Handbook of
Mathematical Formulas and Integrals

FOURTH EDITION

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Preface

This book contains a collection of general mathematical results, formulas, and integrals that occur throughout applications of mathematics. Many of the entries are based on the updated fifth edition of Gradshteyn and Ryzhik's "Tables of Integrals, Series, and Products," though during the preparation of the book, results were also taken from various other reference works. The material has been arranged in a straightforward manner, and for the convenience of the user a quick reference list of the simplest and most frequently used results is to be found in Chapter 0 at the front of the book. Tab marks have been added to pages to identify the twelve main subject areas into which the entries have been divided and also to indicate the main interconnections that exist between them. Keys to the tab marks are to be found inside the front and back covers.

The Table of Contents at the front of the book is sufficiently detailed to enable rapid location of the section in which a specific entry is to be found, and this information is supplemented by a detailed index at the end of the book. In the chapters listing integrals, instead of displaying them in their canonical form, as is customary in reference works, in order to make the tables more convenient to use, the integrands are presented in the more general form in which they are likely to arise. It is hoped that this will save the user the necessity of reducing a result to a canonical form before consulting the tables. Wherever it might be helpful, material has been added explaining the idea underlying a section or describing simple techniques that are often useful in the application of its results.

Standard notations have been used for functions, and a list of these together with their names and a reference to the section in which they occur or are defined is to be found at the front of the book. As is customary with tables of indefinite integrals, the additive arbitrary constant of integration has always been omitted. The result of an integration may take more than one form, often depending on the method used for its evaluation, so only the most common forms are listed.

A user requiring more extensive tables, or results involving the less familiar special functions, is referred to the short classified reference list at the end of the book. The list contains works the author found to be most useful and which a user is likely to find readily accessible in a library, but it is in no sense a comprehensive bibliography. Further specialist references are to be found in the bibliographies contained in these reference works.

Every effort has been made to ensure the accuracy of these tables and, whenever possible, results have been checked by means of computer symbolic algebra and integration programs, but the final responsibility for errors must rest with the author.

Preface to the Fourth Edition

The preparation of the fourth edition of this handbook provided the opportunity to enlarge the sections on special functions and orthogonal polynomials, as suggested by many users of the third edition. A number of substantial additions have also been made elsewhere, like the enhancement of the description of spherical harmonics, but a major change is the inclusion of a completely new chapter on conformal mapping. Some minor changes that have been made are correcting of a few typographical errors and rearranging the last four chapters of the third edition into a more convenient form. A significant development that occurred during the later stages of preparation of this fourth edition was that my friend and colleague Dr. Hui-Hui Dai joined me as a co-editor.

Chapter 30 on conformal mapping has been included because of its relevance to the solution of the Laplace equation in the plane. To demonstrate the connection with the Laplace equation, the chapter is preceded by a brief introduction that demonstrates the relevance of conformal mapping to the solution of boundary value problems for real harmonic functions in the plane. Chapter 30 contains an extensive atlas of useful mappings that display, in the usual diagrammatic way, how given analytic functions $w = f(z)$ map regions of interest in the complex z-plane onto corresponding regions in the complex w-plane, and conversely. By forming composite mappings, the basic atlas of mappings can be extended to more complicated regions than those that have been listed. The development of a typical composite mapping is illustrated by using mappings from the atlas to construct a mapping with the property that a region of complicated shape in the z-plane is mapped onto the much simpler region comprising the upper half of the w-plane. By combining this result with the Poisson integral formula, described in another section of the handbook, a boundary value problem for the original, more complicated region can be solved in terms of a corresponding boundary value problem in the simpler region comprising the upper half of the w-plane.

The chapter on ordinary differential equations has been enhanced by the inclusion of material describing the construction and use of the Green's function when solving initial and boundary value problems for linear second order ordinary differential equations. More has been added about the properties of the Laplace transform and the Laplace and Fourier convolution theorems, and the list of Laplace transform pairs has been enlarged. Furthermore, because of their use with special techniques in numerical analysis when solving differential equations, a new section has been included describing the Jacobi orthogonal polynomials. The section on the Poisson integral formulas has also been enlarged, and its use is illustrated by an example. A brief description of the Riemann method for the solution of hyperbolic equations has been included because of the important theoretical role it plays when examining general properties of wave-type equations, such as their domains of dependence.

For the convenience of users, a new feature of the handbook is a CD-ROM that contains the classified lists of integrals found in the book. These lists can be searched manually, and when results of interest have been located, they can be either printed out or used in papers or worksheets as required. This electronic material is introduced by a set of notes (also included in the following pages) intended to help users of the handbook by drawing attention to different notations and conventions that are in current use. If these are not properly understood, they can cause confusion when results from some other sources are combined with results from this handbook. Typically, confusion can occur when dealing with Laplace's equation and other second order linear partial differential equations using spherical polar coordinates because of the occurrence of differing notations for the angles involved and also when working with Fourier transforms for which definitions and normalizations differ. Some explanatory notes and examples have also been provided to interpret the meaning and use of the inversion integrals for Laplace and Fourier transforms.

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Notes for Handbook Users

The material contained in the fourth edition of the *Handbook of Mathematical Formulas and Integrals* was selected because it covers the main areas of mathematics that find frequent use in applied mathematics, physics, engineering, and other subjects that use mathematics. The material contained in the handbook includes, among other topics, algebra, calculus, indefinite and definite integrals, differential equations, integral transforms, and special functions.

For the convenience of the user, the most frequently consulted chapters of the book are to be found on the accompanying CD that allows individual results of interest to be printed out, included in a work sheet, or in a manuscript.

A major part of the handbook concerns integrals, so it is appropriate that mention of these should be made first. As is customary, when listing indefinite integrals, the arbitrary additive constant of integration has always been omitted. The results concerning integrals that are available in the mathematical literature are so numerous that a strict selection process had to be adopted when compiling this work. The criterion used amounted to choosing those results that experience suggested were likely to be the most useful in everyday applications of mathematics. To economize on space, when a simple transformation can convert an integral containing several parameters into one or more integrals with fewer parameters, only these simpler integrals have been listed.

For example, instead of listing indefinite integrals like $\int e^{ax}\sin(bx+c)dx$ and $\int e^{ax}\cos(bx+c)dx$, each containing the three parameters a, b, and c, the simpler indefinite integrals $\int e^{ax}\sin bx dx$ and $\int e^{ax}\cos bx dx$ contained in entries **5.1.3.1**(1) and **5.1.3.1**(4) have been listed. The results containing the parameter c then follow after using additive property of integrals with these tabulated entries, together with the trigonometric identities $\sin(bx+c) = \sin bx\cos c + \cos bx \sin c$ and $\cos(bx+c) = \cos bx \cos c - \sin bx \sin c$.

The order in which integrals are listed can be seen from the various section headings. If a required integral is not found in the appropriate section, it is possible that it can be transformed into an entry contained in the book by using one of the following elementary methods:

1. Representing the integrand in terms of partial fractions.
2. Completing the square in denominators containing quadratic factors.
3. Integration using a substitution.
4. Integration by parts.
5. Integration using a recurrence relation (recursion formula),

or by a combination of these. It must, however, always be remembered that not all integrals can be evaluated in terms of elementary functions. Consequently, many simple looking integrals cannot be evaluated analytically, as is the case with

$$\int \frac{\sin x}{a + be^x} dx.$$

A Comment on the Use of Substitutions

When using substitutions, it is important to ensure the substitution is both continuous and one-to-one, and to remember to incorporate the substitution into the dx term in the integrand. When a definite integral is involved the substitution must also be incorporated into the limits of the integral.

