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Article

Hankel and Symmetric Toeplitz Determinants for a New Subclass of q-Starlike Functions

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Abstract: This paper considers the basic concepts of q-calculus and the principle of subordination. We define a new subclass of q-starlike functions related to the Salagean q-differential operator. For this class, we investigate initial coefficient estimates, Hankel determinants, Toeplitz matrices, and Fekete-Szegö problem. Moreover, we consider the q-Bernardi integral operator to discuss some applications in the form of some results.

Keywords: analytic functions; quantum calculus; q-derivative operator; salagean q-differential operator; q-starlike functions; Hankel determinants; Toeplitz matrices

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1. Introduction and Definitions

Let the set of all analytic functions g in the open unit disk

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

be denoted by $\mathcal{A}$ and every $g \in \mathcal{A}$ can be expressed as

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subset of $\mathcal{A}$, whose functions are univalent in E. A function $g \in \mathcal{A}$ is known as a starlike function (denoted $g \in \mathcal{S}^*$) and a convex function (denoted $g \in \mathcal{K}$) if it satisfies the following inequalities.

$$Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \ (z \in E)$$

and

$$Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0, \ (z \in E).$$

For $0 \leq \alpha < 1$, define the set $\mathcal{S}^*(\alpha)$ of starlike functions of order α and the set $\mathcal{K}(\alpha)$ of convex functions of order α as follows:

$$\mathcal{S}^*(\alpha) = \left\{ g \in \mathcal{A} : Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha, \ (z \in E) \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ g \in \mathcal{A} : Re\left(1 + \frac{zg^{''}(z)}{g^{'}(z)} \right) > \alpha,\ (z \in E) \right\}.$$

In particular,

$$\mathcal{S}^*(0) = \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) = \mathcal{K}.$$

The class $k - \mathcal{UCV}$ of k-uniformly convex functions and the class $k - \mathcal{US}^*$ of k-uniformly starlike functions were introduced by Kanas and Wisniowska [1,2], which are defined by

$$k - \mathcal{US}^* = \left\{ g \in \mathcal{A} : k\left| \frac{zg'(z)}{g(z)} - 1 \right| < Re\left(\frac{zg'(z)}{g(z)} \right),\ z \in E,\ k \geq 0 \right\}$$

and

$$k - \mathcal{UCV} = \left\{ g \in \mathcal{A} : k\left| \frac{(zg'(z))'}{g'(z)} - 1 \right| < Re\left(\frac{(zg'(z))'}{g'(z)} \right),\ z \in E,\ k \geq 0 \right\}.$$

In particular, if we take $k = 0$, then $k - \mathcal{US}^* = \mathcal{US}^*$ and $k - \mathcal{UCV} = \mathcal{UCV}$ introduced by Goodman [3]. Moreover, Wang et al. [4] defined and investigated the subclasses $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{K}(\alpha, \beta)$ of analytic functions satisfy the following conditions, respectively.

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| < \beta \left| \alpha \frac{zg'(z)}{g(z)} + 1 \right|,\ \ z \in E.$$

and

$$\left| \frac{(zg'(z))'}{g'(z)} - 1 \right| < \beta \left| \alpha \frac{(zg'(z))'}{g'(z)} + 1 \right|,\ \ z \in E,$$

where, $0 < \alpha \leq 1,\ 0 < \beta \leq 1$.

Let $g, h \in \mathcal{A}$ define their convolution by

$$(g * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (h * g)(z),$$

where, g is given by (1) and

$$h(z) = \sum_{n=2}^{\infty} b_n z^n, \quad (z \in E).$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions. An analytic function $p \in \mathcal{P}$ if it has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2}$$

and satisfies

$$Re(p(z)) > 0.$$

The study of q-calculus has recently inspired researchers because of its many applications in mathematics and physics, especially in quantum physics. Jackson [5,6] was the first who introduced the q-analogues of derivatives by applying the q-calculus theory. He defined the q-derivative operator (D_q) for analytic function g in the open unit disk U. Furthermore, in [7], Ismail et al. defined q-starlike functions by using the quantum (or q-) calculus operator theory, and many researchers studied q-calculus in the perspective of Geometric Functions Theory (GFT). In 2014, Kanas and Raducanu [8] introduced Ruscheweyh q-differential operators and discussed some of its applications in a class of analytic functions related to conic domains. After that, many q-differential and integral operators have been defined so far (see for details [9,10]. Arif et al. [11,12] studied q-derivative operator for multivalent functions, and in [13] Zang et al. gave the generalizations of the conic domain by using q-calculus. Srivastava [14] used fractional q-calculus operators to discuss some properties of analytic functions. Recently, Srivastava [15] published a review article that benefits new researchers and scholars that are working in GFT and q-calculus. Khan et al. [16,17] studied the q-derivative operator and defined a new subclass of q-starlike functions, while in [18] Mahmood et al. investigated a third Hankel determinant for the class of q-starlike functions.

