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1. Introduction.

We consider the problem of finding solutions $u \varepsilon L_2(\Omega, \mathbb{R}) = L_2(\Omega)$ to the Hammerstein integral equation

$$u(t) = \int_\Omega K(s,t)g(u(s),s)ds, \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^q$, $K(s,t) = K(t,s) \varepsilon L_2(\Omega \times \Omega)$, and $g\colon \mathbb{R} \times \Omega \to \mathbb{R}$ is a continuous function. We let $K_1\colon L_2(\Omega) \to L_2(\Omega)$ be the operator defined by $K_1(u)(t) = \int_\Omega K(s,t)u(s)ds$. We denote by $\{\lambda_i\}_{i\varepsilon Z}$ and $\{\psi_i\}_{i\varepsilon Z}$ the sequences of eigenvalues and corresponding eigenfunctions of

$$u = \lambda K_1(u) \tag{1.2}$$

The equation (1.1) was discussed by Dolph in [2] and the following results were obtained. If K_1 is positive definite the problem of finding the solutions of (1.1) can be reduced to the problem of finding the critical points of certain functional J defined on $L_2(\Omega)$. Moreover, if: there exist two consecutive eigenvalues of (1.2), λ_N and λ_{N+1}, and real numbers γ, γ' and C such that

$$\begin{aligned} &(g(u,x) - g(v,x))/(u-v) \leq \gamma' < \lambda_{N+1} \text{ for all } (u,x) \in \mathbb{R} \times \Omega \\ &G(u,x) \equiv \int_0^u g(s,x)ds \geq (\gamma/2)u^2 + C \text{ for all } (u,x) \in \mathbb{R} \times \Omega \end{aligned} \tag{1.3}$$

with $\gamma > \lambda_N$; and for every ξ in $X = \text{span}\,\{\psi_i | \lambda_i \leq \lambda_N\}$ J has a unique minimum on the linear manifold $\xi + X^{\perp}$; then (1.1) has a solution.

Here we prove that even in the case that K_1 is indefinite the problem of finding the solutions of (1.1) can be reduced to the study of the critical points of certain functional defined on $L_2(\Omega)$. We show that (1.3) alone implies the existence of a solution of (1.1). We also apply a result on Liusternik-Schnierelmann theory due to Clark [2,p.71] to the study of (1.1) when g is odd in the first variable, mildly nonlinear and again K_1 not necessarily positive definite.

In [5,ch.VI] and [6,ch.V] the equation (1.1) is studied when (1.2) has a finite number of negative eigenvalues. For a historical account concerning (1.1) see [3].

2. A Max-min Principle.

Throughout this section H is a real separable Hilbert space, with inner product <,>, and f: $H \to \mathbb{R}$ is a function of class C^1. For each $u \in H$ we denote by $\nabla f(u)$ the unique element of H such that

$$\lim_{t\to 0} \frac{f(u+tv) - f(u)}{t} = \langle \nabla f(u), v\rangle \quad \text{for } v \in H.$$

If ∇f is differentiable we denote by $D^2f(u)$ the differential of ∇f at u. If $\{x_n\}$ is a sequence in a Hilbert space which converges to some element x we denote $x_n \to x$, if $\{x_n\}$ converges weakly to x we denote $x_n \rightharpoonup x$.

We say that a function is continuous with respect to weak convergence (CWC) if it takes weakly convergent sequences into convergent sequences. We say that a function w: $H \to \mathbb{R}$ is weakly lower semicontinuous (WLSC) if $x_n \rightharpoonup x$ implies $w(x) \leq \underline{\lim}\, w(x_n)$.

Lemma 2.1.: Suppose there exist X and Y, which are closed subspaces of H, such that $H = X \oplus Y$ and for some $m > 0$

$$\langle \nabla f(x+y_1) - \nabla f(x+y_2), y_1-y_2\rangle \geq m\|y_1-y_2\|^2 \tag{2.1}$$

for every $x \in X$, $y_1 \in Y$, and $y_2 \in Y$.

Assertion: There exists a continuous function ϕ: $X \to Y$ satisfying:

i) $f(x+\phi(x)) = \min_{y \in Y} f(x+y)$

ii) The function $\tilde{f}$: $X \to \mathbb{R}$, $x \to f(x+\phi(x))$ is of class C^1

iii) If, in addition, there exist an isomorphism $A: H \to H$ such that

$A(X) \subset X$, $A(Y) \subset Y$, for some $m_1 > 0$ $\langle A(y), y\rangle \geq m_1\|y\|^2$ for all $y \in Y$, and $\nabla f - A = F$ is continuous with respect to weak convergence (CWC), then ϕ is CWC when either $\dim Y < \infty$ or X and Y are orthogonal.