When an integrand involves an expression of the form $\sqrt{a^2 - x^2}$, it is usual to use the substitution $x = |a \sin\theta|$ which is equivalent to $\theta = \arcsin(x/|a|)$, though the substitution $x = |a| \cos\theta$ would serve equally well. The occurrence of an expression of the form $\sqrt{a^2 + x^2}$ in an integrand can be treated by making the substitution $x = |a| \tan\theta$, when $\theta = \arctan(x/|a|)$ (see also Section **9.1.1**). If an expression of the form $\sqrt{x^2 - a^2}$ occurs in an integrand, the substitution $x = |a| \sec\theta$ can be used. Notice that whenever the square root occurs the *positive* square root is always implied, to ensure that the function is single valued.

If a substitution involving either $\sin\theta$ or $\cos\theta$ is used, it is necessary to restrict θ to a suitable interval to ensure the substitution remains one-to-one. For example, by restricting θ to the interval $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$, the function $\sin\theta$ becomes one-to-one, whereas by restricting θ to the interval $0 \le \theta \le \pi$, the function $\cos\theta$ becomes one-to-one. Similarly, when the inverse trigonometric function $y = \arcsin x$ is involved, equivalent to $x = \sin y$, the function becomes one-to-one in its principal branch $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$, so $\arcsin(\sin x) = x$ for $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$ and $\sin(\arcsin x) = x$ for $-1 \le x \le 1$. Correspondingly, the inverse trigonometric function $y = \arccos x$, equivalently $x = \cos y$, becomes one-to-one in its principal branch $0 \le y \le \pi$, so $\arccos(\cos x) = x$ for $0 \le x \le \pi$ and $\sin(\arccos x) = x$ for $-1 \le x \le 1$.

It is important to recognize that a given integral may have more than one representation, because the form of the result is often determined by the method used to evaluate the integral. Some representations are more convenient to use than others so, where appropriate, integrals of this type are listed using their simplest representation. A typical example of this type is

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \begin{cases} \operatorname{arcsinh}(x/a) \\ \ln\left(x + \sqrt{a^2 + x^2}\right) \end{cases}$$

where the result involving the logarithmic function is usually the more convenient of the two forms. In this handbook, both the inverse trigonometric and inverse hyperbolic functions all carry the prefix "arc." So, for example, the inverse sine function is written arcsin x and the inverse hyperbolic sine function is written arcsinh x, with corresponding notational conventions for the other inverse trigonometric and hyperbolic functions. However, many other works denote the inverse of these functions by adding the superscript $^{-1}$ to the name of the function, in which case $\arcsin x$ becomes $\sin^{-1} x$ and $\operatorname{arcsinh} x$ becomes $\sinh^{-1} x$. Elsewhere yet another notation is in use where, instead of using the prefix "arc" to denote an inverse hyperbolic function, the prefix “arg” is used, so that arcsinh x becomes argsinh x, with the corresponding use of the prefix “arg” to denote the other inverse hyperbolic functions. This notation is preferred by some authors because they consider that the prefix “arc” implies an angle is involved, whereas this is not the case with hyperbolic functions. So, instead, they use the prefix “arg” when working with inverse hyperbolic functions.

Example: Find $I = \int \frac{x^5}{\sqrt{a^2-x^2}} dx$.
Of the two obvious substitutions $x = |a| \sin\theta$ and $x = |a| \cos\theta$ that can be used, we will make use of the first one, while remembering to restrict θ to the interval $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ to ensure the transformation is one-to-one. We have $dx = |a| \cos\theta d\theta$, while $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = |a| \sqrt{1-\sin^2\theta} = |a\cos\theta|$. However $\cos\theta$ is positive in the interval $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$, so we may set $\sqrt{a^2-x^2} = |a| \cos\theta$. Substituting these results into the integrand of I gives

$$I = \int \frac{|a|^5 \sin^5\theta\, |a| \cos\theta d\theta}{|a| \cos\theta} = a^4\, |a| \int \sin^5\theta d\theta,$$

and this trigonometric integral can be found using entry **9.2.2.2**, 5. This result can be expressed in terms of x by using the fact that $\theta = \arcsin\left(x/\,|a|\right)$, so that after some manipulation we find that

$$I = -\frac{1}{5}x^4\sqrt{a^2-x^2} - \frac{4a^2}{15}\sqrt{a^2-x^2}\left(2a^2+x^2\right).$$

A Comment on Integration by Parts

Integration by parts can often be used to express an integral in a simpler form, but it also has another important property because it also leads to the derivation of a **reduction formula**, also called a **recursion relation**. A reduction formula expresses an integral involving one or more parameters in terms of a simpler integral of the same form, but with the parameters having smaller values. Let us consider two examples in some detail, the second of which given a brief mention in Section **1.15.3**.

Example:

(a) Find a reduction formula for

$$I_m = \int \cos^m\theta d\theta,$$

and hence find an expression for I_5.

(b) Modify the result to find a recurrence relation for

$$J_m = \int_0^{\pi/2} \cos^m\theta d\theta,$$

and use it to find expressions for J_m when m is even and when it is odd.

To derive the result for (a), write

$$\begin{aligned}
I_m &= \int \cos^{m-1}\theta \frac{d(\sin\theta)}{d\theta} d\theta \\
&= \cos^{m-1}\theta \sin\theta - \int \sin\theta (m-1)\cos^{m-2}\theta(-\sin\theta) d\theta \\
&= \cos^{m-1}\theta \sin\theta + (m-1)\int \cos^{m-2}\theta(1-\cos^2\theta) d\theta \\
&= \cos^{m-1}\theta \sin\theta + (m-1)\int \cos^{m-2}\theta d\theta - (m-1)\int \cos^{m}\theta d\theta.
\end{aligned}$$

Combining terms and using the form of I_m, this gives the reduction formula

$$I_m = \frac{\cos^{m-1}\theta \sin\theta}{m} + \left(\frac{m-1}{m}\right) I_{m-2.}$$

we have $I_1 = \int \cos\theta d\theta = \sin\theta$. So using the expression for I_1, setting $m = 5$ and using the recurrence relation to step up in intervals of 2, we find that

$$I_3 = \frac{1}{3}\cos^2\theta \sin\theta + \frac{2}{3} I_1 = \frac{1}{3}\cos^2\theta + \frac{2}{3}\sin\theta,$$

and hence that

$$\begin{aligned}
I_5 &= \frac{1}{5}\cos^4\theta \sin\theta + \frac{4}{5} I_3 \\
&= \frac{1}{5}\cos^4\theta \sin\theta - \frac{4}{15}\sin^3\theta + \frac{4}{5}\sin\theta.
\end{aligned}$$

The derivation of a result for (b) uses the same reasoning as in (a), apart from the fact that the limits must be applied to both the integral, and also to the uv term in $\int u dv = uv - \int v du$, so the result becomes $\int_a^b u dv = (uv)_a^b - \int_a^b v du$. When this is done it leads to the result

$$J_m = \left(\frac{\cos^{m-1}\theta \sin\theta}{m}\right)_{\theta=0}^{\pi/2} + \left(\frac{m-1}{m}\right) J_{m-2} = \left(\frac{m-1}{m}\right) J_{m-2}.$$

When m is even, this recurrence relation links J_m to $J_0 = \int_0^{\pi/2} 1 d\theta = \frac{1}{2}\pi$, and when m is odd, it links J_m to $J_1 = \int_0^{\pi/2} \cos\theta d\theta = 1$. Using these results sequentially in the recurrence relation, we find that

$$J_{2n} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \frac{1}{2}\pi, \quad (m = 2n \text{ is even})$$

and

$$J_{2n+1} = \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)} \quad (m = 2n+1 \text{ is odd}).$$

Example: The following is an example of a recurrence formula that contains two parameters. If $I_{m,n} = \int \sin^m \theta \cos^n \theta d\theta$, an argument along the lines of the one used in the previous example, but writing

$$I_{m,n} = \int \sin^{m-1} \theta \cos^n \theta d(-\cos\theta),$$

leads to the result

$$(m+n)I_{m,n} = -\sin^{m-1}\theta\cos^{n+1}\theta + (m-1)I_{m-2,n},$$

in which n remains unchanged, but m decreases by 2.

