2}\right)+\frac{b}{x^{2}}+\frac{c}{y^{2}}$ | $\left\{L, p_{x} p_{y}\right\}$ |
| $V_{c}=a\left(x^{2}+y^{2}\right)+\frac{\hbar^{2}}{x^{2}}+\frac{\hbar^{2}}{y^{2}}$ | $L^{3} ;\left\{L, p_{x} p_{y}\right\}$ |
| $V_{d}=a\left(x^{2}+y^{2}\right)+\frac{\hbar^{2}}{y^{2}}$ | $L^{3} ;\left\{L, p_{x} p_{y}\right\} ;\left\{L, p_{y}^{2}\right\}$ |
| $\left.V_{e}=\frac{\hbar^{2}}{8 \alpha^{4}} x^{2}+y^{2}\right)+\frac{\hbar^{2}}{(x-\alpha)^{2}}+\frac{\hbar^{2}}{(x+\alpha)^{2}}$ | $2 L^{3}-3 \alpha^{2}\left\{L, p_{y}^{2}\right\} ;\left\{L, p_{x}^{2}\right\}$ |
| $\left.V_{f}=\frac{\hbar^{2}}{8 \alpha^{4}} x^{2}+y^{2}\right)+\frac{\hbar^{2}}{y^{2}}+\frac{\hbar^{2}}{(x+\alpha)^{2}}+\frac{\hbar^{2}}{(x-\alpha)^{2}}$ | $2 L^{3}-3 \alpha^{2}\left\{L, p_{y}^{2}\right\}$ |
| $V_{g}=\frac{\hbar^{2}}{8 \alpha^{4}}\left(x^{2}+y^{2}\right)$ |  |
| $+\frac{\hbar^{2}}{(y-\alpha)^{2}}+\frac{\hbar^{2}}{(x-\alpha)^{2}}+\frac{\hbar^{2}}{(y+\alpha)^{2}}+\frac{\hbar^{2}}{(x+\alpha)^{2}}$ | $2 L^{3}-3 \alpha^{2}\left(\left\{L, p_{x}^{2}\right\}+\left\{L, p_{y}^{2}\right\}\right)$ |
| $\star V_{h}=a\left(4 x^{2}+y^{2}\right)+\frac{b}{y^{2}}+c x$ | $p_{x} p_{y}^{2}$ |
| $\star V_{i}=a\left(9 x^{2}+y^{2}\right)$ | $\left\{L, p_{y}^{2}\right\}$ |
| $V_{j}=a\left(9 x^{2}+y^{2}\right)+\frac{\hbar^{2}}{y^{2}}$ | $\left\{L, p_{y}^{2}\right\}$ |
| $\left.V_{k}=\frac{\hbar^{2}}{8 \alpha^{4}} 9 x^{2}+y^{2}\right)+\frac{\hbar^{2}}{(y+\alpha)^{2}}+\frac{\hbar^{2}}{(y-\alpha)^{2}}$ | $\left\{L, p_{y}^{2}\right\}$ |
| $V_{l}=\frac{\hbar^{2}}{x^{2}}+\frac{a}{y^{2}}$ | $\left\{L^{2}, p_{x}\right\} ;\left\{L, p_{x} p_{y}\right\} ; p_{x}^{3}$ |
| $V_{m}=\frac{\hbar^{2}}{x^{2}}+\frac{\hbar^{2}}{y^{2}}$ | $L^{3} ;\left\{L^{2}, p_{x}\right\} ;\left\{L^{2}, p_{y}\right\}$ |
| $V_{n}=a x+\frac{\hbar^{2}}{y^{2}}$ | $\left\{L, p_{x} p_{y}\right\} ; p_{x}^{3} ; p_{y}^{3}$ |
| $V_{o}=\frac{\hbar^{2}}{y^{2}}+V(x)$ | $\left\{L, p_{y}^{2}\right\} ; p_{y}^{3} ; p_{x} p_{y}^{2}$ |

Table 1: Superintegrable potentials that satisfy linear compatibility conditions for nonzero parameters.
(A.3) $f_{3}=\frac{c_{1}}{x^{2}}+c_{2} x^{3}+c_{3} x^{2}+c_{4} x$
(A.4) $f_{4}=c_{1} x^{4}+c_{2} x^{2}+c_{3} x$
(A.5) $f_{5}=c_{1} x^{3}+c_{2} x$
(A.6) $f_{6}=c_{1} x^{2}$
(A.7) $f_{7}=c_{1} x$,
and then solve (7) to (10). These long but rather straightforward calculations yield the 15 superintegrable potentials included in Table 1. Some of them are obviously particular cases of others, but we listed them separately to account for their additional integrals. Only the third-order integrals are listed, some of them being trivial consequences of lower-order ones. With the exception of the harmonic oscillator, potentials that have first-order integrals are not listed here for they were already presented in 12 with all their third-order integrals. The complete integrals of motion can be found in Appendix I.

Many of these potentials were not known. The only classical potentials among these are indicated with a $\star$. These are well-known superintegrable
potentials (see e.g. (9) , and all of them, except $V_{i}$, are in fact quadratically superintegrable.

All the potentials are superintegrable in the quantum case. We therefore notice that classical nontrivial potentials can have many different quantum equivalents. The classical harmonic oscillator $V_{a}$ can be seen as a limiting case of the quantum potentials $V_{a}, V_{c}$ and $V_{d}$, and also, if we set $\alpha=\sqrt{\hbar} / \omega$, of $V_{e}, V_{f}$ and $V_{g}$, not to mention the similar potentials that can be obtained by permutations of $x$ and $y$. The anisotropic harmonic oscillator with ratio $1: 3$ also admits many quantum deformations but, interestingly, the anisotropic oscillator with ratio $1: 2$ does not admit such deformations. Notice also that if we want to deal with real potentials only, $\alpha$ must be either real or purely imaginary in potentials $V_{e}, V_{f}, V_{g}$ and $V_{k}$. Therefore these have as a classical limit harmonic oscillators with $a>0$.

All quantum superintegrable potentials reduce to classical ones when the classical limit is considered, sometimes in more than one way. For example, potentials $V_{e}, V_{f}$ and $V_{g}$ give the free motion potential instead of the harmonic oscillator if $\alpha$ remains constant as $\hbar \rightarrow 0$.

These potentials all satisfy the linear equations (11) and (12), and can be expressed as sums of simple superintegrable potentials.

