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Hardy's inequalities and integral operators on Herz-Morrey spaces

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Abstract: We obtain some estimates for the operator norms of the dilation operators on Herz-Morrey spaces. These results give us the Hardy's inequalities and the mapping properties of the integral operators on Herz-Morrey spaces. As applications of this general result, we have the boundedness of the Hadamard fractional integrals on Herz-Morrey spaces. We also obtain the Hilbert inequality on Herz-Morrey spaces.

Keywords: Herz spaces, Morrey spaces, central Morrey space, Hardy's inequalities, integral operators, Boyd's indices, Hadamard fractional integrals, Hilbert inequality

MSC 2010: 42B35, 44A15, 46E30

1 Introduction

This paper focuses on the Hardy's inequalities and the boundedness of the integral operators on Herz-Morrey spaces.

Herz-Morrey spaces are extensions of Herz spaces [1] and Morrey spaces [2]. They also include the central Morrey spaces [3–6]. One of the pioneer studies on the Herz-Morrey spaces is from Lu and Xu [7] on the mapping properties of the singular integral operators on the Herz-Morrey spaces. Since then, the study of Herz-Morrey spaces inspires the introduction of a number of new function spaces including Herz-Morrey-Hardy spaces [8, 9], Herz-Morrey spaces with variable exponents [10–15] and the Herz-Morrey-Besov spaces [16].

In this paper, we study the mapping properties of integral operators on Herz-Morrey spaces. In particular, we are interested in Hadamard fractional integrals, the Hardy operator and the Hilbert operator. We find that the mapping properties of these operators rely on the operator norms of dilation operators on Herz-Morrey spaces.

The use of the dilation operators to study the mapping properties of integral operators is well studied, especially for the rearrangement-invariant Banach function spaces [17]. The study in [17, Chapter 3, Section 5] relies on the notion of Boyd's indices. The Boyd indices are also used in the study of the mapping properties of Fourier transform and the Hankel transform, see [18–20].

In this paper, we give some estimates for the operator norms of the dilation operators on Herz-Morrey spaces. With these estimates, we define and obtain the Boyd indices of Herz-Morrey spaces.

By using these indices, we establish the a general result on the mapping properties of integral operators on Herz-Morrey spaces. This general result yields the boundedness of Hadamard fractional integrals, the

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Hardy operator and the Hilbert operator on Herz-Morrey spaces. We are interested in Hadamard fractional integrals because they are fractional integrals for Mellin transform [21]. The reader is referred to [21] for the relation between Mellin transform and Hadamard fractional integrals.

It is well known that the Hardy operator is not bounded on $L^1(\mathbb{R}) = M\dot{K}^{0,0}_{1,1}(\mathbb{R})$. On the other hand, our result shows that the Hardy operator is bounded on the Herz-Morrey $M\dot{K}^{\alpha,\lambda}_{1,1}(\mathbb{R})$ when $\alpha < \lambda$. The reader is referred to Definition 2.1 for the definition of the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

Since Herz spaces and central Morrey spaces are members of Herz-Morrey spaces, our results yield the mapping properties of Hadamard fractional integrals, the Hardy operator and the Hilbert operator on Herz spaces and central Morrey spaces.

This paper is organized as follows. Section 2 contains the definition of Herz-Morrey spaces. The Boyd's indices of the Herz-Morrey spaces are obtained in this section. The main result for the Hardy's inequalities and the boundedness of integral operators on Herz-Morrey spaces is established in Section 3. As applications for the general results on the boundedness of integral operators, we also obtain the boundedness of Hadamard fractional integrals and the Hilbert inequalities on Herz-Morrey spaces. Notice that in this paper, the results on dilation operators and Boyd's indices are on Herz-Morrey spaces over $\mathbb{R}^n$ while the remaining results are on Herz-Morrey spaces over $\mathbb{R}$.

2 Herz-Morrey spaces and Boyd's indices

We give the definition of Herz-Morrey spaces in this section. We also obtain some estimates for the operator norms of the dilation operators on Herz-Morrey spaces. These estimates give the Boyd indices of the Herz-Morrey spaces.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $R_k = B_k \backslash B_{k-1}$, $k \in \mathbb{Z}$. Define $\chi_k = \chi_{R_k}$.