Had integration by parts been used differently with $I_{m,n}$ written as

$$I_{m,n} = \int \sin^m \theta \cos^{n-1} \theta d(\sin\theta)$$

a different reduction formula would have been obtained in which m remains unchanged but n decreases by 2.

Some Comments on Definite Integrals

Definite integrals evaluated over the semi-infinite interval $[0,\infty)$ or over the infinite interval $(-\infty,\infty)$ are improper integrals and when they are convergent they can often be evaluated by means of contour integration. However, when considering these improper integrals, it is desirable to know in advance if they are convergent, or if they only have a finite value in the sense of a Cauchy principal value. (see Section **1.15.4**). A geometrical interpretation of a Cauchy principal value for an integral of a function $f(x)$ over the interval $(-\infty,\infty)$ follows by regarding an area between the curve $y = f(x)$ and the x-axis as positive if it lies above the x-axis and negative if it lies below it. Then, when finding a Cauchy principal value, the areas to the left and right of the y-axis are paired off symmetrically as the limits of integration approach $\pm\infty$. If the result is a finite number, this is the Cauchy principal value to be attributed to the definite integral $\int_{-\infty}^{\infty} f(x)dx$, otherwise the integral is divergent. When an improper integral is convergent, its value and its Cauchy principal value coincide.

There are various tests for the convergence of improper integrals, but the ones due to Abel and Dirichlet given in Section **1.15.4** are the main ones. Convergent integrals exist that do not satisfy all of the conditions of the theorems, showing that although these tests represent *sufficient* conditions for convergence, they are *not necessary* ones.

Example: Let us establish the convergence of the improper integral $\int_a^{\infty} \frac{\sin mx}{x^p} dx$, given that $a, p > 0$.

To use the Dirichlet test we set $f(x) = \sin x$ and $g(x) = 1/x^p$. Then $\lim_{x\to\infty} g(x) = 0$ and $\int_a^{\infty} |g'(x)|dx = 1/a^p$ is finite, so this integral involving $g(x)$ converges. We also have $F(b) = \int_a^b \sin mx dx = (\cos ma - \cos mb)/m$, from which it follows that $|F(b)| \leq 2$ for all

$a \leq x \leq b < \infty$. Thus the conditions of the Dirichlet test are satisfied showing that $\int_a^\infty \frac{\sin x}{x^p} dx$ is convergent for $a, p > 0$.

It is necessary to exercise caution when using the fundamental theorem of calculus to evaluate an improper integral in case the integrand has a singularity (becomes infinite) inside the interval of integration. If this occurs the use of the fundamental theorem of calculus is invalid.

Example: The improper integral $\int_{-a}^{a} \frac{dx}{x^2}$ with $a > 0$ has a singularity at the origin and is, in fact, divergent. This follows because if $\varepsilon, \delta > 0$, we have $\lim\limits_{\varepsilon \to 0} \int_{-a}^{-\varepsilon} \frac{dx}{x^2} + \lim\limits_{\delta \to 0} \int_{\delta}^{b} \frac{dx}{x^2} = \infty$. However, an incorrect application of the fundamental theorem of calculus gives $\int_{-a}^{a} \frac{dx}{x^2} = \left(-\frac{1}{x}\right)_{x=-a}^{a} = -\frac{2}{a}$. Although this result is finite, it is obviously incorrect because the integrand is positive over the interval of integration, so the definite integral must also be positive, but this is not the case here because $a > 0$ so $-2/a < 0$.

Two simple results that often save time concern the integration of even and odd functions $f(x)$ over an interval $-a \leq x \leq a$ that is symmetrical about the origin.

We have the obvious result that when $f(x)$ is *odd*, that is when $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x) dx = 0,$$

and when $f(x)$ is even, that is when $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx.$$

These simple results have many uses as, for example, when working with Fourier series and elsewhere.

Some Comments on Notations, the Choice of Symbols, and Normalization

Unfortunately there is no universal agreement on the choice of symbols used to identify a point P in cylindrical and spherical polar coordinates. Nor is there universal agreement on the choice of symbols used to represent some special functions, or on the normalization of Fourier transforms. Accordingly, before using results derived from other sources with those given in this handbook, it is necessary to check the notations, symbols, and normalization used elsewhere prior to combining the results.

Symbols Used with Curvilinear Coordinates

To avoid confusion, the symbols used in this handbook relating to plane polar coordinates, cylindrical polar coordinates, and spherical polar coordinates are shown in the diagrams in Section **24.3**.

The plane polar coordinates (r, θ) that identify a point P in the $(x,\, y)$-plane are shown in Figure 1(a). The angle θ is the **azimuthal angle** measured counterclockwise from the x-axis in the $(x,\, y)$-plane to the radius vector r drawn from the origin to the point P. The connection between the Cartesian and the plane polar coordinates of P is given by $x = r\cos\theta,\;\; y = r\sin\theta$, with $0 \le \theta < 2\pi$.

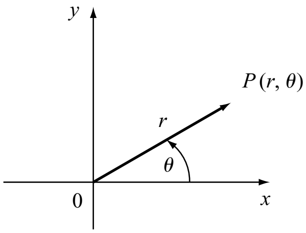

Figure 1(a)

We mention here that a different convention denotes the **azimuthal angle** in plane polar coordinates by θ, instead of by ϕ.

The cylindrical polar coordinates (r, θ, z) that identify a point P in space are shown in Figure 1(b). The angle θ is again the **azimuthal angle** measured as in plane polar coordinates, r is the radial distance measured from the origin in the (x, y)-plane to the projection of P onto the (x, y)-plane, and z is the perpendicular distance of P above the (x, y)-plane. The connection between cartesian and cylindrical polar coordinates used in this handbook is given by $x = r\cos\theta,\;\; y = r\sin\theta$ and $z = z$, with $0 \le \theta < 2\pi$.

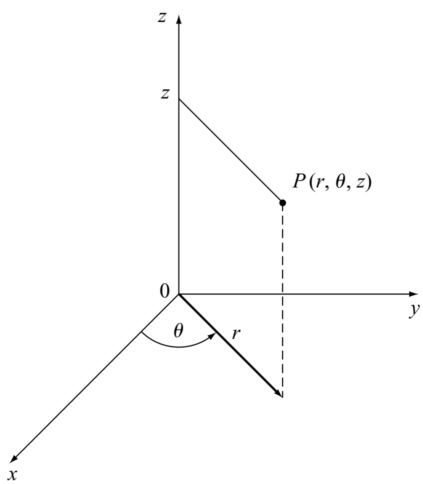

Figure 1(b)

Here also, in a different convention involving cylindrical polar coordinates, the azimuthal angle is denoted by ϕ instead of by θ.

The spherical polar coordinates (r, θ, ϕ) that identify a point P in space are shown in Figure 1(c). Here, differently from plane cylindrical coordinates, the **azimuthal angle** measured as in plane cylindrical coordinates is denoted by ϕ, the radius r is measured from the origin to point P, and the **polar angle** measured from the z-axis to the radius vector OP is denoted by θ, with $0 \leq \phi < 2\pi$, and $0 \leq \theta \leq \pi$. The cartesian and spherical polar coordinates used in this handbook are connected by $x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\theta$.

