Tohoku Math. J.
58 (2006), 475–491

HANKEL OPERATORS ON HARMONIC BERGMAN SPACES

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(Received February 16, 2005, revised October 13, 2005)

Abstract. We study Hankel operators on the harmonic Bergman spaces on bounded smooth domains, and obtain a necessary and sufficient condition for Hankel operators to be bounded or compact on both harmonic Bergman space and its dual space.

1. Introduction. Let Ω be a bounded domain with C^∞-boundary in $\boldsymbol{R}^n$, $n \geq 2$, and V be the Lebesgue measure on $\boldsymbol{R}^n$. For $1 \leq p < \infty$, the L^p harmonic Bergman space $b^p = b^p(\Omega)$ is the set of all complex-valued harmonic functions u on Ω for which

$$\|u\|_p = \left(\int_\Omega |u|^p dV \right)^{1/p} < \infty .$$

Also, by b^∞ we denote the space of all bounded harmonic functions on Ω. It is known that b^∞ is dense in each b^p.

As is well-known, b^p is a closed subspace of $L^p = L^p(\Omega, V)$ and hence a Banach space. In particular, b^2 is a Hilbert space. Each point evaluation is a bounded linear functional on b^2. Hence, for each $x \in \Omega$, there exists a unique function $R(x, \cdot) \in b^2$ having the following reproducing property:

$$f(x) = \int_\Omega f(y)\overline{R(x, y)}dy$$

for all $f \in b^2$, where $dy = dV(y)$. The reproducing kernels $R(x, \cdot)$ are known to be symmetric and real-valued. Let Q be the Hilbert space orthogonal projection from L^2 onto b^2. Then, the following integral formula holds:

$$Q[f](x) = \int_\Omega R(x, y)f(y)dy, \quad x \in \Omega \tag{1.1}$$

for all $f \in L^2$. For each fixed $x \in \Omega$, the function $R(x, \cdot)$ is known to be bounded on Ω. Thus, the operator Q defined by (1.1) extends to an integral operator from L^1 into the space of all harmonic functions on Ω. Moreover, for $1 < p < \infty$, it is known that Q is a bounded projection from L^p onto b^p.

Let $1 < p < \infty$ and $f \in L^1$. The *Hankel operator* H_f with symbol f is densely defined on b^p by

$$H_f u = (I - Q)M_f u \tag{1.2}$$

for $u \in b^\infty$, where M_f is the multiplication operator defined by $M_f g = fg$.

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 47B32.
Key words and phrases. Hankel operators, harmonic Bergman spaces.

Let $f \in L^1$. The *commutator* with symbol f is defined by $C_f = M_f Q - Q M_f$. If $g \in L^\infty$, then it is easy to see that $C_f g$ is well-defined. Since L^∞ is dense in every L^p, C_f is densely defined on L^p for each $1 < p < \infty$. As we will see, there is a close relationship between Hankel operators and commutators.

In this paper, we study Hankel operators on harmonic Bergman spaces b^p defined on a bounded smooth domain in $\boldsymbol{R}^n$ for $1 < p < \infty$. We present a necessary and sufficient condition for H_f to be bounded or compact on both b^p and its dual space. The results of this paper extend those in [6] on the unit ball to general bounded smooth domains in $\boldsymbol{R}^n$.

This paper is organized as follows. In Section 2, we state our main results. In Section 3, we collect some preliminary results that we will need. In the last section, we prove our main result.

The author would like to thank Professor Hitoshi Arai for his help, encouragement, and advice. He also thanks Dai Wakisaka for useful discussions during the preparation of this paper.

NOTATION. Throughout the paper, the exponent p' will always denote the conjugate exponent of p, i.e., $1/p + 1/p' = 1$ for $1 < p < \infty$. χ_S denotes the characteristic function of a set $S \subset \boldsymbol{R}^n$. We also use the notation $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main results. Let $1 \leq p < \infty$ and $\delta \in (0, 1)$. For $x \in \Omega$, let $r(x) = \operatorname{dist}(x, \partial\Omega)$ and

$$E_\delta(x) = \{y \in \Omega\ ;\ |y - x| < \delta r(x)\}\,.$$

Since $\delta < 1$, $E_\delta(x)$ is actually the euclidean ball with center at x and radius $\delta r(x)$.

For $f \in L^p$, we define

$$\widehat{f_\delta}(x) = \frac{1}{V(E_\delta(x))} \int_{E_\delta(x)} f(y)dy\,,$$

$$\mathrm{MV}_\delta^p(f; x) = \frac{1}{V(E_\delta(x))} \int_{E_\delta(x)} |f(y)|^p dy\,,$$

$$\mathrm{MO}_\delta^p(f; x) = \frac{1}{V(E_\delta(x))} \int_{E_\delta(x)} |f(y) - \widehat{f_\delta}(x)|^p dy\,.$$

The *Bloch space* $\mathcal{B}$ and *little Bloch space* $\mathcal{B}_0$ are defined by

$$\mathcal{B} = \{f \in C^1(\Omega)\ ;\ \sup_{x \in \Omega} r(x)|\nabla f(x)| < \infty\}\,,$$

$$\mathcal{B}_0 = \{f \in C^1(\Omega)\ ;\ r(x)|\nabla f(x)| \to 0 \text{ as } x \to \partial\Omega\}\,.$$

The space BM_δ^p and its subspace VM_δ^p are defined by

$$\mathrm{BM}_\delta^p = \{f \in L^p\ ;\ \sup_{x \in \Omega} \mathrm{MV}_\delta^p(f; x) < \infty\}\,,$$

$$\mathrm{VM}_\delta^p = \{f \in L^p\ ;\ \mathrm{MV}_\delta^p(f; x) \to 0 \text{ as } x \to \partial\Omega\}\,.$$

Since $\mathrm{MV}^p_\delta(f;x) = \widehat{(|f|^p)}_\delta(x)$, Theorem 3.5 and Theorem 3.11 of [2] indicate that BM^p_δ and VM^p_δ are independent of the choice of δ. So, we may drop δ and simply write $\mathrm{BM}^p = \mathrm{BM}^p_\delta$ and $\mathrm{VM}^p = \mathrm{VM}^p_\delta$.

The space BMO^p_δ and its subspace VMO^p_δ are defined by

$$\mathrm{BMO}^p_\delta = \{f \in L^p\ ;\ \sup_{x\in\Omega} \mathrm{MO}^p_\delta(f;x) < \infty\},$$
$$\mathrm{VMO}^p_\delta = \{f \in L^p\ ;\ \mathrm{MO}^p_\delta(f;x) \to 0 \text{ as } x \to \partial\Omega\}.$$

We will see later that BMO^p_δ and VMO^p_δ are independent of the choice of δ. Therefore δ will be dropped in the future references to these two spaces.

Let $1 \le p < q < \infty$. A simple computation using Hölder's inequality gives

$$\mathrm{MV}^p_\delta(f;x)^{1/p} \le \mathrm{MV}^q_\delta(f;x)^{1/q}, \quad \mathrm{MO}^p_\delta(f;x)^{1/p} \le \mathrm{MO}^q_\delta(f;x)^{1/q}. \tag{2.1}$$

Thus, we have

$$\mathrm{BM}^q \subset \mathrm{BM}^p, \quad \mathrm{VM}^q \subset \mathrm{VM}^p, \quad \mathrm{BMO}^q \subset \mathrm{BMO}^p, \quad \mathrm{VMO}^q \subset \mathrm{VMO}^p.$$

Furthermore, it is easy to see that these inclusions are proper. For example, if f is a function with compact support in Ω such that f is in L^p but not in L^q, then f is in VMO^p but not in VMO^q.

