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LETTER TO THE EDITOR

Hard hexagons: exact solution

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Abstract. The hard-hexagon model in lattice statistics (i.e. the triangular lattice gas with nearest-neighbour exclusion) has been solved exactly. It has a critical point when the activity z has the value $\frac{1}{2}(11+5\sqrt{5})=11{\cdot}09017\ldots$, with exponents $\alpha=\frac{1}{3}$, $\beta=\frac{1}{9}$. More generally, a restricted class of square-lattice models with nearest-neighbour exclusion and non-zero diagonal interactions can be solved.

Various models of systems of rigid molecules have been studied. In general they are expected to undergo a transition from a disordered fluid state to an ordered solid state as the density increases. For dimensions greater than one, the only non-trivial exact results are for some two-dimensional lattice models, either at close-packing (dimers and some colouring problems: Kasteleyn (1961), Fisher (1961), Baxter (1970)), or with special next-nearest neighbour interactions added to make the model solvable (Fisher 1963). Here I indicate that the hard-hexagon model (the triangular lattice gas with nearest-neighbour exclusion) can be solved for all densities, and give the main results. A full derivation will appear later.

The hard-hexagon model has been studied numerically (Runnels and Combs 1966, Gaunt 1967) and found to have an order–disorder transition at $z\simeq 11{\cdot}09$, where z is the activity. Metcalf and Yang (1978) made further approximate numerical studies for $z=1$, and Baxter and Tsang (1980) extended these by using the corner-transfer matrix (CTM) method.

Some intriguing properties emerged from this last approximate calculation. If $a_1, a_2, a_3, \ldots$ and $a_1', a_2', a_3', \ldots$ are the eigenvalues of the CTMs $A(+)$ and $A(-)$ (arranged in numerically decreasing order), then it turned out that a_1a_3/a_2^2 was very close to one, and became closer the higher the approximation. More generally, the limiting values of $a_1, a_2, \ldots, a_1', a_2', \ldots$ appeared to satisfy

$$a_n = a_1 x^l, \qquad a_n' = a_1' x^m, \tag{1}$$

where $x = a_2/a_1$ and l, m are integers dependent on n.

Very similar properties occur in the Ising and eight-vertex models (Baxter 1976, 1977, Tsang 1977), so this suggested to me that the model should be solvable. I have now established that it is. Here I shall give the results and a brief outline of the derivation. A full account will be published later.

Set

$$g(x) = \prod_{n=1}^{\infty} \frac{(1-x^{5n-4})(1-x^{5n-1})}{(1-x^{5n-3})(1-x^{5n-2})}. \tag{2}$$

Let z be the activity and $\kappa = Z^{1/N}$ the partition function per site. Then, for z less than some critical value z_c,

$$z = -x[g(x)]^5, \tag{3}$$

$$\kappa = \prod_{n=1}^{\infty} \frac{(1-x^{6n-4})(1-x^{6n-3})^2(1-x^{6n-2})(1-x^{5n-4})^2(1-x^{5n-1})^2(1-x^{5n})^2}{(1-x^{6n-5})(1-x^{6n-1})(1-x^{6n})^2(1-x^{5n-3})^3(1-x^{5n-2})^3}; \tag{4}$$

while for $z > z_c$

$$z^{-1} = x[g(x)]^5, \tag{5}$$

$$\kappa = x^{-1/3} \prod_{n=1}^{\infty} \frac{(1-x^{3n-2})(1-x^{3n-1})(1-x^{5n-3})^2(1-x^{5n-2})^2(1-x^{5n})^2}{(1-x^{3n})^2(1-x^{5n-4})^3(1-x^{5n-1})^3}. \tag{6}$$

Eliminating x gives κ as a function of z. For $0 < z < z_c$, x is negative; for $z > z_c$, x is positive. In both cases $|x| < 1$. As x decreases from 0 to -1, z in (3) increases from 0 to

$$z_c = \tfrac{1}{2}(11 + 5\sqrt{5}) = 11{\cdot}09017\ldots, \tag{7}$$

while as x decreases from 1 to 0, z in (5) increases from this same z_c to ∞. It follows that $\kappa(z)$ is analytic except at $z = z_c$, so $z = z_c$ is the critical point. (Gaunt actually conjectured (7) in 1967, using his numerical results, but did not include this conjecture in his paper.)