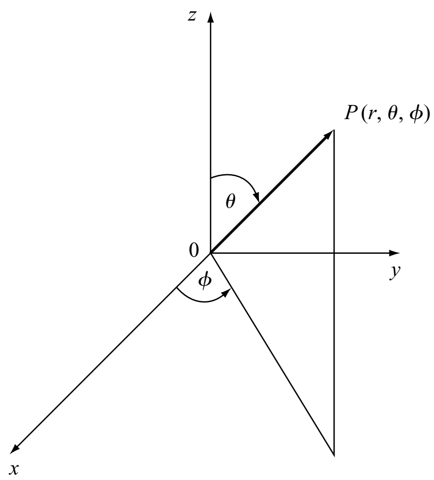

Figure 1(c)

In a different convention the roles of θ and ϕ are interchanged, so the azimuthal angle is denoted by θ, and the polar angle is denoted by ϕ.

Bessel Functions

There is general agreement that the **Bessel function of the first kind of order ν** is denoted by $J_\nu(x)$, though sometimes the symbol ν is reserved for orders that are not integral, in which case n is used to denote integral orders. However, notations differ about the representation of the **Bessel function of the second kind of order ν**. In this handbook, a definition of the Bessel function of the second kind is adopted that is true for *all* orders ν (both integral and fractional) and it is denoted by $Y_\nu(x)$. However, a widely used alternative notation for this same Bessel function of the second kind of order ν uses the notation $N_\nu(x)$. This choice of notation, sometimes called the **Neumann form of the Bessel function of the second kind of order ν**, is used in recognition of the fact that it was defined and introduced by the German mathematician Carl Neumann. His definition, but with $Y_\nu(x)$ in place of $N_\nu(x)$, is given in Section **17.2.2**. The reason for the rather strange form of this definition is because when the second linearly independent solution of Bessel's equation is derived using the Frobenius method, the nature of the solution takes one form when ν is an integer and a different one when ν is not an integer. The form of definition of $Y_\nu(x)$ used here overcomes this difficulty because it is valid for all ν.

The recurrence relations for all Bessel functions can be written as

$$\begin{aligned} &Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x), \\ &Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z_\nu'(x), \\ &Z_\nu'(x) = Z_{\nu-1}(x) - \frac{\nu}{x} Z_\nu(x)' \\ &Z_\nu'(x) = -Z_{\nu+1}(x) + \frac{\nu}{x} Z_\nu(x), \end{aligned} \tag{1}$$

where $Z_\nu(x)$ can be either $J_\nu(x)$ or $Y_\nu(x)$. Thus any recurrence relation derived from these results will apply to all Bessel functions. Similar general results exist for the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$.

Normalization of Fourier Transforms

The convention adopted in this handbook is to define the **Fourier transform** of a function $f(x)$ as the function $F(\omega)$ where

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx, \tag{2}$$

when the **inverse Fourier transform** becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \tag{3}$$

where the normalization factor multiplying each integral in this Fourier transform pair is $1/\sqrt{2\pi}$. However other conventions for the normalization are in common use, and they follow from the requirement that the product of the two normalization factors in the Fourier and inverse Fourier transforms must equal $1/(2\pi)$.

Thus another convention that is used defines the Fourier transform of $f(x)$ as

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{4}$$

and the inverse Fourier transform as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega. \tag{5}$$

To complicate matters still further, in some conventions the factor $e^{i\omega x}$ in the integral defining $F(\omega)$ is replaced by $e^{-i\omega x}$ and to compensate the factor $e^{-i\omega x}$ in the integral defining $f(x)$ is replaced by $e^{i\omega x}$.

If a Fourier transform is defined in terms of an angular frequency, the ambiguity concerning the choice of normalization factors disappears because the Fourier transform of $f(x)$ becomes

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixs}dx \tag{6}$$

and the inverse Fourier transform becomes

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-2\pi ix\omega}d\omega. \tag{7}$$

Nevertheless, the difference between definitions still continues because sometimes the exponential factor in $F(s)$ is replaced by $e^{-2\pi ixs}$, in which case the corresponding factor in the inverse Fourier transform becomes $e^{2\pi ixs}$. These remarks should suffice to convince a reader of the necessity to check the convention used before combining a Fourier transform pair from another source with results from this handbook.

Some Remarks Concerning Elementary Ways of Finding Inverse Laplace Transforms

The Laplace transform $F(s)$ of a suitably integrable function $f(x)$ is defined by the improper integral

$$F(s) = \int_{0}^{\infty} f(x)e^{-xs}dx. \tag{8}$$

Let a Laplace transform $F(s)$ be the quotient $F(s) = P(s)/Q(s)$ of two polynomials $P(s)$ and $Q(s)$. Finding the inverse transform $\mathcal{L}^{-1}\{F(s)\} = f(x)$ can be accomplished by simplifying $F(s)$ using partial fractions, and then using the Laplace transform pairs in Table 19.1 together with the operational properties of the transform given in **19.1.2.1**. Notice that the degree of $P(s)$ must be less than the degree of $Q(s)$ because from the limiting condition in **19.11.2.1**(10), if $F(s)$ is to be a Laplace transform of some function $f(x)$, it is necessary that $\lim_{s\to\infty} F(s) = 0$. The same approach is valid if exponential terms of the type e^{-as} occur in the numerator $P(s)$ because depending on the form of the partial fraction representation of $F(s)$, such terms will simply introduce either a Heaviside step function $H(x-a)$, or a Dirac delta function $\delta(x-a)$ into the resulting expression for $f(x)$.

On occasions, if a Laplace transform can be expressed as the product of two simpler Laplace transforms, the convolution theorem can be used to simplify the task of inverting the Laplace transform. However, when factoring the transform before using the convolution theorem, care must be taken to ensure that each factor is in fact a Laplace transform of a function of x. This is easily accomplished by appeal to the limiting condition in **19.11.2.1**(10), because if $F(s)$ is factored as $F(s) = F_1(s)F_2(s)$, the functions $F_1(s)$ and $F_2(s)$ will only be the Laplace transforms of some functions $f_1(x)$ and $f_2(x)$ if $\lim_{s\to\infty} F_1(s) = 0$ and $\lim_{s\to\infty} F_2(s) = 0$.

Example: (a) Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \frac{s^3+3s^2+5s+15}{(s^2+1)(s^2+4s+13)}$. (b) Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \frac{s^2}{(s^2+a^2)^2}$.

To solve (a) using partial fractions we write $F(s)$ as $F(s) = \frac{1}{s^2+1} + \frac{s+2}{s^2+4s+13}$. Taking the inverse Laplace transform of $F(s)$ and using entry 26 in Table 19.1 gives

$$\mathcal{L}^{-1}\{F(s)\} = \sin x + \mathcal{L}^{-1}\left(\frac{s+2}{s^2+4s+13}\right).$$

Completing the square in the denominator of the second term and writing, $\frac{s+2}{s^2+4s+13} = \frac{s+2}{(s+2)^2+3^2}$, we see from the first shift theorem in **19.1.2.1**(4) and entry 27 in Table 19.1 that $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} = e^{-2x}\cos 3x$. Finally, combining results, we have

$$\mathcal{L}^{-1}\{F(s)\} = \sin x + e^{-2x}\cos 3x.$$

To solve (b) by the convolution transform, $F(s)$ must be expressed as the product of two factors. The transform $F(s)$ can be factored in two obvious ways, the first being $F(s) = \frac{s^2}{(s^2+a^2)}\frac{1}{(s^2+a^2)}$ and the second being $F(s) = \frac{s}{(s^2+a^2)}\frac{s}{(s^2+a^2)}$.

Of these two expressions, only the second is the product of two Laplace transforms, namely the product of the Laplace transforms of cos ax. The first result cannot be used because the factor $s^2/(s^2+a^2)$ fails the limiting condition in **19.11.2.1**(10), and so is not the Laplace transform of a function of x.

