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HARDY-LITTLEWOOD-SOBOLEV AND STEIN-WEISS INEQUALITIES ON HOMOGENEOUS LIE GROUPS

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ABSTRACT. In this note we prove the Stein-Weiss inequality on general homogeneous Lie groups. The obtained results extend previously known inequalities. Special properties of homogeneous norms play a key role in our proofs. Also, we give a simple proof of the Hardy-Littlewood-Sobolev inequality on general homogeneous Lie groups.

1. INTRODUCTION

Historically, in [10], Hardy and Littlewood considered the one dimensional fractional integral operator on $(0, \infty)$ given by

$$T_\lambda u(x) = \int_0^\infty \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < 1, \quad (1.1)$$

and proved the following theorem:

Theorem 1.1. *Let $1 < p < q < \infty$ and $u \in L^p(0, \infty)$ with $\frac{1}{q} = \frac{1}{p} + \lambda - 1$, then*

$$\|T_\lambda u\|_{L^q(0, \infty)} \leq C \|u\|_{L^p(0, \infty)}, \quad (1.2)$$

where C is a positive constant independent of u .

The N -dimensional analogue of (1.1) can be written by the formula:

$$I_\lambda u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < N. \quad (1.3)$$

The N -dimensional case of Theorem 1.1 was extended by Sobolev in [17]:

Theorem 1.2. *Let $1 < p < q < \infty$, $u \in L^p(\mathbb{R}^N)$ with $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$, then*

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{L^p(\mathbb{R}^N)}, \quad (1.4)$$

where C is a positive constant independent of u .

Later, in [19] Stein and Weiss obtained the following two-weight extension of the Hardy-Littlewood-Sobolev inequality, which is known as the Stein-Weiss inequality.

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Theorem 1.3. *Let $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{N} - 1$. If $1 < p \leq q < \infty$, then*

$$\| |x|^{-\beta} I_\lambda u \|_{L^q(\mathbb{R}^N)} \leq C \| |x|^\alpha u \|_{L^p(\mathbb{R}^N)}, \quad (1.5)$$

where C is a positive constant independent of u .

The Hardy-Littlewood-Sobolev inequality on the Heisenberg group was obtained by Folland and Stein in [5]. In [8] the authors studied the Stein-Weiss inequality on the Carnot groups. Note that in [9] the authors also proved an analogue of the Stein-Weiss inequality on the Heisenberg groups. In [20] author proved Stein-Weiss inequality on product spaces. In [1] author proved the Stein-Weiss inequality on the Euclidean half-space. In the works [2], [3], [12] and [13] authors were studied the regularity of fractional integrals on Euclidean spaces. In this note we first give a simple proof for the Hardy-Littlewood-Sobolev inequality on general homogeneous groups, recapturing the result of [16, Theorem 4.1] where a much heavier machinery was used. In the proof we follow the method of Stein and Weiss, however, special properties of homogeneous norms of the homogeneous Lie groups play a key role in our calculations. Furthermore, in Theorem 2.7 we establish the Stein-Weiss on general homogeneous groups based on the integral Hardy inequalities established in [16]. Of course, the obtained result recovers the previously known results of Abelian (Euclidean), Heisenberg, Carnot groups since the class of the homogeneous Lie groups contains those and since we can work with an arbitrary homogeneous quasi-norm. Note that in this direction systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the paper [14]. We refer to this and other papers by the authors (e.g. [15]) for further discussions.

We also note that the best constant in the Hardy-Littlewood-Sobolev inequality on the Heisenberg group is now known, see Frank and Lieb [7] (in the Euclidean case this was done earlier by Lieb in [11]). The expression for the best constant depends on the particular quasi-norm used and may change for a different choice of the quasi-norm.

The main results of this paper are as follows:

- **Hardy-Littlewood-Sobolev inequality:** Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on \mathbb{G} . Let $1 < p < q < \infty$, $0 < \lambda < Q$, $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1$. Then for all $u \in L^p(\mathbb{G})$ and $h \in L^{q'}(\mathbb{G})$ we have

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{u(y)h(x)}{|y^{-1}x|^\lambda} dx dy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \quad (1.6)$$

where C is a positive constant independent of u and h .

For the formulation similar to that of Theorem 1.2 see Theorem 2.1.