Let us now set $A_{210}=A_{111}=A_{012}=0$ so that (11) vanishes trivially. We may also assume that $V_{1}$ does not take one of the forms (A) to (A), for we have already worked these cases out. This is quite useful, for if we set $y=1$ in (6), we obtain for $V_{1}$ an equation of the same form as (11) with different coefficients. These coefficients must therefore vanish, so $A_{300}=A_{201}=A_{102}=$ 0 . This is a significant simplification that allows us to restrict our attention, when considering equation (12), to the following three cases:
i) $V_{2}=a y^{2}$;
ii) $V_{2}=a y$;
iii) $A_{120}=A_{021}=0$

Before we consider each case separately, it is worth noticing that potentials of the form $V=V_{1}(x)$ that admit third-order integrals independent of $y$ and $p_{y}$ should appear as solutions here, for the integral remains if we add a function $V_{2}(y)$ to these potentials. These potentials were found in [12] and [16] to satisfy equation

$$
\begin{equation*}
\hbar^{2} V_{1}^{\prime 2}=4 V_{1}^{3}-g_{2} V_{1}-g_{3}, \tag{13}
\end{equation*}
$$

and can therefore be written as

$$
\begin{equation*}
V_{1}=\hbar^{2} \mathcal{P}(x) \tag{14}
\end{equation*}
$$

where $\mathcal{P}(x)$ is the Weierstrass elliptic function. Since the $y$ variable plays no role here and these potentials admit integrals with leading-order terms proportional to $p_{x}^{3}$, these solutions will appear in cases (ii) to (iiii).
A. Case i; $V=V_{1}(x)+a y^{2}$

When $V=V_{1}(x)+a y^{2}$ and $a \neq 0$, we find that $A_{300}=0$, and the following two equations must be satisfied:

$$
\begin{align*}
0= & A_{030}\left(\hbar^{2} V_{1}^{(3)}-6\left(V_{1}^{2}\right)^{\prime}\right)+\gamma_{1} V_{1}^{\prime}  \tag{15}\\
0= & A_{120}\left(-\hbar^{2} V_{1}^{(4)}-24 a\left(x V_{1}\right)^{\prime}+6\left(V_{1}^{2}\right)^{\prime \prime}-4 a x^{2} V_{1}^{\prime \prime}+8 a^{2} x^{2}\right)  \tag{16}\\
& +8 a A_{021}\left(2 a x-\left(x V_{1}^{\prime}\right)^{\prime}-2 V_{1}^{\prime}\right)+4 \eta\left(2 a-V_{1}^{\prime \prime}\right) .
\end{align*}
$$

where $\gamma_{1}$ and $\eta$ are arbitrary constants. When $A_{030} \neq 0$, equation (15) is equivalent to (13) (up to a translation of $V_{1}$ to get rid of $\gamma_{1}$ ), hence its solutions are of the form (14). These potentials cannot satisfy simultaneously equation (16) for nontrivial parameters. This can be observed by expanding (13) in series around $x=0$ and substituting the result in (16). Therefore solutions given by $A_{030} \neq 0$ are of no special interest here.

Let us now set $A_{030}=\gamma_{1}=0$. Equation (16) can be greatly simplified. We assume that $A_{120} \neq 0$, for otherwise equation (16) can be solved to give potential (III). Then by an appropriate translation of $x$ and $V$, we can get rid of the terms involving $A_{021}$ and $\eta$ and finally divide by $A_{120}$;

$$
\begin{equation*}
0=-\hbar^{2} V_{1}^{(4)}-24 a\left(x V_{1}\right)^{\prime}+6\left(V_{1}^{2}\right)^{\prime \prime}-4 a x^{2} V_{1}^{\prime \prime}+8 a^{2} x^{2} . \tag{17}
\end{equation*}
$$

This equation admits a first integral, namely

$$
\begin{equation*}
k=\hbar^{2}\left(x V_{1}^{\prime \prime \prime}-V_{1}^{\prime \prime}\right)+4 x\left(a x^{2}-3 V_{1}\right) V_{1}^{\prime}+6 V_{1}^{2}+12 a x^{2} V_{1}-2 a^{2} x^{4} . \tag{18}
\end{equation*}
$$

Both (17) and (18) can be simplified by setting $V_{1}=W(x)+a x^{2} / 3$. Then

$$
\begin{equation*}
\hbar^{2} W^{(4)}=12 W W^{\prime \prime}+12\left(W^{\prime}\right)^{2}+b x W^{\prime}+2 b W-\frac{1}{6} b^{2} x^{2} \tag{19}
\end{equation*}
$$

with $b=-8 a \neq 0$ for (17), and

$$
\begin{equation*}
k_{2}=3 \hbar^{2}\left(x W^{\prime \prime \prime}-W^{\prime \prime}\right)-18 x\left(W^{2}\right)^{\prime}+2\left(2 a x^{2}+3 W\right)^{2} . \tag{20}
\end{equation*}
$$

for the first integral. Equation (19) is well known. It is equivalent to equations (3.16) in [4] and (2.17) in [20, which were obtained by nonclassical reduction of the Boussinesq Equation. It was also shown in 3 to be a nonclassical reduction of the Kadomtsev-Petviashvili equation. It has the Painlevé property, and, when $b \neq 0$, its solution, given in [5] (equation 2.88) may be written in terms of the fourth transcendant function of Painlevé, namely

$$
\begin{equation*}
W=\frac{\hbar}{2} b_{1} P_{4}^{\prime}\left(x, \frac{b}{\hbar^{2}}\right)-\frac{1}{2} b P_{4}^{2}\left(x, \frac{b}{\hbar^{2}}\right)-\frac{1}{2} b x P_{4}\left(x, \frac{b}{\hbar^{2}}\right)-\frac{1}{6}\left(\frac{b}{2} x^{2}+\hbar^{2} K_{1}-\hbar b_{1}\right) \tag{21}
\end{equation*}
$$

where $b_{1} \equiv \pm \sqrt{-b}= \pm \sqrt{8 a}$ and $P_{4}\left(x, \frac{b}{\hbar^{2}}\right)=P_{4}\left(x, \frac{b}{\hbar^{2}}, K_{1}, K_{2}\right)$ is the fourth transcendant function of Painlevé, and therefore satisfies equation

$$
\begin{align*}
P_{4}^{\prime \prime}(x, \alpha)= & \frac{\left(P_{4}^{\prime}(x, \alpha)\right)^{2}}{2 P_{4}(x, \alpha)}-\frac{3 \alpha}{2} P_{4}(x, \alpha)^{3}-2 \alpha x P_{4}(x, \alpha)^{2}  \tag{22}\\
& -\left(\frac{\alpha}{2} x^{2}+K_{1}\right) P_{4}(x, \alpha)+\frac{K_{2}}{P_{4}(x, \alpha)}
\end{align*}
$$