Presently, we recall some definitions and details about q-calculus, which will help us to understand this new article.

Definition 1 ([19])**.** *The q-number $[t]_q$ for $q \in (0,1)$ is defined as*

$$[t]_q = \frac{1-q^t}{1-q}, \qquad (t \in \mathbb{C}).$$

In particular, $t = n \in \mathbb{N}$,

$$[n]_q = \sum_{k=0}^{n-1} q^k.$$

The q-factorial $[n]_q!$ can be defined as

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad (n \in \mathbb{N}).$$

In particular, $[0]_q! = 1$.

Definition 2. *The q-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as*

$$[t]_{n,q} = [t]_q [t+1]_q [t+2]_q \cdots [t+n-1]_q, \quad (n \in \mathbb{N}).$$

In particular, if $n = 0$, then $[t]_{0,q} = 1$.

Definition 3 (Jackson [6])**.** *The q-integral for a function g is defined by*

$$\int g(z) d_q z = (1-q) z \sum_{n=0}^{\infty} g(q^n z) q^n.$$

Definition 4 ([5])**.** *For $g \in \mathcal{A}$, the q-derivative operator or q-difference operator is defined by*

$$\begin{aligned} D_q g(z) &= \frac{g(z) - g(qz)}{(1-q)z}, \quad z \neq 0, q \neq 1, \\ &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \end{aligned} \tag{3}$$

Definition 5 ([20])**.** *The Salagean q-differential operator for g is defined by*

$$\begin{aligned} \mathcal{S}_q^0 g(z) &= g(z),\ \mathcal{S}_q^1 g(z) = z D_q g(z) = \frac{g(qz) - g(z)}{q-1}, \cdots, \\ \mathcal{S}_q^m g(z) &= z D_q \left(\mathcal{S}_q^{m-1} g(z) \right) = g(z) * \left(z + \sum_{n=2}^{\infty} [n]_q^m z^n \right) \\ &= z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n. \end{aligned}$$

Motivated by the work of Kanas and Raducanu [8] and Govindaraj and Sivasubramanian [20], we define the following class of functions with the help of q-calculus.

Definition 6. *An analytic function g is said to be in the class $\mathcal{S}^*(m, q, \alpha, \beta)$ if*

$$\left|\frac{\mathcal{S}_q^m g(z)}{g(z)} - 1\right| < \beta \left|\frac{\alpha \mathcal{S}_q^m g(z)}{g(z)} + 1\right|, \quad z \in E$$

where $0 < \alpha \leq 1$ and $0 < \beta \leq 1$.

Equivalently,

$$\frac{\mathcal{S}_q^m g(z)}{g(z)} \prec \varphi(z), \tag{4}$$

where

$$\varphi(z) = \frac{1 + \beta z}{1 - \alpha \beta z}.$$

Note that

$$\mathcal{S}^*(m, q, \alpha, \beta) \subset \mathcal{S}^* \subset \mathcal{S}.$$

Remark 1. *If $m = 1$ and $q \to 1-$, then $\mathcal{S}^*(m, q, \alpha, \beta) = \mathcal{S}^*(\alpha, \beta)$, which is introduced by Liu et al. in [21].*

If $q \to 1-$, $m = 1$, $\alpha = 1$, and $\beta = 1$, then $\mathcal{S}^*(m, q, \alpha, \beta) = \mathcal{S}^*$, which is the well known class of starlike functions.

Noonan and Thomas [22] introduced the following jth Hankel determinants, where $n \geq 0$, $j \geq 1$, and $a_1 = 1$.

$$\mathcal{H}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+j-1} & a_{n+j} & \cdots & a_{n+2j-2} \end{vmatrix}.$$

The Hankel determinant plays an important role in the theory of singularities [23] and are helpful in the study of power series with integer coefficients (see [24–26]). Note that the number of authors found the sharp upper bounds on $\mathcal{H}_2(2)$ (see, for example, [27–31] for numerous classes of functions.

If $j = 2$ and $n = 1$, we then obtain a well-known fact for the Fekete-Szegö functional that:

$$\mathcal{H}_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

This functional was further generalized as follows,

$$\left|a_3 - \mu a_2^2\right|$$

for some real or complex number μ.

If $j = 2$ and $n = 2$, then Janteng [32] defined the following Hankel determinant and studied it for starlike functions.

$$\mathcal{H}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

Babalola [33] studied the Hankel determinant $\mathcal{H}_3(1)$ for some subclasses of analytic functions.

Recently, Thomas and Halim [34] introduced the symmetric Toeplitz determinant $\mathcal{T}_j(n)$ for $f \in \mathcal{A}$, defined by:

$$\mathcal{T}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_n & \cdots & a_{n+j-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_n \end{vmatrix}, \quad (5)$$

where $n \geq 1$, $j \geq 1$ and $a_1 = 1$. In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \; T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$
$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \; T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

Very recently, a large list of authors investigated estimates of the Toeplitz determinant $\mathcal{T}_j(n)$ for functions belonging to different families of univalent functions (see, for example, [34–40].