- **Stein-Weiss inequality:** Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on \mathbb{G} . Let $0 < \lambda < Q$, $1 < p \leq q < \infty$, $\alpha < \frac{Q}{p'}$, $\beta < \frac{Q}{q}$, $\alpha + \beta \geq 0$, $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then we have

$$\left| \int_{\mathbb{G}} \frac{u(y)h(x)}{|x|^\beta |y^{-1}x|^\lambda |y|^\alpha} dx dy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \quad (1.7)$$

where C is a positive constant independent of u and h .

For the formulation similar to that of Theorem 1.3 see Theorem 2.7.

Although (1.6) is clearly contained in (1.7), we still keep them as separate statements since the Hardy-Littlewood-Sobolev inequality (1.6) allows for a simple proof which is much more transparent than that of the Stein-Weiss inequality (1.7). The present proof is also much simpler than the original proof of (1.6) in [16].

Finally, let us note that the heavier machinery developed in [16] also yielded a differential version of the Stein-Weiss inequality (which may be also called Stein-Weiss-Sobolev inequality), however, in a more special case of graded groups as follows (see [16, Theorem 5.12] for details and the proof):

- **Differential Stein-Weiss (or Stein-Weiss-Sobolev) inequality:** Let \mathbb{G} be a graded Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $(Q - ap)/(pQ) + (Q - q(b - \beta))/(qQ) + (\alpha + \lambda)/Q = 2$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha, \beta, a, b)$ such that

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda |y|^\beta} dx dy \right| \leq C \|f\|_{\dot{L}_a^p(\mathbb{G})} \|g\|_{\dot{L}_b^q(\mathbb{G})} \quad (1.8)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$ and $g \in \dot{L}_b^q(\mathbb{G})$, where $\dot{L}_a^p(\mathbb{G})$ stands for a homogeneous Sobolev space of order a over L^p on the graded Lie group \mathbb{G} .

2. STEIN-WEISS INEQUALITY ON HOMOGENEOUS GROUP

Let us recall that a Lie group (on \mathbb{R}^N) \mathbb{G} with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1} x_1, \dots, \lambda^{\nu_N} x_N), \quad \nu_1, \dots, \nu_N > 0, \quad D_\lambda : \mathbb{R}^N \rightarrow \mathbb{R}^N,$$

which is an automorphism of the group \mathbb{G} for each $\lambda > 0$, is called a *homogeneous (Lie) group*. For simplicity, throughout this paper we use the notation λx for the dilation D_λ . The homogeneous dimension of the homogeneous group \mathbb{G} is denoted by $Q := \nu_1 + \dots + \nu_N$. Also, in this note we denote a homogeneous quasi-norm on \mathbb{G} by $|x|$, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \quad (2.1)$$

with the properties

- i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
- ii) $|\lambda x| = \lambda |x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
- iii) $|x| = 0$ iff $x = 0$.

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure σ on the unit quasi-sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$, so that for every $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x) dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \quad (2.2)$$

The quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ can be defined by

$$B(x, R) := \{y \in \mathbb{G} : |x^{-1}y| < R\}. \quad (2.3)$$

We refer to [6] for the original appearance of such groups, and to [4] for a recent comprehensive treatment.

Let us consider the integral operator

$$I_{\lambda, |\cdot|} u(x) = \int_{\mathbb{G}} \frac{u(y)}{|y^{-1}x|^\lambda} dy, \quad 0 < \lambda < Q. \quad (2.4)$$

Note that when $Q > \alpha > 0$ and $\lambda = Q - \alpha$ we get the Riesz potential $I_{\lambda, |\cdot|} = I_{Q-\alpha, |\cdot|}$. First we give a short proof of a version of the Hardy-Littlewood-Sobolev inequality on \mathbb{G} .

Theorem 2.1. *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on \mathbb{G} . Let $1 < p < q < \infty$, $0 < \lambda < Q$, $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1$, and $u \in L^p(\mathbb{G})$. Then we have*

$$\|I_{\lambda, |\cdot|} u\|_{L^q(\mathbb{G})} \leq C \|u\|_{L^p(\mathbb{G})}, \quad (2.5)$$

where C is a positive constant independent of u .

Remark 2.2. With the assumptions of Theorem 2.1 and $h \in L^{q'}(\mathbb{G})$, we have the following Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{u(y)h(x)}{|y^{-1}x|^\lambda} dx dy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \quad (2.6)$$

where C is a positive constant independent of u and h . This gives (1.6).