$K_{1}$ and $K_{2}$ are integration constants. The potential therefore reads

$$
\begin{align*}
V(x, y)= & a\left(x^{2}+y^{2}\right)+\frac{\hbar}{2} b_{1} P_{4}^{\prime}\left(x, \frac{-8 a}{\hbar^{2}}\right)+4 a P_{4}^{2}\left(x, \frac{-8 a}{\hbar^{2}}\right) \\
& +4 a x P_{4}\left(x, \frac{-8 a}{\hbar^{2}}\right)+\frac{1}{6}\left(-\hbar^{2} K_{1}+\hbar b_{1}\right) . \tag{23}
\end{align*}
$$

This potential admits as special cases two anisotropic harmonic oscillators, $V=a\left(x^{2}+y^{2}\right)$ when $K_{2}=0\left(\right.$ and $\left.P_{4}=0\right)$, and $V=a\left(x^{2} / 9+y^{2}\right)$ when $K_{1}=0$ and $K_{2}=-1 / 18$ (and $P_{4}=-x / 3$ ), as well as all their quantum deformations that have the form $V=a\left(p^{2} x^{2}+y^{2}\right)+f(x)$, that is potentials $V_{d}, V_{e}, V_{j}$ and $V_{k}$ (up to a permutation of $x$ and $y$ ).

The constant term in the potential, $\left(-\hbar^{2} K_{1}+\hbar b_{1}\right)$, can be set to zero, but we will keep it in order to be able to write the quantum and classical integrals in a unified way.

In classical mechanics, the equation (18) with $\hbar=0$ admits a first integral, which reads

$$
c=\frac{\left(9 V_{1}-a x^{2}\right)\left(V_{1}-a x^{2}\right)^{3}+\frac{k^{2}}{4}-k\left(V_{1}-a x^{2}\right)\left(3 V_{1}+a x^{2}\right)}{x^{2}} .
$$

We may therefore write the solution for $V_{1}$ implicitly as

$$
\begin{equation*}
c x^{2}-d^{2}+2 d\left(V_{1}-a x^{2}\right)\left(3 V_{1}+a x^{2}\right)=\left(9 V_{1}-a x^{2}\right)\left(V_{1}-a x^{2}\right)^{3} \tag{24}
\end{equation*}
$$

where $c$ and $d$ are arbitrary constants. If $c=d=0$, we find either the familiar anisotropic harmonic oscillators, or a potential obtained by joining at $x=0$ two halves of anisotropic harmonic oscillators with different ratios. Even though this potential does not have a continuous second derivative, it can be obtained as a limiting case of the family of smooth superintegrable potentials (26).

In the general case the potential may be expressed as the root of a fourthorder polynomial with three arbitrary parameters ( $a, c$ and $d$ ), although we can set $a=1$ and one of the other two coefficients to $\pm 1$ by scaling $x$ and $V_{1}$.

Since equation (24) may describe new two-dimensional potentials with bounded motion, it is worth studying for interesting special cases. Indeed, if we assume $a>0$ and $d \geq 0$, we can consider the case $c=128 a^{4} \tilde{d}^{3} / 27^{2}$ and $d=4 a^{2} \tilde{d}^{2} / 27$. Four solutions exist in that case, two of which are translated harmonic oscillators. The remaining two potentials are

$$
\begin{equation*}
V=a y^{2}+\frac{a}{9}\left(2 \tilde{d}+5 x^{2} \pm 4 x \sqrt{\left(\tilde{d}+x^{2}\right)}\right) \tag{25}
\end{equation*}
$$

Up to an additive constant, potentials (25) are equal to

$$
\begin{equation*}
V=a y^{2}+\frac{a}{9}\left(x \pm 2 \sqrt{\tilde{d}+x^{2}}\right)^{2} \tag{26}
\end{equation*}
$$

Those potentials are smooth interpolations between anisotropic harmonic oscillators with ratio $1: 1$ and $1: 3$. When $\tilde{d}=0$ (and therefore $c=d=0$ ) they are the junctions of the two halves of harmonic oscillators mentioned above.

The integral of motion is similar in quantum and classical mechanics, and reads

$$
\begin{equation*}
X=\left\{L, p_{x}^{2}\right\}+\left\{a x^{2} y-3 y V_{1}(x), p_{x}\right\}-\frac{1}{2 a}\left\{\frac{\hbar^{2}}{4} V_{1 x x x}+\left(a x^{2}-3 V_{1}\right) V_{1 x}, p_{y}\right\} \tag{27}
\end{equation*}
$$

where $V_{1}$ is a solution to equation (24). Therefore the integral for (26) must be slightly modified to take into account the constant term we removed.

## B. Case ii: $V=a y+V_{1}(x)$

Here we find again two equations:

$$
\begin{align*}
0= & \hbar^{2} A_{120} V_{1}^{(3)}-6 A_{120}\left(V_{1}^{2}\right)^{\prime}+12 a^{2} A_{003}+\gamma_{1} V_{1}^{\prime}, \\
0= & \frac{A_{030}}{4}\left(-6\left(V_{1}^{2}\right)^{\prime}+\hbar^{2} V_{1}^{(3)}\right)^{\prime}-\frac{a}{2} A_{120}\left(6\left(x V_{1}\right)^{\prime}+x^{2} V_{1}^{\prime \prime}\right)  \tag{28}\\
& -a A_{021}\left(\left(x V_{1}^{\prime}\right)^{\prime}+2 V_{1}^{\prime}\right)-\gamma_{2} V_{1}^{\prime \prime}+a \gamma_{1} .
\end{align*}
$$

This time we cannot treat the equations separately, but we can, when $A_{120} \neq$ 0 , translate $V$ to annihilate $\gamma_{1}$. Then we can substitute the first equation in the second to get rid of $A_{030}$, and finally translate $x$ to get rid of $A_{021}$. We can then solve the remaining system, first by solving the second, linear equation, and then by substituting the result in the first one. The only solution remaining is then potential $V_{n}=a y+\frac{\hbar^{2}}{x^{2}}$. Hence we can set $A_{120}=0$, and therefore also $A_{003}=\gamma_{1}=0\left(\right.$ as $V_{1} \neq b x$ and $\left.a \neq 0\right)$. In the second equation we can set $\gamma_{2}=0$. If then $A_{030}=0$, we find $V_{1}=a / x^{2}$ which was already classified. If $A_{030} \neq 0$ and $A_{021}=0$, though, the potential $V_{1}(x)$ is solution to

$$
\begin{equation*}
\hbar^{2} V_{1}^{\prime \prime}=6 V_{1}^{2}+\lambda x+k \tag{29}
\end{equation*}
$$