The main result of this paper is the following theorem, which extends the results obtained by J. Miao in [6].

THEOREM 2.1. *Let $p \in [2,\infty)$ and $f \in L^p$.*
(a) *H_f is bounded on both b^p and $b^{p'}$ if and only if $f \in \mathrm{BMO}^p$.*
(b) *H_f is compact on both b^p and $b^{p'}$ if and only if $f \in \mathrm{VMO}^p$.*

The following two corollaries are immediate consequences of the theorem above.

COROLLARY 2.2. *Let $f \in L^2$.*
(a) *H_f is bounded on b^2 if and only if $f \in \mathrm{BMO}^2$.*
(b) *H_f is compact on b^2 if and only if $f \in \mathrm{VMO}^2$.*

COROLLARY 2.3. *Let $p \in [2,\infty)$ and $f \in b^p$.*
(a) *H_f is bounded on both b^p and $b^{p'}$ if and only if $f \in \mathcal{B}$.*
(b) *H_f is compact on both b^p and $b^{p'}$ if and only if $f \in \mathcal{B}_0$.*

3. Lemmas. Recall that $r(x) = \mathrm{dist}(x,\partial\Omega)$ for $x \in \Omega$. For $\varepsilon > 0$, we set

$$\Omega_\varepsilon = \{y \in \Omega\ ;\ r(y) \ge \varepsilon\},$$

and $D_\varepsilon = \Omega \setminus \Omega_\varepsilon$. Let π be the normal projection to $\partial\Omega$, namely, for $x \in \Omega$ near $\partial\Omega$, $\pi(x)$ is the closest point of $\partial\Omega$ to x. Then the smoothness of the boundary $\partial\Omega$ implies that there exists $\varepsilon_0 > 0$ such that the following hold.

(a) r is a smooth function on D_{ε_0}.
(b) The projection $\pi : D_{\varepsilon_0} \to \partial\Omega$ is well-defined and smooth.

(c) For $t > 0$ with $t \leq \varepsilon_0$, the projection $\pi|_{\partial\Omega_t} : \partial\Omega_t \to \partial\Omega$ is one-to-one and onto, and $\eta \in \partial\Omega_t$ can be written as $\eta = \pi(\eta) + t\boldsymbol{n}_{\pi(\eta)}$. Here and elsewhere, $\boldsymbol{n}_\zeta$ denotes the inward unit normal to $\partial\Omega$ at $\zeta \in \partial\Omega$.

(d) $\nabla r(\eta) = \boldsymbol{n}_{\pi(\eta)}$ for $\eta \in D_{\varepsilon_0}$.

(e) For all $0 < \varepsilon \leq \varepsilon_0$ and nonnegative continuous functions f on D_ε,

$$\int_{D_\varepsilon} f(x)dx \approx \int_{\partial\Omega} \int_0^\varepsilon f(\zeta + t\boldsymbol{n}_\zeta)dt d\sigma(\zeta)\,, \tag{3.1}$$

where σ denotes the surface area measure on $\partial\Omega$.
We refer to [5] and [3] for more information and proofs.

LEMMA 3.1. *Let $\delta \in (0, 1)$. Then we have*

$$(1-\delta)r(x) < r(y) < (1+\delta)r(x) \tag{3.2}$$

for all $x \in \Omega$ and $y \in E_\delta(x)$.

PROOF. See Lemma 3.1 of [2]. □

LEMMA 3.2. *Let $\delta \in (0, 1)$ and $x \in \Omega$. If $y \in E_{\delta/3}(x)$, then $E_{\delta/3}(y) \subset E_\delta(x)$ and $E_{\delta/3}(x) \subset E_\delta(y)$.*

PROOF. The proof is essentially the same as that of Lemma 5 of [6]. □

LEMMA 3.3. (a) *There is a constant C_0 depending only on Ω such that*

$$C_0^{-1} \leq R(x, x)r(x)^n \leq C_0 \tag{3.3}$$

for all $x \in \Omega$.

(b) *Let $\delta \in (0, 1)$. Then there is a constant C_1 depending only on Ω such that*

$$\frac{|R(y, z) - R(x, x)|}{|R(x, x)|} \leq \frac{C_1\delta}{(1-\delta)^{n+1}} \tag{3.4}$$

for all $x \in \Omega$ and $y, z \in E_\delta(x)$.

PROOF. Part (a) is an easy consequence of Theorem 1.1 of [4]. Now we prove (b). By Theorem 1.1 of [4], there is a constant C such that

$$|\nabla_y R(y, z)| \leq \frac{C}{d(y, z)^{n+1}} \leq \frac{C}{r(y)^{n+1}}\,,$$
$$|\nabla_z R(y, z)| \leq \frac{C}{d(y, z)^{n+1}} \leq \frac{C}{r(y)^{n+1}}$$

for all $y, z \in \Omega$, where $d(y, z) = |y - z| + r(y) + r(z)$. For $y \in E_\delta(x)$, (3.2) shows that $r(y) > (1-\delta)r(x)$. Thus, for $y, z \in E_\delta(x)$, we have

$$|\nabla_y R(y, z)| \leq \frac{C}{r(y)^{n+1}} \leq \frac{C}{(1-\delta)^{n+1}r(x)^{n+1}}\,,$$
$$|\nabla_z R(y, z)| \leq \frac{C}{r(y)^{n+1}} \leq \frac{C}{(1-\delta)^{n+1}r(x)^{n+1}}\,.$$

If $y, z \in E_\delta(x)$, then Mean Value Theorem gives

$$
\begin{aligned}
|R(y,z) - R(x,x)| &\le \sup_{u,v \in E_\delta(x)} (|\nabla_u R(u,v)||y-x| + |\nabla_v R(u,v)||z-x|) \\
&\le \frac{2C\delta}{(1-\delta)^{n+1} r(x)^n} .
\end{aligned}
$$

Combining this with (a), we obtain (b). □

LEMMA 3.4. *Let $\delta \in (0,1)$ and $p \in [1,\infty)$. If $f \in L^p$, then*

$$
\mathrm{MO}_\delta^p(f;x) \le \frac{1}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} |f(y) - f(z)|^p dz dy \le 2^p \mathrm{MO}_\delta^p(f;x) \tag{3.5}
$$

for all $x \in \Omega$.