In recent years, studies on estimating the coefficient bounds for the Toeplitz determinants for the class of univalent functions and its subclasses have been conducted by numerous researchers, such as Srivastava et al. [39], Ramachand and Kavita [38], Al-Khafaji et al. [41], Radnika et al. [36,37], Sivasupramanian et al. [42], Zhang et al. [43] and Ali et al. [44].

The problem of determining the sharp upper bounds for the functional $|a_2 - \mu a_2^2|$ for a given compact family $\mathcal{F}$ of functions in the normalized analytic class $\mathcal{A}$ is often called the Fekete-Szegö problem for $\mathcal{F}$. Many researchers have investigated the Fekete-Szegö problem for analytic functions (see [45–47]).

Aleman and Constantin [48] produced an admirable connection between univalent function theory and fluid dynamics. They found explicit solutions to the incompressible two-dimensional Euler equations by means of a univalent harmonic mapping. More accurately, the problem of finding all solutions describing the particle paths of the flow in Lagrangian variables was reduced to finding harmonic functions satisfying an explicit nonlinear differential system in C^n with $n = 3$ or $n = 4$ (see also [49]). The problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see [50]). For more details about the symmetric Toeplitz determinants, see [51,52].

2. A Set of Lemmas

In this section, we give some lemmas to investigate the main results of this paper.

Lemma 1 ([50])**.** *Let the function $p(z)$ be given by (2), then*

$$using|c_n| \leq 2, \;\; n \geq 1.$$

The inequality is sharp for the following function,

$$g(z) = (1+z)(1-z)^{-1}.$$

Lemma 2 ([53,54])**.** *Let for some $x, z \in \mathbb{C}$, with $|z| \leq 1$ and $|x| \leq 1$. Let the function $p(z)$ be analytic in E and given by (2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - \left|x^2\right|)z.$$

Lemma 3 ([55]). *Let the function* $p(z)$ *be given by* (2) *and*

$$Re(p(z)) > 0,$$

and let $\mu \in \mathbb{C}$, *then*

$$|c_n - \mu c_k c_{n-k}| \leq 2\max(1, |2\mu - 1|),\ 1 \leq k \leq n-k.$$

In this section, we investigate initial coefficient estimates, Hankel determinants, Toeplitz matrices and Fekete-Szegö problems.

3. Main Results

In the following theorem, we will find initial coefficients bounds, which will help out to prove other results.

Theorem 1. *Let the function* g *of the form* (1) *be in the class* $\mathcal{S}^*(m, q, \alpha, \beta)$. *Then*

$$\begin{aligned} |a_2| &\leq \frac{\beta(1+\alpha)}{[2]_q^m - 1}, \\ |a_3| &\leq \frac{\beta^2(1+\alpha)}{[3]_q^m - 1}\left\{\alpha + \frac{1+\alpha}{[2]_q^m - 1}\right\}, \\ |a_4| &\leq \frac{\beta(1+\alpha)}{[4]_q^m - 1}\{1 + 2\Lambda_1(\alpha,\beta,m,q) + 4\Lambda_2(\alpha,\beta,m,q)\}. \end{aligned}$$

where

$$\Lambda_1(\alpha,\beta,m,q) = \alpha\beta - 1 + \frac{\beta(1+\alpha)}{2}\frac{[2]_q^m + [3]_q^m - 2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)}, \tag{6}$$

$$\Lambda_2(\alpha,\beta,m,q) = \left\{ \begin{array}{c} \frac{1}{4} - \frac{\alpha\beta}{2} + \frac{\alpha^2\beta^2}{4} + \frac{\beta(1+\alpha)}{2}\frac{[2]_q^m + [3]_q^m - 2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)} \\ \times\left(\frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right) - \left(\frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right)^2 \end{array} \right\}. \tag{7}$$

Proof. Let $g \in \mathcal{S}^*(m, q, \alpha, \beta)$, then we have

$$\frac{\mathcal{S}_q^m g(z)}{g(z)} \prec \varphi(z), \tag{8}$$

or

$$\frac{\mathcal{S}_q^m g(z)}{g(z)} = \varphi(u(z)), \tag{9}$$

where,

$$\varphi(z) = \frac{1+\beta z}{1-\alpha\beta z}.$$

After some simple calculations, we obtain

$$\varphi(z) = 1 + \beta(1+\alpha)z + \alpha\beta^2(1+\alpha)z^2 + \alpha^2\beta^3(1+\alpha)z^3 + \cdots. \tag{10}$$