If $\lambda \neq 0$ we can set $k=0$, and we find that $V_{1}$ can be expressed in terms of the first Painlevé transcendent:

Comme $\lambda$ est une constante arbitraire, il est plus pratique d'crire $\omega^{5}=\frac{\lambda}{\hbar^{4}}$,

$$
\begin{equation*}
V=a y+\hbar^{2} \omega^{2} P_{1}(\omega x) \tag{30}
\end{equation*}
$$

with $\omega^{5}=\frac{\lambda}{\hbar^{4}}$.
The integral of motion is

$$
X=2 p_{x}^{3}+3\left\{V_{2}(x), p_{x}\right\}+\left\{\frac{\hbar^{4} \omega^{5}}{4 a}, p_{y}\right\}
$$

Notice that in order to consider the limiting case $a=0$, we can multiply the integral by $a$, and thus we find a potential that depends only on $x$ with a trivial $p_{y}$ integral. We can also set $\omega^{5}=a \omega^{5^{\prime}}$ first, in which case we find the potentials (14). If we look for the classical limit, we find

$$
\begin{equation*}
V=a y+b \sqrt{x} \tag{31}
\end{equation*}
$$

with the integral

$$
X=2 p_{x}^{3}+3 b\left\{\sqrt{x}, p_{x}\right\}-\left\{\frac{3 b^{2}}{2 a}, p_{y}\right\}
$$

Returning to equation (29) and assuming $\lambda=0$, we find again that $V_{1}$ has the form (14), and the integral depends only on $x$ and $p_{x}$.

Finally, if we set $A_{030} A_{021} \neq 0$, we have to solve the equation

$$
\begin{equation*}
0=\left(-\frac{3}{2}\left(V_{1}^{2}\right)^{\prime}+\frac{\hbar^{2}}{4} V_{1}^{(3)}\right)^{\prime}+b\left(\left(x V_{1}^{\prime}\right)^{\prime}+2 V_{1}^{\prime}\right), \quad b=\frac{-a A_{021}}{A_{030}} \neq 0 \tag{32}
\end{equation*}
$$

It can be integrated once to give

$$
C_{1}=-\frac{3}{2}\left(V_{1}^{2}\right)^{\prime}+\frac{\hbar^{2}}{4} V_{1}^{(3)}+b\left(\left(x V_{1}^{\prime}\right)+2 V_{1}\right)
$$

We can set $C_{1}=0$ by translations in $x$ and $V$. The equation admits a first integral which reads
$2 b \hbar^{2}\left(V_{1}(x)-b x\right) V_{1}^{\prime \prime}(x)+b \hbar^{2}\left(2 b-V_{1}^{\prime}(x)\right) V_{1}^{\prime}(x)-8 b V_{1}(x)\left(V_{1}(x)-b x\right)^{2}=k_{1}$.
In the classical case, $\hbar=0$, this is enough to solve. The solution, which can be written implicitly in a more compact form, is given by

$$
\begin{equation*}
d=V_{1}\left(V_{1}-b x\right)^{2} \tag{34}
\end{equation*}
$$

where $d$ is an arbitrary constant. When $d=0$ we find the familiar case $V=$ $b x+a y$ which admits a first-order integral. We may notice that the implicit form of the solution is somewhat similar to (24).

In the quantum case, we notice that the transformation $W(x)=V_{1}(x)-b x$, that preserves the Painlevé property, simplifies equation (33) that becomes

$$
\begin{equation*}
\hbar^{2}\left(2 W W^{\prime \prime}-W^{\prime 2}\right)-8(W+b x) W^{2}=k_{2} \tag{35}
\end{equation*}
$$

or

$$
\begin{equation*}
W^{\prime \prime}=\frac{W^{\prime 2}}{2 W}+\frac{4 W^{2}}{\hbar^{2}}+\frac{4 b x W}{\hbar^{2}}+\frac{k_{2}}{2 \hbar^{2} W} \tag{36}
\end{equation*}
$$

We can also substitute $Y(x)=\sqrt{W(x)}$ to find

$$
\hbar^{2} Y^{\prime \prime}=2\left(Y^{2}+b x\right) Y+\frac{k_{2}}{4 Y^{3}}
$$

If $k_{2}=0$, we can set $\hbar=1$ and $b=1 / 2$ by the change of variables

$$
Y=(2 \hbar b)^{\frac{1}{3}} Z, x=\left(\frac{\hbar^{2}}{2 b}\right)^{\frac{1}{3}} \xi
$$

The solution for $Z(\xi)$ is then a special case of the second Painlevé transcendant, defined by the equation

$$
P_{2}^{\prime \prime}(x, \alpha)=2 P_{2}(x, \alpha)^{3}+x P_{2}(x, \alpha)+\alpha
$$

The solution for $V$ is

$$
\begin{equation*}
V(x, y)=b x+a y+(2 \hbar b)^{\frac{2}{3}} P_{2}^{2}\left(\left(\frac{2 b}{\hbar^{2}}\right)^{\frac{1}{3}} x, 0\right) \tag{37}
\end{equation*}
$$

with the particular case $V=b x+a y$.
Now if $k_{2} \neq 0$, we use (36), that can be normalized by a change of variables,

$$
x=-\left(\frac{\hbar^{2}}{(4 b)}\right)^{\frac{1}{3}} \xi, W=-\frac{\sqrt{-k_{2}}}{(4 \hbar b)^{\frac{1}{3}}} Y
$$

We then find

$$
\begin{equation*}
Y^{\prime \prime}=\frac{Y^{\prime 2}}{2 Y}+4 \beta Y^{2}-\xi Y-\frac{1}{2 Y} \tag{38}
\end{equation*}
$$

where $\beta=-\sqrt{-k_{2}} /(4 \hbar b) \neq 0$ is an arbitrary constant. Equation (38) corresponds to case XXXIV, p. 340, in [17]. The solution for $Y(\xi)$ reads

$$
\begin{equation*}
2 \beta Y=P_{2}^{\prime}\left(\xi,-2 \beta-\frac{1}{2}\right)+P_{2}\left(\xi,-2 \beta-\frac{1}{2}\right)^{2}+\frac{\xi}{2} \tag{39}
\end{equation*}
$$

where $P_{2}$ is once again the second Painlevé transcendant. Since $\beta$ is an arbitrary constant, we can set $\kappa=-2 \beta-\frac{1}{2}$.