Proof: For each $x \in X$ we define f_x: $Y \to \mathbb{R}$ by $f_x(y) = f(x+y)$. From (2.1) we have $\langle \nabla f_x(y_1) - \nabla f_x(y_2), y_1 - y_2\rangle \geq m\|y_1-y_2\|^2$. Thus, for each

$x \in X$ f_x has a unique critical point $\phi(x)$ (see[6,p.80]). Moreover, $f_x(\phi(x)) = \min_{y \in Y} f_x(y) = \min_{y \in Y} (f(x+y))$. Therefore, $\phi(x)$ is the only element of Y such that

$$
\begin{aligned}
0 &= \langle \nabla f_x(\phi(x)), K\rangle \\
&= \langle \nabla f(x+\phi(x)), K\rangle \quad \text{for all } K \in Y.
\end{aligned}
\tag{2.2}
$$

Let us see now that ϕ is continuous. Suppose, on the contrary, that there exist $\delta > 0$ and a sequence $\{x_n\}_n$ in X such that $x_n \to x \in X$ and $\| \phi(x_n) - \phi(x) \| \geq \delta$. Since ∇f is continuous, for n sufficiently large

$$\| P^*(\nabla f(x_n+\phi(x))) \| < \delta m, \tag{2.3}$$

where $P: H \to Y$ is defined by $P(x+y) = y$, $x \in X$, $y \in Y$, and P^* is the adjoint of P.

Because of (2.1) we obtain using Schwarz's inequality

$$\| P^*(\nabla f(x_n+\phi(x_n))) - P^*(\nabla f(x_n+\phi(x))) \| \geq m \| \phi(x) - \phi(x_n) \| \geq m\delta. \tag{2.4}$$

Since, by (2.2) $P^*(\nabla f(x_n+\phi(x_n))) = 0$, the inequality (2.4) contradicts (2.3). Thus ϕ is continuous, and this proves part i).

For $t > 0$ and $h \in X$, we have

$$
\begin{aligned}
\frac{\tilde{f}(x+th) - \tilde{f}(x)}{t} &= \frac{f(x+th+\phi(x+th)) - f(x+\phi(x))}{t} \\
&\leq \frac{f(x+th+\phi(x)) - f(x+\phi(x))}{t} \\
&= \int_0^1 \langle \nabla f(x+\phi(h) + sth), h\rangle ds
\end{aligned}
\tag{2.5}
$$

In a similar manner, we can see that

$$\frac{f(x+th) - f(x)}{t} \geq \int_0^1 \langle \nabla f(x+\phi(x+th)+sth, h\rangle ds.$$

Therefore, since ∇f and ϕ are continuous we have

$$\lim_{t\to 0+} \frac{\tilde{f}(x+th) - \tilde{f}(x)}{t} = \langle \nabla f(x+\phi(x)),h \rangle .$$

This shows that $\tilde{f}$ has a continuous Gateaux derivative and hence is of class C^1 (see [6,p.42]). From the above

$$\langle \nabla\tilde{f}(x),h \rangle = \langle \nabla f(x+\phi(x)),h \rangle \quad \text{for all } h\varepsilon X. \tag{2.6}$$

We prove now part iii). By (2.1) we have

$$\langle \nabla f(x+\phi(x)) - \nabla f(x),\phi(x) \rangle \geq m\| \phi(x) \|^2 . \tag{2.7}$$

Since $\langle \nabla f(x+\phi(x)),\phi(x) \rangle = 0$, by Schwarz's inequality we obtain

$$\| \nabla f(x) \| \geq m\| \phi(x) \| . \tag{2.8}$$

By iii), ∇f is bounded on bounded sets; hence (2.8) implies that ϕ is bounded on bounded sets. In particular, if $\{x_n\}$ is a sequence in X such that $x_n \to \bar{x}$ then $\{\phi(x_n)\}$ is bounded. Let $\{\phi(x_{n_j})\}$ be a subsequence of $\{\phi(x_n)\}$ such that $\phi(x_{n_j}) \rightharpoonup y\varepsilon Y$. Since F is CWC, $F(x_{n_j}+\phi(x_{n_j})) \to F(\bar{x}+y)$.
Therefore, for any $K\varepsilon Y$

$$0 = \lim_{j\to\infty} \langle A(x_{n_j}) + A(\phi(x_{n_j})) + F(x_{n_j}+\phi(x_{n_j})),\ K \rangle$$

$$= \langle A(\bar{x}) + A(y) + F(\bar{x}+y),\ K \rangle .$$

Consequently, by the characterization (2.2), we have $y = \phi(\bar{x})$. This implies $\phi(x_n) \rightharpoonup \phi(\bar{x})$. In particular, if $\dim Y < \infty$ then $\phi(x_n) \to \phi(\bar{x})$;i.e. ϕ is CWC.