PROOF. For every $y, z \in \Omega$,

$$
|f(y) - f(z)| \le |f(y) - \widehat{f_\delta}(x)| + |f(z) - \widehat{f_\delta}(x)| ,
$$

and therefore

$$
|f(y) - f(z)|^p \le 2^{p-1}(|f(y) - \widehat{f_\delta}(x)|^p + |f(z) - \widehat{f_\delta}(x)|^p) .
$$

Thus

$$
\begin{aligned}
&\frac{1}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} |f(y) - f(z)|^p dz dy \\
&\quad \le \frac{2^{p-1}}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} (|f(y) - \widehat{f_\delta}(x)|^p + |f(z) - \widehat{f_\delta}(x)|^p) dz dy \\
&\quad = \frac{2^{p-1}}{V(E_\delta(x))^2} \int_{E_\delta(x)} |f(y) - \widehat{f_\delta}(x)|^p dy \int_{E_\delta(x)} dz \times 2 = 2^p\, \mathrm{MO}_\delta^p(f;x) .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathrm{MO}_\delta^p(f;x) &= \frac{1}{V(E_\delta(x))} \int_{E_\delta(x)} \left| f(y) - \frac{1}{V(E_\delta(x))} \int_{E_\delta(x)} f(z) dz \right|^p dy \\
&\le \frac{1}{V(E_\delta(x))^{1+p}} \int_{E_\delta(x)} \left(\int_{E_\delta(x)} |f(y) - f(z)| dz \right)^p dy .
\end{aligned}
$$

Applying Hölder's inequality, we get

$$
\mathrm{MO}_\delta^p(f;x) \le \frac{1}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} |f(y) - f(z)|^p dz dy . \qquad \square
$$

REMARK. If $p = 2$, then it follows from a direct computation that

$$
\frac{1}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} |f(y) - f(z)|^2 dz dy = 2\, \mathrm{MO}_\delta^2(f;x)
$$

for all $x \in \Omega$.

LEMMA 3.5. *Let $p \in [1,\infty)$ and $\delta \in (0,1)$. Then $\mathcal{B} \subset \mathrm{BMO}_\delta^p$ and $\mathcal{B}_0 \subset \mathrm{VMO}_\delta^p$.*

PROOF. Suppose $f \in \mathcal{B}$ and $x \in \Omega$. For $y \in E_\delta(x)$, it follows from Mean Value Theorem that

$$\begin{aligned}|f(y)-f(x)| &\le \left(\sup_{z\in E_\delta(x)} |\nabla f(z)|\right)|y-x| \le \left(\sup_{z\in E_\delta(x)} r(z)|\nabla f(z)|\right)\frac{|y-x|}{(1-\delta)r(x)}\\ &\le \frac{\delta}{1-\delta}\left(\sup_{z\in E_\delta(x)} r(z)|\nabla f(z)|\right).\end{aligned}$$

The second inequality above comes from (3.2). It is easy to see that

$$\begin{aligned}\mathrm{MO}_\delta^p(f;x) &\le \frac{1}{V(E_\delta(x))^2}\int_{E_\delta(x)}\int_{E_\delta(x)} |f(y)-f(z)|^p dz dy\\ &\le \frac{2^{p-1}}{V(E_\delta(x))^2}\int_{E_\delta(x)}\int_{E_\delta(x)} (|f(y)-f(x)|^p+|f(z)-f(x)|^p) dz dy\\ &= \frac{2^p}{V(E_\delta(x))}\int_{E_\delta(x)} |f(y)-f(x)|^p dy\\ &\le \left(\frac{2\delta}{1-\delta}\right)^p\left(\sup_{z\in E_\delta(x)} r(z)|\nabla f(z)|\right)^p.\end{aligned}$$

This shows that $f \in \mathrm{BMO}_\delta^p$ as desired.

Suppose $f \in \mathcal{B}_0$. Then, for any $\varepsilon > 0$, there exists $\rho > 0$ such that $r(z)|\nabla f(z)| < \varepsilon$ for all $z \in \Omega$ with $r(z) < \rho$. For $x \in \Omega$ with $r(x) < \rho/(1+\delta)$, we obtain by (3.2)

$$r(z) < (1+\delta)r(x) < \rho \quad \text{for } z \in E_\delta(x).$$

Therefore we have

$$\sup_{z\in E_\delta(x)} r(z)|\nabla f(z)| < \varepsilon$$

for all $x \in \Omega$ with $r(x) < \rho/(1+\delta)$, which implies that

$$\mathrm{MO}_\delta^p(f;x) \le \left(\frac{2\delta}{1-\delta}\right)^p\left(\sup_{z\in E_\delta(x)} r(z)|\nabla f(z)|\right)^p \to 0$$

as $x \to \partial\Omega$. Thus $f \in \mathrm{VMO}_\delta^p$ and we are done. □

LEMMA 3.6. *Let* $p \in [1,\infty)$ *and* $\delta \in (0,1)$. *Then* $\mathrm{BM}^p \subset \mathrm{BMO}_\delta^p$ *and* $\mathrm{VM}^p \subset \mathrm{VMO}_\delta^p$.

PROOF. Suppose $f \in \mathrm{BM}^p$. By Hölder's inequality, we have

$$\begin{aligned}|\widehat{f_\delta}(x)|^p &= \left|\frac{1}{V(E_\delta(x))}\int_{E_\delta(x)} f(y)dy\right|^p\\ &\le \frac{1}{V(E_\delta(x))^p}\left(\int_{E_\delta(x)} |f(y)|^p dy\right)\left(\int_{E_\delta(x)} dy\right)^{p-1}\\ &= \frac{1}{V(E_\delta(x))}\int_{E_\delta(x)} |f(y)|^p dy = \mathrm{MV}_\delta^p(f;x).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{MO}^p_\delta(f;x) &= \frac{1}{V(E_\delta(x))}\int_{E_\delta(x)} |f(y)-\widehat{f_\delta}(x)|^p dy \\ &\le \frac{2^{p-1}}{V(E_\delta(x))}\int_{E_\delta(x)} (|f(y)|^p + |\widehat{f_\delta}(x)|^p) dy \\ &\le 2^{p-1}\,(\mathrm{MV}^p_\delta(f;x) + |\widehat{f_\delta}(x)|^p) \le 2^p\,\mathrm{MV}^p_\delta(f;x)\,.\end{aligned}$$

This shows that $\mathrm{BM}^p \subset \mathrm{BMO}^p_\delta$ and $\mathrm{VM}^p \subset \mathrm{VMO}^p_\delta$. □

We have shown that $\mathcal{B} + \mathrm{BM}^p \subset \mathrm{BMO}^p_\delta$ and $\mathcal{B}_0 + \mathrm{VM}^p \subset \mathrm{VMO}^p_\delta$. We wish to show the converse inclusions $\mathrm{BMO}^p_\delta \subset \mathcal{B} + \mathrm{BM}^p$ and $\mathrm{VMO}^p_\delta \subset \mathcal{B}_0 + \mathrm{VM}^p$. This also means that BMO^p_δ and VMO^p_δ are independent of the choice of δ. To prove this, we need the following lemma.

LEMMA 3.7. *Let $\delta \in (0,1)$. Then there exists a smooth nonnegative function ψ on $\Omega \times \Omega$ which satisfies the following conditions*:

(a) *For each $x \in \Omega$, $\psi(x,y) = 0$ if $y \notin E_{\delta/3}(x)$ and*

$$\int_\Omega \psi(x,y)dy = 1\,. \tag{3.6}$$

(b) *There are constants C_0, C_1 depending only on Ω and δ such that*

$$|\psi(x,y)| \le C_0\, r(x)^{-n}\,, \tag{3.7}$$

$$|\nabla_x \psi(x,y)| \le C_1\, r(x)^{-n-1} \tag{3.8}$$

for all $x, y \in \Omega$.