Back to the original variables, we get

$$
\begin{equation*}
V(x, y)=a y+\left(2 \hbar^{2} b^{2}\right)^{\frac{1}{3}}\left(P_{2}^{\prime}\left(-\left(4 b / \hbar^{2}\right)^{\frac{1}{3}} x, \kappa\right)+P_{2}^{2}\left(-\left(4 b / \hbar^{2}\right)^{\frac{1}{3}} x, \kappa\right)\right) \tag{40}
\end{equation*}
$$

This potential admits $V=a y$ and $V=a y+\hbar^{2} / x^{2}$ as particular cases.
In all cases with $V=a y+V_{1}(x)$ and $A_{012} A_{030} \neq 0$ the integral of motion is

$$
\begin{equation*}
2 a p_{x}^{3}-2 b p_{x}^{2} p_{y}+a\left\{3 V_{1}(x)-b x, p_{x}\right\}-2 b\left\{V_{1}(x), p_{y}\right\} . \tag{41}
\end{equation*}
$$

Limiting values for $a, b$ and $\hbar$ give either known potentials or trivial integrals.
C. Case iiii) $A_{120}=A_{021}=0$

Since we set here $A_{120}=A_{021}=0$, we assume $A_{030} A_{003} \neq 0$ (otherwise we would find potentials of the form (14)). The conditions for the existence of a third-order operator then read

$$
\begin{align*}
& \hbar^{2} V_{1}^{\prime \prime}(x)=6 V_{1}^{2}(x)+A_{003} \sigma x \\
& \hbar^{2} V_{2}^{\prime \prime}(y)=6 V_{2}^{2}(y)-A_{030} \sigma y \tag{42}
\end{align*}
$$

If $\sigma$ vanishes, we find the potential

$$
V=\hbar^{2}(\mathcal{P}(x)+\mathcal{P}(y))
$$

with two integrals that each depend on only one variable, as could have been predicted from the results of equation (13). If $\sigma$ does not vanish, let us set $b_{1}=A_{003} \sigma$ and $b_{2}=-A_{030} \sigma$.

If $\hbar=0$, we find that

$$
V= \pm \sqrt{\beta_{1} x} \pm \sqrt{\beta_{2} y}
$$

is superintegrable, where the $\beta_{i}=b_{i} / 6$ are arbitrary constants.
This defines similar real potentials on each quadrant of the plane. We can again patchwork the pieces to find real continuous potentials defined everywhere, e.g. $V=c_{1} \sqrt{|x|}+c_{2} \sqrt{|y|}$, although in that case neither the Hamiltonian nor the integral are differentiable at the equilibrium point.

Finally, if $\hbar \neq 0$, we find that $V_{1}$ and $V_{2}$ can be both written using the first Painlevé transcendent.

$$
\begin{equation*}
V=\hbar^{2} \omega_{1}^{2} P_{1}\left(\omega_{1} x\right)+\hbar^{2} \omega_{2}^{2} P_{1}\left(\omega_{2} x\right) \tag{43}
\end{equation*}
$$

where the $\omega_{i}=\left(b_{i} / \hbar^{4}\right)^{1 / 5}$ are arbitrary constants.
The integral of motion is

$$
\begin{equation*}
X=2 b_{2} p_{x}^{3}-2 b_{1} p_{y}^{3}+3 b_{2}\left\{V_{1}\left(x, b_{1}\right), p_{x}\right\}-3 b_{1}\left\{V_{2}\left(y, b_{2}\right), p_{y}\right\} \tag{44}
\end{equation*}
$$

both in quantum and classical mechanics.

## IV Conclusion

We have found all systems in two-dimensional euclidian space that admit separation of variables in cartesian coordinates and at least one third-order integral, both in quantum and classical mechanics. Many new superintegrable potentials were found, and, interestingly, all the quantum superintegrable potential are found as solutions of equations having the Painlevé property, and this is probably not accidental. Many of the quantum potentials can in fact be written in terms of different transcendent functions of Painlevé, and many are related to group invariant solutions of the Korteweg-de Vries equation, and to reductions
of the Boussinesq and KP equations. All classical integrable potentials were found to be limiting cases of quantum ones. They do not obey to equations having the Painlevé property, though, for many classical integrable potentials have movable branch points of the form $\sqrt{x-b}$. Thus, in that respect, quantum integrable potentials behave more regularly than classical ones. A natural question is what does the Painlevé property, in quantum mechanics, tell us about classical integrable potentials. Since this paper deals mostly with the classification of superintegrable systems, the consideration of such questions regarding their properties and their solutions will be postponed to a future article.

Our investigation provided interesting new examples of the differences between quantum and classical integrability. A systematic search for systems with higher-order integrals is therefore a useful task since we still know little about such systems, and they are likely to share interesting properties.

Note added in proof: The future article mentioned earlier (11]) indeed shows that superintegrability of separable systems tells a lot about the physical properties of the separated systems. Even though the systems considered here can all be separated in two independent one-dimensional systems, two-dimensional superintegrability provides information on them that is far from obvious from a one-dimensional perspective.

Equations (2) to (5) are the necessary and sufficient conditions for the existence of third-order integrals. The general solution to these equation is highly nontrivial, and it is not likely that a direct approach will lead to such a solution for integrals of order higher than four, without the use of new methods. In order to develop such methods, it would be useful to have a rigorous definition of quantum integrability.

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## V Annex I

Here is the complete list of two-dimensional Hamiltonians separable in cartesian coordinates that admit at least one third-order integral. We also gave all their third-order integrals. Potentials depending on only one cartesian coordinates are not listed here, for they were classified in 12. Many of the potentials listed below were already known to be superintegrable, but we listed them here for completeness and in some cases to take into account additional third-order integrals. Potentials in boxes are those who do not admit enough first- or second-order integrals to make them superintegrable. Most of them were not known before.

Some potentials (namely $V_{p}$ and $V_{q}$ ) are not superintegrable according to
the usual definition, for their integrals have simple though nontrivial relations with the Hamiltonian. For definitions of parameters the reader is referred to the body of the article. In order to write the integrals of motion in a compact form we often use the notation $V(x, y)=V_{1}(x)+V_{2}(y)=V_{1}+V_{2}$.