Let

$$\begin{aligned} p(z) &= \frac{1+u(z)}{1-u(z)} \\ &= 1 + c_1 z + c_2 z^2 + \cdots, \end{aligned} \tag{11}$$

then

$$\begin{aligned} u(z) &= (p(z)-1)(p(z)+1)^{-1} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{1}{4}c_1^3\right)z^3 + \cdots. \end{aligned} \tag{12}$$

In view of (9), (10) and (12), we have

$$\begin{aligned} &\varphi(u(z)) \\ =\ & 1 + \frac{1}{2}\beta(1+\alpha)c_1 z + \left\{\left(\frac{1}{2}\beta(1+\alpha)\right)\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}\alpha\beta^2(1+\alpha)c_1^2\right\}z^2 \\ & + \frac{1}{2}\beta(1+\alpha)\left\{c_3 + \alpha\beta c_1c_2 - c_1c_2 + \frac{1}{4}c_1^3 - \frac{\alpha\beta}{2}c_1^3 + \frac{\alpha^2\beta^2}{4}c_1^3\right\}z^3. \end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned} &\frac{\mathcal{S}_q^m g(z)}{g(z)} \\ =\ & 1 + \left([2]_q^m - 1\right)a_2 z + \left\{\left([3]_q^m - 1\right)a_3 - \left([2]_q^m - 1\right)a_2^2\right\}z^2 \\ & + \left\{\left([4]_q^m - 1\right)a_4 - \left\{[2]_q^m + [3]_q^m - 2\right\}a_2a_3 + \left([2]_q^m - 1\right)a_2^3\right\}z^3 \cdots. \end{aligned} \tag{14}$$

Equating the corresponding coefficients of (13) and (14), we have

$$a_2 = \frac{\beta(1+\alpha)c_1}{2\left([2]_q^m - 1\right)}, \tag{15}$$

$$a_3 = \frac{\beta(1+\alpha)}{2\left([3]_q^m - 1\right)}\left\{c_2 + \left(\frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right)c_1^2\right\}, \tag{16}$$

$$a_4 = \frac{\beta(1+\alpha)}{2\left([4]_q^m - 1\right)}\left\{c_3 + \Lambda_1(\alpha,\beta,m,q)c_1c_2 + \Lambda_2(\alpha,\beta,m,q)c_1^3\right\}, \tag{17}$$

where

$$\Lambda_1(\alpha,\beta,m,q) = \alpha\beta - 1 + \frac{\beta(1+\alpha)}{2}\frac{[2]_q^m + [3]_q^m - 2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)}, \tag{18}$$

$$\Lambda_2(\alpha,\beta,m,q) = \left\{\begin{array}{c} \frac{1}{4} - \frac{\alpha\beta}{2} + \frac{\alpha^2\beta^2}{4} + \frac{\beta(1+\alpha)}{2}\frac{[2]_q^m + [3]_q^m - 2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)} \\ \times\left(\frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right) - \left(\frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right)^2 \end{array}\right\}. \tag{19}$$

Applying the Lemma 1 on (15)–(17), we obtained the desired result after some simplification. □

In Theorem 2, we will investigate symmetric Toeplitz determinant $T_3(2)$.

Theorem 2. *Let the function g of the form (1) be in the class $\mathcal{S}^*(m,q,\alpha,\beta)$. Then*

$$\begin{aligned} T_3((2) &\leq 4\beta(1+\alpha)\{\Omega_1+\Omega_2(1+\Omega_3)\} \\ &\quad \times\left(\Omega_4+4\Omega_5+\Omega_7+\Omega_8\left|1-\frac{2\Omega_6}{\Omega_8}\right|\right), \end{aligned}$$

where

$$\Omega_1 = \frac{1}{[2]_q^m-1}, \ \Omega_2=\frac{1}{[4]_q^m-1}, \tag{20}$$

$$\Omega_3 = 2\Lambda_1(\alpha,\beta,m,q)+4\Lambda_2(\alpha,\beta,m,q), \tag{21}$$

$$\Omega_4 = \left(\frac{\beta(1+\alpha)}{2\left([2]_q^m-1\right)}\right)^2, \tag{22}$$

$$\Omega_5 = \left(\frac{\beta(1+\alpha)}{2}\right)^2\left(2\left(\frac{\Lambda_3}{[3]_q^m-1}\right)^2-\frac{\Lambda_2}{\left([2]_q^m-1\right)\left([4]_q^m-1\right)}\right),$$

$$\Omega_6 = \left(\frac{\beta(1+\alpha)}{2}\right)^2\left\{\frac{4\Lambda_3}{\left([3]_q^m-1\right)^2}-\frac{\Lambda_1}{\left([2]_q^m-1\right)\left([4]_q^m-1\right)}\right\},$$

$$\Omega_7 = 2\left(\frac{\beta(1+\alpha)}{2\left([3]_q^m-1\right)}\right)^2, \tag{23}$$

$$\Omega_8 = \left(\frac{\beta(1+\alpha)}{2}\right)^2\left(\frac{1}{\left([2]_q^m-1\right)\left([4]_q^m-1\right)}\right), \tag{24}$$

$$\Lambda_3 = \frac{\alpha\beta-1}{2}+\frac{\beta(1+\alpha)}{2\left([2]_q^m-1\right)}. \tag{25}$$

Proof. As we know that $T_3(2)$ is given by

$$T_3(2)=(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right),$$

where, a_2, a_3, and a_4 are given by (15), (16), and (17).