If $\dim Y = +\infty$ and X and Y orthogonal then by (2.2) and iii)

$$0 = \langle A(x_n) + A(\phi(x_n)) + F(x_n+\phi(x_n)),\phi(x_n) - \phi(\bar{x}) \rangle \tag{2.9}$$

$= \langle A(\phi(x_n)) + F(x_n+\phi(x_n)), \phi(x_n) - \phi(\bar{x})\rangle$

Also, $0 = \langle A(\phi(x)) + F(x+\phi(x+\phi(x)), \phi(x_n) - \phi(x)\rangle$. Therefore, subtracting the last equation from (2.9) we obtain

$$\langle A(\phi(x_n)) - A(\phi(\bar{x})), \phi(x_n) - \phi(\bar{x})\rangle =$$

$$= \langle F(x+\phi(x)) - F(x_n+\phi(x_n), \phi(x_n) - \phi(\bar{x})\rangle. \tag{2.10}$$

Since $x_n+\phi(x_n) \rightharpoonup \bar{x}+\phi(\bar{x})$, $F(x_n+\phi(x_n)) \to F(x+\phi(\bar{x}))$. This together with the fact that ϕ is bounded on bounded sets and relation (2.10) imply that $\langle A(\phi(x_n)) - A(\phi(\bar{x})), \phi(x_n) - \phi(x)\rangle \to 0$ as $n \to \infty$. But, according to iii) $m_1 \| \phi(x_n) - \phi(\bar{x}) \|^2 \le \langle A(\phi(x_n)) - A(\phi(x)), \phi(x_n) - \phi(x)\rangle$. This proves that $\phi(x_n)$ converges strongly to $\phi(\bar{x})$. Therefore, ϕ is CWC and the lemma is proved.

Now we are ready to prove our <u>variational principle</u>.

Theorem 2.2.: <u>Let</u> f,X,Y,H <u>and</u> m <u>be as in lemma</u> 2.1. <u>If, in addition,</u> $-\tilde{f}$ <u>is WLSC, and</u>

$$f(x) \to -\infty \text{ as } \|x\| \to \infty \quad x \in X, \tag{2.11}$$

<u>then there exists</u> $u_0 \in H$ <u>such that</u>

$$\nabla f(u_0) = 0 \tag{2.12}$$

<u>and</u>

$$f(u_0) = \max_{x \in X} \left(\min_{y \in Y} f(x+y)\right) \tag{2.13}$$

Proof: Let $\tilde{f}$ and ϕ be as in lemma 2.1. We proved that $\tilde{f}$ is of class C^1 and ϕ is continuous. Since $-\tilde{f}(x) = -f(x+\phi(x)) \ge -f(x)$ and $f(x) \to -\infty$ as $\|x\| \to \infty$ then

$$-f(x) \to +\infty \text{ as } \|x\| \to +\infty. \tag{2.14}$$

Therefore, since $-\tilde{f}$ is WLSC, then there exists $x_0 \in X$(see[6,p.80]) such that $-\tilde{f}(x_0) = \min_{x\in X} -\tilde{f}(x)$. Thus,

$$\tilde{f}(x_0) = \max_{x\in X} \tilde{f}(x) = \max_{x\in X} \left(\min_{y\in Y} f(x+y)\right) \tag{2.15}$$

$$= f(x_0 + \phi(x_0)).$$

Also, since $\tilde{f}$ is of class C^1, for $h\in X$

$$<\nabla\tilde{f}(x_0),h> = 0 \tag{2.16}$$

Therefore, if $w = h+k\in H = X\oplus Y$, then

$$<\nabla f(x_0+\phi(x_0)),h+k> = <\nabla f(x_0+\phi(x_0)),h> + <\nabla f(x_0+\phi(x_0)),k>. \tag{2.17}$$

Because of (2.2), the second term of the right hand side of (2.17) is equal to zero. Because of (2.6) and (2.16), the first term of the right hand side of (2.17) is also equal to zero. Thus, if $u_0 = x_0+\phi(x_0)$, we have $\nabla f(u_0) = 0$ and, by (2.15), $f(u_0) = \max_{x\in X} \left(\min_{y\in Y} f(x+y)\right)$.