Definition 2.1. Let $\alpha \in \mathbb{R}$, $\lambda \ge 0$, $0 < p \le \infty$ and $0 < q < \infty$. The Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k\in\mathbb{Z}} 2^{-k\lambda} \left(\sum_{j=-\infty}^{k} 2^{j\alpha p} \|f\chi_j\|^p_{L^q} \right)^{\frac{1}{p}} < \infty.$$

When $\lambda = 0$, the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ becomes the Herz space $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ studied in [22]. In addition, when $\alpha = 0$ and $p = q$, the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ reduces to the Lebesgue space L^q.

Furthermore, when $\alpha = 0$, $p = q$ and $\lambda = \frac{n\theta}{q}$ with $0 < \theta < 1$, the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ is the central Morrey space $\dot{B}^{q,\theta}(\mathbb{R}^n)$ [3–6, 23, 24]. Recall that the central Morrey space $\dot{B}^{q,\theta}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{\dot{B}^{q,\theta}(\mathbb{R}^n)} = \left(\sup_{R>0} \frac{1}{|B(0,R)|^\theta} \int_{B(0,R)} |f(y)|^q \, dy \right)^{\frac{1}{q}} < \infty.$$

The reader is referred to [3–5] for the studies of central Morrey spaces. We use the definition of central Morrey spaces from [5, Definition 2] while we use the notion for the central Morrey spaces from [3, 4].

The study of Herz-Morrey spaces had been extended to Herz-Morrey-Hardy spaces [8, 9], Herz-Morrey-Besov spaces and the Herz-Morrey-Triebel-Lizorkin spaces [16]. Moreover, the Herz-Morrey spaces had been further generalized to the Herz-Morrey spaces built on Lebesgue spaces with variable exponents in [11, 13, 14, 16].

For $\beta \in \mathbb{R}$, $L \in \mathbb{Z}$, define $f_{\beta,L}(x) = |x|^{\beta}\chi_L(x)$. For any $j \in \mathbb{Z}$, we have

$$\|f_{\beta,L}\chi_j\|_{L^q} = \left(\int_{2^{j-1}}^{2^j} r^{\beta q} r^{n-1}\, dr\right)^{\frac{1}{q}} = C2^{\min(j,L)(\beta+\frac{n}{q})},$$

where C is independent of j.

When $\alpha + \beta + \frac{n}{q} > \lambda > \alpha$, we have

$$2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f_{\beta,L}\chi_j\|_{L^q}^p\right)^{\frac{1}{p}} = C2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}2^{p\min(j,L)(\beta+\frac{n}{q})}\right)^{\frac{1}{p}} \le C2^{-k\lambda}2^{k\alpha}2^{\min(k,L)(\beta+\frac{n}{q})}$$

for some $C > 0$ independent of k. For the case $k \ge L$, we have

$$2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f_{\beta,L}\chi_j\|_{L^q}^p\right)^{\frac{1}{p}} \le C2^{-k(\lambda-\alpha)}. \tag{2.1}$$

For the case $k < L$, we find that

$$2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f_{\beta,L}\chi_j\|_{L^q}^p\right)^{\frac{1}{p}} \le C2^{k(-\lambda+\alpha+\beta+\frac{n}{q})}. \tag{2.2}$$

Therefore, (2.1) and (2.2) conclude that

$$\|f_{\beta,L}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k\in\mathbb{Z}} 2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f_{\beta,L}\chi_j\|_{L^q}^p\right)^{\frac{1}{p}} < \infty.$$

That is, if $\alpha + \beta + \frac{n}{q} > \lambda > \alpha$, $f_{\beta,L} \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

We establish the Minkowski inequality for the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ in the following.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $\lambda \ge 0$ and $1 \le p, q < \infty$. Let m be the Lebesgue measure and μ be a signed σ-finite measure on $\mathbb{R}$. For any $m \times \mu$ measurable function $f(x,s)$ on $\mathbb{R}^n \times \mathbb{R}$, we have*

$$\left\|\int_{\mathbb{R}} f(\cdot,s)\, d\mu\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \le \int_{\mathbb{R}} \|f(\cdot,s)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\, d|\mu|.$$

Proof. The Minkowski inequality for L^q guarantees that

$$\left\|\int_{\mathbb{R}} f(\cdot,s)\chi_j(\cdot)\, d\mu\right\|_{L^q} \le \int_{\mathbb{R}} \|f(\cdot,s)\chi_j(\cdot)\|_{L^q}\, d|\mu|.$$