The behaviour near z_c can be obtained by using identities between elliptic functions of conjugate moduli. Doing this, setting

$$t = (z - z_c)/(5\sqrt{5} z_c), \tag{8}$$

gives

$$\kappa = \kappa_c[1 + \tfrac{1}{2}5(\sqrt{5} - 1)t + 3|t|^{5/3} + O(t^2)], \tag{9}$$

as $t \to \pm 0$, where

$$\kappa_c = [27(25 + 11\sqrt{5})/250]^{1/2} = 2{\cdot}3144\ldots. \tag{10}$$

The density $\rho = z\,\partial(\ln \kappa)/\partial z$ is therefore continuous at z_c, with value

$$\rho_c = (5 - \sqrt{5})/10 = 0{\cdot}276393\ldots, \tag{11}$$

while the compressibility diverges as $|z - z_c|^{-1/3}$, so the critical exponent α has the value $\frac{1}{3}$.

At high densities one sublattice (say 1) is occupied preferentially over the other two (2 and 3). Let ρ_k be the mean density on sublattice k (at close packing $\rho_1 = 1$, $\rho_2 = \rho_3 = 0$). Then the order parameter is

$$R = \rho_1 - \rho_2 = \rho_1 - \rho_3. \tag{12}$$

Expressions for ρ_1, ρ_2, ρ_3 are given in (40). From these one can prove that near z_c

$$R = (3/\sqrt{5})t^{1/9}[1 + O(t)], \tag{13}$$

so the exponent β has the value $\frac{1}{9}$. It is interesting that these exponents α, β differ from those of the Ising model $(0, \frac{1}{8})$ and of hard squares $(0{\cdot}09 \pm 0{\cdot}05, \frac{1}{8})$ (Baxter *et al* 1980). Enting has suggested to me that both hard hexagons and hard squares may have $\delta = 14$.

Star-triangle relation

In solving the model I was guided by the eight-vertex model. I first looked for models whose transfer matrices commute with that of hard hexagons. This led me to regard hard hexagons as a square lattice gas in which nearest-neighbour sites, and next-nearest neighbour sites on NW–SE diagonals, cannot be simultaneously occupied. Thus the partition function for a lattice of N sites is

$$\kappa^N = Z = \sum_\sigma \prod W(\sigma_i, \sigma_j, \sigma_k, \sigma_l), \tag{14}$$

where σ_i is the occupation number at site i, the sum is over all values of $\sigma_1, \ldots, \sigma_N$, the product is over all faces (i, j, k, l) of the square lattice (i, j, k, l being the four sites round the face, starting at the bottom-left and going anti-clockwise), and $W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ is the Boltzmann weight of the interactions within a face.

For the moment let $W(a, b, c, d)$ be an arbitrary function. Consider two models, one with function W, the other with function W'. Proceeding similarly to Appendix B of Baxter (1972), one can verify that the row-to-row transfer matrices of the models commute if there exists a third function W'' such that

$$\sum_g W(b, c, g, a) W'(a, g, e, f) W''(g, c, d, e) = \sum_g W''(a, b, g, f) W'(b, c, d, g) W(g, d, e, f), \tag{15}$$

for all values of the six spins $a, \ldots, f$. (This is a generalisation of equation (4.3) of Baxter (1978a), which is in turn a generalisation of the star-triangle relation of the Ising model.)

Now let W correspond to a model with nearest-neighbour exclusion plus diagonal interactions. Let $\sigma_i = 0$ if site i is empty, $= 1$ if site i is full. Sharing out the site activity z between the four adjacent faces then gives

$$\begin{aligned} W(a, b, c, d) &= m z^{(a+b+c+d)/4} \mathrm{e}^{Lac+Mbd} t^{-a+b-c+d} \qquad \text{if } ab = bc = cd = da = 0 \\ &= 0 \text{ otherwise.} \end{aligned} \tag{16}$$

(This t cancels out of (14), but is needed in (15); m is a trivial normalisation factor. For the original hard-hexagon model $m = 1$, $L = 0$ and $M = -\infty$.)