Presently, if $g\in\mathcal{S}^*(m,q,\alpha,\beta)$, then

$$\begin{aligned} |a_2-a_4| &\leq |a_2|+|a_4|, \\ &\leq \beta(1+\alpha)(\Omega_1+\Omega_2(1+\Omega_3)), \end{aligned} \tag{26}$$

where, Ω_1, Ω_2, Ω_3 are given by (20) and (21).

We need to maximize $\left|a_2^2-2a_3^2+a_2a_4\right|$ for $g\in\mathcal{S}^*(m,q,\alpha,\beta)$, so by writing a_2, a_3, a_4 in terms of c_1, c_2, c_3, with the help of (15)–(17), we obtain

$$\begin{aligned} &\left|a_2^2-2a_3^2+a_2a_4\right| \\ \leq\ &\left|\Omega_4c_1^2-\Omega_5c_1^4-\Omega_6c_1^2c_2-\Omega_7c_2^2+\Omega_8c_1c_3\right|, \\ \leq\ &\Omega_4c_1^2+\Omega_5c_1^4+\Omega_7c_2^2+\Omega_8c_1\left|c_3-\frac{\Omega_6c_1c_2}{\Omega_8}\right|. \end{aligned} \tag{27}$$

Using the Lemmas 1 and 3 along with (26) and (27), we have the required result. □

We take $q \to 1-$, $m = 1$, $\beta = 1$, and $\alpha = 1$, we then have the following corollary proved in [44].

Corollary 1 ([44])**.** *Let the function g of the form (1) be in the class $\mathcal{S}^*$. Then*

$$T_3(2) \leq 84.$$

In Theorem 3, we will investigate the second Hankel determinant $H_2(2)$.

Theorem 3. *Let the function g of the form (1) be in the class $\mathcal{S}^*(m, q, \alpha, \beta)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \left(\frac{\beta(1+\alpha)}{[3]_q^m - 1}\right)^2.$$

Proof. Making use of (15), (16), and (17), we obtain

$$\begin{aligned} & a_2a_4 - a_3^2 \\ = & \frac{\beta^2(1+\alpha)^2}{4}\left\{\begin{array}{c} \Omega_9c_1c_3 + (\Omega_9\Lambda_1 - 2\Omega_{10}\Lambda_3)c_1^2c_2 \\ -\Omega_{10}c_2^2 + \left(\Omega_9\Lambda_2 - \Omega_{10}\Lambda_3^2\right)c_1^4 \end{array}\right\}, \end{aligned}$$

where

$$\Omega_9 = \frac{1}{\left([2]_q^m - 1\right)\left([4]_q^m - 1\right)}, \quad \Omega_{10} = \frac{1}{\left([3]_q^m - 1\right)^2}.$$

By using Lemma 2 and taking $\Upsilon = 4 - c_1^2$ and $\mathcal{Z} = \left(1 - |x|^2\right)z$. Without loss of generality, we assume that $c = c_1$, $(0 \leq c \leq 2)$, so that

$$\begin{aligned} & a_2a_4 - a_3^2 \\ = & \left(\frac{\beta(1+\alpha)}{2}\right)^2\left(\lambda_1c^4 + \lambda_2\Upsilon c^2x - \frac{\Omega_9}{4}\Upsilon c^2x^2 - \frac{\Omega_{10}}{4}\Upsilon^2x^2 + \frac{\Omega_9}{2}\Upsilon c\mathcal{Z}\right), \end{aligned} \tag{28}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\Omega_9}{4} - \frac{\Omega_{10}}{4} + \Omega_9\Lambda_2 - \Omega_{10}\Lambda_3^2 + \frac{\Omega_9\Lambda_1 - 2\Omega_{10}\Lambda_3}{2}, \\ \lambda_2 &= \frac{\Omega_9}{2} + \frac{\Omega_9\Lambda_1 - 2\Omega_{10}\Lambda_3}{2} - \frac{\Omega_{10}}{2}. \end{aligned}$$

Applying for the modulus on both sides of (28) and using a triangle inequality,

$$\begin{aligned} & \left|a_2a_4 - a_3^2\right| \\ \leq & \left(\frac{\beta(1+\alpha)}{2}\right)^2\left(\begin{array}{c} |\lambda_1|c^4 + |\lambda_2|\Upsilon c^2|x| + \left|\frac{\Omega_9}{4}\right|\Upsilon c^2|x|^2 + \left|\frac{\Omega_{10}}{4}\right|\Upsilon^2|x|^2 \\ +\left|\frac{\Omega_9}{2}\right|\left(1 - |x|^2\right)c\Upsilon \end{array}\right) \\ = & \mathcal{Q}(c, |x|). \end{aligned}$$

