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HARD LEFSCHETZ THEOREM FOR SASAKIAN MANIFOLDS

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Abstract: We prove that on a compact Sasakian manifold (M, η, g) of dimension $2n+1$, for any $0 \leq p \leq n$ the wedge product with $\eta \wedge (d\eta)^p$ defines an isomorphism between the spaces of harmonic forms $\Omega_{\Delta}^{n-p}(M)$ and $\Omega_{\Delta}^{n+p+1}(M)$. Therefore it induces an isomorphism between the de Rham cohomology spaces $H^{n-p}(M)$ and $H^{n+p+1}(M)$. Such isomorphism is proven to be independent of the choice of a compatible Sasakian metric on a given contact manifold. As a consequence, an obstruction for a contact manifold to admit Sasakian structures is found.

1. Introduction

Sasakian manifolds, introduced by Sasaki [22] in 1960, can be described as an odd-dimensional counterpart of Kähler manifolds. Starting from the 90s, a renewed interest in Sasakian geometry was stimulated by the new findings in theoretical physics (see e.g. [11, 16, 19, 20]), especially after the Maldacena conjecture [18] on the duality between conformal field theory and the anti-de-Sitter space. As a consequence, many important geometric and topological properties of Sasakian manifolds were discovered, see e.g. [3, 4, 10, 15].

It is well known that Sasakian geometry is naturally related to Kähler geometry from two sides: on the one hand Sasakian manifolds can be defined as those manifolds whose metric cone is Kähler, on the other hand the 1-dimensional foliation defined by the Reeb vector field is transversally Kähler.

A remarkable property of compact Kähler manifolds is given by the celebrated Hard Lefschetz Theorem, stating that the cup product with the suitable powers of the symplectic form gives isomorphisms between the de Rham cohomology groups of complementary degrees. This result was first obtained by Lefschetz [17] but the first complete proof was given by Hodge in [13]. One of the consequences of the Hard Lefschetz Theorem is an obstruction for a symplectic manifold to admit a Kähler structure.

Later on, an odd dimensional version of the Hard Lefschetz Theorem was proven for compact cosymplectic manifolds in [6]. Thus one may ask whether a similar property also holds in the context of Sasakian geometry. So far,

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the only result that can be considered to be in this direction is the transversal Hard Lefschetz Theorem, proved by El Kacimi-Alaoui [7], which holds for the basic cohomology with respect to any homologically orientable transversely Kähler foliation. In this paper, our aim is to prove that a version of Hard Lefschetz Theorem holds for the de Rham cohomology of any compact Sasakian manifold.

Our approach needs to be different from the one adopted in Kähler geometry. Indeed, although Sasakian and Kähler manifolds share many properties, in this case the picture for Sasakian manifolds shows deep peculiarities. Let (M, ω, g) be a compact Kähler manifold of dimension $2n$ and $\Omega^p_\triangle(M)$ the space of harmonic p-forms on M. We recall that the Hard Lefschetz Theorem states that for any $0 \le p \le n$, the maps

$$\begin{aligned} \omega^p \wedge - \colon \Omega^{n-p}_\triangle(M) &\to \Omega^{n+p}_\triangle(M) \\ \alpha &\mapsto \omega^p \wedge \alpha \end{aligned}$$

are isomorphisms. Now, let (M, η, g) be a compact Sasakian manifold of dimension $2n+1$. As a natural generalization of the above Lefschetz isomorphism, one can consider, for each $0 \le p \le n$, the maps

$$\begin{aligned} \eta \wedge (d\eta)^p \wedge - \colon \Omega^{n-p}_\triangle(M) &\to \Omega^{n+p+1}_\triangle(M) \\ \alpha &\mapsto \eta \wedge (d\eta)^p \wedge \alpha. \end{aligned} \tag{1.1}$$

However at this step a serious problem already arises. Namely, differently from the Kähler case, it is not true that the wedge multiplication by either $d\eta$ or $\eta \wedge d\eta$ maps harmonic forms into harmonic forms, so in principle the definition of the above maps may happen to be ill posed.

Nevertheless, we discover some spectral properties of the Laplace operator on differential forms which allow to overcome this obstacle. More precisely, given $\alpha \in \Omega^{n-p}_\triangle(M)$, we show that the forms $\eta \wedge (d\eta)^k \wedge \alpha$ and $(d\eta)^{k-1} \wedge \alpha$ are eigenforms of the Laplacian with positive integer eigenvalues for all $0 \le k \le p-1$. These eigenforms and their eigenvalues are visualized in Figure 1 at page 12 in the case $n = 5$. In the figure the points on the horizontal axis represent the spaces of harmonic forms on M. All other points with coordinates (p, ν) denote suitable subspaces of

$$\{\, \beta \in \Omega^p(M) \mid \triangle\, \beta = 4\nu\beta \}\,.$$

The segments represent the isomorphisms (3.1) and (3.7) between the corresponding vector spaces. As shown in the figure, the Δ-eigenvalues of $\eta \wedge (d\eta)^k \wedge \alpha$ initially increase with k up to degree n and then decrease until to reach zero, for $k = p$. Thus each mapping in (1.1) is actually well defined, in the sense that its target space is $\Omega_\Delta^{n+p+1}(M)$. Moreover, in Theorem 3.6 we prove that such map is in fact an isomorphism by explicitly giving its inverse map. This isomorphism obviously induces via Hodge theory an isomorphism between the corresponding de Rham cohomology groups. Namely, for each $0 \leq p \leq n$ we obtain the isomorphism

$$\begin{aligned} \mathit{Lef}_{n-p}\colon H^{n-p}(M) &\to H^{n+p+1}(M) \\ [\,\beta\,] &\mapsto [\,\eta \wedge (d\eta)^p \wedge \Pi_\Delta\, \beta\,]\,, \end{aligned}$$

where $\Pi_\Delta \beta$ denotes the orthogonal projection of β on the space of harmonic forms. Note that, contrary to the symplectic case, we are forced to use the metric structure in the definition of Lef_p. Thus, a priori, one could expect that different Sasakian metrics on M could lead to different Lefschetz isomorphisms. To the utter surprise of the authors, this is not the case. In Theorem 4.5 we prove that the Lefschetz isomorphism is independent of the metric. This provides an obstruction for a contact manifold to admit Sasakian structures. In the last section, we introduce the notion of Lefschetz contact manifold and we prove that their odd Betti numbers up to the middle dimension are odd.

2. Preliminaries

In this section we recall the definition of Sasakian manifolds and list some of their properties. For further details we refer the reader to [1] or [2].

Let M be a smooth manifold of dimension $2n+1$. A 1-form η on M is called a *contact form* if $\eta \wedge d\eta^n$ nowhere vanishes. Then the pair (M, η) is called a (strict) *contact manifold*. We write Φ for $\frac{1}{2} d\eta$ and we denote by ξ the *Reeb vector field*, that is the unique vector field on M such that $i_\xi \eta = 1$ and $i_\xi d\eta = 0$.

Let (M, η) be a contact manifold and g a Riemannian metric on M. We define the endomorphism $\phi\colon TM \to TM$ by $\Phi(X, Y) = g(X, \phi Y)$.

Then (M, η, g) is called a *Sasakian manifold* if the following conditions hold.

(*i*) $\phi^2 = -I + \eta \otimes \xi$, where I is the identity operator;
(*ii*) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X and Y on M;

(iii) The normality condition is satisfied, namely

$$[\phi, \phi]_{FN} + 2d\eta \otimes \xi = 0,$$

where $[-,-]_{FN}$ is the Frölicher-Nijenhuis bracket as defined in [14].