The inverse of the convolution theorem asserts that if $F(s)$ and $G(s)$ are Laplace transforms of the functions $f(x)$ and $g(x)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^x f(\tau)g(x-\tau)d\tau. \tag{9}$$

So setting $F(s) = G(s) =$ cos ax, it follows that

$$f(x) = \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \int_0^x \cos\tau\cos(x-\tau)d\tau = \frac{\sin ax}{2a} + \frac{x\cos ax}{2}.$$

When more complicated Laplace transforms occur, it is necessary to find the inverse Laplace transform by using contour integration to evaluate the inversion integral in **19.1.1.1**(5). More will be said about this, and about the use of the Fourier inversion integral, after a brief review of some key results from complex analysis.

Using the Fourier and Laplace Inversion Integrals

As a preliminary to discussing the Fourier and Laplace inversion integrals, it is necessary to record some key results from complex analysis that will be used.

An analytic function A complex valued function $f(z)$ of the complex variable $z = x + iy$ is said to be **analytic** on an open domain G (an area in the z-plane without its boundary points) if it has a derivative at each point of G. Other names used in place of analytic are *holomorphic* and *regular*. A function $f(z) = u(x, y) + \nu(x, y)$ will be analytic in a domain G if at every point of G it satisfies the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial \nu}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial \nu}{\partial x}. \tag{10}$$

These conditions are sufficient to ensure that $f(z)$ had a derivative at every point of G, in which case

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial \nu}{\partial x} = \frac{\partial \nu}{\partial y} - i\frac{\partial u}{\partial y}. \tag{11}$$

A pole of $f(z)$ An analytic function $f(z)$ is said to have a **pole** of order p at $z = z_0$ if in some neighborhood the point z_0 of a domain G where $f(z)$ is defined,

$$f(z) = \frac{g(z)}{(z - z_0)^p}, \tag{12}$$

where the function $g(z)$ is analytic at z_0. When $p = 1$, the function $f(z)$ is said to have **simple pole** at $z = z_0$.

A meromorphic function A function $f(z)$ is said to be **meromorphic** if it is analytic everywhere in a domain G except for isolated points where its only singularities are poles. For example, the function $f(z) = 1/(z^2 + a^2) = 1/\left[(z - ia)(z + ia)\right]$ is a meromorphic function with simple poles at $z = \pm\, ia$.

The residue of $f(z)$ at a pole If a function has a pole of order p at $z = z_0$, then its **residue** at $z = z_0$ is given by

$$\text{Residue } (f(z) : z = z_0) = \lim_{z \to z_0} \left[\frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} (z - z_0)^p f(z) \right].$$

For example, the residues of $f(z) = 1/(z^2 + a^2)$ at its poles located at $z = \pm\, ia$ are

$$\text{Residue } (1/(z^2 + a^2) : z = ia) = -i/(2a)$$

and

$$\text{Residue } (1/(z^2 + a^2) : z = -ia) = i/(2a).$$

The Cauchy residue theorem Let Γ be a simple closed curve in the z-plane (a non-intersecting curve in the form of a simple loop). Denoting by $\int_\Gamma f(z)dz$ the integral of $f(z)$ around Γ in the counter-clockwise (positive) sense, the **Cauchy residue theorem** asserts that

$$\int_\Gamma f(z)dz = 2\pi i \times (\text{sum of residues of } f(z) \text{ inside } \Gamma). \tag{13}$$

So, for example, if Γ is *any* simple closed curve that contains *only* the residue of $f(z) = 1/(z^2+a^2)$ located at $z = ia$, then

$$\int_\Gamma 1/(z^2+a^2)dz = 2\pi i \times (-i/(2a)) = \pi/a.$$

Jordan's Lemma in Integral Form, and Its Consequences

This lemma take various forms, the most useful of which are as follows:

(i) Let C_+ be a circular arc of radius R located in the first and/or second quadrants, with its center at the origin of the z-plane. Then if $f(z) \to 0$ uniformly as $R \to \infty$,

$$\lim_{R\to\infty} \int_{C_+} f(z)e^{imz}dz = 0, \qquad \text{where } m > 0.$$

(ii) Let C_- be a circular arc of radius R located in the third and/or fourth quadrant with its center at the origin of the z plane. Then if $f(z) \to 0$ uniformly as $R \to \infty$,

$$\lim_{R\to\infty} \int_{C_-} f(z)e^{-imz}dz = 0, \qquad \text{where } m > 0.$$

(iii) In a somewhat different form the lemma takes the form $\int_0^{\pi/2} e^{-k\sin\theta}d\theta \le \frac{\pi}{2k}\left(1 - e^{-k}\right)$.

The first two forms of Jordan's lemma are useful in general contour integration when establishing that the integral of an analytic function around a circular arc of radius R centered on the origin vanishes in the limit as $R \to \infty$. The third form is often used when estimating the magnitude of a complex function that is integrated around a quadrant. The form of Jordan's lemma to be used depends on the nature of the integrand to which it is to be applied. Later, result (iii) will be used when determining an inverse Laplace transform by means of the Laplace inversion integral.

The Fourier Transform and Its Inverse

In this handbook, the Fourier transform $F(\omega)$ of a suitably integrable function $f(x)$ is defined as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx, \tag{14}$$

while the **inverse Fourier transform** becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \tag{15}$$

it being understood that when $f(x)$ is piecewise continuous with a piecewise continuous first derivative in any finite interval, that this last result is to be interpreted as

$$\frac{f(x_-) + f(x_+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \tag{16}$$

with $f(x_\pm)$ the values of $f(x)$ on either side of a discontinuity in $f(x)$. Notice first that although $f(x)$ is real, its Fourier transform $F(\omega)$ may be complex. Although $F(\omega)$ may often be found by direct integration care is necessary, and it is often simpler to find it by converting the line integral defining $F(\omega)$ into a contour integral. The necessary steps involve (i) integrating $f(x)$ along the real axis from $-R$ to R, (ii) joining the two ends of this segment of the real axis by a semicircle of radius R with its center at the origin where the semicircle is either located in the upper half-plane, or in the lower half-plane, (iii) denoting this contour by Γ_R, and (iv) using the limiting form Γ of the contour Γ_R as $R \to \infty$ as the contour around which integration is to be performed. The choice of contour in the upper or lower half of the z-plane to be used will depend on the sign of the transform variable ω.

This same procedure is usually necessary when finding the inverse Fourier transform, because when $F(\omega)$ is complex direct integration of the inversion integral is not possible. The example that follows will illustrate the fact that considerable care is necessary when working with Fourier transforms. This is because when finding a Fourier transform, the transform variable ω often occurs in the form $|\omega|$, causing the transform to take one form when ω is positive, and another when it is negative.

Example: Let us find the Fourier transform of $f(x) = 1/(x^2 + a^2)$ where $a > 0$, the result of which is given in entry 1 of Table 20.1.

Replacing x by the complex variable z, the function $f(z) = e^{i\omega z}/(z^2 + a^2)$, the integrand in the Fourier transform, is seen to have simple poles at $z = ia$ and $z = -ia$, where the residues are, respectively, $-ie^{-\omega a}/(2a)$ and $ie^{\omega a}/(2a)$. For the time being, allowing C_R to be a semicircle in either the upper or the lower half of the z-plane with its center at the origin, we have

$$F(\omega) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{e^{i\omega x}}{(x^2 + a^2)} dx + \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{C_R} \frac{e^{i\omega z}}{(z^2 + a^2)} dz.$$

To use the residue theorem we need to show the second integral vanishes in the limit as $R \to \infty$. On C_R we can set $z = Re^{i\theta}$, so $dz = iRe^{i\theta} d\theta$, showing that

$$\frac{1}{\sqrt{2\kappa}} \int_{C_R} \frac{e^{i\omega z}}{(z^2 + a^2)} dz = \frac{1}{\sqrt{2\pi}} \int_{C_R} \frac{e^{i\omega R(\cos\theta + i\sin\theta)} iR\, e^{i\theta}}{(R^2 e^{2i\theta} + a^2)} e^{-\omega R \sin\theta} d\theta.$$

We now estimate the magnitude of the integral on the right by the result

$$\left|\frac{1}{\sqrt{2\pi}}\int_{C_R}\frac{e^{i\omega z}}{(z^2+a^2)}dz\right| \le \frac{1}{\sqrt{2\pi}}\frac{R}{|R^2-a^2|}\int_{C_R} e^{-\omega R\sin\theta}d\theta.$$

The multiplicative factor involving R on the right will vanish as $R \to \infty$, so the integral around C_R will vanish if the integral on the right around C_R remains finite or vanishes as $R \to \infty$. There are two cases to consider, the first being when $\omega > 0$, and the second when $\omega < 0$. If $\omega = 0$ the integral will certainly vanish as $R \to \infty$, because then the integral around C_R becomes $\int_{C_R} d\theta = \pi$.