By applying the Minkowski inequality for ℓ^p, we obtain that for any $k \in \mathbb{Z}$

$$\begin{aligned}2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\left\|\int_{\mathbb{R}} f(\cdot,s)\chi_j(\cdot)\, d\mu\right\|_{L^q}^p\right)^{\frac{1}{p}} &\le 2^{-k\lambda}\left(\sum_{j=-\infty}^{k}\left(\int_{\mathbb{R}} 2^{j\alpha}\|f(\cdot,s)\chi_j(\cdot)\|_{L^q}\, d|\mu|\right)^p\right)^{\frac{1}{p}}\\ &\le \int_{\mathbb{R}} 2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f(\cdot,s)\|_{L^q}^p\right)^{\frac{1}{p}} d|\mu|\end{aligned}$$

$$\leq \int_{\mathbb{R}} \|f(\cdot,s)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\, d|\mu|.$$

Finally, by taking the supremum for $k \in \mathbb{Z}$ on both sides of the above inequality, we get

$$\begin{aligned}\left\|\int_{\mathbb{R}} f(\cdot,s)\, d\mu\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} &= \sup_{k\in\mathbb{Z}} 2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\left\|\int_{\mathbb{R}} f(\cdot,s)\chi_j(\cdot)\, d\mu\right\|_{L^q}^p\right)^{\frac{1}{p}}\\ &\leq \int_{\mathbb{R}} \|f(\cdot,s)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\, d|\mu|.\end{aligned}$$

□

As a special case of Theorem 2.1, we obtain

$$\left\|\int_{\mathbb{R}} f(\cdot,s)\, d\mu\right\|_{\dot{B}^{q,\theta}(\mathbb{R}^n)} \leq \int_{\mathbb{R}} \|f(\cdot,s)\|_{\dot{B}^{q,\theta}(\mathbb{R}^n)}\, d|\mu|$$

which is the Minkowski inequality for the central Morrey space $\dot{B}^{q,\theta}(\mathbb{R}^n)$.

In addition, Theorem 2.1 gives the Minkowski inequality for the Herz space. That is,

$$\left\|\int_{\mathbb{R}} f(\cdot,s)\, d\mu\right\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq \int_{\mathbb{R}} \|f(\cdot,s)\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)}\, d|\mu|.$$

Next, we study dilation operators on Herz-Morrey spaces. For any $s \in \mathbb{R}\backslash\{0\}$ and Lebesgue measurable function f, the dilation operator D_s is defined as

$$(D_s f)(x) = f(x/s), \quad x \in \mathbb{R}^n.$$

The following theorem gives us some estimates for the operator norms of D_s on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

Theorem 2.2. *Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$, $0 < p \leq \infty$ and $0 < q < \infty$. There is a $C > 0$ such that for any $s \in \mathbb{R}\backslash\{0\}$*

$$\|D_s f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leq C s^{\frac{n}{q}+\alpha-\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}. \tag{2.3}$$

Proof. It suffices to consider $s > 0$ since

$$\|f(\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \|f(-\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

For any $s > 0$, there is a unique $J \in \mathbb{Z}$ such that $2^J \leq \frac{1}{s} < 2^{J+1}$.

As $D_{1/s}\chi_j \leq \chi_{j+J-1} + \chi_{j+J} + \chi_{j+J+1}$, $j \in \mathbb{Z}$ and

$$\|(D_s f)\chi_j\|_{L^q} = s^{\frac{n}{q}}\|f(D_{1/s}\chi_j)\|_{L^q},$$

we have

$$\|(D_s f)\chi_j\|_{L^q} \leq C s^{\frac{n}{q}} \sum_{i=-1}^{1} \|f\chi_{j+J+i}\|_{L^q}, \quad \forall j \in \mathbb{Z}$$

for some $C > 0$ because $\|\cdot\|_{L^q}$ is a norm when $1 \leq q < \infty$ and $\|\cdot\|_{L^q}$ is a quasi-norm when $0 < q < 1$.