It is well known that in any Sasakian manifold

$$\phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.1}$$

Now we will introduce notation for some linear operators on the exterior algebra $\Omega^*(M)$. For a p-form α on M, we denote by ϵ_α the operator given by

$$\epsilon_\alpha \beta = \alpha \wedge \beta,$$

where $\beta \in \Omega^*(M)$. If (M, η) is a compact contact manifold, then ϵ_η is adjoint to i_ξ with respect to the usual global scalar product on $\Omega^*(M)$, that is $\epsilon_\eta = i_\xi^*$. Since $i_\xi \eta = 1$, for any $\omega \in \Omega^*(M)$ we have

$$\{i_\xi, \epsilon_\eta\}\, \omega = \omega, \tag{2.2}$$

where the curly brackets are used to denote the anti-commutator of two operators.

Let (M, η, g) be a compact Sasakian manifold. We define the operators L and Λ on $\Omega^*(M)$ by

$$L = \epsilon_\Phi, \qquad \Lambda = L^*.$$

Then, since d is a graded derivation on $\Omega^*(M)$ of degree 1 and η is a 1-form, we get

$$\{d, \epsilon_\eta\} = \epsilon_{d\eta} = 2\epsilon_\Phi = 2L. \tag{2.3}$$

Therefore

$$\{i_\xi, \delta\} = \{d, \epsilon_\eta\}^* = 2L^* = 2\Lambda. \tag{2.4}$$

Hereafter, we will use some elements of Frölicher-Nijenhuis calculus, developed in [8] (see also [14, Section 8]). In this framework, for every vector valued k-form ψ, the graded derivation i_ψ of degree $(k-1)$ on $\Omega^*(M)$ is defined. It acts on $\omega \in \Omega^p(M)$ by

$$(i_\psi \omega)(X_1, \dots, X_{p+k-1}) = \sum_\sigma (-1)^\sigma \omega(\psi(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \dots, X_{\sigma(p+k-1)}),$$

where the summation is taken over all $(k, p-1)$-shuffles. In the case $k = 0$, that is when $\psi = X$ is a vector field, we reobtain the usual interior product

i_X. If ψ is an endomorphism of TM, that is when $k=1$, the formula above can be rewritten as

$$i_\psi\omega\left(X_1,\ldots,X_p\right)=\sum_{s=1}^{p}\omega\left(X_1,\ldots,\psi X_s,\ldots,X_p\right).$$

For any vector valued k-form ψ the operator $\mathcal{L}_\psi$ is defined to be $i_\psi d-(-1)^{k-1}di_\psi$. Note that every $\mathcal{L}_\psi$ is a graded derivation of degree k on $\Omega^*(M)$. If $\psi=X$ is a vector field, then $\mathcal{L}_X$ is the usual Lie derivative.

When ψ is the identity operator $I\colon TM\to TM$, the operator i_ψ will be denoted by deg, motivated by the fact that for any p-form ω we have

$$\deg\omega=i_I\omega=p\omega.$$

Now we recall from [9] some results on commutators between some operators on Sasakian manifolds (note that Fujitani uses Φ for i_ϕ, φ for Φ, λ for i_ξ, and l for ϵ_η).

Theorem 2.1. *Let M be a compact Sasakian manifold of dimension $2n+1$. Then the operator i_ϕ commutes with ϵ_η, i_ξ, L, and Λ. The Lie derivative $\mathcal{L}_\xi$ commutes with d, δ, ϵ_η, i_ξ, L, Λ, and i_ϕ. Furthermore*

$$[d,\Lambda]=[i_\phi,\delta]-2\left(n-\deg\right)i_\xi \tag{2.5}$$
$$[\triangle,i_\xi]=2\left[i_\phi,\delta\right]-4\left(n-\deg\right)i_\xi \tag{2.6}$$
$$[\triangle,i_\phi]=-2\left(\mathcal{L}_\xi-i_\xi d+\epsilon_\eta\delta\right) \tag{2.7}$$
$$[\triangle,\epsilon_\eta]=-2\mathcal{L}_\phi+4\epsilon_\eta\left(n-\deg\right). \tag{2.8}$$

Proof: See Propositions 1.1, 1.2, 3.3 and Theorem 3.2 in [9]. ■

Using the equalities (2.5)–(2.8), in [9, Theorem 4.1] Fujitani reobtained a few results of Tachibana [23] and complemented them with the dual ones in [9, Corollary 4.2]. Below is the summary of these results.

Proposition 2.2. *Let M be a compact Sasakian manifold of dimension $2n+1$ and ω a harmonic p-form. Then $i_\phi\omega$ is a also harmonic and, moreover,*

(i) if $p\le n$, then $i_\xi\omega=0$;
(ii) if $p\ge n+1$, then $\epsilon_\eta\omega=0$;
(iii) if $p\le n+1$, then $\Lambda\omega=0$;
(iv) if $p\ge n$, then $L\omega=0$.

In the following proposition we prove some other useful identities.

Proposition 2.3. *In any Sasakian manifold we have*

$$[i_\phi, \epsilon_\eta] = 0 \tag{2.9}$$
$$[i_\phi, i_\xi] = 0 \tag{2.10}$$
$$\mathcal{L}_\phi^2 = -2L\mathcal{L}_\xi \tag{2.11}$$
$$\{\delta, \epsilon_\eta\} = -\mathcal{L}_\xi. \tag{2.12}$$

Proof: Since i_ϕ is a derivation of degree zero, we have $[i_\phi, \epsilon_\eta] = \epsilon_{i_\phi \eta} = 0$, as $i_\phi \eta = 0$ by (2.1). Next, it is easy to check that for any $\psi\colon TM \to TM$ and any vector field X we have $[i_\psi, i_X] = -i_{\psi X}$. Thus $[i_\phi, i_\xi]$ is zero by (2.1). Further, according to Frölicher-Nijenhuis calculus, we have $\{\mathcal{L}_\phi, \mathcal{L}_\phi\} = \mathcal{L}_{[\phi,\phi]_{FN}}$. Thus, from the normality condition for Sasakian manifolds, we get

$$\mathcal{L}_\phi^2 = \frac{1}{2}\mathcal{L}_{[\phi,\phi]_{FN}} = -\mathcal{L}_{d\eta\otimes\xi} = -\{i_{d\eta\otimes\xi}, d\} = -\{d\eta \wedge i_\xi, d\} = -d\eta \wedge \mathcal{L}_\xi = -2L\mathcal{L}_\xi.$$

Finally, it was shown on page 109 of [12] that, for any Killing vector field X, one has $\mathcal{L}_X + \left\{\delta, \epsilon_{g(X,-)}\right\} = 0$. Since in any Sasakian manifold the Reeb vector field ξ is Killing, the equation (2.12) holds.

■

3. Hard Lefschetz isomorphism for harmonic forms

In this section we establish the hard Lefschetz isomorphism between the spaces of harmonic p-forms and harmonic $(2n+1-p)$-forms in a compact Sasakian manifold M of dimension $2n+1$.

We start by introducing some notation. Let $A_1, \dots, A_k$ be operators on $\Omega^*(M)$. We shall denote by $\Omega^p_{A_1,\dots,A_k}(M)$ the set of forms ω such that $A_1\omega = \dots = A_k\omega = 0$ and we shall use $\Omega^{p,\nu}_{A_1,\dots,A_k}(M)$ as a shorthand for $\Omega^p_{\Delta-\nu I, A_1,\dots,A_k}(M)$, where ν is a real number. Of course these spaces are empty if $\nu < 0$ and M is compact, since all Δ-eigenvalues are non-negative in this case.