3. Applications to Hammerstein Integral Equations.

Throughout this section Ω is a bounded region in $\mathbb{R}^q$, $K(s,t) = K(t,s) \in L_2(\Omega\times\Omega)$ and $g: \mathbb{R}\times\Omega \to \mathbb{R}$ is a continuous function such that the Nemytsky operators $u(\xi) \to g(u(\xi),\xi)$ and $u(\xi) \to \int_0^{u(\xi)} g(s,\xi)ds$ are continuous with domain in $L_2(\Omega)$ and range contained in $L_2(\Omega)$. From the theory of linear operators we know that $K_1:L_2(\Omega) \to L_2(\Omega)$, $u(t) \to \int_\Omega K(s,t)u(s)ds$ is a compact operator.

We want to consider the problem

$$u(t) = K_1(g(u(t),t)) \qquad t\in\Omega. \tag{3.1}$$

Let $\{\lambda_i;\ i = \pm1, \pm2,\dots\}$ be the sequence of eigenvalues of $u = \lambda K_1(u)$. Let $\{\psi_i\}_i$ be an orthonormal sequence of eigenfuctions corresponding to the sequence of eigenvalues $\{\lambda_i\}_i$. We assume that $\dots\ \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \dots$

and that the $\{\psi_i\}_i$ form a complete set in $L_2(\Omega)$. Let $X = \text{span}\{\psi_{-1},\psi_{-2},\ldots\}$ and $Y = \text{span}\{\psi_1,\psi_2,\ldots\}$. It follows that $K_1(X)\subset X$ and $K_1(Y)\subset Y$, and Y is the orthogonal complement of X. Moreover, K_1 restricted to Y is positive definite and K_1 restricted to X is negative definite.

The completeness of $\{\psi_i\}_i$ implies that for any $u\epsilon L_2(\Omega)$

$$\begin{aligned} K_1(u) &= \sum_{j=-\infty}^{-1} <K_1(u),\psi_j>\psi_j + \sum_{j=1}^{\infty} <K_1(u),\psi_j>\psi_j \\ &= \sum_{j=-\infty}^{-1} <u,K_1(\psi_j)>\psi_j + \sum_{j=1}^{\infty} <u,K_1(\psi_j)>\psi_j \\ &= \sum_{j=-\infty}^{-1} \frac{1}{\lambda_j} <u,\psi_j>\psi_j + \sum_{j=1}^{\infty} \frac{1}{\lambda_j} <u,\psi_j>\psi_j, \end{aligned}$$

where $<\,,\,>$ is the usual inner product in $L_2(\Omega)$.

Now we define the operators Q,Q_1: $L_2(\Omega)$ by $Q(u) = \sum_{j=-\infty}^{-1} \frac{1}{\sqrt{-\lambda_j}} <u,\psi_j>\psi_j$ and $Q_1(u) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} <u,\psi_j>\psi_j$. It is easily seen that Q and Q_1 are compact linear operators. Suppose $u = x+y$ with $x\epsilon X$, $y\epsilon Y$,

$$\begin{aligned} Q(Q(u)) &= Q\Big(\sum_{j=-\infty}^{-1} \frac{1}{\sqrt{-\lambda_j}} <u,\psi_j>\psi_j\Big) \qquad (3.2) \\ &= \sum_{j=-\infty}^{-1} \frac{1}{\sqrt{-\lambda_j}} <u,\psi_j>Q(\psi_j) \\ &= \sum_{j=-\infty}^{-1} \frac{1}{-\lambda_j} <x,\psi_j>\psi_j \\ &= -K_1(x). \end{aligned}$$

Similary

$$Q_1(Q_1(u)) = K_1(y). \qquad (3.3)$$

That is, Q_1 is a square root of K_1 on Y and Q a square root of $-K_1$ on X. From the definition of Q and Q_1 it is easy to see that Q and Q_1 are selfadjoint.

Next we reduce the problem (3.1) to an operator equation envolving Q and Q_1. Let $P: L_2(\Omega) \to L_2(\Omega)$ be the orthogonal projection onto X and $P_1 = I - P$. Clearly $PQ = Q$ and $PQ_1 = 0$. We denote $\tilde{g}(u)(t) = g(u(t),t)$, for u $L_2(\Omega)$.

Lemma 3.1.: <u>The problem</u> (3.1) <u>has a solution iff the problem</u>

$$-x(t)+y(t) = Q(\tilde{g}(Q_1(y) + Q(x))(t)) + Q_1(\tilde{g}(Q_1(y) + Q(x))(t)) \quad (3.4)$$

has a solution $(x,y) \in X\times Y$.