The case $\omega > 0$. The integral on the right around C_R will vanish in the limit as $R \to \infty$ provided $\sin\theta \ge 0$ because its integrand vanishes. This happens when C_R becomes the semicircle C_{R+} located in the upper half of the z-plane.

The case $\omega < 0$. The integral around C_R will vanish in the limit as $R \to \infty$, provided $\sin\theta \le 0$ because its integrand vanishes. This happens when C_R becomes the semicircle C_{R-} located in the lower half of the z-plane.

We may now apply the residue theorem after proceeding to the limit as $R \to \infty$. When $\omega > 0$ we have $C_R = C_{R+}$, in which case only the pole at $z = ia$ lies inside the contour at which the residue is $-ie^{-\omega a}/(2a)$, so

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{(x^2+a^2)}dx = 2\pi i \times \frac{1}{\sqrt{2\pi}}\left[-\frac{ie^{-\omega a}}{2a}\right] = \sqrt{\frac{\pi}{2}}\frac{e^{-\omega a}}{a}, \quad (\omega > 0).$$

Similarly, when $\omega < 0$ we have $C_R = C_{R-}$, in which case only the pole at $z = -ia$ lies inside the contour at which the residue is $ie^{\omega a}/(2a)$. However, when integrating around C_{R-} in the positive (counterclockwise) sense, the integration along the x-axis occurs in the negative sense, that is from $x = R$ to $x = -R$, leading to the result

$$\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}\frac{e^{i\omega x}}{(x^2+a^2)}dx = 2\pi i \times \frac{1}{\sqrt{2\pi}}\left[\frac{ie^{\omega a}}{2a}\right] = -\sqrt{\frac{\pi}{2}}\frac{e^{\omega a}}{a}, \quad (\omega < 0).$$

Reversing the order of the limits in the integral, and compensating by reversing its sign, we arrive at the result

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{(x^2+a^2)}dx = \sqrt{\frac{\pi}{2}}\frac{e^{\omega a}}{a}, \quad (\omega < 0).$$

Combining the two results for positive and negative ω we have shown the Fourier transform $F(\omega)$ of $f(x) = 1/(x^2+a^2)$ is

$$F(\omega) = \sqrt{\frac{\pi}{2}}\frac{e^{-a|\omega|}}{a}, \quad (a > 0).$$

The function $f(x)$ can be recovered from its Fourier transform $F(\omega)$ by means of the inversion integral, though this case is sufficiently simplest that direct integration can be used.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \frac{e^{-i\omega x} e^{-a|\omega|}}{a} d\omega = \frac{1}{2a} \int_{-\infty}^{\infty} e^{-a|\omega|} \left(\cos(\omega x) - i\sin(\omega x)\right) d\omega.$$

The imaginary part of the integrand is an odd function, so its integral vanishes. The real part of the integrand is an even function, so the interval of integration can be halved and replaced by $0 \le \omega < \infty$, while the resulting integral is doubled, with the result that

$$f(x) = \frac{1}{a} \int_0^{\infty} e^{-a\omega} \cos(\omega x) d\omega = \frac{1}{x^2 + a^2}.$$

The Inverse Laplace Transform

Given an elementary function $f(x)$ for which the Laplace transform $F(s)$ exists, the determination of the form of $F(s)$ is usually a matter of routine integration. However, when finding $f(x)$ from $F(s)$ cannot be accomplished by use of a table of Laplace transform pairs and the properties of the transform, it becomes necessary to make use of the Laplace inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{sx} ds. \tag{17}$$

Here the real number γ must be chosen such that all the poles of the integrand lie to the left of the line $s = \gamma$ in the complex s-plane. This integral is to be interpreted as the limit as $R \to \infty$ of a contour integral around the contour shown in Figure 2. This is called the **Bromwich contour** after the Cambridge mathematician T.J.I'A. Bromwich who introduced it at the beginning of the last century.

Example: To illustrate the application of the Laplace inversion integral it will suffice to consider finding $f(x) = \mathcal{L}^{-1}\left\{1/\sqrt{s}\right\}$.

The function $1\sqrt{s}$ has a branch point at the origin, so the Bromwich contour must be modified to make the function single valued inside the contour. We will use the contour shown in Figure 3, where the branch point is enclosed in a small circle about the origin while the complex s-plane is cut along the negative real axis to make the function single valued inside the contour.

Let C_{R1} denote the large circular arc and C_{R2} denote the small circle around the origin. Then on C_{R1} $s = \gamma + Re^{i\theta}$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, and for subsequent use we now set $\theta = \frac{\pi}{2} + \phi$, so $s = \gamma + iRe^{i\phi}$ with $0 \le \phi \le \pi$. Consequently, $ds = -Re^{i\phi} d\phi$, with the result that $|ds| = Rd\phi$. Thus, when R is sufficiently large $|s| = \left|\gamma + iRe^{i\phi}\right| \ge \left|\left|Re^{i\phi}\right| - |\gamma|\right| = R - \gamma$.

Also for subsequent use, we need the result that

$$|e^{sx}| = \left|\exp\Big[x\left[(\gamma - R\sin\phi) + iR\cos\phi\right]\Big]\right| = e^{\gamma x} \exp\left[-Rx\sin\phi\right].$$

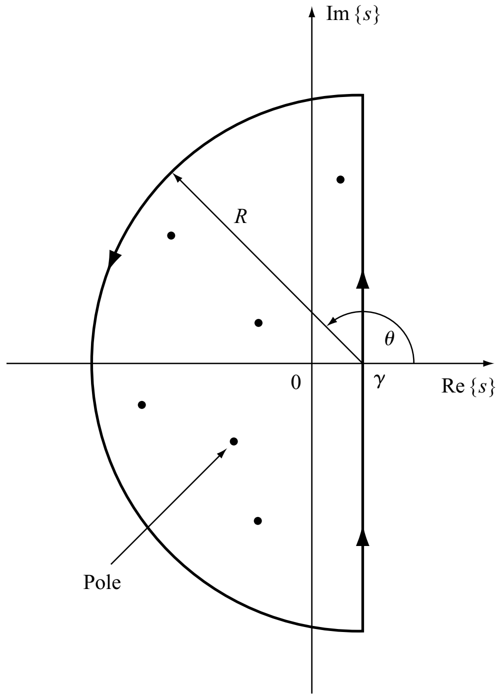

Figure 2. The Bromwich contour for the inversion of a Laplace transform.

The integral around the modified Bromwich contour is the sum of the integrals along each of its separate parts, so we now estimate the magnitudes of the respective integrals.