Proof of Theorem 2.1. As in the Euclidean case we will show that there is a constant $C > 0$, such that

$$m\{x : |K * u(x)| > \zeta\} \leq C \frac{\|u\|_{L^p(\mathbb{G})}^q}{\zeta^q}, \quad (2.7)$$

where m is the Haar measure on \mathbb{G} , $K(x) = |x|^{-\lambda}$ and $I_{\lambda, |\cdot|} u(x) = K * u(x)$, where $*$ is convolution. This implies inequality (2.5) via the Marcinkiewicz interpolation theorem. Let $K(x) = K_1(x) + K_2(x)$, where

$$K_1(x) := \begin{cases} K(x), & \text{if } |x| \leq \mu, \\ 0, & \text{if } |x| > \mu, \end{cases} \quad \text{and} \quad K_2(x) := \begin{cases} K(x), & \text{if } |x| > \mu, \\ 0, & \text{if } |x| \leq \mu. \end{cases} \quad (2.8)$$

Here μ is a positive constant. We have $I_{\lambda, |\cdot|} u(x) = K * u(x) = K_1 * u(x) + K_2 * u(x)$, so

$$m\{x : |K * u(x)| > 2\zeta\} \leq m\{x : |K_1 * u(x)| > \zeta\} + m\{x : |K_2 * u(x)| > \zeta\}. \quad (2.9)$$

It is enough to prove inequality (2.7) with 2ζ instead of ζ in the left-hand side of the inequality. Without loss of generality we can assume $\|u\|_{L^p(\mathbb{G})} = 1$ and by using

Chebychev's and Minkowski's inequalities, we get

$$\begin{aligned} m\{x : |K_1 * u(x)| > \zeta\} &\leq \frac{\int_{|K_1 * u| > \zeta} |K_1 * u|^p dx}{\zeta^p} \\ &\leq \frac{\|K_1 * u\|_{L^p(\mathbb{G})}^p}{\zeta^p} \leq \frac{\|K_1\|_{L^1(\mathbb{G})}^p \|u\|_{L^p(\mathbb{G})}^p}{\zeta^p} = \frac{\|K_1\|_{L^1(\mathbb{G})}^p}{\zeta^p}. \end{aligned} \quad (2.10)$$

By using (2.2) and (2.8), we compute

$$\begin{aligned} \|K_1\|_{L^1(\mathbb{G})} &= \int_{0 < |x| \leq \mu} |x|^{-\lambda} dx = \int_0^\mu r^{Q-1} r^{-\lambda} dR \int_{\mathfrak{S}} |y|^{-\lambda} d\sigma(y) \\ &= |\mathfrak{S}| \int_0^\mu r^{Q-\lambda-1} dr = \frac{|\mathfrak{S}|}{Q-\lambda} (r^{Q-\lambda}|_0^\mu) = \frac{|\mathfrak{S}|}{Q-\lambda} \mu^{Q-\lambda}, \end{aligned} \quad (2.11)$$

where $|\mathfrak{S}|$ is the $Q-1$ dimensional surface measure of the unit quasi-sphere \mathfrak{S} . By using this in (2.10), we obtain

$$m\{x : |K_1 * u(x)| > \zeta\} \leq \left(\frac{|\mathfrak{S}|}{Q-\lambda} \right)^p \frac{\mu^{(Q-\lambda)p}}{\zeta^p}. \quad (2.12)$$

Similarly by using Young's inequality, (2.2) and the assumptions, we get

$$\begin{aligned} \|K_2 * u\|_{L^\infty(\mathbb{G})} &\leq \|K_2\|_{L^{p'}(\mathbb{G})} \|u\|_{L^p(\mathbb{G})} = \left(\int_\mu^\infty r^{-\lambda p'} r^{Q-1} dr \int_{\mathfrak{S}} |y|^{-\lambda p'} d\sigma(y) \right)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{Q-\lambda p'} \right)^{\frac{1}{p'}} \left(\int_\mu^\infty r^{Q-\lambda p'-1} dr \right)^{\frac{1}{p'}} = \left(\frac{|\mathfrak{S}|}{Q-\lambda p'} \right)^{\frac{1}{p'}} (r^{Q-\lambda p'}|_\mu^\infty)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}}, \end{aligned} \quad (2.13)$$

since from the assumptions, we get $\frac{Q-\lambda p'}{p'} = \frac{Q}{p'} - \lambda = Q(1 - \frac{1}{p} - \frac{\lambda}{Q}) = -\frac{Q}{q}$. Moreover, if $\left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}} = \zeta$, then $\mu = \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{-\frac{q}{Q p'}} \zeta^{-\frac{q}{Q}}$, so we have $\|K_2 * u\|_{L^\infty(\mathbb{G})} \leq \zeta$. Thus, we have $m\{x : |K_2 * u| > \zeta\} = 0$. Combining these facts with (2.9), $\|u\|_{L^p(\mathbb{G})} = 1$ and the assumptions we establish