To construct a function satisfying the above lemma, we need a smooth defining function by which the distance function r is bounded from above and below. Let ρ be a smooth defining function for Ω such that $\rho(x) = r(x)$ for $x \in \Omega$ close enough to $\partial\Omega$ (see Section 1.2 of [5]). Then it is easy to see that there exists a constant R such that

$$R^{-1} \le \frac{\rho(x)}{r(x)} \le R \quad \text{for all } x \in \Omega\,.$$

We can also take a constant M satisfying $|\nabla\rho(x)| \le M$ for all $x \in \Omega$.

PROOF OF LEMMA 3.7. Let $\phi \in C_0^\infty(\boldsymbol{R}^n)$ be a nonnegative function on $\boldsymbol{R}^n$ with support inside $B(0,\delta/3) = \{y \in \boldsymbol{R}^n\ ;\ |y| < \delta/3\}$ such that $\int_{\boldsymbol{R}^n} \phi dV = 1$. For $x, y \in \Omega$, we define

$$\psi(x,y) = \left(\frac{R}{\rho(x)}\right)^n \phi\left(\frac{R(y-x)}{\rho(x)}\right). \tag{3.9}$$

We are going to prove that ψ thus defined satisfies (a) and (b).

Let $x \in \Omega$. If $y \notin E_{\delta/3}(x)$, then

$$|y-x| \ge \frac{\delta}{3} r(x) \ge \frac{\delta}{3R}\rho(x), \quad \text{and we have} \quad \frac{R|y-x|}{\rho(x)} \ge \frac{\delta}{3}\,.$$

Since $\operatorname{supp}\phi \subset B(0, \delta/3)$, it follows that $\psi(x, y) = 0$ if $y \notin E_{\delta/3}(x)$. Next, by change of variables, we get

$$\int_{\Omega} \psi(x, y)dy = \int_{\boldsymbol{R}^n} \phi dV = 1\,.$$

To prove (b), let

$$C(\phi) = \sup_{z\in\boldsymbol{R}^n} \max\{|\phi(z)|, |D_1\phi(z)|, \dots, |D_n\phi(z)|\}\,.$$

Then

$$|\psi(x, y)| \le C(\phi)\left(\frac{R}{r(x)}\right)^n,$$

and we get (3.7). Note that for $x \in \Omega$ and $y \in E_{\delta/3}(x)$,

$$\nabla_x\psi(x, y) = \left(\frac{R}{\rho(x)}\right)^n \nabla_x\left(\phi\left(\frac{R(y-x)}{\rho(x)}\right)\right) + R^n\nabla_x(\rho(x)^{-n})\phi\left(\frac{R(y-x)}{\rho(x)}\right).$$

Since $\nabla(\rho(x)^{-n}) = -n\rho(x)^{-n-1}\nabla\rho(x)$, we have

$$\begin{aligned} R^n\left|\nabla(\rho(x)^{-n})\phi\left(\frac{R(y-x)}{\rho(x)}\right)\right| &\le nR^nC(\phi)\left(\sup_{z\in\Omega}|\nabla\rho(z)|\right)\rho(x)^{-n-1} \\ &\le nR^{2n+1}C(\phi)Mr(x)^{-n-1}\,. \end{aligned}$$

For each $j = 1, 2, \dots, n$, the chain rule then gives

$$\left|\frac{\partial}{\partial x_j}\phi\left(\frac{R(y-x)}{\rho(x)}\right)\right| \le \sum_{k=1}^n \left|(D_k\phi)\left(\frac{R(y-x)}{\rho(x)}\right)\right|\left|\frac{\partial}{\partial x_j}\left(\frac{R(y_k-x_k)}{\rho(x)}\right)\right|.$$

Since

$$\begin{aligned} \left|\frac{\partial}{\partial x_j}\left(\frac{R(y_k-x_k)}{\rho(x)}\right)\right| &\le \frac{R}{\rho(x)}\left|\frac{\partial x_k}{\partial x_j}\right| + R\,|y-x|\rho(x)^{-2}\left|\frac{\partial\rho}{\partial x_j}(x)\right| \\ &\le \frac{R}{\rho(x)} + RM\frac{|y-x|}{\rho(x)^2}\,, \end{aligned}$$

we obtain

$$\begin{aligned} \left|\frac{\partial}{\partial x_j}\phi\left(\frac{R(y-x)}{\rho(x)}\right)\right| &\le \sum_{k=1}^n \left|(D_k\phi)\left(\frac{R(y-x)}{\rho(x)}\right)\right|\left(\frac{R}{\rho(x)} + RM\frac{|y-x|}{\rho(x)^2}\right) \\ &\le nC(\phi)\left(R + R^2M\frac{\delta}{3}\right)\frac{1}{\rho(x)} \le nC(\phi)\left(R + R^2M\frac{\delta}{3}\right)\frac{R}{r(x)}\,. \end{aligned}$$

The second inequality follows from the fact that $(D_k\phi)(R(y-x)/\rho(x)) = 0$ if $|y-x| \ge (\delta\rho(x))/(3R)$. Thus,

$$\left|\nabla_x\phi\left(\frac{R(y-x)}{\rho(x)}\right)\right| \le n\sqrt{n}\,C(\phi)\left(R + R^2M\frac{\delta}{3}\right)\frac{R}{r(x)}\,.$$

Therefore, we obtain

$$
\begin{aligned}
|\nabla_x \psi(x,y)| &\le \left(\frac{R^2}{r(x)}\right)^n n\sqrt{n}\, C(\phi)\left(R + R^2 M\frac{\delta}{3}\right)\frac{R}{r(x)} + nR^{2n+1}C(\phi)M\, r(x)^{-n-1} \\
&= nR^{2n+1}C(\phi)\left(\sqrt{n}\left(R + R^2 M\frac{\delta}{3}\right) + M\right) r(x)^{-n-1},
\end{aligned}
$$

and we are done. □

Now, we can prove the following lemma.

LEMMA 3.8. *Let* $p \in [1,\infty)$ *and* $\delta \in (0,1)$. *Then*
(a) $\mathrm{BMO}^p_\delta = \mathcal{B} + \mathrm{BM}^p$,
(b) $\mathrm{VMO}^p_\delta = \mathcal{B}_0 + \mathrm{VM}^p$.

PROOF. We have already seen that $\mathcal{B} + \mathrm{BM}^p \subset \mathrm{BMO}^p_\delta$ and $\mathcal{B}_0 + \mathrm{VM}^p \subset \mathrm{VMO}^p_\delta$.