Consequently,

$$\begin{aligned}2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|(D_s f)\chi_j\|_{L^q}^p\right)^{\frac{1}{p}} &\leq C s^{\frac{n}{q}} \sum_{i=-1}^{1} 2^{-k\lambda}\left(\sum_{j=-\infty}^{k} 2^{j\alpha p}\|f\chi_{j+J+i}\|_{L^q}^p\right)^{\frac{1}{p}}\\ &\leq C s^{\frac{n}{q}} 2^{J\lambda} \sum_{i=-1}^{1} 2^{-(k+J+i)\lambda} 2^{-(J+i)\alpha}\left(\sum_{j=-\infty}^{k+J+i} 2^{j\alpha p}\|f\chi_j\|_{L^q}^p\right)^{\frac{1}{p}}\\ &\leq C s^{\frac{n}{q}+\alpha-\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\end{aligned}$$

for some $C > 0$ independent of f and s. □

We modify the definition of Boyd's indices for rearrangement-invariant Banach function spaces from [17, Chapter 3, Definition 5.12] to define the Boyd indices for Herz-Morrey spaces.

Definition 2.2. Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$, $0 < p \leq \infty$ and $0 < q < \infty$. Define

$$\underline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \lim_{s\to\infty} \frac{\log \|D_s\|}{\log s}, \quad \overline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \lim_{s\to 0^+} \frac{\log \|D_s\|}{\log s},$$

where $\|D_s\|$ is the operator norm of $D_s : M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n) \to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

We also have the corresponding definitions of the Boyd indices for central Morrey spaces $\dot{B}^{q,\theta}(\mathbb{R}^n)$ and Herz spaces $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$.

Theorem 2.2 yields the formula for the Boyd indices of Herz-Morrey spaces.

Theorem 2.3. *Let* $\alpha \in \mathbb{R}$, $\lambda \geq 0$, $0 < p \leq \infty$ *and* $0 < q < \infty$. *We have*

$$\underline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \overline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \frac{n}{q} + \alpha - \lambda.$$

Proof. Since $D_{1/s}D_s f = f$, $\forall s > 0$, (2.3) gives

$$\|D_{1/s}D_s f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leq Cs^{-\frac{n}{q}-\alpha+\lambda}\|D_s f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

That is,

$$C^{-1}s^{\frac{n}{q}+\alpha-\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leq \|D_s f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}. \tag{2.4}$$

The above inequality and (2.3) yield

$$C^{-1}s^{\frac{n}{q}+\frac{1}{p}-\lambda} \leq \|D_s\| \leq Cs^{\frac{n}{q}+\alpha-\lambda}.$$

Consequently, by applying the logarithm and, then, dividing by $\log s$ on the above inequalities, we obtain

$$-\frac{\log C}{\log s} + \frac{n}{q} + \alpha - \lambda \leq \frac{\log \|D_s\|}{\log s} \leq \frac{\log C}{\log s} + \frac{n}{q} + \alpha - \lambda$$

when $s > 1$. By taking $\lim_{s\to\infty}$ on the above inequalities, we find that the limit $\lim_{s\to\infty} \frac{\log \|D_s\|}{\log s}$ exists and

$$\underline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \frac{n}{q} + \alpha - \lambda.$$

Similarly, when $0 < s < 1$, we have

$$\frac{\log C}{\log s} + \frac{n}{q} + \alpha - \lambda \leq \frac{\log \|D_s\|}{\log s} \leq -\frac{\log C}{\log s} + \frac{n}{q} + \alpha - \lambda. \tag{2.5}$$

The above inequalities ensure the existence of the limit $\lim_{s\to 0^+} \frac{\log \|D_s\|}{\log s}$ and give

$$\overline{\alpha}_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \frac{n}{q} + \alpha - \lambda.$$

□

Since $\dot{B}^{q,\theta}(\mathbb{R}^n) = M\dot{K}^{0,\frac{n\theta}{q}}_{q,q}$, we have

$$\underline{\alpha}_{\dot{B}^{q,\theta}(\mathbb{R}^n)} = \overline{\alpha}_{\dot{B}^{q,\theta}(\mathbb{R}^n)} = \frac{n(1-\theta)}{q}.$$

We can also calculate the Boyd indices for $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$, namely,

$$\underline{\alpha}_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} = \overline{\alpha}_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} = \frac{n}{q} + \alpha.$$

Moreover, for any $s \in \mathbb{R}\backslash\{0\}$, we also have

$$\|D_s f\|_{\dot{B}^{q,\theta}(\mathbb{R}^n)} \leq Cs^{\frac{n(1-\theta)}{q}}\|f\|_{\dot{B}^{q,\theta}(\mathbb{R}^n)},$$
$$\|D_s f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq Cs^{\frac{n}{q}+\alpha}\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)}.$$

The above inequalities give estimates for the dilation operators on central Morrey spaces and Herz spaces.