$$\begin{aligned} m\{x : |K * u| > 2\zeta\} &\leq \left(\frac{|\mathfrak{S}|}{Q-\lambda} \right)^p \frac{\mu^{(Q-\lambda)p}}{\zeta^p} \\ &= \left(\frac{|\mathfrak{S}|}{Q-\lambda} \right)^p \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{-\frac{q(Q-\lambda)p}{Q p'}} \zeta^{-\frac{(Q-\lambda)pq}{Q} - p} \leq C \zeta^{-\frac{(Q-\lambda)pq}{Q} - p} = C \zeta^{(\frac{\lambda}{Q} - 1)pq - p} \\ &= C \zeta^{(\frac{1}{q} - \frac{1}{p})pq - p} = C \zeta^{p-q-p} = C \frac{\|u\|_{L^p(\mathbb{G})}^q}{\zeta^q}. \end{aligned} \quad (2.14)$$

For completeness, let us recall two well-known ingredients.

Definition 2.3 ([18]). Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $V : L^p(\mathbb{G}) \rightarrow L^q(\mathbb{G})$ be a operator, then V is called an operator of *weak type* (p, q) if

$$m\{x : |Vu| > \zeta\} \leq C \left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta} \right)^q, \quad \zeta > 0, \quad (2.15)$$

where C is a positive constant and independent by u .

Let us also recall the classical Marcinkiewicz interpolation theorem:

Theorem 2.4. *Let V be sublinear operator of weak type (p_k, q_k) with $1 \leq p_k \leq q_k < \infty$, $k = 0, 1$ and $q_0 < q_1$. Then V is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ with*

$$\frac{1}{p} = \frac{1-\gamma}{p_0} + \frac{\gamma}{p_1}, \quad \frac{1}{q} = \frac{1-\gamma}{q_0} + \frac{\gamma}{q_1}, \quad (2.16)$$

for any $0 < \gamma < 1$, namely,

$$\|Vu\|_{L^q(\mathbb{G})} \leq C\|u\|_{L^p(\mathbb{G})}, \quad (2.17)$$

for any $u \in L^p(\mathbb{G})$ and C is a positive constant.

From assumptions $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1 < \frac{1}{p}$, then $q > p$. According to Definition 2.3, $I_{\lambda, |\cdot|}u$ is of weak type (p, q) , so by using the Marcinkiewicz interpolation theorem, we prove (2.5).

The proof of Theorem 2.1 is complete. \square

The following statements will be useful to prove the homogeneous group version of the Stein-Weiss inequality ([19, Theorem B*]). The next proposition is well-known, see e.g. [4, Theorem 3.1.39 and Proposition 3.1.35] and historical references therein.

Proposition 2.5. *Let \mathbb{G} be a homogeneous Lie group. Then there exists a homogeneous quasi-norm on \mathbb{G} which is a norm, that is, a homogeneous quasi-norm $|\cdot|$ which satisfies the triangle inequality*

$$|xy| \leq |x| + |y|, \quad \forall x, y \in \mathbb{G}. \quad (2.18)$$

Furthermore, all homogeneous quasi-norms on \mathbb{G} are equivalent.

The next theorem is the integral version of Hardy inequalities on general homogeneous groups that will be instrumental in our proof.

Theorem 2.6 ([16]). *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $1 < p \leq q < \infty$. Let $W(x)$ and $U(x)$, be positive functions on \mathbb{G} . Then we have the following properties:*

(1) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q W(x) dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{G}} f^p(x) U(x) dx \right)^{\frac{1}{p}} \quad (2.19)$$

holds for all $f \geq 0$ a.e. on \mathbb{G} if and only if

$$A_1 := \sup_{R>0} \left(\int_{\mathbb{G} \setminus B(0,|x|)} W(x) dx \right)^{\frac{1}{q}} \left(\int_{B(0,|x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} < \infty. \quad (2.20)$$

(2) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} f(z) dz \right)^q W(x) dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{G}} f^p(x) U(x) dx \right)^{\frac{1}{p}}, \quad (2.21)$$

holds for all $f \geq 0$ if and only if

$$A_2 := \sup_{R>0} \left(\int_{B(0,|x|)} W(x) dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} < \infty. \quad (2.22)$$

(3) If $\{C_i\}_{i=1}^2$ are the smallest constants for which (2.19) and (2.21) hold, then

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_i, \quad i = 1, 2. \quad (2.23)$$

Now we formulate the Stein-Weiss inequality on \mathbb{G} .