To prove that $\mathrm{BMO}^p_\delta \subset \mathcal{B}+\mathrm{BM}^p$, let $f \in \mathrm{BMO}^p_\delta$ and let $\psi(x,y)$ be given by Lemma 3.7. Define

$$
f_1(x) = \int_\Omega f(y)\psi(x,y)dy
$$

and $f_2 = f - f_1$. Note that f_1 is continuously differentiable. For $y \in \Omega$, we have by (3.6) and (3.7),

$$
\begin{aligned}
|f_2(y)|^p &= \left|\int_\Omega (f(y)-f(z))\psi(y,z)dz\right|^p \le \frac{C_0{}^p}{r(y)^{np}}\left(\int_{E_{\delta/3}(y)} |f(y)-f(z)|dz\right)^p \\
&\le \frac{C_0{}^p}{r(y)^{np}}\left(\int_{E_{\delta/3}(y)} |f(y)-f(z)|^p dz\right)\left(\int_{E_{\delta/3}(y)} dz\right)^{p-1} \\
&\approx \frac{1}{V(E_{\delta/3}(y))}\int_{E_{\delta/3}(y)} |f(y)-f(z)|^p dz .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{MV}^p_{\delta/3}(f_2;x) &= \frac{1}{V(E_{\delta/3}(x))}\int_{E_{\delta/3}(x)} |f_2(y)|^p dy \\
&\lesssim \frac{1}{V(E_{\delta/3}(x))}\int_{E_{\delta/3}(x)} \frac{1}{V(E_{\delta/3}(y))}\int_{E_{\delta/3}(y)} |f(y)-f(z)|^p dzdy .
\end{aligned}
$$

Because $r(x) \approx r(y)$ and $E_{\delta/3}(y) \subset E_\delta(x)$ if $y \in E_{\delta/3}(x)$, we have

$$
\mathrm{MV}^p_{\delta/3}(f_2;x) \lesssim \frac{1}{V(E_\delta(x))^2}\int_{E_\delta(x)}\int_{E_\delta(x)} |f(y)-f(z)|^p dzdy \le 2^p \mathrm{MO}^p_\delta(f;x) . \tag{3.10}
$$

This implies that $f_2 \in \mathrm{BM}^p$.

Next we prove that $f_1 \in \mathcal{B}$. For every $x \in \Omega$ and $a \in E_{\delta/3}(x)$, we have by (3.6),

$$
f_1(x) = \int_\Omega (f(y) - \widehat{f}_{\delta/3}(a))\psi(x,y)dy + \widehat{f}_{\delta/3}(a) ,
$$

and it follows from (3.8) that

$$r(x)|\nabla f_1(x)| \le C_1 r(x)^{-n} \int_{E_{\delta/3}(x)} |f(y) - \widehat{f_{\delta/3}}(a)|dy\,.$$

Since

$$|f(y) - \widehat{f_{\delta/3}}(a)| \le \frac{1}{V(E_{\delta/3}(a))} \int_{E_{\delta/3}(a)} |f(y) - f(z)|dz\,,$$

we have by Lemmas 3.1 and 3.2

$$\begin{aligned} r(x)|\nabla f_1(x)| &\lesssim \frac{1}{V(E_\delta(x))} \int_{E_{\delta/3}(x)} \frac{1}{V(E_{\delta/3}(a))} \int_{E_{\delta/3}(a)} |f(y) - f(z)|dzdy \\ &\lesssim \frac{1}{V(E_\delta(x))^2} \int_{E_\delta(x)} \int_{E_\delta(x)} |f(y) - f(z)|dzdy \le 2\mathrm{MO}^1_\delta(f; x)\,. \end{aligned}$$

By (2.1),

$$r(x)|\nabla f_1(x)| \lesssim \mathrm{MO}^1_\delta(f; x) \le \mathrm{MO}^p_\delta(f; x)^{1/p}\,. \tag{3.11}$$

This shows that $f_1 \in \mathcal{B}$ and finishes the proof that $\mathrm{BMO}^p_\delta \subset \mathcal{B} + \mathrm{BM}^p$.

If we let $f \in \mathrm{VMO}^p_\delta$, then (3.10) indicates that $f_2 \in \mathrm{VM}^p$. By (3.11), we have $f_1 \in \mathcal{B}_0$. □

It follows from the above lemma that BMO^p_δ and VMO^p_δ are independent of the choice of δ.

Let $\mathcal{H}$ be the set of all complex-valued harmonic functions on Ω.

LEMMA 3.9. *Let $p \in [1, \infty)$. Then*
(a) $\mathrm{BMO}^p \cap \mathcal{H} = \mathcal{B} \cap \mathcal{H}$,
(b) $\mathrm{VMO}^p \cap \mathcal{H} = \mathcal{B}_0 \cap \mathcal{H}$.

PROOF. By Lemma 3.5, we have

$$\mathcal{B} \cap \mathcal{H} \subset \mathrm{BMO}^p \cap \mathcal{H} \quad \text{and} \quad \mathcal{B}_0 \cap \mathcal{H} \subset \mathrm{VMO}^p \cap \mathcal{H}\,.$$

To show the converse, let $a \in \Omega$ and $x \in E_{\delta/3}(a)$. Then, by Lemmas 3.1 and 3.2, $V(E_{\delta(a)}) \lesssim V(E_{\delta/3(x)})$ and $E_{\delta/3}(x) \subset E_\delta(a)$. For $f \in \mathcal{H}$,

$$|f(x) - f(a)| \le \frac{1}{V(E_{\delta/3}(x))} \int_{E_{\delta/3}(x)} |f(y) - f(a)|dy\,.$$

Since $f(a) = \widehat{f_\delta}(a)$ by the mean-value property, we have

$$\begin{aligned} |f(x) - f(a)| &\lesssim \frac{1}{V(E_\delta(a))} \int_{E_\delta(a)} |f(y) - \widehat{f_\delta}(a)|dy \\ &= \mathrm{MO}^1_\delta(f; a) \le \mathrm{MO}^p_\delta(f; a)^{1/p}\,. \end{aligned}$$

By Cauchy's Estimates (see, for example, 2.4 of [1]),

$$|\nabla f(a)| \le \sup_{x \in E_{\delta/3}(a)} |\nabla f(x)| = \sup_{x \in E_{\delta/3}(a)} |\nabla (f(x) - f(a))| \lesssim \frac{\mathrm{MO}^p_\delta(f; a)^{1/p}}{r(a)}\,.$$

Since $a \in \Omega$ is arbitrary, we conclude that $\mathrm{BMO}^p \cap \mathcal{H} \subset \mathcal{B} \cap \mathcal{H}$ and $\mathrm{VMO}^p \cap \mathcal{H} \subset \mathcal{B}_0 \cap \mathcal{H}$. □

In order to prove Theorem 2.1, we need the following lemma which indicates the relationship between Hankel operators and commutators.

LEMMA 3.10. *Let $p \in (1, \infty)$ and $f \in L^1$.*
(a) *C_f is bounded on L^p if and only if H_f is bounded on both b^p and $b^{p'}$.*
(b) *C_f is compact on L^p if and only if H_f is compact on both b^p and $b^{p'}$.*

PROOF. Let $p \in (1, \infty)$ and $f \in L^1$. Suppose that H_f is bounded on both b^p and $b^{p'}$. If we let $\widetilde{H}_f = H_f Q$, then $\widetilde{H}_f$ is bounded on L^p. Since $H_{\bar{f}} u = \overline{H_f \bar{u}}$, the boundedness of H_f on $b^{p'}$ yields that $\widetilde{H}_{\bar{f}}$ is bounded on $L^{p'}$. Thus the adjoint operator $\widetilde{H}^*_{\bar{f}}$ is bounded on L^p. Let $u \in C_0^\infty(\Omega)$ and write $u = Q[u] + (I - Q)[u]$. Then

$$\begin{aligned} C_f u &= M_f Q[Q[u] + (I - Q)[u]] - Q[M_f(Q[u] + (I - Q)[u])] \\ &= (I - Q)[M_f Q[u]] - Q M_f (I - Q)[u] = \widetilde{H}_f u - \widetilde{H}^*_{\bar{f}} u \,. \end{aligned}$$

Now $\widetilde{H}_f$ and $\widetilde{H}^*_{\bar{f}}$ are bounded on L^p. Thus C_f is bounded on L^p, as desired.