We will be mainly concerned with two families of spaces of differential forms on a compact Sasakian manifold, namely, $\Omega^{p,\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{p,\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$, where $0 \le p \le 2n+1$ and ν are non-negative real numbers. It follows from (2.12) that the spaces of both families are included in $\Omega^*_{\mathcal{L}_\xi}(M)$. Let us show that these two families are related to each other by the Hodge star operator $*$.

Proposition 3.1. *Let M be a compact Sasakian manifold of dimension $2n+1$. Then for any p-form ω, we have $\omega \in \Omega^{p,\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$ if and only if $*\omega \in \Omega^{2n+1-p,\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$.*

Proof: It is easy to check that for any p-form ω, one has $\delta * \omega = (-1)^{p+1} * d\omega$, $\epsilon_\eta * \omega = (-1)^{p+1} * i_\xi\omega$, hence $i_\xi d * \omega = - * \epsilon_\eta \delta\omega$ and $\triangle *\omega = * \triangle \omega$. Therefore the claim holds. ∎

Now we show that the spaces of harmonic forms on a compact Sasakian manifold are included in the above families of subspaces.

Proposition 3.2. *Let M be a compact Sasakian manifold of dimension $2n+1$.*

(i) For $p \leq n$, we have $\Omega^p_\triangle(M) = \Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M) = 0$.
(ii) For $p \geq n+1$, we have $\Omega^p_\triangle(M) = \Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M)$ and $\Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M) = 0$.

Proof: It is obvious that $\Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M) \subset \Omega^p_\triangle(M)$. Let $\omega \in \Omega^p_\triangle(M)$ with $p \leq n$. Since ω is harmonic, we have $d\omega = 0$ and $\epsilon_\eta\delta\omega = 0$. Moreover, by Proposition 2.2 we have that $i_\xi\omega = 0$, since $p \leq n$. Thus $\Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M) = \Omega^p_\triangle(M)$.

In order to prove that $\Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M) = 0$, notice that

$$\Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M) \subset \Omega^p_\triangle(M) = \Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M)$$

implies $\epsilon_\eta\omega = 0$ and $i_\xi\omega = 0$ for any $\omega \in \Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M)$. Therefore by (2.2)

$$\omega = i_\xi\epsilon_\eta\omega + \epsilon_\eta i_\xi\omega = 0.$$

Hence $\Omega^{p,0}_{\delta,\epsilon_\eta,i_\xi d}(M) = 0$. The second part of the proposition can be proved by duality considerations, using Proposition 3.1. ∎

Proposition 3.3. *Let M be a compact Sasakian manifold.*

(i) If $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$ then $d\omega \in \Omega^{p+1,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$. If moreover $\nu \neq 0$, then $d\omega \neq 0$.
(ii) If $\omega \in \Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$, then $\delta\omega \in \Omega^{p-1,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$. If moreover $\nu \neq 0$, then $\delta\omega \neq 0$.

Proof: By duality, it is enough to prove just (i). Let $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$. Then

$$d\,(d\omega) = 0, \qquad i_\xi d\omega = 0, \qquad \triangle\, d\omega = d \triangle \omega = 4\nu d\omega.$$

It is left to show that $\epsilon_\eta \delta d\omega = 0$. Since $\delta\omega = 0$ and $\epsilon_\eta\omega = 0$, we get

$$\epsilon_\eta \delta d\omega = \epsilon_\eta \delta d\omega + \epsilon_\eta d\delta\omega = \epsilon_\eta \vartriangle \omega = 4\nu\epsilon_\eta\omega = 0.$$

If moreover $\nu \neq 0$, then $\delta d\omega = \vartriangle \omega = 4\nu\omega \neq 0$. Thus also $d\omega \neq 0$.

■

Proposition 3.3 shows that for $\nu \neq 0$, we have the pair of isomorphisms

$$\Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M) \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{p+1,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)\,, \tag{3.1}$$

for any $0 \leq p \leq 2n$.

Theorem 3.4. *Let M be a compact Sasakian manifold of dimension $2n+1$.*

(i) If $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$ then $i_\xi\omega \in \Omega^{p-1,4(\nu+p-n-1)}_{d,i_\xi,\epsilon_\eta\delta}(M)$.

(ii) If $\omega \in \Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$ then $\epsilon_\eta\omega \in \Omega^{p+1,4(\nu-p+n)}_{\delta,\epsilon_\eta,i_\xi d}(M)$.

Proof: We will prove just (i) as (ii) can be obtained from (i) by using Hodge duality and Proposition 3.1. Let $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$. We write ν' for $\nu+p-n-1$. We have to show that $i_\xi\omega \in \Omega^{p-1,4\nu'}_{d,i_\xi,\epsilon_\eta\delta}(M)$. First of all, from (2.12), we get

$$d i_\xi\omega = \mathcal{L}_\xi\omega - i_\xi d\omega = -\{\epsilon_\eta, \delta\}\,\omega = 0. \tag{3.2}$$

Next, it is obvious that $i_\xi i_\xi\omega = 0$. Moreover, by using (2.12), (2.2) and (3.2) we get

$$\epsilon_\eta\delta i_\xi\omega = -\mathcal{L}_\xi i_\xi\omega - \delta\epsilon_\eta i_\xi\omega = -i_\xi d i_\xi\omega - \delta\left(\omega - i_\xi\epsilon_\eta\omega\right) = 0.$$

Thus $i_\xi\omega \in \Omega^{p-1}_{d,i_\xi,\epsilon_\eta\delta}(M)$. It is left to prove that $\vartriangle i_\xi\omega = 4\nu' i_\xi\omega$. From the equation (2.6) for $[\vartriangle, i_\xi]$, using that $\delta\omega = 0$, we get

$$\vartriangle i_\xi\omega - i_\xi \vartriangle \omega = -2\delta i_\phi\omega - 4(n-p+1)i_\xi\omega.$$

Since $\vartriangle \omega = 4\nu\omega$ and $\nu - n + p - 1 = \nu'$, we get

$$\vartriangle i_\xi\omega = 4\nu' i_\xi\omega - 2\delta i_\phi\omega. \tag{3.3}$$

To get that $\delta i_\phi\omega = 0$, we proceed as follows. First we apply $d\delta$ to (3.3). Using that $d\delta$ commutes with $\vartriangle$, we get

$$\vartriangle d\delta i_\xi\omega = 4\nu' d\delta i_\xi\omega.$$

As $d i_\xi\omega = 0$ by (3.2), we have $d\delta i_\xi\omega = \vartriangle i_\xi\omega$. Therefore

$$\vartriangle\vartriangle i_\xi\omega = 4\nu' \vartriangle i_\xi\omega. \tag{3.4}$$

Comparing (3.3) and (3.4), we see that $\triangle\,\delta i_\phi\omega = 0$, thence also

$$\delta \triangle i_\phi\omega = 0. \tag{3.5}$$

Using the equation (2.7) for $[\triangle, i_\phi]$, we get

$$\triangle i_\phi\omega = i_\phi \triangle \omega - 2\mathcal{L}_\xi\omega + 2i_\xi d\omega - 2\epsilon_\eta\delta\omega = 4\nu i_\phi\omega, \tag{3.6}$$

since $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$ and hence $\mathcal{L}_\xi\omega = 0$. From (3.5) and (3.6) it follows that $4\nu\delta i_\phi\omega = \delta \triangle i_\phi\omega = 0$. If $\nu \neq 0$, this implies that $\delta i_\phi\omega = 0$. On the other hand if $\nu = 0$, then ω is harmonic. Thus by Proposition 2.2 the form $i_\phi\omega$ is also harmonic. Hence $\delta i_\phi\omega = 0$. Therefore from (3.3) we finally get that $\triangle i_\xi\omega = 4\nu' i_\xi\omega$. ■

Since $\{\epsilon_\eta, i_\xi\} = I$, we get from Theorem 3.4 that ϵ_η and i_ξ induce the pair of inverse isomorphisms

$$\Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M) \underset{\epsilon_\eta}{\overset{i_\xi}{\rightleftarrows}} \Omega^{p-1,4\nu'}_{d,i_\xi,\epsilon_\eta\delta}(M), \tag{3.7}$$

where $\nu' = \nu + p - n - 1$. In fact, if ω is on the left hand side, then $\epsilon_\eta\omega = 0$ and $\epsilon_\eta i_\xi\omega = \omega - i_\xi\epsilon_\eta\omega = \omega$. Similarly, if ω is on the right hand side, then $i_\xi\omega = 0$ and $i_\xi\epsilon_\eta\omega = \omega$.