3 Integral operators

In this section, we establish the main result of this paper, a general principle on the boundedness of integral operators and Hardy's inequalities on Herz-Morrey spaces on $\mathbb{R}$. As applications of this principle, we get the boundedness of the Hadamard fractional integrals on Herz-Morrey spaces. We also obtain the Hilbert inequalities on Herz-Morrey spaces.

We consider the integral operator

$$Tf(t) = \int_0^\infty K(s,t)f(s)\,ds, \quad t \geq 0$$

and $Tf(t) = 0$, $t < 0$ where f is a Lebesgue measurable function on $\mathbb{R}$ and K is a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$. The mapping property of this operator on Lebesgue space is named as the Hardy-Littlewood-Pólya inequalities [25, Chapter IX].

The following theorem gives the boundedness of T on Herz-Morrey spaces.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p, q < \infty$ and $K : (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. Suppose that*

$$K(\lambda s, \lambda t) = \lambda^{-1}K(s,t), \tag{3.1}$$

$$\int_0^\infty |K(x,1)|x^{-\frac{1}{q}-\alpha+\lambda}\,dx < \infty. \tag{3.2}$$

There exists a constant $C > 0$ such that for any $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$

$$\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}.$$

Proof. By using the substitution $u = \frac{s}{t}$, we find that

$$|Tf(t)| \leq \int_0^\infty |K(ut,t)||(D_{\frac{1}{u}}f)(t)|\,t du = \int_0^\infty |K(u,1)||(D_{\frac{1}{u}}f)(t)|\,du$$

because $K(\cdot,\cdot)$ satisfies (3.1).

Theorem 2.1 guarantees that

$$\begin{aligned}\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} &\leq \left\|\int_0^\infty |K(u,1)||(D_{\frac{1}{u}}f)(\cdot)|\,du\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \\ &\leq \int_0^\infty |K(u,1)|\|(D_{\frac{1}{u}}f)(\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}\,du.\end{aligned}$$

Theorem 2.3 yields

$$\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}\int_0^\infty |K(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du$$

for some $C > 0$ because $K(\cdot,\cdot)$ fulfills (3.2). □

We give an estimate for the lower bound of the operator norm $T : M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}) \to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$ when $\alpha + \frac{1}{q} > \lambda > \alpha$ and K is a nonnegative Lebesgue measurable function satisfying (3.1), (3.2) and

$$\int_0^\infty K(u,1)u^{-\frac{1}{q}-\alpha+\lambda}\,du > 0.$$

Let $\beta \in (\lambda - \alpha - \frac{1}{q}, 0)$. As $\alpha + \beta + \frac{1}{q} > \lambda > \alpha$, $f_{\beta,0} \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$. For any $M \in \mathbb{N}$, by using the substitution $u = \frac{s}{t}$, we find that

$$\begin{aligned} Tf_{\beta,0}(t) &= \int_0^\infty K(u,1)(D_{\frac{1}{u}} f_{\beta,0})(t)\,du \\ &\geq \int_0^{2^M} K(u,1)(ut)^\beta \chi_{(0,1)}(ut)\,du \\ &\geq t^\beta \chi_{(0,2^{-M})}(t) \int_0^{2^M} K(u,1)u^\beta\,du \\ &= 2^{-M\beta} D_{2^{-M}}(f_{\beta,0}(t)) \int_0^{2^M} K(u,1)u^\beta\,du. \end{aligned}$$

Consequently, (2.4) yields

$$\begin{aligned} \|Tf_{\beta,0}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} &\geq 2^{-M\beta} \|D_{2^{-M}} f_{\beta,0}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \int_0^{2^M} K(u,1)u^\beta\,du \\ &\geq C 2^{-M(\frac{1}{q}+\alpha-\lambda+\beta)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \int_0^{2^M} K(u,1)u^\beta\,du \end{aligned}$$

for some $C > 0$ independent of M and β. For any $\beta \in (\lambda - \alpha - \frac{1}{q}, 0)$, we have

$$\begin{aligned} \|T\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}) \to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} &= \sup_{\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq 1} \frac{\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}}{\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}} \\ &\geq C 2^{-M(\frac{1}{q}+\alpha-\lambda+\beta)} \int_0^{2^M} K(u,1)u^\beta\,du \end{aligned}$$

for some $C > 0$ independent of M and β.