Define $W'(W'')$ similarly, replacing z, L, M, t by $z', L', M', t'(z'', L'', M'', t'')$. For convenience, interchange L' and M', and set $s = (zz'z'')^{1/4}/(tt't'')$. Then (15) gives

$$(z'z'')^{1/2} = s + s^2 \mathrm{e}^L \tag{17}$$

$$z(z'z'')^{1/2} \mathrm{e}^M = s^2 + s^3 \mathrm{e}^{L'+L''} \tag{18}$$

$$zz'z'' \mathrm{e}^{M+M'+M''} = s^3 + s^4 \mathrm{e}^{L+L'+L''}, \tag{19}$$

together with four other equations obtained from (17) and (18) by permuting the unprimed, primed and double-primed sets of variables. With an obvious notation these can be called (17′), (17″), (18′), (18″). Eliminating s, z'', L'', M'' from all seven equations leaves

$$\Delta_i = \Delta_i', \qquad i = 1, 2, 3, \tag{20}$$

where

$$\begin{aligned} \Delta_1 &= z^{-1/2}(1 - z\,\mathrm{e}^{L+M}) \\ \Delta_2 &= z^{1/2}(\mathrm{e}^{L} + \mathrm{e}^{M} - \mathrm{e}^{L+M}) \\ \Delta_3 &= z^{-1/2}(\mathrm{e}^{-L} + \mathrm{e}^{-M} - \mathrm{e}^{-L-M} - z\,\mathrm{e}^{L+M}), \end{aligned} \tag{21}$$

and Δ'_1, Δ'_2, Δ'_3 are defined similarly, z, L, M being replaced by z', L', M'.

[Let $(\dagger) = \mathrm{e}^{L'}(17) - \mathrm{e}^{L}(17')$. Then (17), (17′), (18), (18′) give $\Delta_1 = \Delta'_1$; (†), (17″), (18), (18′) give $\Delta_2 = \Delta'_2$; (†), (18), (18′), (18″), (19) give $\Delta_3 = \Delta'_3$.]

In general the only solutions of (20) are $z', L', M' = z, L, M$ or z, M, L, implying merely that the transfer matrix commutes with its transpose or itself. However, a corollary of (21) is

$$\Delta_1\Delta_2 - 1 = (\Delta_3 - \Delta_1 - \Delta_2)\,\mathrm{e}^{L+M}, \tag{22}$$

so if the values of Δ_1, Δ_2, Δ_3 satisfy

$$\Delta_2 = \Delta_1^{-1}, \qquad \Delta_3 = \Delta_1 + \Delta_1^{-1}, \tag{23}$$

then (22) is satisfied identically, and (20) reduces to only two equations, say $i = 1, 2$. It follows that all transfer matrices commute for which

$$z = (1 - \mathrm{e}^{-L})(1 - \mathrm{e}^{-M})/(\mathrm{e}^{L+M} - \mathrm{e}^{L} - \mathrm{e}^{M}), \tag{24}$$

and have the same value of

$$\Delta = z^{-1/2}(1 - z\,\mathrm{e}^{L+M}). \tag{25}$$

The restriction (24) is satisfied for all z if $L \to 0$ and $M \to -\infty$, corresponding to the hard-hexagon model. It is *not* so satisfied if $L, M \to 0$, i.e. by hard squares: indeed, recent series results for hard squares (Baxter *et al* 1980) gave no indication of any simple properties like (1).

The next step is to find a parametrisation of z, L, M as single-valued functions of some complex variable w such that (24), (25) are automatically satisfied and Δ is independent of w. As with the original star-triangle relation (Appendix 2 of Onsager (1944)), this introduces elliptic functions. Define the function $f(w, q)$, or simply $f(w)$, by

$$f(w, q) = f(w) = \prod_{n=1}^{\infty} (1 - q^{n-1}w)(1 - q^{n}w^{-1}), \tag{26}$$

and let $\omega_1, \ldots, \omega_5$ be the Boltzmann weights of the allowed spin configurations round a face:

$$\begin{aligned} \omega_1 &= W(0000) = m, \\ \omega_2 &= W(1000) = W(0010) = mz^{1/4}t^{-1}, \\ \omega_3 &= W(0100) = W(0001) = mz^{1/4}t, \\ \omega_4 &= W(1010) = mz^{1/2}t^{-2}\,\mathrm{e}^{L}, \\ \omega_5 &= W(0101) = mz^{1/2}t^{2}\,\mathrm{e}^{M}. \end{aligned} \tag{27}$$