Proof: Let $u = x+y$, $x\in X$, $y\in Y$ satisfy (3.1); therefore $x = P(u) = P(K_1(\tilde{g}(u))) = K_1(P(\tilde{g}(u))) = -Q^2(P(\tilde{g}(u)))$. Thus, there exists $\tilde{x}\in X$ such that $Q(\tilde{x}) = x$. A similar argument shows the existence of $\tilde{y}\in Y$ such that $y = Q_1(\tilde{y})$. Therefore,

$$Q(\tilde{x}) = x = Q(-Q(\tilde{g}(Q(\tilde{x}) + Q_1(\tilde{y})))). \quad (3.5)$$

Since 0 is not an eigenvalue of Q restricted to X and $-Q(\tilde{g}(Q(\tilde{x})+Q_1(\tilde{y})))$ belongs to X, we have

$$\tilde{x} = -Q(\tilde{g}(Q(\tilde{x}) + Q_1(\tilde{y}))). \quad (3.6)$$

A parallel argument shows that

$$\tilde{y} = Q_1(\tilde{g}(Q(\tilde{x}) + Q_1(\tilde{y}))). \quad (3.7)$$

Subtracting (3.6) from (3.7) we obtain

$$-\tilde{x}+\tilde{y} = Q(\tilde{g}(Q(\tilde{x}) + Q_1(\tilde{y}))) + Q_1(\tilde{g}(Q(\tilde{x}) + Q_1(\tilde{y})))$$

So (3.4) has a solution. This proves the necessity of (3.4).

Conversely, suppose $(x,y)\in X\times Y$ satisfies (3.4). Let us call $u = Q_1(y) + Q(x)$. Applying Q and Q_1 to (3.4) we obtain respectively

$$Q(x) = -Q^2(\tilde{g}(u)) = K_1(P(\tilde{g}(u)) \quad (3.8)$$

$$Q_1(y) = K_1(P_1(\tilde{g}(u)). \quad (3.9)$$

Adding the last two equations we get

$$u = K_1(P(\tilde{g}(u)) + K_1(P_1(\tilde{g}(u)))) = K_1(\tilde{g}(u)).$$

Therefore (3.1) has a solution, and the lemma is proved.

Theorem 3.2.: Let $g: \mathbb{R}\times\bar{\Omega} \to \mathbb{R}$ be a continuous function. If there exist real numbers γ, γ' and C such that for some integer N

a) $\lambda_N < \gamma \leq \gamma' < \lambda_{N+1}$, if N=-1 we let $\lambda_{-1} < \gamma \leq \gamma' < \lambda_1$

b) For $u\varepsilon\mathbb{R}$, $v\varepsilon\mathbb{R}$, $t\varepsilon\Omega$ $(g(u,t)-g(v,t))/(u-v) \leq \gamma'$

and

c) For $u\varepsilon\mathbb{R}$, $t\varepsilon\bar{\Omega}$ $G(u,t) = \int_0^u g(s,t)ds \geq (\gamma/2)u^2 - C$

then the problem (3.1) has at least one solution.

Proof: We define $J:L_2(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega((P_1u)^2/2) - ((Pu)^2/2) - \tilde{G}(Q(P(u)) + Q_1(P(u))),$$

where $\tilde{G}(u)(t) = G(u(t),t)$. Conditions b) and c) imply the existence of constants A,B > 0 such that

$$|g(s,t)| \leq A + B|s| \quad \text{and} \quad G(s,t) \leq A + B|s|^2 \tag{3.10}$$

for any $(s,t)\varepsilon\mathbb{R}\times\Omega$. Therefore, J is a functional of class C^1. An elementary computation shows that if $u\varepsilon L_2(\Omega)$, $u = x+y$, $x\varepsilon X$, $y\varepsilon Y$ then

$$<\nabla J(u),x_1> = \int_\Omega -x\cdot x_1 - \tilde{g}(Q(x) + Q_1(y))Q(x_1) \quad \text{for} \quad x_1\varepsilon X \tag{3.11}$$

and

$$<\nabla J(u),y_1> = \int_\Omega yy_1 - \tilde{g}(Q(x) + Q_1(y))Q_1(y_1) \quad \text{for} \quad y_1\varepsilon Y. \tag{3.12}$$

From (3.11) and (3.12) we see that $u\varepsilon L_2(\Omega)$ is a critical point of J iff $(Pu,P_1(u))$ is a solution of (3.4). Thus, by lemma 3.1, J has a critical point iff (3.1) has a solution.

