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# Hankel Determinant for Starlike and Convex Functions

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**Abstract**

Denote $\mathcal{S}$ to be the class of functions which are analytic, normalised and univalent in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$. The important subclasses of $\mathcal{S}$ are the class of starlike and convex functions, which we denote by $\mathcal{S}^\star$ and $\mathcal{C}$. This paper focuses on attaining sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to $\mathcal{S}^\star$ and $\mathcal{C}$.

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# 1 Introduction

Let $\mathcal{S}$ denote the class of normalised analytic univalent functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where $z \in \mathcal{D} = \{z : |z| < 1\}$. In [5], the $qth$ Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor in [6] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1) with bounded boundary. Ehrenborg in [1] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [4].

Easily, one can observe that the Fekete and Szegö functional is $\mathcal{H}_2(1)$. Fekete and Szegö then further generalised the estimate $|a_3 - \mu a_2^2|$ where $\mu$ is real and $f \in \mathcal{S}$. For our discussion in this paper, we consider the Hankel determinant in the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

We seek upper bound for the functional $|a_2 a_4 - a_3^2|$ for functions $f$ belongs to the class $\mathcal{S}^\star$ and $\mathcal{C}$. The class $\mathcal{S}^\star$ and $\mathcal{C}$ are defined as follows.

**Definition 1.1** *Let $f$ be given by (1). Then $f \in \mathcal{S}^\star$ if and only if*

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathcal{D}. \tag{2}$$

**Definition 1.2** *Let $f$ be given by (1). Then $f \in \mathcal{C}$ if and only if*

$$Re \left\{ \frac{(zf'(z))'}{f'(z)} \right\} > 0, \quad z \in \mathcal{D}. \tag{3}$$

It follows that $f \in C$ if and only if $zf'(z) \in S^\star$.

First, some preliminary lemmas.

## 2 Preliminary Results

Let $\mathcal{P}$ be the family of all functions $p$ analytic in $\mathcal{D}$ for which $Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{4}$$

for $z \in \mathcal{D}$.

**Lemma 2.1** *([7]) If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each $k$.*

**Lemma 2.2** *([3]) The power series for $p$ given in (4) converges in $\mathcal{D}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 \dots & c_n \\ c_{-1} & 2 & c_1 \dots & c_{n-1} \\ \vdots & & & \\ c_{-n} & c_{-n+1} & c_{-n+2} \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, ... \tag{5}$$

*and $c_{-k} = \bar{c}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_o(e^{it_k} z),\ \rho_k > 0,\ t_k$ real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.*

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [3].

## 3 Main Result

**Theorem 3.1** *Let $f \in \mathcal{S}^\star$. Then*

$$|a_2 a_4 - a_3^2| \leq 1.$$

*The result obtained is sharp.*

**Proof.**

Since $f \in \mathcal{S}^\star$, it follows from (2) that $\exists p \in \mathcal{P}$ such that

$$zf'(z) = f(z)p(z) \tag{6}$$

for some $z \in \mathcal{D}$. Equating coefficients in (6) yields

$$\left.\begin{aligned} a_2 &= c_1 \\ a_3 &= \tfrac{c_2}{2} + \tfrac{c_1^2}{2} \\ a_4 &= \tfrac{c_3}{3} + \tfrac{c_1 c_2}{2} + \tfrac{c_1^3}{6} \end{aligned}\right\}. \tag{7}$$

From (7), it is easily established that

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12}\right|. \tag{8}$$

Lemma 2.2 can then be used to obtain the proper bound on (8). We may assume without restriction that $c_1 \geq 0$. Rewriting (5) for the cases n=2 and n=3, result in

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c}_2 & c_1 & 2 \end{vmatrix} = 8 + 2\ Re\{c_1^2c_2\} - 2|c_2|^2 - 4c_1^2 \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{9}$$

for some $x, |x| \leq 1$.

Further, $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2\ ;$$

and this, with (9), provides the relation

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{10}$$

for some value of $z$, $|z| \leq 1$.