Then in the regimes I and IV of figure 1, a convenient parametrisation is

$$\begin{aligned}
&\omega_1 = f(xw)/f(x), && \omega_2 = r^{-1}(-x)^{1/2} f(w)/[f(x)f(x^2)]^{1/2},\\
&\omega_3 = rf(x^2w)/f(x^2), && \omega_4 = r^{-2} wf(xw^{-1})/f(x), \qquad (28a)\\
&\omega_5 = r^2 f(x^2w^{-1})/f(x^2), && \Delta^2 = -x[f(x^2)/f(x)]^5,
\end{aligned}$$

where $f(w)$ is defined by (26) with $q = x^5$, and x, w are real and satisfy

$$\begin{aligned}
&\text{Regime I:} && 0 > x > -1, && 1 > w > x^2,\\
&\text{Regime IV:} && 0 > x > -1, && x^{-2} > w > 1. \qquad (29a)
\end{aligned}$$

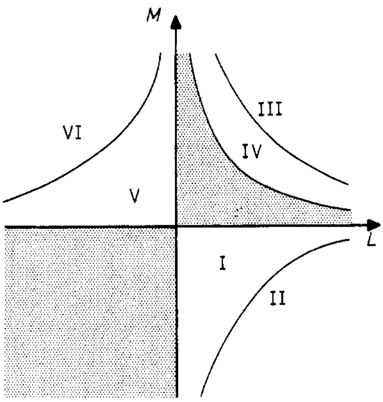

Figure 1. Regimes in the (L, M) plane. Shaded areas are unphysical, since (24) gives z to be negative. Regimes I, III, V are disordered, II and VI have triangular ordering, IV has square ordering. Inter-regime boundaries are given by (24), (25) and (30). Regimes V and VI differ from I and II only by a 90° lattice rotation, so they are not further considered herein.

In regimes II and III of figure 1 it is more convenient to use the parametrisation

$$\begin{aligned}
&\omega_1 = f(x^2w)/f(x^2), && \omega_2 = r^{-1} x^{1/2} f(w)/[f(x)f(x^2)]^{1/2},\\
&\omega_3 = rf(xw^{-1})/f(x), && \omega_4 = r^{-2} wf(x^2w^{-1})/f(x^2), \qquad (28b)\\
&\omega_5 = r^2 w^{-1} f(xw)/f(x), && \Delta^2 = x[f(x)/f(x^2)]^5,
\end{aligned}$$

$$\begin{aligned}
&\text{Regime II:} && 0 < x < 1, && x^{-1} > w > 1,\\
&\text{Regime III:} && 0 < x < 1, && 1 > w > x, \qquad (29b)
\end{aligned}$$

and again $f(w)$ is defined by (26) with $q = x^5$. In both cases $\omega_1, \ldots, \omega_5$ are entire functions of w; Δ is independent of w.

As $x \to 0$, z, $z\mathrm{e}^L$, $z\mathrm{e}^M \to 0$ in regimes I and III, which is the low-density limit; in II, e^L and e^{-M} become large (while $z \sim 1$), in which limit the system adopts a triangular ordered state with every third site in a row (or column) occupied; in IV, z becomes large while $L, M \sim 1$, so the system adopts the usual square-lattice close packing, every other site being occupied. Thus in every case the limit $x \to 0$ is one of extreme disorder or order.

The other boundaries of the regimes are $|x| \to 1$. In every case this corresponds to

$$\Delta^{-2} = [\tfrac{1}{2}(\sqrt{5}+1)]^5 = \tfrac{1}{2}(11+5\sqrt{5}). \tag{30}$$

This is the equation of the lines separating I from II, III from IV, V from VI in figure 1.

In the Ising and eight-vertex models one can obtain tractable equations for the eigenvalues of the row-to-row transfer matrix, for a lattice with a finite number of columns. I have not found a way to do this for the present model. Instead I have considered the infinite lattice and used the following argument (which is correct for the Ising and eight-vertex models).