Let us prove that J has at least one critical point. First we assume N = -1. Let $x\varepsilon X$, $y\varepsilon Y$, and $y_1\varepsilon Y$; so, by condition b), we have

$$<\nabla J(x+y_1) - \nabla J(x+y),y_1-y> \geq \int_\Omega(y_1-y)^2 - \gamma'(Q_1(y_1-y))^2. \tag{3.13}$$

Using Parseval's formula and the definition of Q_1 we infer

$$<\nabla J(x+y_1) - J(x+y),y_1-y> \geq (1-(\gamma'/\lambda_1))\|y_1-y\|^2 \tag{3.14}$$

where, by a), $(1-(\gamma'/\lambda_1)) > 0$.

From (3.11) and (3.12) we see that

$$\nabla J(u) = P(\nabla J(u)) + P_1(\nabla J(u)) \tag{3.15}$$
$$= -P(u) - Q(\tilde{g}(Q(u)+Q_1(u))) + P_1(u) - Q_1(\tilde{g}(Q(u)+Q_1(u))).$$

The relations (3.14) and (3.15) imply that X,Y,J and $(1-(\gamma'/\lambda_1))$ satisfy the conditions of lemma 2.1. Consequently, there exists a CWC function $\phi:X \to Y$ such that $J(x+\phi(x)) = \min_{y\varepsilon Y} J(x+y)$.

Since $-\tilde{J}(x) = -J(x+\phi(x))$ can be written as

$$-\tilde{J}(x) = ((1/2)\int_\Omega x^2) - [\int_\Omega((\phi(x))^2/2) + \tilde{G}(Q(x) + Q_1(\phi(x)))], \tag{3.16}$$

since the function $x \to (1/2)\int_\Omega x$ is convex, and since the expression in the bracketed integral is CWC, $-\tilde{J}$ is WLSC. Therefore, by theorem 2.2, J has a critical point if $J(x) \to -\infty$ as $\|x\| \to \infty$. Let us verify this condition. Since

$$J(x) \leq \int_\Omega\{-(x^2(\xi)/2) - G(Q(x(\xi)),\xi)\}d\xi,$$

it follows from c) that

$$J(x) \leq -\int_\Omega\{(x^2(\xi)/2) + (\gamma/2)(Q(x(\xi)))^2 - C\}d\xi. \tag{3.17}$$

From (3.17) we see that $J(x) \to -\infty$ as $\|x\| \to \infty$ when $\gamma > 0$. If $\gamma \leq 0$, then by Parseval's formula we have $-(\gamma/2)\int_\Omega Q^2(x(\xi))d\xi \leq (\gamma/\lambda_1)\int_\Omega x^2(\xi)d\xi$. Substituting this in (3.17) we have

$$J(x) \leq (1/2)(-1+(\gamma/\lambda_1))\|x\|^2 + C\text{meas}(\Omega).$$

Therefore, $J(x) \to -\infty$ as $\|x\| \to \infty$. So (3.1) has a solution.

Suppose now $N \neq -1$. Let ρ be a real number such that $\lambda_N < \rho < \gamma$. Thus, $\lambda_N-\rho < 0$ and $\lambda_{N+1}-\rho > 0$. Let

$$K_2(s,t) = \sum_{j=-\infty}^{\infty} (\phi_j(s)\phi_j(t))(\lambda_j-\rho)^{-1}$$

Therefore, the eigenvalues of the problem

$$u(t) = \lambda\int_\Omega K_2(s,t)u(s)ds \tag{3.18}$$

are $\{\lambda_j-\rho\}_{j=-\infty}^{\infty}$, where λ_j is as before. The greatest negative eigenvalue of (3.18) is $\lambda_N-\rho$, and the smallest positive eigenvalue of (3.18) is

$\lambda_{N+1}-\rho$. Consequently, by the reasoning for the case $N = -1$, if $f_1(u,\xi) = g(u,\xi) - \rho u$, then there exists $u_0 \varepsilon L_2(\Omega)$ such that

$$u_0(t) = \int_\Omega K_2(s,t)(g(u_0(s),s) - \rho u_0(s))ds.$$

An elementary computation shows that u_0 is a solution of (3.1). And the theorem is proved.

Remark: If we change (b) and (c) by

(b') For $u\varepsilon\mathbb{R}$, $v\varepsilon\mathbb{R}$, $t\varepsilon\bar{\Omega}$, $(g(u,t)-g(b,t))/(u-v) \geq \gamma$

and

(c') For $u\varepsilon\mathbb{R}$, $t\varepsilon\bar{\Omega}$, $G(u,t) = \int_0^u g(s,t)ds \leq (\gamma' u^2)/2 - C$,

respectively, then (3.1) has at least one solution. This comes from applying theorem 3.3 to $u(t) = \int_\Omega(-K(s,t))(-g(u(s),s))ds$.