Corollary 3.5. *Let M be a compact Sasakian manifold of dimension $2n+1$. Then for $\nu \neq 0$ and $p \leq 2n-1$*

$$\Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M) \underset{\Lambda}{\overset{L}{\rightleftarrows}} \Omega^{p+2,4(\nu-p-1+n)}_{\delta,\epsilon_\eta,i_\xi d}(M), \tag{3.8}$$

is a pair of isomorphisms such that $\Lambda L = \nu I$ and $L\Lambda = \nu I$. Moreover, for $\nu \neq p-n$

$$\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \underset{\Lambda}{\overset{L}{\rightleftarrows}} \Omega^{p+2,4(\nu-p+n)}_{d,i_\xi,\epsilon_\eta\delta}(M), \tag{3.9}$$

is a pair of isomorphisms, such that $L\Lambda = (\nu-p+n)I$ and $\Lambda L = (\nu-p+n)I$.

Proof: Notice that from (2.3), we have that for every $\omega \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$

$$2L\omega = \epsilon_\eta d\omega. \tag{3.10}$$

Similarly from (2.4), for every $\omega \in \Omega^{p+2,4(\nu-p-1+n)}_{\delta,\epsilon_\eta,i_\xi d}(M)$, we get

$$2\Lambda\omega = \delta i_\xi\omega. \tag{3.11}$$

Therefore, using the isomorphisms (3.1) and (3.7), we can construct the diagram

$$\begin{tikzcd} \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M) \arrow[r, shift left, "d"] \arrow[rr, bend left, "2L"] & \Omega^{p+1,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\epsilon_\eta"] & \Omega^{p+2,4(\nu-p-1+n)}_{\delta,\epsilon_\eta,i_\xi d}(M). \arrow[l, shift left, "i_\xi"] \arrow[ll, bend left, "2\Lambda"] \end{tikzcd}$$

This shows that L and Λ induce the isomorphisms between the spaces in (3.8). For every $\alpha \in \Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$ and every $\beta \in \Omega^{p+2,4(\nu-p-1+n)}_{\delta,\epsilon_\eta,i_\xi d}(M)$, we have

$$\begin{aligned} \Lambda L\alpha &= \frac{1}{4}\delta i_\xi \epsilon_\eta d\alpha = \frac{1}{4}\delta d\alpha = \frac{1}{4} \triangle \alpha = \nu\alpha \\ L\Lambda\beta &= \frac{1}{4}\epsilon_\eta d\delta i_\xi \beta = \frac{1}{4}\epsilon_\eta \triangle i_\xi\beta = \nu\epsilon_\eta i_\xi \beta = \nu\beta. \end{aligned}$$

The second part of the corollary is proved by the same line of reasoning from the diagram

$$\begin{tikzcd} \Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \arrow[r, shift left, "\epsilon_\eta"] \arrow[rr, bend left, "2L"] & \Omega^{p+1,4(\nu-p+n)}_{\delta,\epsilon_\eta,i_\xi d}(M) \arrow[l, shift left, "i_\xi"] \arrow[r, shift left, "d"] & \Omega^{p+2,4(\nu-p+n)}_{d,i_\xi,\epsilon_\eta\delta}(M). \arrow[l, shift left, "\delta"] \arrow[ll, bend left, "2\Lambda"] \end{tikzcd}$$

■

Now, we are prepared to prove the Hard Lefschetz Theorem on the level of harmonic forms.

Theorem 3.6. *Let M a compact Sasakian manifold of dimension $2n+1$ and $p \leq n$. Then the map*

$$\epsilon_\eta L^{n-p} \colon \Omega^p(M) \to \Omega^{2n+1-p}(M)$$

induces an isomorphism $F_p \colon \Omega^p_\triangle(M) \to \Omega^{2n+1-p}_\triangle(M)$. Similarly, the map

$$\Lambda^{n-p} i_\xi \colon \Omega^{2n+1-p}(M) \to \Omega^p(M)$$

induces an isomorphism $G_p \colon \Omega^{2n+1-p}_\triangle(M) \to \Omega^p_\triangle(M)$. Moreover,

$$F_p G_p = (n-p)!^2, \qquad G_p F_p = (n-p)!^2.$$

Proof: We have by Proposition 3.2, that

$$\Omega^p_\triangle(M) = \Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M)$$

and $\Omega^{2n+1-p}_\triangle(M) = \Omega^{2n+1-p,0}_{\delta,\epsilon_\eta,i_\xi d}(M)$. Let us define the numbers $\nu_{p,k}$ by

$$\nu_{p,k} := k(n-p-k+1), \ k \in \mathbb{Z}. \tag{3.12}$$

To make the notation less heavy, we will write ν_k instead of $\nu_{p,k}$ along the proof. It is easy to check that $\nu_{k+1} = \nu_k - (p+2k) + n$, and that $\nu_k \neq 0$ for $1 \leq k \leq n-p$. Thus by Corollary 3.5, the operators L and Λ induce isomorphisms between $\Omega^{p+2k,4\nu_k}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{p+2(k+1),4\nu_{k+1}}_{d,i_\xi,\epsilon_\eta\delta}(M)$. Moreover, for $\omega \in \Omega^{p+2k,4\nu_k}_{d,i_\xi,\epsilon_\eta\delta}(M)$, it holds

$$\Lambda L\omega = \nu_{k+1}\omega. \tag{3.13}$$

Note that $\nu_0 = 0$ and $\nu_{n-p} = n-p$. Then we get the chain of isomorphisms

$$\Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M) \xrightarrow{L} \Omega^{p+2,4\nu_1}_{d,i_\xi,\epsilon_\eta\delta}(M) \xrightarrow{L} \cdots \xrightarrow{L} \Omega^{2n-p,4(n-p)}_{d,i_\xi,\epsilon_\eta\delta}(M)\,. \tag{3.14}$$

Thus L^{n-p} induces an isomorphism between $\Omega^{p,0}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{2n-p,4(n-p)}_{d,i_\xi,\epsilon_\eta\delta}(M)$. From (3.7), we have that ϵ_η and i_ξ induce two mutually inverse maps between $\Omega^{2n-p,4(n-p)}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{2n-p+1,0}_{\delta,\epsilon_\eta,i_\xi d}(M) = \Omega^{2n-p+1}_\triangle(M)$. This shows, that F_p is an isomorphism between $\Omega^p_\triangle(M)$ and $\Omega^{2n-p+1}_\triangle(M)$. Similarly, one can check that also $G_p\colon \Omega^{2n-p+1}_\triangle(M) \to \Omega^p_\triangle(M)$ is an isomorphism. Iterating (3.13), we see that for all $\omega \in \Omega^p_\triangle(M)$

$$G_pF_p\omega = \Lambda^{n-p}L^{n-p}\omega = \left(\prod_{k=1}^{n-p}\nu_k\right)\omega = \left(\prod_{k=1}^{n-p}k(n-p-k+1)\right)\omega = (n-p)!^2\omega.$$

Similarly, for $\omega \in \Omega^{2n-p+1}_\triangle(M)$, we have $F_pG_p\omega = (n-p)!^2\omega$. ■

From (3.14) it follows that the spaces $\Omega^{p+2k,4\nu_{p,k}}_{d,i_\xi,\epsilon_\eta\delta}(M)$, where $\nu_{p,k}$ are defined by (3.12), are isomorphic to $\Omega^p_\triangle(M)$ for all $1 \leq k \leq n-p$. Thus, if $H^p(M) \neq 0$, the same is true for $\Omega^{p+2k,4\nu_{p,k}}_{d,i_\xi,\epsilon_\eta\delta}(M)$. The following proposition shows that the only pairs (p,ν) such that $\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \neq 0$ are necessarily of the form $(p+2k, \nu_{p,k})$.