The magnitude of the integral around the large circular arc C_{R1} can be estimated as

$$I_R = \left| \int_{ABEF} \frac{e^{sx}}{\sqrt{s}} ds \right| \leq \int_{ABEF} \frac{|e^{sx}|}{|s|^{1/2}} |ds| \leq \frac{e^{\gamma x} R}{(R-\gamma)^{1/2}} \int_0^{\pi} \exp\left[-Rx \sin\phi\right] d\phi.$$

The symmetry of $\sin\phi$ about $\phi = \frac{1}{2}\pi$ allows the inequality to be rewritten as

$$I_R \leq \frac{2e^{\gamma x} R}{(R-\gamma)^{1/2}} \int_0^{\pi/2} \exp\left[-Rx \sin\phi\right] d\phi,$$

so after use of the Jordan inequality in form (iii), this becomes

$$I_R \leq \frac{\pi e^{\gamma x}}{(R-\gamma)^{1/2} x} \left(1 - e^{-Rx}\right), \qquad \text{when } x > 0.$$

This shows that when $x > 0$, $\lim\limits_{R\to\infty} I_R = 0$, so that the integral around C_{R1} vanishes in the limit as $R \to \infty$.

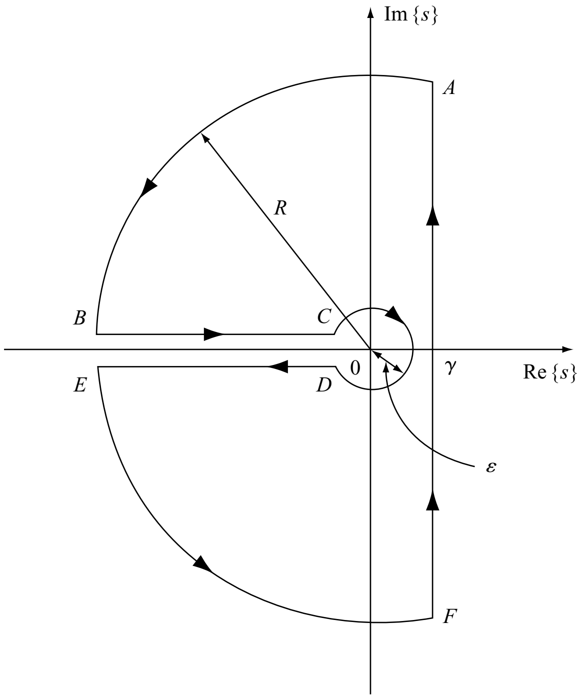

Figure 3. The modified Bromwich contour with an indentation and a cut.

On the small circle C_{R2} with radius ε we have $s = \varepsilon e^{i\theta}$, so $ds = i\varepsilon e^{i\theta} d\theta$ and $s^{1/2} = e^{i\theta/2}\sqrt{\varepsilon}$, so the integral around C_{R2} becomes

$$\int_{-\pi}^{\pi} \frac{1}{e^{i\theta/2}\sqrt{\varepsilon}} \exp\left[\varepsilon x\left(\cos\theta + i\sin\theta\right)\right] i\varepsilon e^{i\theta} d\theta,$$

but this vanishes as $\varepsilon \to 0$, so in the limit the integral around C_{R2} also vanishes.

Along the top BC of the branch cut $s = re^{\pi i} = -r$, so $\sqrt{s} = e^{\pi i/2}\sqrt{r} = i\sqrt{r}$, so that $ds = -dr$. Along the bottom BC of the branch cut the situation is different, because there $s = re^{-\pi i} = -r$, so $\sqrt{s} = e^{-\pi i/2}\sqrt{r} = -i\sqrt{r}$, where again $ds = -dr$.

The construction of the Bromwich contour has ensured that no poles lie inside it, so from the Cauchy residue theorem, in the limit as $R \to \infty$ and $\varepsilon \to 0$, the only contributions to the contour integral come from integration along opposite sides of the branch cut, so we arrive at the result

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sx}}{\sqrt{s}} ds = \frac{1}{2\pi i}\left\{-\int_{\infty}^{0} \frac{ie^{-rx}}{\sqrt{r}} dr + \int_{0}^{\infty} \frac{ie^{-rx}}{\sqrt{r}} dr\right\} = \frac{1}{\pi}\int_{0}^{\infty} \frac{e^{-rx}}{\sqrt{r}} dr.$$

Finally, the change of variable $r = u^2$, followed by setting $\nu = u\sqrt{x}$, changes this result to

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{sx}}{\sqrt{s}}ds = \frac{2}{\pi\sqrt{x}}\int_0^\infty e^{-\nu^2}d\nu.$$

This last definite integral is a standard integral, and from entry **15.3.1**(29) we have $\int_0^\infty e^{-\nu^2}d\nu = \sqrt{\pi}/2$, so we have shown that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi x}}, \qquad \text{for } \mathrm{Re}\{s\} > 0.$$

The inversion integral can generate an infinite series if an infinite number of isolated poles lie along a line parallel to the imaginary s-axis. This happens with $\mathcal{L}^{-1}\left\{\dfrac{1}{s\cosh s}\right\}$, where the poles are actually located on the imaginary axis.

We omit the details, but straightforward reasoning using the standard Bromwich contour shows that

$$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{s\cosh s}\right\} = 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\cos\left[(2n+1)\pi x/2\right]}{2n+1}.$$

To understand why this periodic representation of $f(x)$ has occurred, notice that $F(s) = 1/[s\cosh s]$ is the Laplace transform of the piecewise continuous function

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 2, & 1 < x < 3 \\ 0, & 3 < x < 4, \end{cases}$$

that is periodic with period 4 and defined for $x \geq 0$. So $f(x)$ is in fact the Fourier series representation of this function with period 4 when it is defined for all x. Here the term **period** is used in the usual sense that X is the period of $f(x)$ if $f(X + x) = f(x)$ is true for all x and X is the smallest value for which this result is true.

Index of Special Functions and Notations

Notation	*Name*	*Section of formula containing its definition*
$\lvert a \rvert$	Absolute value of the real number a	1.1.2.1
am u	Amplitude of an elliptic function	12.2.1.1.2
$\sim$	Asymptotic relationship	1.14.2.1
α	Modular angle of an elliptic integral	12.1.2
$\arg z$	Argument of complex number z	2.1.1.1
$A(x)$	$A(x) = 2P(x) - 1$; probability function	13.1.1.1.7
$\mathbf{A}$	Matrix	
$\mathbf{A}^{-1}$	Multiplicative inverse of a square matrix $\mathbf{A}$	1.5.1.1.9
$\mathbf{A}^{\mathrm{T}}$	Transpose of matrix $\mathbf{A}$	1.5.1.1.7
$\lvert\mathbf{A}\rvert$	Determinant associated with a square matrix $\mathbf{A}$	1.4.1.1
B_n	Bernoulli number	1.3.1.1
B_n^*	Alternative Bernoulli number	1.3.1.1.6
$B_n(x)$	Bernoulli polynomial	1.3.2.1.1
$B(x, y)$	Beta function	11.1.7.1
$\binom{n}{k}$	Binomial coefficient	1.2.1.1
	$\binom{n}{k} = \dfrac{n!}{k!(n-k)!}, \qquad \binom{n}{0} = 1$	
$(a)_n$	Pochhammer symbol	
	$(a)_n = a(a+1)(a+2)\cdots(a+n-1)$	0.3
$C(x)$	Fresnel cosine integral	14.1.1.1.1
C_{ij}	Cofactor of element a_{ij} in a square matrix $\mathbf{A}$	1.4.2
${}^nC_m$ or ${}_nC_m$	Combination symbol ${}^nC_m = \binom{n}{m}$	1.6.2.1
$\mathrm{cn}\, u$	Jacobian elliptic function	12.2.1.1.4
$\mathrm{cn}^{-1} u$	Inverse Jacobian elliptic function	12.4.1.1.4
$\mathrm{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$	Curl of vector $\mathbf{F}$	23.8.1.1.6
$\delta(x)$	Dirac delta function	19.1.3
δ_{ij}	Kronecker delta symbol	1.4.2.11
$D_n(x)$	Dirichlet kernel	1.13.1.10.3
$\mathrm{dn}\, u$	Jacobian elliptic function	12.2.1.1.5
$\mathrm{dn}^{-1} u$	Inverse Jacobian elliptic function	12.4.1.1.5
$\mathrm{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$	Divergence of vector $\mathbf{F}$	23.8.1.1.4
$e^{i\theta}$	Euler formula; $e^{i\theta} = \cos\theta + i \sin\theta$	2.1.1.2.1
e	Euler's constant	0.3
$Ei(x)$	Exponential integral	5.1.2.2
$E(\varphi, k)$	Incomplete elliptic integral of the second kind	12.1.1.1.5
$E(k), E'(k)$	Complete ellipitic integrals of the second kind	13.1.1.1.8, 13.1.1.1.10