From here on we consider the equation (3.1) assuming $g(u,t) = g(u) = -g(-u)$. In our next theorem we make use of the following result which is a specialization of a theorem due to Clark [1,p.71].

Lemma 3.3.: *Let H be a real Hilbert space and f an even, real valued C^2 function defined on H. Suppose that f has the property that whenever $\{x_n\} \subset H$ is a bounded sequence such that $f(x_n) > 0$ and $f(x_n) \to 0$, then $\{x_n\}$ contains a convergent subsequence. Suppose that $f(0) = 0$, f is bounded above, there exists a subspace M of H of dimension $\ell > 0$ such that $\langle D^2 f(0)h,h\rangle > 0$ if $h\varepsilon M$ with $h \neq 0$, and $f(x) \geq 0$ for $\|x\|$ sufficiently large. Then there exist at least $2\ell + 1$ solutions of $\nabla f(x) = 0$.*

Theorem 3.4.: *Suppose g is a function of class C^1 and g' is bounded, if there exist two integers N and r, $N < r$, such that:*

i') *there exists a real number γ with $g'(u) \leq \gamma < \lambda_r$ for all $u\varepsilon\mathbb{R}$,* ii') $\lambda_N < g'(0) < \lambda_{N+1}$,

iii') <u>there exists real numbers</u> γ' <u>and</u> C, $\gamma' > \lambda_{r-1}$ <u>such that</u>

$$G(u) = \int_0^u g(s)ds \geq (\gamma'/2)u^2 + C,$$

then the equation (3.1) has at least $2(r-N) + 1$ solutions.

Proof: As shown in theorem 3.2 it is sufficient to prove that J has at least $2(r-N) + 1$ critical points. Also, arguing as in the previous theorem, we can assume $r = 1$ and $\lambda_1 \leq -\lambda_{-1}$. If X and Y are as in lemma 3.1 then condition i') implies that for $x\varepsilon X$, $y\varepsilon Y$, and $y_1\varepsilon Y$, we have

$$<\nabla J(x+y_1) - \nabla J(x+y), y_1-y> \geq (1-(\gamma/\lambda_1))\|y_1-y\|^2. \tag{3.20}$$

So by (3.15) and lemma 2.1 there is a CWC function $\phi: X \to Y$ such that for each $x\varepsilon X$

$$J(x+\phi(x)) = \min_{y\varepsilon Y} J(x+y) \equiv \tilde{J}(x).$$

Since g is of class C^1 and g' is bounded it follows that J is of class C^2. Hence, following a reasoning based on the implicit function theorem, as in [4,theorem 1], it can be seen that ϕ is of class C^1 and $\tilde{J}$ is of class C^2. Moreover, for $x\varepsilon X$, $h\varepsilon X$ we have

$$<D^2\tilde{J}(x)h,h> = <D^2J(x+\phi(x))(h+\phi'(x)h),h>. \tag{3.21}$$

Now we show that $\tilde{J}$ satisfy the hypothesis of lemma 3.3. Because $\tilde{J}$ is even ϕ is odd. So $\tilde{J}$ is even, and $\phi(0) = 0$. Thus, from (3.21) we obtain

$$<D^2J(0)h,h> = \int_\Omega -h^2 - g'(0)(Q(h))^2 \tag{3.22}$$

for $h\varepsilon X$. From (3.22) and ii') it is easy to see that D^2J is positive definite in a subspace of dimension $(r-N)$, namely $\text{span}\{\psi_{-1},\psi_{-2},\ldots,\psi_{N+1}\}$. From iii') and the definition of $\tilde{J}$ we have

$$\begin{aligned}\tilde{J}(x) &\leq J(x) \\ &\leq \int_\Omega -x^2 - (\gamma'/2)(Q(x))^2 + C.\end{aligned} \tag{3.23}$$

Thus, since $\gamma' > \lambda_{-1}$ and $\lambda_1 \leq -\lambda_{-1}$ we infer that $\tilde{J}$ is bounded above and $\tilde{J}(x) \to -\infty$ as $\|x\| \to \infty$. Let $\{x_n\}$ is a bounded sequence in X such that

$\nabla f(x_n) \to 0$. By (2.6) and (3.15) we have

$$\nabla \tilde{J}(x_n) = -x_n - Q(\tilde{g}(Q(x_n) + Q_1(\phi(x_n))) \to 0. \tag{3.24}$$

Let $\{x_{n_j}\}$ be a weakly convergent subsequence of $\{x_n\}$. In consequence, since Q and ϕ are CWC hence the sequence $\{Q(\tilde{g}(Q(x_n) + Q_1(\phi(x_n))))\}$ converges strongly. Therefore, by (3.24) $\{x_{n_j}\}$ converges strongly. Thus, by lemma 3.3, $\tilde{J}$ has at least $2(r-N)+1$ critical points. Since by (2.2) and (2.6) every critical point of J is of the form $x+\phi(x)$, where x is a critical point of $\tilde{J}$ the theorem is proved.