Proposition 3.7. *Let M be a compact Sasakian manifold of dimension $2n+1$.*

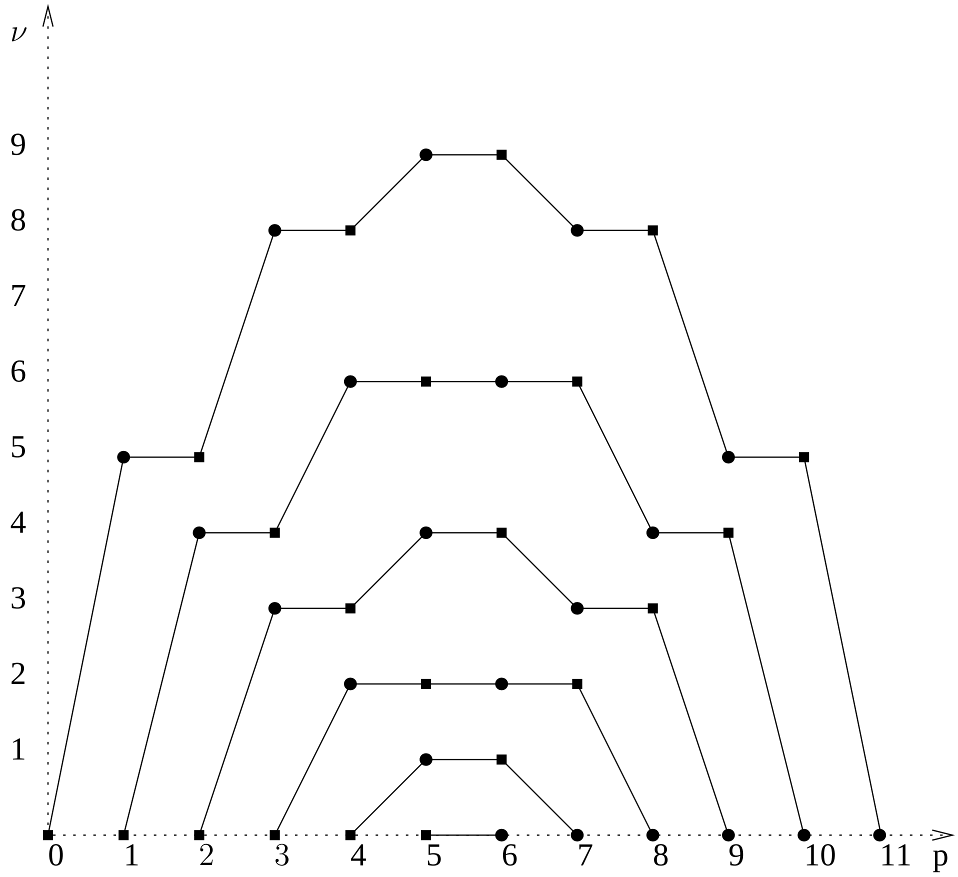

FIGURE 1.

(i) If $\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \neq 0$ *then* $\nu = \nu_{p-2k,k}$ *for some integer* k *such that*

$$\max\{0, (p-n)/2\} \le k \le p/2.$$

(ii) If $\Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M) \neq 0$ *then* $\nu = \nu_{p+1-2k,k}$ *for some integer* k *such that*

$$\max\{0, (p+1-n)/2\} \le k \le (p+1)/2.$$

Proof: Note that the second part of the proposition follows from the first part and the isomorphism (3.1). We will prove the first part by induction on p. Suppose $p = 0$ and $\Omega^{0,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \neq 0$. Let $f \in \Omega^{0,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$, $f \neq 0$. Then $df = 0$ and thus f is a harmonic function. Since 4ν is a $\triangle$-eigenvalue of f, we get that necessarily $\nu = 0$. Thus $\nu = \nu_{0,0}$.

Suppose we proved the claim for all $p' < p$ and $\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M) \neq 0$. If $\nu = 0$, then by Proposition 3.2 we get that $p \le n$. Therefore 0 is in the allowed range of values for k, and we are done since $\nu_{p,0} = 0$. Now consider the case

$\nu \neq 0$. From Corollary 3.5, we get that the spaces $\Omega^{p-2,4(\nu+p-2-n)}_{d,i_\xi,\epsilon_\eta\delta}(M)$ and $\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$ are isomorphic. From the induction assumption, we get that

$$\nu + p - 2 - n = \nu_{(p-2)-2k',k'}$$

for some k' between $\max\{0, (p-2-n)/2\}$ and $(p-2)/2$. Define $k = k'+1$. Then

$$\begin{aligned} \nu &= \nu_{p-2k,k-1} - p + 2 + n \\ &= (k-1)(n-p+k+2) + n - p + 2 \\ &= k(n-p+k+1) = \nu_{p-2k,k}. \end{aligned}$$

Moreover k lies between $\max\{1, (p-n)/2\}$ and $p/2$. This completes the induction argument. ■

In Figure 1, we give an illustration of what happens for a compact Sasakian manifold M of dimension 11. In the picture the circles mark the places (p,ν) such that $\Omega^{p,4\nu}_{\delta,\epsilon_\eta,i_\xi d}(M)$ can be non-zero. Similarly, the squares mark the points (p,ν) such that the spaces $\Omega^{p,4\nu}_{d,i_\xi,\epsilon_\eta\delta}(M)$ can be non-zero. The horizontal segments with the circle at the left edge and the square at the right edge represent the isomorphisms of the type (3.1), and all the other segments correspond to the isomorphisms of the type (3.7). Thus, we can see that if we start with a harmonic p-form ω for $p \leq n$ and move it along the segments representing the isomorphisms of the types (3.1) and (3.7), we get in intermediate steps $\triangle$-eigenforms with non-zero eigenvalues, and we will eventually end up with the harmonic $(2n+1-p)$-form $F_p\omega$.

4. Hard Lefschetz isomorphism in de Rham cohomology

Let M be a compact Sasakian manifold of dimension $2n+1$. Let us denote by $\Pi_\triangle$ the orthogonal projection from $\Omega^*(M)$ onto $\Omega^*_\triangle(M)$. Then we can define

$$\begin{aligned} \mathit{Lef}_p \colon H^p(M) &\to H^{2n+1-p}(M) \\ [\,\beta\,] &\mapsto [\,F_p \Pi_\triangle \beta\,]\,. \end{aligned} \tag{4.1}$$

Due to Hodge theory and Theorem 3.6 the map Lef_p is an isomorphism.

Suppose (η, g') is another Sasakian structure on M with the same contact form η. Denote by $\triangle'$ the corresponding Laplacian. For a closed p-form β it can happen that

$$\Pi_\triangle \beta \neq \Pi_{\triangle'} \beta.$$

Thus it is not a priori clear whether Lef_p depends on the full Sasakian structure or just on its contact form. Note that this is rather different from the Kähler case, where it is obvious from the definition of the Lefschetz map that it is fully determined by the symplectic structure and does not depend on chosen Kähler metric.