## A. Quantum potentials

(Q.1) $V=a\left(x^{2}+y^{2}\right)$
$X_{1}=L^{3}$
$X_{2}=\left\{L, p_{x} p_{y}\right\}+a\left\{2 x^{2} y, p_{y}\right\}-a\left\{2 x y^{2}, p_{x}\right\}$
$X_{3}=\left\{L, p_{y}^{2}\right\}+2 a\left(\left\{x y^{2}, p_{y}\right\}-\left\{y^{3}, p_{x}\right\}\right)$
$X_{4}=\left\{L, p_{x}^{2}\right\}-2 a\left(\left\{x^{2} y, p_{x}\right\}-\left\{x^{3}, p_{y}\right\}\right)$
(Q.2) $V=a\left(x^{2}+y^{2}\right)+\frac{b}{x^{2}}+\frac{c}{y^{2}}$
$X_{1}=\left\{L, p_{x} p_{y}\right\}+\left\{x y\left(-\frac{2 b}{x^{3}}+2 a x\right), p_{y}\right\}-\left\{x y\left(-\frac{2 c}{y^{3}}+2 a y\right), p_{x}\right\}$
(Q.3) $V=a\left(x^{2}+y^{2}\right)+\frac{\hbar^{2}}{x^{2}}+\frac{\hbar^{2}}{y^{2}}$
$X_{1}=2 L^{3}-\hbar^{2}\left\{\frac{3 x^{2}}{y}+2 y+\frac{3 y^{3}}{x^{2}}, p_{x}\right\}+\hbar^{2}\left\{\frac{3 y^{2}}{x}+2 x+\frac{3 x^{3}}{y^{2}}, p_{y}\right\}$
$X_{2}=\left\{L, p_{x} p_{y}\right\}+\left\{x y\left(-\frac{2 \hbar^{2}}{x^{3}}+2 a x\right), p_{y}\right\}-\left\{x y\left(-\frac{2 \hbar^{2}}{y^{3}}+2 a y\right), p_{x}\right\}$
(Q.4) $V=a\left(x^{2}+y^{2}\right)+\frac{\hbar^{2}}{y^{2}}$
$X_{1}=2 L^{3}-\hbar^{2}\left\{2 y+\frac{3 x^{2}}{y}, p_{x}\right\}+\hbar^{2}\left\{\frac{3 x^{3}}{y^{2}}+2 x, p_{y}\right\}$
$X_{2}=\left\{L, p_{x} p_{y}\right\}-2\left\{a x y^{2}-\frac{\hbar^{2} x}{y^{2}}, p_{x}\right\}+2 a\left\{x^{2} y, p_{y}\right\}$
$X_{3}=\left\{L, p_{y}^{2}\right\}-\left\{2 a y^{3}+\hbar^{2} \frac{1}{y}, p_{x}\right\}+\left\{\hbar^{2} \frac{3 x}{y^{2}}+2 a x y^{2}, p_{y}\right\}$
(Q.5)
$V=\hbar^{2}\left(\frac{1}{8 \alpha^{4}}\left(x^{2}+y^{2}\right)+\frac{1}{(x-\alpha)^{2}}+\frac{1}{(x+\alpha)^{2}}\right)$
$X_{1}=2 L^{3}-3 \alpha^{2}\left\{L, p_{y}^{2}\right\}+\frac{\hbar^{2}}{4}\left\{y\left(-8+\frac{3 y^{2}}{\alpha^{2}}-\frac{24 y^{2}\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}\right), p_{x}\right\}$
$+\frac{\hbar^{2}}{4}\left\{x\left(8-\frac{3 y^{2}\left(x^{4}-10 \alpha^{2} x^{2}-24 \alpha^{4}\right)}{\alpha^{2}(x-\alpha)^{2}(x+\alpha)^{2}}\right), p_{y}\right\}$
$X_{2}=\left\{L, p_{x}^{2}\right\}+\hbar^{2}\left\{y\left(\frac{4 \alpha^{2}-x^{2}}{4 \alpha^{4}}-\frac{\left.6\left(x^{2}+\alpha^{2}\right)\right)}{\left(x^{2}-\alpha^{2}\right)^{2}}\right), p_{x}\right\}$
$+\hbar^{2}\left\{\frac{x\left(x^{2}-4 \alpha^{2}\right)}{4 \alpha^{4}}-\frac{2 x}{x^{2}-\alpha^{2}}+\frac{4 x\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}, p_{y}\right\}$
(Q.6)
$V=\hbar^{2}\left(\frac{1}{8 \alpha^{4}}\left(x^{2}+y^{2}\right)+\frac{1}{y^{2}}+\frac{1}{(x+\alpha)^{2}}+\frac{1}{(x-\alpha)^{2}}\right)$
$X_{1}=2 L^{3}-3 \alpha^{2}\left\{L, p_{y}^{2}\right\}+\hbar^{2}\left\{\frac{3 y^{3}}{4 \alpha^{2}}+\frac{6 y^{3}\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}-\frac{3\left(x^{2}-\alpha^{2}\right)}{y}-2 y, p_{x}\right\}+$ $3 \hbar^{2}\left\{x\left(\frac{x^{2}-3 \alpha^{2}}{y^{2}}-\frac{3 y^{2}-8 \alpha^{2}}{12 \alpha^{2}}-\frac{2 y^{2}}{x^{2}-\alpha^{2}}+\frac{4 y^{2}\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}\right), p_{y}\right\}$
(Q.7)
$V=\hbar^{2}\left(\frac{1}{8 \alpha^{4}}\left(x^{2}+y^{2}\right)+\frac{1}{(y-\alpha)^{2}}+\frac{1}{(x-\alpha)^{2}}+\frac{1}{(y+\alpha)^{2}}+\frac{1}{(x+\alpha)^{2}}\right)$
$X_{1}=2 L^{3}-3 \alpha^{2}\left(\left\{L, p_{x}^{2}\right\}+\left\{L, p_{y}^{2}\right\}\right)+\frac{\hbar^{2}}{4}\left\{y\left(124+\frac{3\left(x^{2}+y^{2}\right)}{\alpha^{2}}+\frac{24\left(x^{2}-5 y^{2}\right)}{y^{2}-\alpha^{2}}-\right.\right.$
$\left.\left.\frac{144 x^{2}}{x^{2}-\alpha^{2}}+\frac{24\left(3 x^{2}-y^{2}\right)\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}+\frac{48\left(y^{2}-x^{2}\right)\left(y^{2}+\alpha^{2}\right)}{(y-\alpha)^{2}(y+\alpha)^{2}}\right), p_{x}\right\}-\frac{\hbar^{2}}{4}\left\{x\left(124+\frac{3\left(x^{2}+y^{2}\right)}{\alpha^{2}}-\right.\right.$
$\left.\left.\frac{24\left(5 x^{2}-y^{2}\right)}{x^{2}-\alpha^{2}}-\frac{144 x^{2}}{y^{2}-\alpha^{2}}-\frac{24\left(x^{2}-3 y^{2}\right)\left(y^{2}+\alpha^{2}\right)}{(y-\alpha)^{2}(y+\alpha)^{2}}+\frac{48\left(x^{2}-y^{2}\right)\left(x^{2}+\alpha^{2}\right)}{(x-\alpha)^{2}(x+\alpha)^{2}}\right), p_{y}\right\}$
(Q.8) $V=a\left(4 x^{2}+y^{2}\right)+\frac{b}{y^{2}}+c x$
$X_{1}=2 p_{x} p_{y}^{2}+\left\{-2 a y^{2}+\frac{2 b}{y^{2}}, p_{x}\right\}+\left\{8 a x y+c y, p_{y}\right\}$
(Q.9)
$V=a\left(9 x^{2}+y^{2}\right)$
