

HANDBOOK OF DIFFERENTIAL EQUATIONS

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* These methods are also applicable to partial differential equations.

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