Since $\mathcal{Q}'(c,|x|) > 0$ on $[0,1]$, $\mathcal{Q}(c,|x|)$ is an increasing function in the interval $[0,1]$, and the maximum value occurs at $x = 1$:

$$Max\,\mathcal{Q}(c,|1|) = \mathcal{Q}(c)$$

and hence

$$\mathcal{Q}(c) = \left(\frac{\beta(1+\alpha)}{2}\right)^2 \left(|\lambda_1|c^4 + |\lambda_2|\Upsilon c^2 + \left|\frac{\Omega_9}{4}\right|\Upsilon c^2 + \left|\frac{\Omega_{10}}{4}\right|\Upsilon^2\right).$$

Putting $\Upsilon = 4 - c_1^2$, after some simplification, we have

$$\mathcal{Q}(c) = \left(\frac{\beta(1+\alpha)}{2}\right)^2 \left(\begin{array}{c} \left(|\lambda_1| - |\lambda_2| - \left|\frac{\Omega_9}{4}\right| + \left|\frac{\Omega_{10}}{4}\right|\right)c^4 \\ +(4|\lambda_2| + |\Omega_9| - 2|\Omega_{10}|)c^2 + 4|\Omega_{10}| \end{array}\right).$$

Let $\mathcal{Q}'(c) = 0$, the optimum value of $\mathcal{Q}(c)$ implies that $c = 0$. So $\mathcal{Q}(c)$ has the maximum value at $c = 0$, which is given by

$$4\left(\frac{\beta(1+\alpha)}{2}\right)^2 |\Omega_{10}|, \tag{29}$$

which occurs at $c = 0$ or

$$c^2 = \frac{(4|\lambda_2| + |\Omega_9| - 2|\Omega_{10}|)}{\left(\frac{\beta(1+\alpha)}{2}\right)^2 \left(|\lambda_1| - |\lambda_2| - \left|\frac{\Omega_9}{4}\right| + \left|\frac{\Omega_{10}}{4}\right|\right)}.$$

By putting $\Omega_{10} = \frac{1}{\left([3]_q^m - 1\right)^2}$ in (29), we obtained the desired result. □

Corollary 2 ([32]). *If an analytic function $g \in \mathcal{S}^*(1,1,1,q \to 1-) = \mathcal{S}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \le 1.$$

3.1. Fekete–Szegö Problem

In this section, we will prove the Fekete-Szegö problem for the class $\mathcal{S}^*(m,q,\alpha,\beta)$ of analytic functions.

Theorem 4. *Let the function g of the form (1) be in the class $\mathcal{S}^*(m,q,\alpha,\beta)$. Then*

$$\left|a_2 - \mu a_2^2\right| \le \begin{cases} \frac{\beta(1+\alpha)}{\left([3]_q^m - 1\right)} \left\{ \begin{array}{c} \left(\frac{\beta(1+\alpha)}{[2]_q^m - 1} + \alpha\beta\right) \\ -\frac{\beta(1+\alpha)\left([3]_q^m - 1\right)}{[2]_q^m - 1}\mu \end{array} \right\}, & \text{if } \rho_1 \le \mu \le \frac{1}{2}, \\ \frac{\beta(1+\alpha)}{\left([3]_q^m - 1\right)}, & \text{if } \frac{1}{2} \le \mu \le \rho_2, \\ \frac{\beta(1+\alpha)}{\left([3]_q^m - 1\right)} \left\{ \begin{array}{c} \frac{\beta(1+\alpha)}{[2]_q^m - 1}\left([3]_q^m - 1\right)\mu \\ -\left(\alpha\beta + \frac{\beta(1+\alpha)}{[2]_q^m - 1}\right), \end{array} \right\} & \text{if } \mu \ge \rho_2, \end{cases} \tag{30}$$

where, ρ_1 and ρ_2 are given by (32) and (33).

Proof. From (15) and (16), we derive

$$\left|a_3 - \mu a_2^2\right| = \frac{\beta(1+\alpha)}{2\left([3]_q^m - 1\right)}\left|(\Lambda_3 - \mu\Lambda_4)c^2 + c_2\right|,$$

where

$$\begin{aligned} \Lambda_3 &= \frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}, \\ \Lambda_4 &= \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\left([3]_q^m - 1\right). \end{aligned} \tag{31}$$

Applying Lemma 2, if $c = c_1$ $(0 \le c \le 2)$, then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &= \frac{\beta(1+\alpha)}{4\left([3]_q^m - 1\right)}\left|(2\Lambda_3 - 2\Lambda_4\mu + 1)c^2 + \left(4 - c^2\right)\rho\right|, \\ &= J(\rho). \end{aligned}$$