Notation	*Name*	*Section of formula containing its definition*
e^{Az}	Matrix exponential	1.5.4.1
$\operatorname{erf} x$	Error function	13.2.1.1
$\operatorname{erfc} x$	Complementary error function	13.2.1.1.4
E_n	Euler number	1.3.1.1
E_n^*	Alternative Euler number	1.3.1.1.6
$E_n(x)$	Euler polynomial	1.3.2.3.1
$f(x)$	A function of x	
$f'(x)$	First derivative df/dx	1.15.1.1.6
$f^{(n)}(x)$	nth derivative $d^n f/dx^n$	1.12.1.1
$f^{(n)}(x_0)$	nth derivative $d^n f/dx^n$ at x_0	1.12.1.1
$F(\varphi, k)$	Incomplete elliptic integral of the first kind	12.1.1.1.4
$\|\|\Phi_n\|\|$	Norm of $\Phi_n(x)$	18.1.1.1
$\operatorname{grad}\phi = \nabla\phi$	Gradient of the scalar function ϕ	23.8.1.6
$\Gamma(x)$	Gamma function	11.1.1.1
$\Gamma(a, x), \gamma(a, x)$	Incomplete gamma functions	11.1.8.9
γ	Euler–Mascheroni constant	1.11.1.1.7
$H(x)$	Heaviside step function	19.1.2.5
$H_n(x)$	Hermite polynomial	18.5.3
i	Imaginary unit	1.1.1.1
$\operatorname{Im}\{z\}$	Imaginary part of $z = x + iy; \operatorname{Im}\{z\} = y$	1.1.1.2
$\mathbf{I}$	Unit (identity) matrix	1.5.1.1.3
$\mathrm{i}^n \operatorname{erfc} x$	nth repeated integral of $\operatorname{erfc} x$	13.2.7.1.1
$I_{\pm\nu}(x)$	Modified Bessel function of the first kind of order ν	17.6.1.1
$\int f(x)dx$	Indefinite integral (antiderivative) of $f(x)$	1.15.2
$\int_a^b f(x)dx$	Definite integral of $f(x)$ from $x = a$ to $x = b$	1.15.2.5
$j_n(x)$	Spherical Bessel function	17.14.1
$J_{\pm\nu}(x)$	Bessel function of the first kind of order ν	17.1.1.1
k	Modulus of an elliptic integral	12.1.1.1
k'	Complementary modulus of an elliptic integral; $k' = \sqrt{1-k^2}$	12.1.1.1
$\mathbf{K}(k)$, $\mathbf{K}'(k)$	Complete elliptic integrals of the first kind	12.1.1.1.7, 12.1.1.1.9
$k_v(x)$	Modified Bessel function of the second kind of order v	17.6.1.1
$\mathcal{L}[f(x); s]$	Laplace transform of $f(x)$	19.1.1
$L_n(x)$	Laguerre polynomial	18.4.1
$L_n^{(\alpha)}$	Generalized Laguerre polynomial	18.4.8.2
$\log_a x$	Logarithm of x to the base a	2.2.1.1
$\operatorname{In} x$	Natural logarithm of x (to the base e)	2.2.1.1
M_{ij}	Minor of element a_{ij} in a square matrix $\mathbf{A}$	1.4.2
$n!$	Factorial $n; n! = 1 \cdot 2 \cdot 3 \cdots n; \quad 0! = 1$	1.2.1.1
$(2n)!!$	Double factorial; $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n)$	15.2.1
$(2n-1)!!$	Double factorial; $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$	15.2.1
$\left[\frac{n}{2}\right]$	Integral part of $n/2$	18.2.4.1.1
${}^nP_m$ or ${}_nP_m$	Permutation symbol; ${}^nP_m = \dfrac{n!}{(n-m)!}$	1.6.1.1.3

Notation	*Name*	*Section of formula containing its definition*
$P_n(x)$	Legendre polynomial	18.2.1
$P_m^n(x)$	First solution of the associated Legendre equation	18.2.10.1
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial of degree n	18.6.1
$P(x)$	Normal probability distribution	13.1.1.1.5
$\prod_{k=1}^{n} u_k$	Product symbol; $\prod_{k=1}^{n} u_k = u_1 u_2 \cdots u_n$	1.9.1.1.1
$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx$	Cauchy principal value of the integral	1.15.4.IV
π	Ratio of the circumference of a circle to its diameter	0.3
$\Pi(x)$	pi function	11.1.1.1
$\Pi(\varphi, n, k)$	Incomplete elliptic integral of the third kind	12.1.1.1.6
$\psi(z)$	psi (digamma) function	11.1.6.1
$Q(x)$	Probability function; $Q(x) = 1 - P(x)$	13.1.1.1.6
$Q(x)$	Quadratic form	1.5.2.1
$Q_n(x)$	Legendre function of the second kind	18.2.7
$Q_m^n(x)$	Second solution of the associated Legendre equation	18.2.10.1
r	Modulus of $z = x + iy; r = (x^2 + y^2)^{1/2}$	
$\text{Re}\{z\}$	Real part of $z = x + iy$; $\text{Re}\{z\} = x$	1.1.1.2
$\text{sgn}(x)$	Sign of x	
$\text{sn}\ u$	Jacobian elliptic function	12.2.1.1.3
$\text{sn}^{-1} u$	Inverse Jacobian elliptic function	12.4.1.1.3
$S(x)$	Fresnel sine integral	14.1.1.1.2
$\text{Si}(x)$, $\text{Ci}(x)$	Sine and cosine integrals	14.2.1
$\sum_{k=m}^{n} a_k$	Summation symbol; $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$ and if $n < m$ we define $\sum_{k=m}^{n} a_k = 0$.	1.2.3
$\sum_{k=m}^{\infty} a_k(x - x_0)^k$	Power series expanded about x_0	1.11.1.1.1
$T_n(x)$	Chebyshev polynomial	18.3.1.1
$\text{tr}\,\mathbf{A}$	Trace of a square matrix $\mathbf{A}$	15.1.1.10
$U_n(x)$	Chebyshev polynomial	18.3.11
$x = f^{-1}(y)$	Function inverse to $y = f(x)$	1.11.1.8
$Y_v(x)$	Bessel function of the second kind of order v	17.1.1.1
$Y_n^m(\theta, \phi)$	Spherical harmonic	18.2.10.1
$y_n(x)$	Spherical Bessel function	17.14.1
z	Complex number $z = x + iy$	1.1.1.1
$\|z\|$	Modulus of $z = x + iy; r = \|z\| = (x^2 + y^2)^{1/2}$	1.1.1.1
$\bar{z}$	Complex conjugate of $z = x + iy$; $\bar{z} = x - iy$	1.1.1.1
$z_b\{x[n]\}$	bilateral z-transform	26.1
$z_u\{x[n]\}$	unilateral z-transform	26.1