Theorem 2.7. *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on \mathbb{G} . Let $0 < \lambda < Q$, $1 < p < \infty$, $\alpha < \frac{Q}{p'}$, $\beta < \frac{Q}{q}$, $\alpha + \beta \geq 0$, $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $1 < p \leq q < \infty$, we have*

$$\| |x|^{-\beta} I_{\lambda, |\cdot|} u \|_{L^q(\mathbb{G})} \leq C \| |x|^{\alpha} u \|_{L^p(\mathbb{G})}. \quad (2.24)$$

where C is positive constant and independent by u .

In inequality (2.24) with $\alpha = 0$ we get the weighted Hardy-Littlewood-Sobolev inequality established in [16, Theorem 4.1]. Thus, by setting $\alpha = \beta = 0$ we get Hardy-Littlewood-Sobolev inequality on the homogeneous Lie groups. In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot|$ can be any homogeneous quasi-norm on \mathbb{R}^N , so with the usual Euclidean distance, i.e. $|\cdot| = \|\cdot\|_E$, Theorem 2.7 gives the classical result of Stein and Weiss (Theorem 1.3).

Proof of Theorem 2.7. Define

$$\| |x|^{-\beta} I_{\lambda, |\cdot|} u \|_{L^q(\mathbb{G})}^q = \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \frac{u(y)}{|x|^{\beta} |y^{-1}x|^{\lambda}} dy \right)^q dx = I_1 + I_2 + I_3, \quad (2.25)$$

where

$$I_1 = \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^{\beta} |y^{-1}x|^{\lambda}} dy \right)^q dx, \quad (2.26)$$

$$I_2 = \int_{\mathbb{G}} \left(\int_{B(0, 2|x|) \setminus B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^{\beta} |y^{-1}x|^{\lambda}} dy \right)^q dx, \quad (2.27)$$

and

$$I_3 = \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0, 2|x|)} \frac{u(y)}{|x|^{\beta} |y^{-1}x|^{\lambda}} dy \right)^q dx. \quad (2.28)$$

From now on, in view of Proposition 2.5 we can assume that our quasi-norm is actually a norm.

Step 1. Let us consider I_1 . By using Proposition 2.5 and the properties of the quasi-norm with $|y| \leq \frac{|x|}{2}$, we get

$$|x| = |x^{-1}| = |x^{-1}yy^{-1}|$$

$$\begin{aligned} &\leq |x^{-1}y| + |y^{-1}| = |y^{-1}x| + |y| \\ &\leq |y^{-1}x| + \frac{|x|}{2}. \end{aligned}$$

Then for any $\lambda > 0$, we have

$$2^\lambda |x|^{-\lambda} \geq |y^{-1}x|^{-\lambda}.$$

Therefore, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^q dx \leq 2^\lambda \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^{\beta+\lambda}} dy \right)^q dx \\ &= 2^\lambda \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} u(y) dy \right)^q |x|^{-(\beta+\lambda)q} dx. \end{aligned} \quad (2.29)$$

If condition (2.20) in Theorem 2.6 with $W(x) = |x|^{-(\beta+\lambda)q}$ and $U(y) = |y|^{\alpha p}$ in (2.19) is satisfied, then we have

$$I_1 \leq 2^\lambda \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} u(y) dy \right)^q |x|^{-(\beta+\lambda)q} dx \leq C_1 \| |x|^\alpha u \|_{L^p(\mathbb{G})}^q. \quad (2.30)$$

Let us verify condition (2.20). So from the assumption we have $\alpha < \frac{Q}{p'}$, then

$$\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \lambda}{Q} - 1 < \frac{1}{p} + \frac{\frac{Q}{p'} + \beta + \lambda}{Q} - 1 = \frac{1}{p} + \frac{1}{p'} + \frac{\beta + \lambda}{Q} - 1 = \frac{\beta + \lambda}{Q},$$

that is, $Q - (\beta + \lambda)q < 0$ and by the using polar decomposition (2.2):

$$\begin{aligned} \left(\int_{\mathbb{G} \setminus B(0, |x|)} W(x) dx \right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G} \setminus B(0, |x|)} |x|^{-(\beta+\lambda)q} dx \right)^{\frac{1}{q}} \\ &= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1} r^{-(\beta+\lambda)q} dr d\sigma(y) \right)^{\frac{1}{q}} = \left(|\mathfrak{S}| \int_R^\infty r^{Q-1-(\beta+\lambda)q} dr \right)^{\frac{1}{q}} \leq CR^{\frac{Q-(\beta+\lambda)q}{q}}. \end{aligned} \quad (2.31)$$