Applying the triangle inequality, we deduce

$$\begin{aligned} J(\rho) &\le \frac{\beta(1+\alpha)}{4\left([3]_q^m - 1\right)}\left\{|(2\Lambda_3 - 2\Lambda_4\mu + 1)|c^2 + \left(4 - c^2\right)\right\} \\ &= \frac{\beta(1+\alpha)}{4\left([3]_q^m - 1\right)}\left[\left\{\alpha\beta + \frac{\beta(1+\alpha)}{[2]_q^m - 1} - \mu\left(\frac{\beta(1+\alpha)\left([3]_q^m - 1\right)}{[2]_q^m - 1}\right)\right\}c^2 + \left(4 - c^2\right)\right]. \end{aligned}$$

It follows that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{\beta(1+\alpha)}{4\left([3]_q^m - 1\right)}\left\{Q_1(\alpha,\beta,\mu)c^2 + 4\right\}, & \text{if } \mu \le \rho_1, \\ \\ \frac{\beta(1+\alpha)}{4\left([3]_q^m - 1\right)}\left\{Q_2(\alpha,\beta,\mu)c^2 + 4\right\}, & \text{if } \mu \ge \rho_1. \end{cases}$$

where

$$\begin{aligned} Q_1(\alpha,\beta,\mu) &= \left\{\frac{\beta(1+\alpha)}{[2]_q^m - 1}\left(1 - \left([3]_q^m - 1\right)\mu\right) + \alpha\beta - 1\right\}, \\ Q_2(\alpha,\beta,\mu) &= \left\{\frac{\beta(1+\alpha)}{[2]_q^m - 1}\left([3]_q^m - 1\right)\mu - \left(\alpha\beta - 1 + \frac{\beta(1+\alpha)}{[2]_q^m - 1}\right)\right\}. \end{aligned}$$

Therefore,

$$\rho_1 = \frac{\alpha\beta\left([2]_q^m - 1\right) + \beta(1+\alpha)}{\beta(1+\alpha)\left([3]_q^m - 1\right)}, \tag{32}$$

$$\rho_2 = \frac{(\alpha\beta - 1)\left([2]_q^m - 1\right) + \beta(1+\alpha)}{[3]_q^m - 1}, \tag{33}$$

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\beta(1+\alpha)}{[3]_q^m-1} Q_3(\alpha,\beta,\mu), & \text{if } \mu \leq \frac{1}{2},\ c=2, \\ \frac{\beta(1+\alpha)}{[3]_q^m-1}, & \text{if } \frac{1}{2} \leq \mu \leq \rho_1,\ c=0, \\ \frac{\beta(1+\alpha)}{[3]_q^m-1}, & \text{if } \rho_1 \leq \mu \leq \rho_2,\ c=0, \\ \frac{\beta(1+\alpha)}{[3]_q^m-1} Q_4(\alpha,\beta,\mu) & \text{if } \mu \geq \rho_2,\ c=2. \end{cases} \tag{34}$$

where,

$$Q_3(\alpha,\beta,\mu) = \left\{\frac{\beta(1+\alpha)}{[2]_q^m-1} + \alpha\beta - \frac{\beta(1+\alpha)\left([3]_q^m-1\right)}{[2]_q^m-1}\mu\right\}, \tag{35}$$

$$Q_4(\alpha,\beta,\mu) = \left\{\frac{\beta(1+\alpha)}{[2]_q^m-1}\left([3]_q^m-1\right)\mu - \left(\alpha\beta + \frac{\beta(1+\alpha)}{[2]_q^m-1}\right)\right\}. \tag{36}$$

So we can obtain the required result (30) by using Equations (35) and (36) in inequality (34). □

For $q \to 1-$, $m=1$, $\beta=1$, and $\alpha=1$ in Theorem 4, we thus obtain the following known result.

Corollary 3 ([56]). *Let the function g of the form (1) be in the class $\mathcal{S}^*$. Then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu, & \text{if } \mu \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1, \\ 4\mu-3, & \text{if } \mu \geq 1. \end{cases}$$

Theorem 5. *Let the function g of the form (1) be in the class $\mathcal{S}^*(m,q,\alpha,\beta)$. Then*

$$\left|a_3-\mu a_2^2\right| \leq \frac{\beta(1+\alpha)}{[3]_q^m-1}\max\{1,|2v-1|\},\ \mu\in\mathbb{C}, \tag{37}$$

where

$$v = \frac{\beta(1+\alpha)\left([3]_q^m-1\right)}{2\left([2]_q^m-1\right)^2}\mu - \Lambda_3. \tag{38}$$

Proof. It follows from (15) and (16) that

$$\left|a_3-\mu a_2^2\right| = \frac{\beta(1+\alpha)}{2\left([3]_q^m-1\right)}\left\{c_2 - vc_1^2\right\}, \tag{39}$$

where, Λ_3 is given by (31). Now by using Lemma 3 on (39), we get the required result. □

3.2. Applications

In this section, we provide q-analogue of the Bernardi integral operator to discuss some applications of our main results.