$X_{1}=\left\{L, p_{y}^{2}\right\}+\frac{2}{3} a\left\{y^{3}, p_{x}\right\}-6 a\left\{x y^{2}, p_{y}\right\}$
(Q.10)
$V=a\left(9 x^{2}+y^{2}\right)+\frac{\hbar^{2}}{y^{2}}$
$X_{1}=\left\{L, p_{y}^{2}\right\}+\left\{\frac{2 a y^{3}}{3}-\frac{\hbar^{2}}{y}, p_{x}\right\}+\left\{3 x\left(-2 a y^{2}+\frac{\hbar^{2}}{y^{2}}\right), p_{y}\right\}$
(Q.11)
$V=\hbar^{2}\left(\frac{1}{8 \alpha^{4}}\left(9 x^{2}+y^{2}\right)+\frac{1}{(y+\alpha)^{2}}+\frac{1}{(y-\alpha)^{2}}\right)$
$X_{1}=\left\{L, p_{y}^{2}\right\}+\hbar^{2}\left\{y\left(\frac{y^{2}}{12 \alpha^{4}}-\frac{8 \alpha^{2}}{\left(y^{2}-\alpha^{2}\right)^{2}}-\frac{2}{y^{2}-\alpha^{2}}\right), p_{x}\right\}+\frac{3 \hbar^{2}}{4}\left\{x\left(\frac{8\left(y^{2}+\alpha^{2}\right)}{\left(y^{2}-\alpha^{2}\right)^{2}}-\right.\right.$
$\left.\left.\frac{y^{2}}{\alpha^{4}}\right), p_{y}\right\}$
(Q.12)
$V=\frac{\hbar^{2}}{x^{2}}+\frac{a}{y^{2}}$
$X_{1}=\left\{L^{2}, p_{x}\right\}-2 \hbar^{2}\left\{\frac{y^{2}}{x^{2}}, p_{y}\right\}+\left\{3 \hbar^{2} \frac{y^{2}}{x^{2}}+2 a \frac{x^{2}}{y^{2}}+\frac{\hbar^{2}}{2}, p_{x}\right\}$
$X_{2}=\left\{L, p_{x} p_{y}\right\}-2 \hbar^{2}\left\{\frac{y}{x^{2}}, p_{y}\right\}+2 a\left\{\frac{x}{y^{2}}, p_{x}\right\}$
$X_{3}=2 p_{x}^{3}+\left\{\frac{3 \hbar^{2}}{x^{2}}, p_{x}\right\}$
(Q.13)
$V=\frac{\hbar^{2}}{x^{2}}+\frac{\hbar^{2}}{y^{2}}$
$X_{1}=2 L^{3}-\hbar^{2}\left\{2 y+\frac{3 x^{2}}{y}+\frac{3 y^{3}}{x^{2}}, p_{x}\right\}+\hbar^{2}\left\{2 x+\frac{3 y^{2}}{x}+\frac{3 x^{3}}{y^{2}}, p_{y}\right\}$
$X_{2}=\left\{L^{2}, p_{x}\right\}-2 \hbar^{2}\left\{\frac{y}{x}, p_{y}\right\}+\left\{3 \hbar^{2} \frac{y^{2}}{x^{2}}+2 \hbar^{2} \frac{x^{2}}{y^{2}}+\frac{\hbar^{2}}{2}, p_{x}\right\}$
$X_{3}=\left\{L^{2}, p_{y}\right\}-2 \hbar^{2}\left\{\frac{x}{y}, p_{x}\right\}+\left\{3 \hbar^{2} \frac{x^{2}}{y^{2}}+2 \hbar^{2} \frac{y^{2}}{x^{2}}+\frac{\hbar^{2}}{2}, p_{x}\right\}$
$X_{4}=\left\{L, p_{x} p_{y}\right\}-2 \hbar^{2}\left\{\frac{y}{x^{2}}, p_{y}\right\}+2 \hbar^{2}\left\{\frac{x}{y^{2}}, p_{x}\right\}$
$X_{5}=2 p_{x}^{3}+\left\{\frac{3 \hbar^{2}}{x^{2}}, p_{x}\right\}$
$X_{6}=2 p_{y}^{3}+\left\{\frac{3 \hbar^{2}}{y^{2}}, p_{y}\right\}$
(Q.14) $V=a x+\frac{\hbar^{2}}{y^{2}}$
$X_{1}=\left\{L, p_{y}^{2}\right\}-\left\{\frac{\hbar^{2}}{y}, p_{x}\right\}+\left\{3 \frac{\hbar^{2} x}{y^{2}}-a \frac{y^{2}}{2}, p_{y}\right\}$
$X_{2}=2 p_{y}^{3}+\left\{3 \frac{\hbar^{2}}{y^{2}}, p_{y}\right\}$
$X_{3}=2 p_{x} p_{y}^{2}+2 \hbar^{2}\left\{\frac{1}{y^{2}}, p_{x}\right\}+a\left\{y, p_{y}\right\}$
(Q.15)
$V=\hbar^{2} \mathcal{P}(y)+V(x)$
$X_{1}=2 p_{y}^{3}+\left\{3 \hbar^{2} \mathcal{P}(y), p_{y}\right\}$
(Q.16)
$V=\hbar^{2}(\mathcal{P}(x)+\mathcal{P}(y))$
$X_{1}=2 p_{x}^{3}+\left\{3 \hbar^{2} \mathcal{P}(x), p_{x}\right\}$
$X_{2}=2 p_{y}^{3}+\left\{3 \hbar^{2} \mathcal{P}(y), p_{y}\right\}$
(Q.17)
$V=\hbar^{2} \omega_{1}^{2} P_{1}\left(\omega_{1} x\right)+\hbar^{2} \omega_{2}^{2} P_{1}\left(\omega_{2} x\right)$
$X_{1}=2 \omega_{2}^{5} p_{x}^{3}-2 \omega_{1}^{5} p_{y}^{3}+3 \omega_{2}^{5}\left\{V_{1}\left(x, \omega_{1}\right), p_{x}\right\}-3 \omega_{1}^{5}\left\{V_{2}\left(y, \omega_{2}\right), p_{y}\right\}$
(Q.18)
$V=a\left(x^{2}+y^{2}\right)+\frac{\hbar}{2} b_{1} P_{4}^{\prime}\left(x, \frac{-8 a}{\hbar^{2}}\right)+4 a P_{4}^{2}\left(x, \frac{-8 a}{\hbar^{2}}\right)+4 a x P_{4}\left(x, \frac{-8 a}{\hbar^{2}}\right)+\frac{1}{6}\left(-\hbar^{2} K_{1}+\hbar b_{1}\right)$
$X_{1}=\left\{L, p_{x}^{2}\right\}+\left\{a x^{2} y-3 y V_{1}, p_{x}\right\}-\frac{1}{2 a}\left\{\frac{\hbar^{2}}{4} V_{1 x x x}+\left(a x^{2}-3 V_{1}\right) V_{1 x}, p_{y}\right\}$
(Q.19)
$V=a y+\hbar^{2} \omega^{2} P_{1}(\omega x)$
$X_{1}=2 p_{x}^{3}+3 \hbar^{2} \omega^{2}\left\{P_{1}(\omega x), p_{x}\right\}+\left\{\frac{\omega^{5} \hbar^{4}}{4 a}, p_{y}\right\}$
(Q.20)
$V=b x+a y+(2 \hbar b)^{\frac{2}{3}} P_{2}^{2}\left(\left(\frac{2 b}{\hbar^{2}}\right)^{\frac{1}{3}} x, 0\right)$
$X_{1}=2 a p_{x}^{3}-2 b p_{x}^{2} p_{y}+a\left\{3 V_{1}(x)-b x, p_{x}\right\}-2 b\left\{V_{1}(x), p_{y}\right\}$
(Q.21)
$V=a y+\left(2 \hbar^{2} b^{2}\right)^{\frac{1}{3}}\left(P_{2}^{\prime}\left(-\left(4 b / \hbar^{2}\right)^{\frac{1}{3}} x, \kappa\right)+P_{2}^{2}\left(-\left(4 b / \hbar^{2}\right)^{\frac{1}{3}} x, \kappa\right)\right)$
$X_{1}=2 a p_{x}^{3}-2 b p_{x}^{2} p_{y}+a\left\{3 V_{1}(x)-b x, p_{x}\right\}-2 b\left\{V_{1}(x), p_{y}\right\}$