Since $\alpha < \frac{Q}{p'}$, we have

$$\alpha p(1 - p') + Q > \alpha p(1 - p') + \alpha p' = \alpha p + \alpha p'(1 - p) = \alpha p - \alpha p = 0.$$

So, $\alpha p(1 - p') + Q > 0$. Then, let us consider

$$\begin{aligned} \left(\int_{B(0, |x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} &= \left(\int_{B(0, |x|)} |x|^{(1-p')\alpha p} dx \right)^{\frac{1}{p'}} \\ &= \left(\int_0^R \int_{\mathfrak{S}} r^{(1-p')\alpha p} r^{Q-1} dr d\sigma(y) \right)^{\frac{1}{p'}} \leq C \left(|\mathfrak{S}| \int_0^R r^{(1-p')\alpha p + Q - 1} dr \right)^{\frac{1}{p'}} \\ &\leq CR^{\frac{(1-p')\alpha p + Q}{p'}} = CR^{\frac{Q - \alpha p'}{p'}}. \end{aligned} \quad (2.32)$$

Moreover, the assumptions imply

$$\begin{aligned} A_1 &= \sup_{R>0} \left(\int_{\mathbb{G} \setminus B(0,|x|)} W(x) dx \right)^{\frac{1}{q}} \left(\int_{B(0,|x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} \leq CR^{\frac{Q-(\beta+\lambda)q}{q} + \frac{Q-\alpha p'}{p'}} \\ &= CR^{Q(\frac{1}{q} - \frac{1}{p} - \frac{\alpha+\beta+\lambda}{Q} + 1)} = C < \infty, \end{aligned}$$

where $C = C(\alpha, \beta, p, \lambda)$ is a positive constant. Then by using (2.19), we obtain

$$I_1 \leq C \int_{\mathbb{G}} \left(\int_{B(0, \frac{|x|}{2})} u(y) dy \right)^q |x|^{-(\beta+\lambda)q} dx \leq C_1 \| |x|^\alpha u \|_{L^p(\mathbb{G})}^q. \quad (2.33)$$

Step 2. As in the previous case I_1 , now we consider I_3 . From $2|x| \leq |y|$, we calculate

$$\begin{aligned} |y| &= |y^{-1}| = |y^{-1}xx^{-1}| \leq |y^{-1}x| + |x| \\ &\leq |y^{-1}x| + \frac{|y|}{2}, \end{aligned}$$

that is,

$$\frac{|y|}{2} \leq |y^{-1}x|.$$

Then, if condition (2.22) with $W(x) = |x|^{-\beta q}$ and $U(y) = |y|^{(\alpha+\lambda)p}$ is satisfied, then we have

$$\begin{aligned} I_3 &= \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,2|x|)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^q dx \leq C \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,2|x|)} \frac{u(y)}{|x|^\beta |y|^\lambda} dy \right)^q dx \\ &= C \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,2|x|)} u(y) |y|^{-\lambda} dy \right)^q |x|^{-\beta q} dx \leq C \| |x|^\alpha u \|_{L^p(\mathbb{G})}^q. \quad (2.34) \end{aligned}$$

Now let us check condition (2.22). We have

$$\begin{aligned} \left(\int_{B(0,|x|)} W(x) dx \right)^{\frac{1}{q}} &= \left(\int_{B(0,|x|)} |x|^{-\beta q} dx \right)^{\frac{1}{q}} \\ &= \left(\int_0^R \int_{\mathfrak{S}} r^{-\beta q} r^{Q-1} dr d\sigma(y) \right)^{\frac{1}{q}} \leq CR^{\frac{Q-\beta q}{q}}, \quad (2.35) \end{aligned}$$

where $Q - \beta q > 0$, and

$$\begin{aligned} \left(\int_{\mathbb{G} \setminus B(0,|x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{G} \setminus B(0,|x|)} |x|^{(\alpha+\lambda)(1-p')p} dx \right)^{\frac{1}{p'}} \\ &= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1} r^{(\alpha+\lambda)(1-p')p} dr d\sigma(y) \right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'}}, \quad (2.36) \end{aligned}$$

where from $\beta < \frac{Q}{q}$, we obtain $Q - p'(\alpha + \lambda) < 0$.