Next we show the "only if" part. Suppose C_f is bounded on L^p. For $u \in b^\infty$, $C_f u = H_f u$, and so H_f is bounded on b^p. Also, $\widetilde{H}^*_{\bar{f}} = \widetilde{H}_f - C_f$ is bounded on L^p and thus $\widetilde{H}_{\bar{f}}$ is bounded on $L^{p'}$. It follows that H_f is bounded on $b^{p'}$.

It is easy to see that the same proof as above also works for compact operators. □

LEMMA 3.11. *Let $1 < p < \infty$. Then*

$$\int_\Omega \frac{|h(x)|^p}{r(x)^p} dx \lesssim \int_\Omega |\nabla h(x)|^p dx, \tag{3.12}$$

$$\int_\Omega \frac{|h(x)|^p}{r(x)^{2p}} dx \lesssim \int_\Omega \frac{|\nabla h(x)|^p}{r(x)^p} dx \tag{3.13}$$

for all $h \in C_0^\infty(\Omega)$.

PROOF. Since the proofs of (3.12) and (3.13) are essentially the same, we only prove (3.13). Let $\varepsilon = \varepsilon_0$, where ε_0 is the number provided by the first part of this section. Then we have

$$\int_{\Omega_\varepsilon} \frac{|h(x)|^p}{r(x)^{2p}} dx \le \frac{1}{\varepsilon^{2p}} \int_{\Omega_\varepsilon} |h(x)|^p dx \,.$$

Poincaré's inequality now shows that

$$\int_\Omega |h(x)|^p dx \lesssim \int_\Omega |\nabla h(x)|^p dx \,.$$

Since $1 \lesssim 1/r(x)$ for $x \in \Omega$, we have

$$\int_{\Omega_\varepsilon} \frac{|h(x)|^p}{r(x)^{2p}} dx \lesssim \int_\Omega |\nabla h(x)|^p dx \lesssim \int_\Omega \frac{|\nabla h(x)|^p}{r(x)^p} dx \,. \tag{3.14}$$

For $\zeta \in \partial\Omega$, let $x = \zeta + s\boldsymbol{n}_\zeta \in D_\varepsilon$, $0 \le s < \varepsilon$, where $\boldsymbol{n}_\zeta$ is the inward unit normal to $\partial\Omega$ at ζ. If we write $w_t = \zeta + t\boldsymbol{n}_\zeta$, $0 \le t \le s$, then

$$\begin{aligned}|h(x)|^p &= \int_0^s \frac{\partial}{\partial t}\{|h(w_t)|^p\}dt \\ &\le \int_0^s p|h(w_t)|^{p-1}\sum_{j=1}^n |D_j h(w_t)|dt \\ &\lesssim \int_0^s |h(w_t)|^{p-1}|\nabla h(w_t)|dt\,.\end{aligned}$$

It follows from Fubini's theorem that

$$\begin{aligned}\int_0^\varepsilon \frac{|h(\zeta+s\boldsymbol{n}_\zeta)|^p}{s^{2p}}ds &\lesssim \int_0^\varepsilon\int_0^s |h(\zeta+t\boldsymbol{n}_\zeta)|^{p-1}|\nabla h(\zeta+t\boldsymbol{n}_\zeta)|dt\frac{1}{s^{2p}}ds \\ &= \int_0^\varepsilon |h(\zeta+t\boldsymbol{n}_\zeta)|^{p-1}|\nabla h(\zeta+t\boldsymbol{n}_\zeta)|\int_t^\varepsilon \frac{1}{s^{2p}}dsdt \\ &\lesssim \int_0^\varepsilon |h(\zeta+t\boldsymbol{n}_\zeta)|^{p-1}|\nabla h(\zeta+t\boldsymbol{n}_\zeta)|\frac{1}{t^{2p-1}}dt\,.\end{aligned}$$

Therefore, we have by (3.1)

$$\begin{aligned}\int_{D_\varepsilon}\frac{|h(x)|^p}{r(x)^{2p}}dx &\approx \int_{\partial\Omega}\int_0^\varepsilon \frac{|h(\zeta+s\boldsymbol{n}_\zeta)|^p}{s^{2p}}dsd\sigma(\zeta) \\ &\lesssim \int_{\partial\Omega}\int_0^\varepsilon |h(\zeta+t\boldsymbol{n}_\zeta)|^{p-1}|\nabla h(\zeta+t\boldsymbol{n}_\zeta)|\frac{1}{t^{2p-1}}dtd\sigma(\zeta) \\ &\approx \int_{D_\varepsilon}|h(x)|^{p-1}|\nabla h(x)|\frac{1}{r(x)^{2p-1}}dx \\ &= \int_{D_\varepsilon}\frac{|h(x)|^{p-1}}{r(x)^{2(p-1)}}\frac{|\nabla h(x)|}{r(x)}dx \\ &\le \left(\int_{D_\varepsilon}\frac{|h(x)|^p}{r(x)^{2p}}dx\right)^{1-1/p}\left(\int_{D_\varepsilon}\frac{|\nabla h(x)|^p}{r(x)^p}dx\right)^{1/p},\end{aligned}$$

which implies that

$$\int_{D_\varepsilon}\frac{|h(x)|^p}{r(x)^{2p}}dx \lesssim \int_{D_\varepsilon}\frac{|\nabla h(x)|^p}{r(x)^p}dx\,.$$

Combining this with (3.14), we obtain (3.13). □

COROLLARY 3.12. *Let $1 < p < \infty$. Then*

$$\int_\Omega \frac{|h(x)|^p}{r(x)^{2p}}dx \lesssim \int_\Omega \frac{|\nabla h(x)|^p}{r(x)^p}dx \lesssim \|\Delta h\|_p^p$$

for all $h \in C_0^\infty(\Omega)$.

PROOF. By (3.13), we have

$$\int_{\Omega} \frac{|h(x)|^p}{r(x)^{2p}} dx \lesssim \int_{\Omega} \frac{|\nabla h(x)|^p}{r(x)^p} dx \, .$$

By (3.12), we have

$$\begin{aligned} \int_{\Omega} \frac{|\nabla h(x)|^p}{r(x)^p} dx &\lesssim \sum_{j=1}^{n} \int_{\Omega} \frac{|D_j h(x)|^p}{r(x)^p} dx \\ &\lesssim \sum_{j=1}^{n} \int_{\Omega} |\nabla(D_j h(x))|^p dx \lesssim \int_{\Omega} |\Delta h(x)|^p dx \, , \end{aligned}$$

where the last inequality comes from Proposition III.1.3 of [8, page 59]. This completes the proof of Corollary 3.12. □

LEMMA 3.13. *Let* $1 < p < \infty$ *and* $(b^p)^{\perp} = \{u \in L^{p'} \, ; \ \langle u, v \rangle = 0$ *for all* $v \in b^p\}$. *Then* $\{\Delta h \, ; \ h \in C_0^{\infty}(\Omega)\}$ *is dense in* $(b^p)^{\perp}$.