The aim of this section is to show that Lef_p is uniquely determined by the contact structure of M. Note that from Proposition 2.2 it follows that for any Sasakian metric g' on (M,η) and any closed p-form β with $p \leq n$, we have

$$\Pi_{\triangle'}\beta \in \Omega^p_{i_\xi,d}(M).$$

Moreover, since d commutes with L, by using (2.3) and Theorem 3.6 we get

$$L^{n-p+1}\Pi_{\triangle'}\beta = \frac{1}{2}d\left(\epsilon_\eta L^{n-p}\Pi_{\triangle'}\beta\right) = \frac{1}{2}d\left(F_p\Pi_{\triangle'}\beta\right) = 0. \tag{4.2}$$

Indeed, we will prove that given an arbitrary p-form γ with $p \leq n$ such that

$$i_\xi\gamma = 0, \qquad d\gamma = 0, \qquad L^{n-p+1}\gamma = 0, \tag{4.3}$$

the cohomology classes of $\epsilon_\eta L^{n-p}\gamma$ and of $\epsilon_\eta L^{n-p}\Pi_\triangle\gamma$ are equal. As a consequence we will get that

$$\left[\epsilon_\eta L^{n-p}\Pi_{\triangle'}\beta\right] = \left[\epsilon_\eta L^{n-p}\Pi_\triangle\Pi_{\triangle'}\beta\right] = \left[\epsilon_\eta L^{n-p}\Pi_\triangle\beta\right],$$

since $\Pi_{\triangle'}\beta$ satisfies (4.3) and $\Pi_\triangle\Pi_{\triangle'}\beta = \Pi_\triangle\beta$ as β is closed.

We start by recalling some facts on Hodge theory for compact manifolds. Let M be a compact Riemannian manifold. Denote by G the Green operator for the de Rham complex $\Omega^*(M)$ of M. Then

$$I - \triangle G = \Pi_\triangle, \qquad I - G\triangle = \Pi_\triangle, \qquad dG = Gd, \qquad \delta G = G\delta. \tag{4.4}$$

Proposition 4.1. *Let (M,g) be a compact Riemannian manifold and ξ a Killing vector field on M. Then $\mathcal{L}_\xi$ commutes with the Green operator G.*

Proof: By Theorem 3.7.1 in [12], we have that $\mathcal{L}_\xi\omega = 0$ for every harmonic form ω. Thus $\mathcal{L}_\xi\Pi_\triangle = 0$. Let us show that $\Pi_\triangle\mathcal{L}_\xi = 0$. Using (2.12) and that $d^* = \delta$, $i_\xi^* = \epsilon_\eta$, we get

$$\mathcal{L}_\xi^* = \{d, i_\xi\}^* = \{\epsilon_\eta, \delta\} = -\mathcal{L}_\xi.$$

Hence for any $\beta \in \Omega^k(M)$ and $\omega \in \Omega^k_\triangle(M)$ we have

$$(\mathcal{L}_\xi\beta, \omega) = -\left(\beta, \mathcal{L}_\xi\omega\right) = 0,$$

where (,) denotes the usual global scalar product between two differential forms on a compact Riemannian manifold. This shows that $\mathcal{L}_\xi\beta$ is orthogonal to the subspace $\Omega^k_\triangle(M)$ of $\Omega^k(M)$ and thus $\Pi_\triangle\mathcal{L}_\xi\beta = 0$. Now, from (4.4) we get

$$\mathcal{L}_\xi - \mathcal{L}_\xi \triangle G = \mathcal{L}_\xi\Pi_\triangle = 0, \qquad \mathcal{L}_\xi - G \triangle \mathcal{L}_\xi = \Pi_\triangle\mathcal{L}_\xi = 0.$$

We know from Theorem 2.1 that $\mathcal{L}_\xi$ commutes with d and δ, and thus with $\triangle = \{d, \delta\}$. Thus

$$G\mathcal{L}_\xi = G\mathcal{L}_\xi \triangle G = G \triangle \mathcal{L}_\xi G = \mathcal{L}_\xi G.$$

■

Let us introduce an auxiliary map

$$\begin{aligned}\mathcal{A}_p \colon \Omega^{p-1}(M) &\to \Omega^{2n-p+1}(M)\\ \alpha &\mapsto (n-p+1)L^{n-p}di_\phi d\alpha + L^{n-p+1} \triangle \alpha.\end{aligned}$$

We will study the interplay between $\mathcal{A}_p$ and δ, d, $\triangle$. The following technical lemma is needed.

Lemma 4.2. *Let M be a Sasakian manifold. Then for any $k \geq 1$, we have*

$$\left[\delta, L^k\right] = -kL^{k-1}\mathcal{L}_\phi + 2k\epsilon_\eta L^{k-1}(n - \deg -(k-1)). \tag{4.5}$$

Proof: For any two linear endomorphisms a and b of an arbitrary vector space V, we denote by $\mathrm{ad}_b(a)$ their commutator $[b, a]$. It can be checked (e.g. by induction) that

$$ab^k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}b^{k-j}\,\mathrm{ad}_b^j(a). \tag{4.6}$$

We will apply formula (4.6) to the case $a = \delta$ and $b = L$. Dualizing (2.5), since $i_\phi^* = -i_\phi$, we get

$$\mathrm{ad}_L(\delta) = [L, \delta] = [d, -i_\phi] - 2\epsilon_\eta(n - \deg) = \mathcal{L}_\phi - 2\epsilon_\eta(n - \deg).$$

Moreover, since $\mathcal{L}_\phi$ is a derivation and $L = \epsilon_\Phi$, we get

$$[\mathcal{L}_\phi, L] = \epsilon_{\mathcal{L}_\phi\Phi} = 0.$$

Therefore

$$\mathrm{ad}_L^2(\delta) = [L, \mathcal{L}_\phi - 2\epsilon_\eta n + 2\epsilon_\eta \deg] = 2\epsilon_\eta\,[L, \deg] = -4\epsilon_\eta L.$$

Thus $\mathrm{ad}_L^j(\delta) = 0$ for all $j \geq 3$. Now the claim of the lemma follows from (4.6).

■

Theorem 4.3. *Let (M, η, g) be a compact Sasakian manifold of dimension $2n+1$, $1 \leq p \leq n$, and $\alpha \in \Omega^{p-1}_{\mathcal{L}_\xi,\delta}(M)$. Then $\mathcal{A}_p\alpha$ is coclosed and*

$$\triangle \mathcal{A}_p\alpha = \mathcal{A}_p \triangle \alpha. \tag{4.7}$$

Proof: To check that

$$\mathcal{A}_p\alpha = (n-p+1)L^{n-p} di_\phi d\alpha + L^{n-p+1} \triangle \alpha \tag{4.8}$$

is coclosed, we will repeatedly use Lemma 4.2. We have

$$\delta \left(L^{n-p} di_\phi d\alpha \right) = \left[\delta, L^{n-p} \right] di_\phi d\alpha + L^{n-p} \delta di_\phi d\alpha. \tag{4.9}$$

From Lemma 4.2 we get

$$\left[\delta, L^{n-p} \right] di_\phi d\alpha = -(n-p) L^{n-p-1} \mathcal{L}_\phi di_\phi d\alpha,$$

since the second summand in (4.5) vanishes in this case. Moreover, by using (2.11) we have $\mathcal{L}_\phi di_\phi d\alpha = \mathcal{L}^2_\phi d\alpha = -2Ld\mathcal{L}_\xi\alpha = 0$. Therefore we get

$$\left[\delta, L^{n-p} \right] di_\phi d\alpha = 0. \tag{4.10}$$

Now we compute the second summand of (4.9). We have

$$\delta di_\phi d\alpha = \triangle i_\phi d\alpha - d\delta i_\phi d\alpha. \tag{4.11}$$