In [57], Noor et al. defined q-analogue of Bernardi integral operator for analytic functions $g \in \mathcal{A}$ as follows:

$$\begin{aligned} \mathcal{B}_{\beta}^{q}(z) &= \frac{[1+\beta]_q}{z^{\beta}} \int_0^z t^{\beta-1} g(t) d_q t \qquad (40) \\ &= z + \sum_{n=2}^{\infty} \frac{[\beta+1]_q}{[n+\beta]_q} a_n z^n, \ z \in E, \ \beta > -1, \\ &= z + \sum_{n=2}^{\infty} \mathcal{B}_n a_n z^n. \qquad (41) \end{aligned}$$

Remark 2. *For $q \to 1-$, we obtain the Bernardi integral operator studied in [58].*

Theorem 6. *Let the function g of the form (1) be in the class $\mathcal{S}^*(m, q, \alpha, \beta)$ and $\mathcal{B}_{\beta}^{q}(z)$ is given by (41). Then*

$$\begin{aligned} |a_2| &\le \frac{\beta(1+\alpha)}{\left([2]_q^m - 1\right)\mathcal{B}_2}, \\ |a_3| &\le \frac{\beta^2(1+\alpha)}{\left([3]_q^m - 1\right)\mathcal{B}_3}\left\{\alpha + \frac{1+\alpha}{\left([2]_q^m - 1\right)\mathcal{B}_2}\right\}, \\ |a_4| &\le \frac{\beta(1+\alpha)}{\left([4]_q^m - 1\right)\mathcal{B}_4}\{1 + 2\mathcal{D}_1(\alpha,\beta,m,q) + 4\mathcal{D}_2(\alpha,\beta,m,q)\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_1(\alpha,\beta,m,q) &= \left\{\alpha\beta - 1 + \frac{\beta(1+\alpha)}{2}\left(\frac{\left([2]_q^m + [3]_q^m - 2\right)\mathcal{B}_3\mathcal{B}_2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)}\right)\right\}, \\ \mathcal{D}_2(\alpha,\beta,m,q) &= \left\{\begin{array}{c} \frac{1}{4} - \frac{\alpha\beta}{2} + \frac{\alpha^2\beta^2}{4} + \frac{\beta(1+\alpha)}{2}\left(\frac{\left([2]_q^m + [3]_q^m - 2\right)\mathcal{B}_3\mathcal{B}_2}{\left([2]_q^m - 1\right)\left([3]_q^m - 1\right)}\right) \\ \times\left(\frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)}\right) - \left(\frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)\mathcal{B}_2}\right)^2 \end{array}\right\}. \end{aligned}$$

Proof. The proof follows easily by using (41) and Theorem 1. □

Theorem 7. *If the function $\mathcal{B}_{\beta}^{q}(z)$ is given by (41) belongs to the class $\mathcal{S}^*(m, q, \alpha, \beta)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \le \left(\frac{\beta(1+\alpha)}{\left([3]_q^m - 1\right)\mathcal{B}_3}\right)^2.$$

Proof. The proof follows easily by using (41) and Theorem 3. □

Theorem 8. *If the function $\mathcal{B}_{\beta}^{q}(z)$ is given by (41) belongs to the class $\mathcal{S}^*(m, q, \alpha, \beta)$. Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{\beta(1+\alpha)}{\left([3]_q^m - 1\right)\mathcal{B}_3}\max\{1, |2v_0 - 1|\}, \ \mu \in \mathbb{C},$$

where

$$\mathcal{D}_3 = \frac{\alpha\beta - 1}{2} + \frac{\beta(1+\alpha)}{2\left([2]_q^m - 1\right)\mathcal{B}_1},$$

$$v_0 = \frac{\beta(1+\alpha)\left([3]_q^m - 1\right)\mathcal{B}_3}{2\left([2]_q^m - 1\right)^2\mathcal{B}_1^2}\mu - \mathcal{D}_3.$$

Proof. The proof follows easily by using (41) and Theorem 5. □

4. Conclusions

The work presented in this paper is motivated by the well-established usage of the basic (or q-) calculus in the context of Geometric Function Theory. For this class, we investigated Hankel determinants, Toeplitz matrices and Fekete-Szegö problems. Moreover, the q-Bernardi integral operator is used to discuss some applications of the main results of this paper. Moreover, for validity of our results, the relevant connections with those in earlier works are also pointed out.

In a review article [15], Srivastava explained that (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus and the additional parameter p being redundant or superfluous (for detail see [37], p. 340). According to this observation of Srivastava [15] will indeed apply to any attempt to produce the rather straightforward and inconsequential (p,q)-variations of the results, which we have proved in this paper.

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