PROOF. If $h \in C_0^{\infty}(\Omega)$, then

$$\langle \Delta h, v \rangle = \langle h, \Delta v \rangle = 0$$

for all $v \in b^p$. So we have

$$\{\Delta h \, ; \ h \in C_0^{\infty}(\Omega)\} \subset (b^p)^{\perp} \, .$$

Next, suppose that $u \in L^p$ and

$$\int_{\Omega} u \Delta h \, dV = 0$$

for all $h \in C_0^{\infty}(\Omega)$. Then Weyl's lemma (see Theorem 2.3.1 of [7]) shows that $u \in b^p$. □

LEMMA 3.14. *Let* $1 \le p < \infty$. *Then*

$$\int_{\Omega} r(x)^p |\nabla u(x)|^p dx \lesssim \int_{\Omega} |u(x)|^p dx$$

for all $u \in b^p$.

PROOF. If $x \in \Omega$, then $r(y) \approx r(x)$ for $y \in E_{1/4}(x)$ or $x \in E_{3/4}(y)$ by (3.2). By Corollary 8.2 of [1], we have

$$|\nabla u(x)|^p \lesssim \frac{1}{r(x)^{n+p}} \int_{E_{1/4}(x)} |u(y)|^p dy \, .$$

It follows easily from Lemma 3.2 that $\chi_{E_{1/4}(x)}(y) \le \chi_{E_{3/4}(y)}(x)$ for all $x, y \in \Omega$. Therefore we get

$$\begin{aligned}\int_\Omega r(x)^p |\nabla u(x)|^p dx &\lesssim \int_\Omega \frac{1}{r(x)^n} \int_\Omega \chi_{E_{1/4}(x)}(y)|u(y)|^p dy dx \\ &\le \int_\Omega |u(y)|^p \int_\Omega \frac{\chi_{E_{3/4}(y)}(x)}{r(x)^n} dx dy \\ &\lesssim \int_\Omega |u(y)|^p \frac{V(E_{3/4}(y))}{r(y)^n} dy \approx \int_\Omega |u(y)|^p dy . \qquad \square\end{aligned}$$

4. Proof of the main result. We divide the proof into three lemmas.

LEMMA 4.1. *Let* $p \in (1, \infty)$.
(a) *If* $f \in \mathcal{B}$, *then* H_f *is bounded on* b^p.
(b) *If* $f \in \mathcal{B}_0$, *then* H_f *is compact on* b^p.

PROOF. First, we prove (a). Let $f \in \mathcal{B}$. By Lemma 3.13, we only need to show

$$|\langle H_f u, \Delta h\rangle| = |\langle fu, \Delta h\rangle| \le C\|u\|_p \|\Delta h\|_{p'}$$

for any $u \in b^\infty$ and $h \in C_0^\infty(\Omega)$ in order to prove the boundedness of H_f, since

$$\|H_f u\|_p = \sup_{\substack{g \in L^{p'} \\ \|g\| \le 1}} \langle H_f u, (I-Q)g + Qg\rangle = \sup_{\substack{g \in L^{p'} \\ \|g\| \le 1}} \langle H_f u, (I-Q)g\rangle = \sup_{\substack{\psi \in (b^p)^\perp \\ \|\psi\| \le 1}} \langle H_f u, \psi\rangle .$$

Using integration by part, we have

$$\langle fu, \Delta h\rangle = -\int_\Omega u(\nabla f)\cdot(\nabla \bar{h}) dV + \int_\Omega (\nabla u)\cdot(\nabla f)\bar{h}\, dV =: I_1 + I_2 .$$

It follows from Hölder's inequality and Corollary 3.12 that

$$\begin{aligned}|I_1| &\le \int_\Omega |u||\nabla f||\nabla h| dV \lesssim \int_\Omega |u(x)| \frac{|\nabla h(x)|}{r(x)} dx \\ &\le \left(\int_\Omega |u(x)|^p dx\right)^{1/p} \left(\int_\Omega \frac{|\nabla h(x)|^{p'}}{r(x)^{p'}} dx\right)^{1/p'} \lesssim \|u\|_p \|\Delta h\|_{p'} .\end{aligned}$$

On the other hand, using Hölder's inequality again, we get

$$\begin{aligned}|I_2| &\le \int_\Omega |\nabla u||\nabla f||h| dV \lesssim \int_\Omega r(x)|\nabla u(x)| \frac{|h(x)|}{r(x)^2} dx \\ &\le \left(\int_\Omega r(x)^p |\nabla u(x)|^p dx\right)^{1/p} \left(\int_\Omega \frac{|h(x)|^{p'}}{r(x)^{2p'}} dx\right)^{1/p'} .\end{aligned}$$

Thus, Lemma 3.14 and Corollary 3.12 yield

$$|I_2| \lesssim \|u\|_p \|\Delta h\|_{p'} .$$

This completes the proof of (a).

To prove (b), let $u_j \to 0$ weakly in b^p. Then it is well-known that there is a constant M satisfying $\|u_j\|_p \le M$ for all j, and u_j goes to 0 uniformly on each compact subset of Ω. For any $\varepsilon > 0$, there is a compact set $K \subset \Omega$ such that $r(x)|\nabla f(x)| < \varepsilon$ for $x \in \Omega \setminus K$. Also, we can choose j_0 such that $|u_j(x)| < \varepsilon$, $|\nabla u_j(x)| < \varepsilon$ for $x \in K$ and $j \geq j_0$, by Theorem 1.23 of [1]. For $j \geq j_0$, we have by Corollary 3.12 and Lemma 3.14

$$\begin{aligned}\int_\Omega |u_j||\nabla f||\nabla h| dV &= \int_{\Omega\setminus K} + \int_K |u_j||\nabla f||\nabla h| dV \\ &\lesssim \int_{\Omega\setminus K} |u_j|\varepsilon \frac{|\nabla h|}{r} dV + \int_K \varepsilon \frac{|\nabla h|}{r} dV \\ &\lesssim \varepsilon \|u_j\|_p \||\nabla h|/r\|_{p'} + \varepsilon \||\nabla h|/r\|_{p'} \lesssim \varepsilon(M+1)\|\Delta h\|_{p'} ,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\nabla u_j||\nabla f||h| dV &= \int_{\Omega\setminus K} + \int_K |\nabla u_j||\nabla f||h| dV \\ &\lesssim \int_{\Omega\setminus K} r|\nabla u_j|\varepsilon \frac{|h|}{r^2} dV + \int_K \varepsilon \frac{|h|}{r^2} dV \\ &\lesssim \varepsilon \|r|\nabla u_j|\|_p \|h/r^2\|_{p'} + \varepsilon \|h/r^2\|_{p'} \\ &\lesssim \varepsilon \|u_j\|_p \|\Delta h\|_{p'} + \varepsilon \|\Delta h\|_{p'} \leq \varepsilon(M+1)\|\Delta h\|_{p'} .\end{aligned}$$