Suppose now that $c_1 = c$ and $0 \leq c \leq 2$. Using (9) along with (10), we obtain

$$\begin{aligned} \left|\frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12}\right| &= \Big|\frac{(4 - c^2)c^2x}{24} - \frac{c^4}{16} \\ &\quad + \frac{(4 - c^2)(1 - |x|^2)cz}{6} - \frac{(4 - c^2)x^2(12 + c^2)}{48}\Big|. \end{aligned}$$

Application of the triangle inequality gives

$$\begin{aligned} \left|\frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12}\right| &\leq \frac{c^4}{16} + \frac{c(4 - c^2)}{6} + \frac{c^2(4 - c^2)\rho}{24} \\ &\quad + \frac{(4 - c^2)(c - 2)(c - 6)\rho^2}{48} \\ &= F(\rho) \end{aligned} \tag{11}$$

with $\rho = |x| \leq 1$. Furthermore,

$$F'(\rho) = \frac{c^2(4 - c^2)}{24} + \frac{(4 - c^2)(c - 2)(c - 6)\rho}{24}$$

and with elementary calculus, one can show that $F'(\rho) > 0$ for $\rho > 0$; implying that $F$ is an increasing function and thus the upper bound for (11) corresponds to $\rho = 1$, in which case

$$\left|\frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12}\right| \leq 1$$

for all $c \in [0,2]$. Equality is attained for functions in $\mathcal{S}^\star$ given by

$$\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$$

and

$$\frac{zf'(z)}{f(z)} = \frac{1+z^2}{1-z^2}\,.$$

This completes the proof of theorem 3.1.

**Theorem 3.2** *Let $f \in \mathcal{C}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

*The result obtained is sharp.*

**Proof.**

Similar approach as in the proof of Theorem 3.1. Since $f \in \mathcal{C}$, it follows from (3) that $\exists p \in \mathcal{P}$ such that

$$(zf'(z))' = f'(z)p(z) \tag{12}$$

for some $z \in \mathcal{D}$. Equating coefficients in (12) yields

$$\left.\begin{array}{r} a_2 = \frac{c_1}{2} \\ \\ a_3 = \frac{c_2}{6} + \frac{c_1^2}{6} \\ \\ a_4 = \frac{c_3}{12} + \frac{c_1c_2}{8} + \frac{c_1^3}{24} \end{array}\right\}. \tag{13}$$

From (13), it is easily established that

$$|a_2a_4 - a_3^2| = \frac{1}{144}\left|6c_1c_3 + c_1^2c_2 - 4c_2^2 - c_1^4\right|. \tag{14}$$

Now, assuming $c_1 = c(0 \leq c \leq 2)$ and using (9) together with (10) we have

$$\left|6c_1c_3 + c_1^2c_2 - 4c_2^2 - c_1^4\right| = \left|\frac{3c^2(4-c^2)x}{2} - \frac{(4-c^2)(8+c^2)x^2}{2} + 3c(4-c^2)(1-|x|^2)z\right|$$

and an application of the triangle inequality shows that

$$\begin{aligned} \left|6c_1c_3 + c_1^2c_2 - 4c_2^2 - c_1^4\right| &\leq 3c(4-c^2) + \frac{3c^2(4-c^2)\rho}{2} \\ &+ \frac{(4-c^2)(c-2)(c-4)\rho^2}{2} \\ &= F(\rho) \end{aligned} \tag{15}$$

with $\rho = |x| \leq 1$. For

$$F'(\rho) = \frac{3c^2(4-c^2)}{2} + (c-2)(c-4)(4-c^2)\rho,$$

it can be shown that $F'(\rho) > 0$ and thus is an increasing function implying $Max_{\rho\leq 1}\, F(\rho) = F(1)$. Now let

$$\begin{aligned} G(c) &= F(1) \\ &= 3c(4-c^2) + \frac{3c^2(4-c^2)}{2} + \frac{(4-c^2)(c-2)(c-4)}{2}. \end{aligned}$$

Trivially, one can show that $G$ has a maximum attained at $c = 1$.The upper bound for (15) corresponds to $\rho = 1$ and $c = 1$, in which case

$$\left|6c_1c_3 + c_1^2c_2 - 4c_2^2 - c_1^4\right| \leq 18.$$

Letting $c_1 = 1,\ c_2 = -1$ and $c_3 = -2$ in (14) shows that the result is sharp. This completes the proof of Theorem 3.2.

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