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ABSTRACT. This paper deals with the problem of finding solutions of the Hammerstein integral equation. It is shown that this problem can be reduced to the study of the critical points of certain functional defined on $L_2(\Omega)$. Existence of a solution of the Hammerstein integral equation is proved. Some other related results of interest are obtained.

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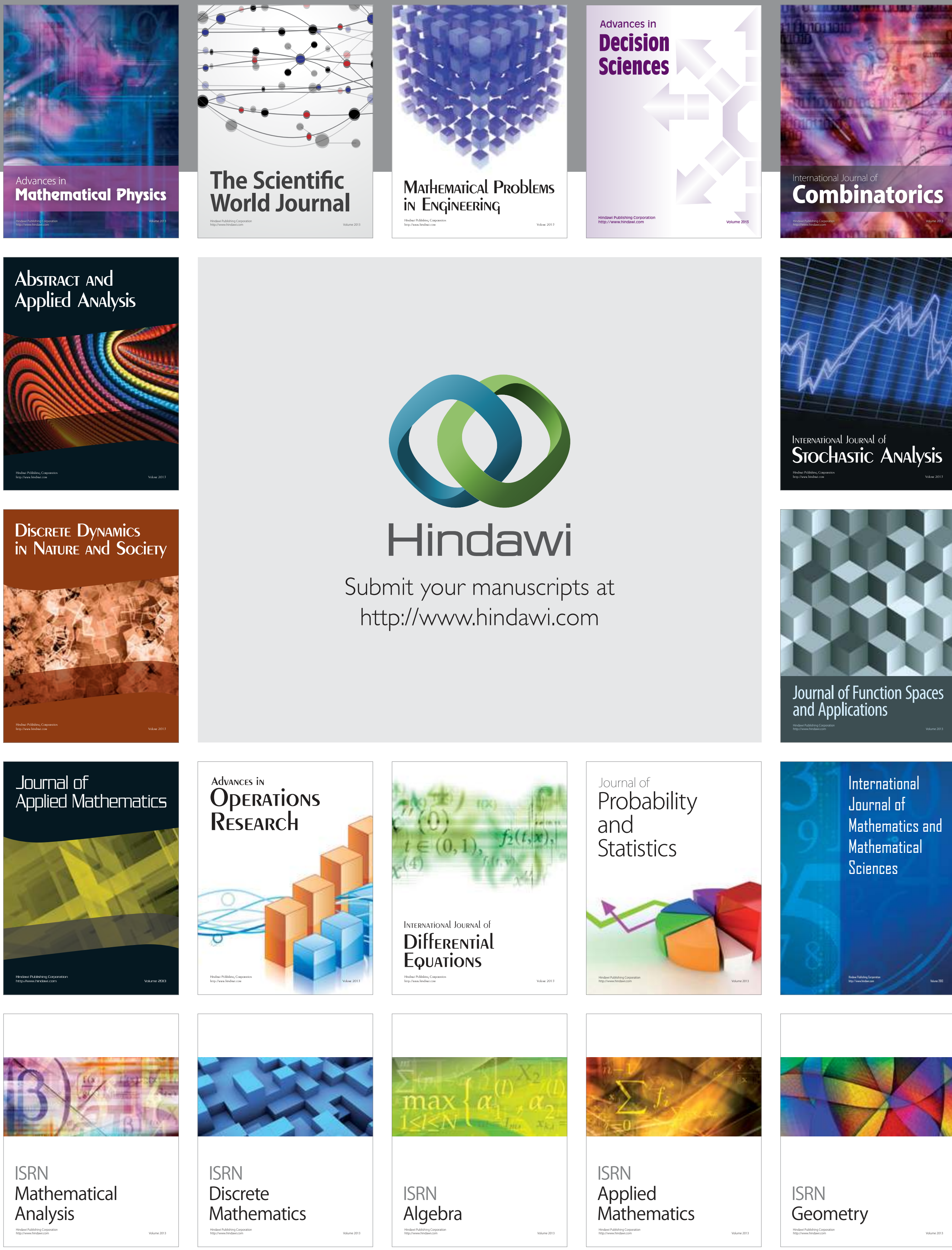

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