Therefore, we have

$$|\langle H_f u_j, \Delta h\rangle| \lesssim \varepsilon(M+1)\|\Delta h\|_{p'}$$

for $j \geq j_0$, and this shows that $\|H_f u_j\|_p \to 0$ as $j \to \infty$. □

LEMMA 4.2. *Let $p \in (1, \infty)$.*

(a) *If $f \in \mathrm{BM}^p$, then H_f is bounded on b^p.*

(b) *If $f \in \mathrm{VM}^p$, then H_f is compact on b^p.*

PROOF. If $f \in \mathrm{BM}^p$ or VM^p, Theorem 3.5 or Theorem 3.11 of [2] implies that the multiplication operator M_f is bounded or compact on b^p, respectively. Thus $H_f = (I - Q)M_f$ is bounded or compact on b^p, respectively. □

LEMMA 4.3. *Let $p \in (1, \infty)$ and $f \in L^p$.*

(a) *If H_f is bounded on both b^p and $b^{p'}$, then $f \in \mathrm{BMO}^p$.*

(b) *If H_f is compact on both b^p and $b^{p'}$, then $f \in \mathrm{VMO}^p$.*

PROOF. (a) Suppose H_f is bounded on both b^p and $b^{p'}$. By part (a) of Lemma 3.10, C_f is bounded on L^p. Let $\delta \in (0, 1)$ and define

$$S(x, y, z) := \frac{R(y, z)}{R(x, x)} - 1 .$$

It follows from Lemma 3.3 that for all $x \in \Omega$ and $y, z \in E_\delta(x)$

$$|S(x, y, z)| = \frac{|R(y, z) - R(x, x)|}{|R(x, x)|} \leq \frac{C_1\delta}{(1-\delta)^{n+1}} . \tag{4.1}$$

By definition, we have

$$1 = \frac{R(y, z)}{R(x, x)} - S(x, y, z) ,$$

which implies that

$$\begin{aligned}\mathrm{MO}_\delta^p(f;x) &= \frac{1}{V(E_\delta(x))^{p+1}}\int_{E_\delta(x)}\left|\int_{E_\delta(x)}(f(y)-f(z))\cdot 1dz\right|^p dy\\ &\leq \frac{2^{p-1}}{V(E_\delta(x))^{p+1}}\int_{E_\delta(x)}\left|\int_{E_\delta(x)}(f(y)-f(z))\frac{R(y,z)}{R(x,x)}dz\right|^p dy\\ &\quad+\frac{2^{p-1}}{V(E_\delta(x))^{p+1}}\int_{E_\delta(x)}\left|\int_{E_\delta(x)}(f(y)-f(z))S(x,y,z)dz\right|^p dy =: I_1+I_2\,.\end{aligned}$$

We can estimate I_1 as follows:

$$\begin{aligned}I_1 &= \frac{2^{p-1}}{V(E_\delta(x))^p|R(x,x)|^p}\int_{E_\delta(x)}\left|\int_{E_\delta(x)}(f(y)-f(z))\frac{R(y,z)}{V(E_\delta(x))^{1/p}}dz\right|^p dy\\ &\leq \frac{2^{p-1}}{V(E_\delta(x))^p|R(x,x)|^p}\int_{\Omega}\left|\int_{\Omega}(f(y)-f(z))R(y,z)h_x(z)dz\right|^p dy\,,\end{aligned}$$

where

$$h_x(z)=\frac{\chi_{E_\delta(x)}(z)}{V(E_\delta(x))^{1/p}}\,.$$

It follows from Lemma 3.3 that there is a constant C_2 independent of δ such that

$$\frac{2^{p-1}}{V(E_\delta(x))^p|R(x,x)|^p}\leq\frac{C_2}{\delta^{np}}\,.$$

Note that for $g\in L^\infty$,

$$C_f g(y)=(M_f Q[g]-Q[fg])(y)=\int_\Omega (f(y)-f(z))R(y,z)g(z)dz\,.$$

Thus we have

$$I_1\leq\frac{C_2}{\delta^{np}}\|C_f h_x\|_p^p\,.$$

Next, we estimate I_2. It follows from (4.1) and Hölder's inequality that

$$\begin{aligned}I_2 &\leq \frac{2^{p-1}C_1^p\delta^p}{(1-\delta)^{p(n+1)}V(E_\delta(x))^{p+1}}\int_{E_\delta(x)}\left(\int_{E_\delta(x)}|f(y)-f(z)|dz\right)^p dy\\ &\leq \frac{2^{p-1}C_1^p\delta^p}{(1-\delta)^{p(n+1)}V(E_\delta(x))^2}\int_{E_\delta(x)}\int_{E_\delta(x)}|f(y)-f(z)|^p dzdy\\ &\leq \frac{2^{2p-1}C_1^p\delta^p}{(1-\delta)^{p(n+1)}}\mathrm{MO}_\delta^p(f;x)\,.\end{aligned}$$

We use Lemma 3.4 for the last inequality. Combining the above two estimates, we obtain

$$\mathrm{MO}_\delta^p(f;x)\leq\frac{C_2}{\delta^{np}}\|C_f h_x\|_p^p+\frac{2^{2p-1}C_1^p\delta^p}{(1-\delta)^{p(n+1)}}\mathrm{MO}_\delta^p(f;x)\,.$$

Now we choose δ small so that

$$\frac{2^{2p-1}C_1^p\delta^p}{(1-\delta)^{p(n+1)}}\leq\frac{1}{2}\,.$$

Then

$$\mathrm{MO}_\delta^p(f;x) \le \frac{2C_2}{\delta^{np}} \|C_f h_x\|_p^p . \tag{4.2}$$

Since $\|h_x\|_p = 1$ for all $x \in \Omega$, we have

$$\sup_{x\in\Omega} \mathrm{MO}_\delta^p(f;x) \le \frac{2C_2}{\delta^{np}} \|C_f\|^p .$$

This shows that $f \in \mathrm{BMO}^p$ and completes the proof of (a).

(b) Suppose H_f is compact on both b^p and $b^{p'}$. By part (b) of Lemma 3.10, C_f is compact on L^p. Because of (4.2), it suffices to show that $h_x \to 0$ weakly in L^p as $x \to \partial\Omega$. For every $g \in L^{p'}$, by Hölder's inequality, we have

$$\left|\int_\Omega h_x \bar{g}\, dV\right| \le \frac{1}{V(E_\delta(x))^{1/p}} \int_{E_\delta(x)} |g|\, dV \le \left(\int_{E_\delta(x)} |g|^{p'} dV\right)^{1/p'} \to 0$$

as $x \to \partial\Omega$. This completes the proof of Lemma 4.3. □

PROOF OF THEOREM 2.1. (a) If $f \in \mathrm{BMO}^p$, then $f \in \mathrm{BMO}^{p'}$, since $p \ge p'$. Thus by Lemmas 3.8, 4.1 and 4.2, H_f is bounded on both b^p and $b^{p'}$. This proves the sufficiency of $f \in \mathrm{BMO}^p$ for (a). The necessity of $f \in \mathrm{BMO}^p$ for (a) has already been proved in Lemma 4.3.

(b) If $f \in \mathrm{VMO}^p$, then $f \in \mathrm{VMO}^{p'}$, since $p \ge p'$. Thus, by Lemmas 3.8, 4.1 and 4.2, H_f is compact on both b^p and $b^{p'}$. This proves the sufficiency of $f \in \mathrm{VMO}^p$ for (b). The necessity of $f \in \mathrm{VMO}^p$ for (b) has already been proved in Lemma 4.3. □

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