Free energy

Let $V_A[V_B]$ be the 'transfer matrix' that adds a single face to the lattice, going in the SE–NW [SW–NE] diagonal direction. It has entries $W(a, b, c, d)[W(d, a, b, c)]$ in row (a, b, c) and column (a, d, c). Let $V'_A = \omega_1 V_A/(\omega_4\omega_5)$ [similarly for v'_B]. From (28), V'_A, V'_B are functions of r and w (regarding x as constant), and

$$V'_A(r, w)V'_A(r_0^2/r, w_0^2/w) = V'_B(r, w)V'_B(r^{-1}, w^{-1}) = 1, \tag{31}$$

where $r_0^2 = -x/g(x), x^{-1}g(x), xg(x), -x^{-1}/g(x)$ and $w_0 = x^3, x^{-3/2}, x, x^{-2}$ in regimes I, II, III, IV, respectively.

In equations (12) and (13) of Baxter (1976) it is shown that the corner transfer matrix (CTM) $A[B]$ is a product of M matrices $V_A[V_B]$, except that it should be divided by κ^M to ensure that the limit $M \to \infty$ exists. For w real and satisfying (29), let y_0 be the maximal eigenvector of $A[B]$. Then κ can be defined as the normalising divisor of $V_A[V_B]$ that ensures that the y_0-eigenvalue of $A[B]$ tends to a finite non-zero limit.

The star-triangle relation (15) implies that all CTMs with the same x commute. Thus y_0 is independent of r and w, and the above definition of κ can be extended beyond the interval (29) appropriate to the regime under consideration. Since κ is independent of r, for fixed x it can be written as $\kappa(w)$. Let

$$\Lambda(w) = \omega_1\kappa(w)/(\omega_4\omega_5). \tag{32}$$

Then (31) and the above definition of $\kappa(w)$ imply that

$$\Lambda(w)\Lambda(w_0^2/w) = \Lambda(w)\Lambda(w^{-1}) = 1. \tag{33}$$

The equations (33) 'almost' define $\Lambda(w)$. For instance, suppose one knew that $\ln \Lambda(w)$ was analytic in an annulus $a < |w| < b$ containing $w = 1$ and $w = w_0$. Then it would have a Laurent expansion. Taking logarithms of (33), it is easily found that all terms in this expansion must be zero, so $\Lambda(w) = 1$. (In regime III, exactly this happens.)

From series expansions it appears that $\ln[w^{-\lambda}\kappa(w)]$ is analytic in such an annulus, where $\lambda = 0, \frac{1}{3}, 0, \frac{1}{2}$ in regimes I, II, III, IV, respectively (corresponding analyticity properties exist for the Ising and eight-vertex models.) Using this, (32) and (28), $\Lambda(w)$ can be factored into a known singular part and an unknown part whose logarithm is Laurent-expandable. The coefficients of this expansion can then be obtained from (33), giving

$$\begin{aligned} &\text{I:} && \Lambda = w^{-1}\frac{f(xw, x^6)f(x^2w, x^6)}{f(x/w, x^6)f(x^2/w, x^6)}, \\ &\text{II:} && \Lambda = w^{1/3}\frac{f(xw^{-1}, x^3)}{f(xw, x^3)}, \\ &\text{III:} && \Lambda = 1, \\ &\text{IV:} && \Lambda = w^{-1/2}\frac{f(xw, x^4)}{f(x/w, x^4)}. \end{aligned} \tag{34}$$

Equations (3)–(6) follow by taking the limit $w \to x^2$ in I and $w \to x^{-1}$ in II, using (32) and (28).

Sublattice densities

The star-triangle relation implies that the CTMs commute and their eigenvalues are of the form

$$c(x)w^n, \tag{35}$$

where $c(x)$ is independent of w and n is an integer (I conjectured this property for the eight-vertex model in 1976 and can now establish it).

The relations (31) for V'_A and V'_B imply the same relations for the CTMs A and B, and these can be used to fix the coefficients $c(x)$. The integers n can then be obtained from the small-x limits.