Combining these facts we have

$$\begin{aligned} A_2 &:= \sup_{R>0} \left(\int_{B(0,|x|)} W(x) dx \right)^{\frac{1}{\theta}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} U^{1-p'}(x) dx \right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'} + \frac{Q-\beta q}{q}} \\ &= CR^{\frac{Q}{p'} - (\alpha+\beta+\lambda) + \frac{Q}{q}} = CR^{Q(\frac{1}{p'} - \frac{\alpha+\beta+\lambda}{Q} + \frac{1}{q})} = C < \infty, \end{aligned} \quad (2.37)$$

where $C = C(\alpha, \beta, p, \lambda)$ is a positive constant. Then we establish

$$I_3 = \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,2|x|)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^q dx \leq C \| |x|^\alpha u \|_{L^p(\mathbb{G})}^q. \quad (2.38)$$

Step 3. Let us estimate I_2 now.

Case 1: $p < q$. From $\frac{|x|}{2} < |y| < 2|x|$, we obtain

$$\frac{|y^{-1}x|}{2} \leq \frac{|x| + |y|}{2} = \frac{|x|}{2} + \frac{|y|}{2} < \frac{3}{2}|y|,$$

that is,

$$|y^{-1}x| < 3|y|.$$

For all $\alpha + \beta \geq 0$, we have

$$|y^{-1}x|^{\alpha+\beta} < 3^{\alpha+\beta} |y|^{\alpha+\beta} = 3^{\alpha+\beta} |y|^\alpha |y|^\beta \leq 3^{\alpha+\beta} 2^{|\beta|} |x|^\beta |y|^\alpha.$$

Therefore,

$$\begin{aligned} I_2 &= \int_{\mathbb{G}} \left(\int_{B(0,2|x|) \setminus B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^q dx \\ &\leq C \int_{\mathbb{G}} \left(\int_{B(0,2|x|) \setminus B(0, \frac{|x|}{2})} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy \right)^q dx \\ &\leq C \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy \right)^q dx = C \| I_{\lambda+\alpha+\beta, |\cdot|} \tilde{u} \|_{L^q(\mathbb{G})}^q, \end{aligned}$$

where $\tilde{u}(x) = |x|^\alpha u(x)$.

By assumption $\frac{1}{q} - \frac{1}{p} = \frac{\lambda+\alpha+\beta}{Q} - 1 < 0$, then $Q > \lambda + \alpha + \beta$ and by using Theorem 2.1 with $p < q$, we establish

$$I_2 \leq C \| I_{\lambda+\alpha+\beta, |\cdot|} \tilde{u} \|_{L^q(\mathbb{G})}^q \leq C \| \tilde{u} \|_{L^p(\mathbb{G})}^q = C \| |x|^\alpha u \|_{L^p(\mathbb{G})}^q. \quad (2.39)$$

Case 2: $p = q$. We decompose I_2 as

$$I_2 = \sum_{k \in \mathbb{Z}} \int_{2^k \leq |x| \leq 2^{k+1}} \left(\int_{B(0,2|x|) \setminus B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^p dx. \quad (2.40)$$

From $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| \leq 2^{k+1}$, we have $2^{k-1} \leq |y| \leq 2^{k+2}$ and $0 \leq |y^{-1}x| \leq 3|x| \leq 3 \cdot 2^{k+1}$.

By using Young's inequality with $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ (our case $p = q$, hence $r = 1$), we calculate

$$\begin{aligned}
I_2 &= \sum_{k \in \mathbb{Z}} \int_{2^k \leq |x| \leq 2^{k+1}} \left(\int_{B(0,2|x|) \setminus B(0, \frac{|x|}{2})} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy \right)^p dx \\
&= \sum_{k \in \mathbb{Z}} \int_{2^k \leq |x| \leq 2^{k+1}} \left(\int_{B(0,2|x|) \setminus B(0, \frac{|x|}{2})} \frac{u(y)}{|y^{-1}x|^\lambda} dy \right)^p \frac{dx}{|x|^{\beta p}} \\
&\leq \sum_{k \in \mathbb{Z}} 2^{-\beta p k} \|u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}} * |x|^{-\lambda}\|_{L^p(\mathbb{G})}^p \\
&\leq \sum_{k \in \mathbb{Z}} 2^{-\beta p k} \| |x|^{-\lambda} \cdot \chi_{\{0 \leq |y| \leq 3 \cdot 2^{k+1}\}} \|_{L^1(\mathbb{G})}^p \|u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}}\|_{L^p(\mathbb{G})}^p \\
&\leq C \sum_{k \in \mathbb{Z}} 2^{(Q-\lambda-\beta)kp} \|u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}}\|_{L^p(\mathbb{G})}^p = C \sum_{k \in \mathbb{Z}} 2^{\alpha k p} \|u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}}\|_{L^p(\mathbb{G})}^p \\
&= C \sum_{k \in \mathbb{Z}} \|2^{\alpha(k-1)} u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}}\|_{L^p(\mathbb{G})}^p \leq C \sum_{k \in \mathbb{Z}} \| |y|^\alpha u \cdot \chi_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}} \|_{L^p(\mathbb{G})}^p \\
&= C \| |x|^\alpha u \|_{L^p(\mathbb{G})}^p.
\end{aligned}$$