From formula (2.7) for $[\triangle, i_\phi]$ we get

$$\begin{aligned} \triangle i_\phi d\alpha &= -2 \left(\mathcal{L}_\xi - i_\xi d + \epsilon_\eta \delta \right) d\alpha + i_\phi d \triangle \alpha \\ &= -2\epsilon_\eta \delta d\alpha + i_\phi d \triangle \alpha \\ &= -2\epsilon_\eta \triangle \alpha + i_\phi d \triangle \alpha, \end{aligned} \tag{4.12}$$

since $\alpha \in \Omega^{p-1}_{\mathcal{L}_\xi,\delta}(M)$ implies $\delta d\alpha = \triangle \alpha$. Now we compute the second summand of (4.11). From (2.6) we get

$$\begin{aligned} d\delta i_\phi d\alpha &= d \left[\delta, i_\phi \right] d\alpha + di_\phi \delta d\alpha \\ &= -\frac{1}{2} d \left[\triangle, i_\xi \right] d\alpha - 2(n-(p-1)) di_\xi d\alpha + di_\phi \triangle \alpha. \end{aligned} \tag{4.13}$$

Note that $d \left[\triangle, i_\xi \right] d\alpha = - \left[\triangle, i_\xi \right] d^2\alpha = 0$, since

$$\{ d, [\triangle, i_\xi] \} = \{ [d, \triangle], i_\xi \} + [\triangle, \mathcal{L}_\xi] = 0.$$

Also $di_\xi d\alpha = -d^2 i_\xi \alpha + d\mathcal{L}_\xi \alpha = 0$. Thus (4.13) becomes

$$d\delta i_\phi d\alpha = di_\phi \triangle \alpha. \tag{4.14}$$

Substituting (4.12) and (4.14) in (4.11), we get

$$\delta di_\phi d\alpha = -2\epsilon_\eta \triangle \alpha + i_\phi d \triangle \alpha - di_\phi \triangle \alpha = -2\epsilon_\eta \triangle \alpha + \mathcal{L}_\phi \triangle \alpha. \tag{4.15}$$

Thus, in view of (4.10), the formula (4.9) becomes

$$\delta\left(L^{n-p} di_\phi d\alpha\right) = -2\epsilon_\eta L^{n-p} \triangle \alpha + L^{n-p}\mathcal{L}_\phi \triangle \alpha. \tag{4.16}$$

Now we compute the value of δ on the second summand of $\mathcal{A}_p\alpha$. Since $\delta \triangle \alpha = \triangle \delta\alpha = 0$, from Lemma 4.2 it follows that

$$\begin{aligned}\delta\left(L^{n-p+1} \triangle \alpha\right) &= \left[\delta, L^{n-p+1}\right] \triangle \alpha \\ &= -(n-p+1)L^{n-p}\mathcal{L}_\phi \triangle \alpha + 2(n-p+1)\epsilon_\eta L^{n-p} \triangle \alpha \\ &= -(n-p+1)(L^{n-p}\mathcal{L}_\phi \triangle \alpha - 2\epsilon_\eta L^{n-p} \triangle \alpha).\end{aligned} \tag{4.17}$$

From (4.8), (4.16) and (4.17), we get that $\delta\mathcal{A}_p\alpha = 0$, that is $\mathcal{A}_p\alpha$ is coclosed.

Now we will prove that $\triangle \mathcal{A}_p\alpha = \mathcal{A}_p \triangle \alpha$. Since $\mathcal{A}_p\alpha$ is coclosed, we have $\triangle \mathcal{A}_p\alpha = \delta d\mathcal{A}_p\alpha$. As d commutes with L, we get from (4.8) that

$$d\mathcal{A}_p\alpha = L^{n-p+1} d \triangle \alpha.$$

Thus

$$\triangle \mathcal{A}_p\alpha = \delta d\mathcal{A}_p\alpha = \left[\delta, L^{n-p+1}\right] d \triangle \alpha + L^{n-p+1}\delta d \triangle \alpha.$$

By Lemma 4.2 we get

$$\left[\delta, L^{n-p+1}\right] d \triangle \alpha = -(n-p+1)L^{n-p}\mathcal{L}_\phi d \triangle \alpha = (n-p+1)L^{n-p} di_\phi d \triangle \alpha.$$

Next,

$$L^{n-p+1}\delta d \triangle \alpha = L^{n-p+1} \triangle \delta d\alpha = L^{n-p+1} \triangle^2 \alpha.$$

Thus,

$$\triangle \mathcal{A}_p\alpha = (n-p+1)L^{n-p} di_\phi d \triangle \alpha + L^{n-p+1} \triangle^2 \alpha = \mathcal{A}_p \triangle \alpha.$$

■

Corollary 4.4. *Let (M, η, g) be a compact Sasakian manifold of dimension $2n+1$ and $\alpha \in \Omega^{p-1}_{\mathcal{L}_\xi,\delta}(M)$ with $p \leq n$, such that $L^{n-p+1} d \triangle \alpha = 0$. Then $\mathcal{A}_p\alpha = 0$.*

Proof: It is easy to check that $\mathcal{A}_p\alpha$ is closed. It is also coclosed by Theorem 4.3. Therefore $\mathcal{A}_p\alpha$ is harmonic. Now we consider the form $\mathcal{A}_p G\alpha$. It follows from (4.4) and Proposition 4.1 that $G\alpha \in \Omega^{p-1}_{\mathcal{L}_\xi,\delta}(M)$. By Theorem 4.3, we have

$$\triangle \mathcal{A}_p G\alpha = \mathcal{A}_p \triangle G\alpha = \mathcal{A}_p\alpha - \mathcal{A}_p \Pi_\triangle \alpha.$$

It is immediate from the definition of $\mathcal{A}_p$ that $\mathcal{A}_p\omega = 0$ for any harmonic $(p-1)$-form ω, in particular for $\Pi_\triangle\alpha$. Thus $\triangle\, \mathcal{A}_pG\alpha = \mathcal{A}_p\alpha$. As $\mathcal{A}_p\alpha$ is harmonic, we obtain

$$0 = (\triangle^2\, \mathcal{A}_pG\alpha, \mathcal{A}_pG\alpha) = (\triangle\, \mathcal{A}_pG\alpha, \triangle\, \mathcal{A}_pG\alpha) = (\mathcal{A}_p\alpha, \mathcal{A}_p\alpha).$$

Thus $\mathcal{A}_p\alpha = 0$. ■

Now we will prove the main result of the article.

Theorem 4.5. *Let (M, η, g) be a compact Sasakian manifold, $p \leq n$, and β a closed p-form. For any $\beta' \in [\beta]$ such that*

$$i_\xi\beta' = 0, \quad L^{n-p+1}\beta' = 0, \tag{4.18}$$

we have $Lef_p([\beta]) = [\epsilon_\eta L^{n-p}\beta']$. In particular, the Lefschetz map Lef_p does not depend on the choice of a compatible Sasakian metric on (M, η).