## B. Classical potentials

(C.1) $V=a\left(x^{2}+y^{2}\right)$
$X_{1}=L^{3}$
$X_{2}=\left\{L, p_{x} p_{y}\right\}+a\left\{2 x^{2} y, p_{y}\right\}-a\left\{2 x y^{2}, p_{x}\right\}$
$X_{3}=\left\{L, p_{y}^{2}\right\}+2 a\left(\left\{x y^{2}, p_{y}\right\}-\left\{y^{3}, p_{x}\right\}\right)$
$X_{4}=\left\{L, p_{x}^{2}\right\}-2 a\left(\left\{x^{2} y, p_{x}\right\}-\left\{x^{3}, p_{y}\right\}\right)$
(C.2) $V=a\left(x^{2}+y^{2}\right)+\frac{b}{x^{2}}+\frac{c}{y^{2}}$
$X_{1}=\left\{L, p_{x} p_{y}\right\}+\left\{x y\left(-\frac{2 b}{x^{3}}+2 a x\right), p_{y}\right\}-\left\{x y\left(-\frac{2 c}{y^{3}}+2 a y\right), p_{x}\right\}$
(C.3) $V=a\left(4 x^{2}+y^{2}\right)+\frac{b}{y^{2}}+c x$
$X_{1}=2 p_{x} p_{y}^{2}+\left\{-2 a y^{2}+\frac{2 b}{y^{2}}, p_{x}\right\}+\left\{8 a x y+c y, p_{y}\right\}$
(C.4)
$V=a\left(9 x^{2}+y^{2}\right)$
$X_{1}=\left\{L, p_{y}^{2}\right\}+\frac{2}{3} a\left\{y^{3}, p_{x}\right\}-6 a\left\{x y^{2}, p_{y}\right\}$
$V= \pm \sqrt{\beta_{1} x} \pm \sqrt{\beta_{2} y}$
$X_{1}=2 \beta_{2} p_{x}^{3}-2 \beta_{1} p_{y}^{3}+3 \beta_{2}\left\{V_{1}\left(x, 6 \beta_{1}\right), p_{x}\right\}-3 \beta_{1}\left\{V_{2}\left(y, 6 b_{2}\right), p_{y}\right\}$
(C.6) $V=a y^{2}+V_{1} \quad$ where $V_{1}$ satisfies equation (24)
$X_{1}=\left\{L, p_{x}^{2}\right\}+\left\{a x^{2} y-3 y V_{1}, p_{x}\right\}-\frac{1}{2 a}\left\{\left(a x^{2}-3 V_{1}\right) V_{1 x}, p_{y}\right\}$
(C.7)
$V=a y+b \sqrt{x}$
$X_{1}=2 p_{x}^{3}+3 b\left\{\sqrt{x}, p_{x}\right\}-\left\{\frac{3 b^{2}}{2 a}, p_{y}\right\}$
(C.8)
$V=a y+V_{1}(x), \quad$ where $\left(V_{1}-b x\right)^{2} V_{1}=d$
$X_{1}=2 a p_{x}^{3}-2 b p_{x}^{2} p_{y}+a\left\{3 V_{1}(x)-b x, p_{x}\right\}-2 b\left\{V_{1}(x), p_{y}\right\}$

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