As in Baxter (1976), the rows and columns of A and B can be labelled by $\tau = \{\sigma_1, \sigma_2, \sigma_3, \dots\}$, where $\sigma_1, \sigma_2, \sigma_3, \dots$ are now occupation numbers (value 0 or 1) and σ_j, σ_{j+1} cannot both be one (for all j). Let $A_d[B_d]$ be the diagonalised matrix $A[B]$, and let $a(\tau)[b(\tau)]$ be the eigenvalue entry in row and column τ. Then the above reasoning gives

$$\begin{aligned} a(\tau) &= (r_0/r)^{\sigma_1}(w_0/w)^{\phi(\tau)}, \\ b(\tau) &= r^{\sigma_1}w^{\phi(\tau)} \end{aligned} \tag{36}$$

where

$$\begin{aligned} \phi(\tau) &= \sum_{j=1}^{\infty} j(\sigma_{j+1} - s_{j+1}) \qquad \text{in I and IV} \\ &= \sum_{j=1}^{\infty} j(\sigma_{j+1} - \sigma_j\sigma_{j+2} - s_{j+1} + s_js_{j+2}) \qquad \text{in II and III,} \end{aligned} \tag{37}$$

and $\sigma_j \to s_j$ as $j \to \infty$.

Here $s_1, s_2, s_3, \dots$ are the ground-state values of $\sigma_1, \sigma_2, \sigma_3, \dots$, corresponding to the maximum eigenvalues of A and B. In the ordered states they depend on the sublattice on which the CTM is centred. They are

$$\begin{aligned} &\text{I:} && s_j = 0, \\ &\text{II:} && s_{3j+k} = 1, \qquad \sigma_{3j+k\pm1} = 0, \qquad k = 1, 2, 3, \\ &\text{III:} && s_j = 0, \\ &\text{IV:} && s_{2j+k} = 1, \qquad s_{2j+k+1} = 0, \qquad k = 1, 2. \end{aligned} \tag{38}$$

Here one first fixes the regime (I to IV) and, if necessary, the value of k. Then (38) applies for all integers j; $a(\tau)$, $b(\tau)$ are given by (36) and (37).

In regimes II and IV, k specifies the sublattice under consideration. Thus there are three matrices $A[B]$ in regime II, and two in regime IV.

Let A_k, B_k be the CTMs for sublattice k. Then from the definition of the CTMs (Baxter 1976, 1978b), the probability of occupancy of a site on sublattice k is

$$\rho_k = \mathrm{Tr}\; S(A_kB_k)^2/\mathrm{Tr}(A_kB_k)^2, \tag{39}$$

where S is a diagonal operator with entries σ_1. Going to a diagonal representation (S, A_k, B_k all commute) and using (36) gives

$$\rho_k = \sum_\tau \sigma_1 r_0^{2\sigma_1} w_0^{2\phi(\tau)} \Big/ \sum_\tau r_0^{2\sigma_1} w_0^{2\phi(\tau)}. \tag{40}$$

These summations are over all states $\tau = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$ such that $\sigma_j\sigma_{j+1} = 0$ for all j, and σ_j tends to the appropriate s_j as $j \to \infty$. Note that ρ_k is independent of r and w.

Critical behaviour

The free energy is singular across the I–II and III–IV boundaries given by (37). Its behaviour near these boundaries can be obtained by using identities between elliptic functions of conjugate moduli. Define ϵ, u, q by

$$\begin{aligned} &\text{I and IV:} \qquad x = -\mathrm{e}^{-\pi^2/5\epsilon}, \qquad w = \mathrm{e}^{2\pi u/\epsilon}, \qquad q^2 = -\mathrm{e}^{-\epsilon}, \\ &\text{II and III:} \qquad x = \mathrm{e}^{-4\pi^2/5\epsilon}, \qquad w = \mathrm{e}^{-4\pi u/\epsilon}, \qquad q^2 = \mathrm{e}^{-\epsilon}. \end{aligned} \tag{41}$$

Then (27) and (28) imply

$$\begin{aligned} &\Delta^2 = \theta_1^5(\pi/5, q)/\theta_1^5(2\pi/5, q), \\ &\mathrm{e}^M = \frac{\theta_1(u+\pi/5, q)\theta_1(u+2\pi/5, q)\theta_1(\pi/5, q)}{\theta_1^2(u-\pi/5, q)\theta_1(2\pi/5, q)}, \end{aligned} \tag{42}$$

where $\theta_1(u, q)$ is the usual elliptic theta function (§ 8.181.3 of Gradshteyn and Ryzhik (1965)): q enters (42) only via q^2.