Proof: Let us define $\gamma = \delta G\,(\beta' - \Pi_\triangle\beta)$. Then, since d and δ commute with G by (4.4) and β' is closed, we get

$$d\gamma = Gd\delta\,(\beta' - \Pi_\triangle\beta) = Gd\delta\beta' = G \triangle \beta' = \beta' - \Pi_\triangle\beta' = \beta' - \Pi_\triangle\beta. \tag{4.19}$$

Since $Lef_p([\beta]) = [\epsilon_\eta L^{n-p}\Pi_\triangle\beta]$, we have to prove that $\epsilon_\eta L^{n-p}\beta'$ and $\epsilon_\eta L^{n-p}\Pi_\triangle\beta$ are in the same cohomology class. Since

$$\epsilon_\eta L^{n-p}d\gamma = \epsilon_\eta L^{n-p}\beta' - \epsilon_\eta L^{n-p}\Pi_\triangle\beta,$$

it is enough to show that $\epsilon_\eta L^{n-p}d\gamma$ is exact. Now, as

$$d\left(\epsilon_\eta L^{n-p}\gamma\right) = 2L^{n-p+1}\gamma - \epsilon_\eta L^{n-p}d\gamma,$$

the form $\epsilon_\eta L^{n-p}d\gamma$ is exact if and only if $L^{n-p+1}\gamma$ is exact.

Let $\alpha = G\gamma$, that is $\alpha = \delta G^2(\beta' - \Pi_\triangle\beta)$. We will now check that α satisfies the hypotheses of Corollary 4.4, namely that $\mathcal{L}_\xi\alpha$, $\delta\alpha$, and $L^{n-p+1}d \triangle \alpha$ are zero. Since $\mathcal{L}_\xi$ commutes with δ by Theorem 2.1 and with G by Proposition 4.1, we get

$$\mathcal{L}_\xi\alpha = \delta G^2\mathcal{L}_\xi\,(\beta' - \Pi_\triangle\beta)\,.$$

As $\Pi_\triangle\beta$ is harmonic and ξ is Killing, we have $\mathcal{L}_\xi\Pi_\triangle\beta = 0$. Moreover, β' is closed and $i_\xi\beta' = 0$ by assumption. Thus $\mathcal{L}_\xi\beta' = 0$. Therefore we get that $\mathcal{L}_\xi\alpha = 0$. As α is in the image of δ, we also have $\delta\alpha = 0$. Now

$$\triangle\, \alpha = \triangle\, G\gamma = \gamma - \Pi_\triangle\gamma = \gamma, \tag{4.20}$$

as γ is coexact and this implies $\Pi_{\triangle}\gamma = 0$. Thus by (4.18) and (4.19), we get

$$L^{n-p+1} d \triangle \alpha = L^{n-p+1} d\gamma = L^{n-p+1}\beta' - L^{n-p+1}\Pi_{\triangle}\beta = -L^{n-p+1}\Pi_{\triangle}\beta.$$

By Theorem 3.6 we know that $\epsilon_\eta L^{n-p}\Pi_{\triangle}\beta$ is harmonic, therefore

$$L^{n-p+1}\Pi_{\triangle}\beta = \frac{1}{2} d\left(\epsilon_\eta L^{n-p}\Pi_{\triangle}\beta\right) = 0.$$

We conclude that $L^{n-p+1} d \triangle \alpha = 0$. Thus all conditions of Corollary 4.4 are satisfied for α. Therefore $\mathcal{A}_p\alpha = 0$ and thus by (4.20) we get that

$$L^{n-p+1}\gamma = L^{n-p+1} \triangle \alpha = -(n-p+1)L^{n-p} d i_\phi d\alpha = d\left(-(n-p+1)L^{n-p} i_\phi d\alpha\right)$$

is an exact form. ■

Corollary 4.6. *Let (M, η) be a compact contact manifold of dimension $2n+1$. Suppose g and g' are two different Sasakian metrics on M, both compatible with η. Then, for any closed p-form β with $p \leq n$, it holds $[\epsilon_\eta L^{n-p}\Pi_{\triangle}\beta] = [\epsilon_\eta L^{n-p}\Pi_{\triangle'}\beta]$.*

Proof: We obviously have $\Pi_{\triangle'}\beta \in [\beta]$. By Proposition 2.2 applied to the Sasakian manifold (M, η, g'), we get $i_\xi \Pi_{\triangle'}\beta = 0$. Next by (4.2) we have $L^{n-p+1}\Pi_{\triangle'}\beta = 0$. Thus the form $\beta' = \Pi_{\triangle'}\beta$ satisfies all the conditions of Theorem 4.5 for the Sasakian manifold (M, η, g). Therefore $[\epsilon_\eta L^{n-p}\Pi_{\triangle'}\beta] = \mathit{Lef}_p([\beta]) = [\epsilon_\eta L^{n-p}\Pi_{\triangle}\beta]$. ■

5. Lefschetz contact manifolds

Let (M, η) be a compact contact manifold of dimension $2n+1$ such that there exists a Sasakian metric on M compatible with the contact form η. Then by choosing an arbitrary such metric g, we can define the maps Lef_p for $p \leq n$ as in (4.1) and these maps are isomorphisms.

Note that there is no obvious way to define similar maps between cohomology spaces of a general contact manifold (M, η) of dimension $2n+1$. To introduce the notion of hard Lefschetz property for a contact manifold, we define the *Lefschetz relation* between cohomology groups $H^p(M)$ and $H^{2n+1-p}(M)$ of (M, η) to be

$$\mathcal{R}_{\mathit{Lef}_p} = \left\{ \left([\beta] \, , [\epsilon_\eta L^{n-p}\beta] \right) \,\middle|\, \beta \in \Omega^p(M),\ d\beta = 0,\ i_\xi\beta = 0,\ L^{n-p+1}\beta = 0 \right\}.$$

Thus if (M, η) admits a compatible Sasakian metric, from Theorem 4.5 it follows that $\mathcal{R}_{\mathit{Lef}_p}$ is the graph of the isomorphism Lef_p. This justifies the following definition.

Definition 5.1. We say that a compact contact manifold (M, η) has the *hard Lefschetz property* if for every $p \leq n$ the relation $\mathcal{R}_{Lef_p}$ is the graph of an isomorphism $Lef_p : H^p(M) \longrightarrow H^{2n+1-p}(M)$. Such manifolds will be called *Lefschetz contact manifolds.*

There is a simple extension to Lefschetz contact manifolds of the well-known property that the odd Betti numbers b_{2k+1} $(0 \leq 2k+1 \leq n)$ of compact Sasakian manifolds are even ([9]).

Theorem 5.2. *Let (M, η) be a Lefschetz contact manifold of dimension $2n+1$. Then the odd Betti numbers b_{2k+1} are even for $0 \leq 2k+1 \leq n$.*

Proof: Let $p \leq n$. Since Lef_p is an isomorphism, using Poincaré duality we can define a nondegenerate bilinear form B on the de Rham cohomology vector space $H^p(M)$ by putting

$$B(x, x') = \int_M Lef_p(x) \smile x'.$$

Note that

$$B(x, x') = \int_M (\epsilon_\eta L^{n-p}\omega) \wedge \omega' = \int_M \eta \wedge \Phi^{n-p} \wedge \omega \wedge \omega', \tag{5.1}$$

where $\omega \in x$ and $\omega' \in x'$ are closed p-forms such that $i_\xi \omega = i_\xi \omega' = 0$ and $L^{n-p+1}\omega = L^{n-p+1}\omega' = 0$. Such ω and ω' always exist since $\mathcal{R}_{Lef_p}$ is the graph of a map. Now, (5.1) implies that $B(x, x') = (-1)^p B(x', x)$. It follows that, when p is odd, the vector space $H^p(M)$ is even dimensional. ■

It would be interesting to find a characterization of Lefschetz contact manifolds (M, η) in the spirit of [5, 21]. It would be also interesting to find explicit examples of Lefschetz contact manifolds which do not admit any Sasakian structure. We will address these matters in our future research.

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