Theorem 2.7 is proved. \square

Remark 2.8. With assumptions Theorem 2.7 and $h \in L^{q'}(\mathbb{G})$, we have the following Stein-Weiss inequality

$$\left| \int_{\mathbb{G}} \frac{u(y)h(x)}{|x|^\beta |y^{-1}x|^\lambda |y|^\alpha} dx dy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \quad (2.41)$$

where C is a positive constant independent of u and h . This gives (1.7).

REFERENCES

- [1] J. Dou. Weighted Hardy-Littlewood-Sobolev inequalities on the upper half space. *Commun. Contemp. Math.*, 18, 1550067, 2016.
- [2] R. R. Coifman and C. Fefferman. Weighted Norm Inequalities for Maximal Functions and Singular Integrals. *Studia Mathematica*, 51: 241-250, 1974.
- [3] C. Fefferman and B. Muckenhoupt. Two Nonequivalent Conditions for Weight Functions. *Proceeding of the American Mathematical Society*, 45: 99-104, 1974.
- [4] V. Fischer, M. Ruzhansky. *Quantization on nilpotent Lie groups*. Progress in Mathematics, Vol. 314, Birkhauser, 2016. (open access book)
- [5] G. B. Folland and E. M. Stein. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429-522, 1974.
- [6] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*. Mathematical Notes, Vol. 28, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.
- [7] R. L. Frank and E. H Lieb. Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math.*, 176:349-381, 2012.
- [8] V. Guliyev, R. Mustafayev and A. Serbetci. Stein-Weiss inequalities for the fractional integral operators in Carnot groups and applications. *Complex Variables and Elliptic Equations*, 55:8-10, 847-863, 2010.
- [9] X. Han, G. Lu and J. Zhu. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities and integral systems on the Heisenberg group. *Nonlinear Analysis*, 75, 4296-4314, 2012.
- [10] G. H. Hardy and J. E. Littlewood. Some properties of fractional integrals. *I. Math.Z.*, 27:565-606, 1928.

- [11] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.*, 118:349–374, 1983.
- [12] B. Muckenhoupt and R. L. Wheeden. Weighted Norm Inequality for Fractional Integrals. *Transactions of the American Mathematical Society* 192: 261-274, 1974.
- [13] C. Perez. Two Weighted Norm Inequalities for Riesz Potentials and Uniform L^p -Weighted Sobolev Inequalities. *Indiana University Mathematics Journal*, 39: 31-44, 1990.
- [14] M. Ruzhansky and D. Suragan. Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.*, 317, 799–822, 2017.
- [15] M. Ruzhansky, D. Suragan and N. Yessirkegenov. Extended Caffarelli-Kohn-Nirenberg inequalities, and remainders, stability, and superweights for L^p -weighted Hardy inequalities. *Trans. Amer. Math. Soc. Ser. B*, 5:32–62, 2018.
- [16] M. Ruzhansky and N. Yessirkegenov. Hypoelliptic functional inequalities. arXiv:1805.01064v1, 2018.
- [17] S. L. Sobolev. On a theorem of functional analysis. *Mat. Sb. (N.S.)*, 4:471–479, 1938, English transl. in *Amer. Math. Soc. Transl. Ser. 2*, 34:39–68, 1963.
- [18] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1970.
- [19] E. M. Stein and G. Weiss. Fractional integrals on n -dimensional Euclidean Space. *J. Math. Mech.*, 7(4):503–514, 1958.
- [20] Z. Wang. Stein-Weiss Inequality on Product Spaces. arXiv:1807.01018v3, 2018.

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