The equations (42) (and corresponding formulae for z, e^L) are true in *all* regimes (with $-\pi/5 < u < 0$ in I and II, $0 < u < \pi/5$ in III and IV). It follows that q^2 and u are analytic across a critical boundary, q^2 having usually a simple zero thereon. From (34), for q^2 small:

$$\text{I and II:} \qquad \Lambda = \frac{\sin(\pi/3 - 5u/3)}{\sin(\pi/3 + 5u/3)}\left[1 - 2\sqrt{3}|q^2|^{5/3} \sin\frac{10u}{3} + \mathrm{O}(q^{20/3})\right] \tag{43a}$$

$$\text{IV:} \qquad \Lambda = 1 - 4(-q^2)^{5/2} \sin 5u + \mathrm{O}(q^{10}). \tag{43b}$$

The critical exponents α, α' therefore are both $\frac{1}{3}$ across the I–II boundary (disorder to triangular ordering), while $\alpha' = -\frac{1}{2}$ in IV (square ordering).

For $u = -\pi/5$ the model becomes that of hard hexagons, for which the total mean density ρ can be obtained by differentiating κ. Since ρ depends on q^2, but not on u, this result can be applied along the entire I–II boundary, giving

$$\rho = \rho_c + 5^{-1/2}\mathrm{sgn}(q^2)|q^2|^{2/3} + \mathrm{O}(q^2), \tag{44}$$

where $\rho_c = (5 - \sqrt{5})/10 = 0{\cdot}27639\ldots$.

Also, using (40), it can be shown that the order parameter in II behaves for small q^2 as

$$R=\rho_1-\rho_2=\rho_1-\rho_3=3q^{2/9}/\sqrt{5}[1+O(q^2)], \tag{45}$$

so $\beta=\frac{1}{9}$. (But note that all exponent values herein apply only to paths in (z, L, M) space lying on the surface (24): only for pure hard hexagons are these necessarily the usual exponents.)

Conjectures

All the above results have been proved, subject only to assumptions such as the thermodynamic limit existing and $\kappa(w)$ being analytic in an appropriate annulus.

The numerator and denominator in (40) can be regarded as 'one-dimensional partition functions' and written as elements of an infinite product of two-by-two, or three-by-three, matrices. The resulting expressions are still unwieldy, but they appear to simplify to tractable infinite products of theta function type. For instance I have used them to expand R to order 80 in a power series in x, and the results agree with

$$\begin{aligned} \text{II:} \qquad & R=\prod_{n=1}^{\infty}(1-x^n)(1-x^{5n})/(1-x^{3n})^2, \\ \text{IV:} \qquad & R=\prod_{n=1}^{\infty}(1-x^{2n})^2(1-x^{5n})/[(1-x^n)(1-x^{4n})^2]. \end{aligned} \tag{46}$$

(The order parameter of the eight-vertex model has a similar product expansion: Barber and Baxter (1973).) The first formula also agrees with (45), so it is a very plausible conjecture (but still a conjecture) that (46) is exactly correct.

There appear to be a number of such mathematical identities, the simplest of which (for $\rho/(1-\rho)$ in regime I) are the Rogers–Ramanujan identities (Ramanujan 1919). Some others are contained in the 130 generalisations of Slater (1951), but at present I can in general only claim them as conjectures. From them I find that

$$\begin{aligned} \text{III:} \qquad & \rho=\rho_c-5^{-1/2}q^{1/2}+O(q), \\ \text{IV:} \qquad & \rho=\rho_c+5^{1/2}(-q^2)+O(q^4), \\ & R=2(-q^2)^{1/4}/\sqrt{5}[1+O(q^2)], \end{aligned} \tag{47}$$

where again $\rho_c=(5-\sqrt{5})/10$. These give the critical exponents across the III–IV boundary to be $\alpha=\frac{3}{4}$, $\beta=\frac{1}{4}$. Since these are calculated on the surface (24), the density values of α, α' do not have to be the same as the free energy values ($\alpha'=-\frac{1}{2}$).

The ordered state in IV is that of the hard-squares model, and the corresponding critical line in (z, L, M) space given by (24) and (30) probably lies on the same critical surface as the hard square model ($z=3{\cdot}7962\ldots$, $L=M=0$). Since the results (43*b*) and (47) apply only to a special surface, crossing the critical surface on a special line, it is not surprising that they give exponents quite different from those expected for hard squares (Baxter *et al* 1980): even so, it is disappointing.

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