By applying the limit $\lim_{\beta \to \lambda - \alpha - \frac{1}{q}}$ on both sides of the above inequalities, the dominated convergence theorem yields

$$\begin{aligned} \|T\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}) \to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} &\geq C \lim_{\beta \to \lambda - \alpha - \frac{1}{q}} \int_0^{2^M} K(u,1)u^\beta\,du \\ &= C \int_0^{2^M} K(u,1)u^{-\frac{1}{q}-\alpha+\lambda}\,du \end{aligned}$$

for some $C > 0$ independent of M because $u^\beta \leq 2^{M(\beta+\frac{1}{q}+\alpha-\lambda)} u^{-\frac{1}{q}-\alpha+\lambda}$ when $u \in (0, 2^M)$.

Finally, by letting M trending to infinity, we have

$$\|T\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}) \to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \geq C \int_0^\infty K(u,1)u^{-\frac{1}{q}-\alpha+\lambda}\,du.$$

For the estimates of the operator norms of integral operators on weighted Morrey spaces, see [26].

The boundedness of T on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$ relies on the integral condition (3.2) where the Boyd's indices of $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, $-\frac{1}{q}-\alpha+\lambda$, involve in (3.2). This is the main reason for the introduction of the Boyd's indices for Herz-Morrey spaces in the previous section.

For the boundedness of the above integral operator on Morrey spaces, block spaces, amalgam spaces, function space of bounded mean oscillation BMO, Campanato spaces and ball Banach function spaces, see [27–31], respectively.

As a consequence of Theorem 3.1, we have the following boundedness result for the integral operator T on central Morrey spaces.

Corollary 3.2. *Let $0<\theta<1$, $1\le q<\infty$ and $K:(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a Lebesgue measurable function. Suppose that K satisfies (3.1) and*

$$\int_0^\infty |K(x,1)|x^{-\frac{1-\theta}{q}}\,dx<\infty.$$

There exists a constant $C>0$ such that for any $f\in\dot{B}^{q,\theta}(\mathbb{R})$

$$\|Tf\|_{\dot{B}^{q,\theta}(\mathbb{R})}\le C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}.$$

Similar to the discussion on the lower estimate of the operator norm of $T: M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, we also have the lower estimate of the operator norm of $T:\dot{B}^{q,\theta}(\mathbb{R})\to\dot{B}^{q,\theta}(\mathbb{R})$. We have

$$\|T\|_{\dot{B}^{q,\theta}(\mathbb{R})\to\dot{B}^{q,\theta}(\mathbb{R})}\ge C\int_0^\infty K(x,1)x^{-\frac{1-\theta}{q}}\,dx$$

when K is nonnegative.

In addition, we have the following result for Herz spaces.

Corollary 3.3. *Let $\alpha\in\mathbb{R}$, $1\le p,q<\infty$ and $K:(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a Lebesgue measurable function. Suppose that K satisfies (3.1) and*

$$\int_0^\infty |K(x,1)|x^{-\frac{1}{q}-\alpha}\,dx<\infty.$$

There exists a constant $C>0$ such that for any $f\in\dot{K}^{\alpha}_{p,q}(\mathbb{R})$

$$\|Tf\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}\le C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}.$$

3.1 Hardy's inequality and Hilbert's inequality

In this section, we present another main result of this paper, the Hardy's inequalities on Herz-Morrey spaces. We also study the Hilbert inequality on the Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$. They are applications of Theorem 3.1.

We begin with the definitions of the Hardy operators

$$Hf(t)=\frac{1}{t}\int_0^t f(s)\,ds,$$

$$\tilde{H}f(t)=\int_t^\infty \frac{f(s)}{s}\,ds.$$

For the history, development and applications of the Hardy's inequality, the reader is referred to [32–34].

For the Hardy's inequalities on non-Lebesgue space such as the Morrey spaces, the block spaces, the amalgam spaces, Hardy type spaces and rearrangement-invariant Banach function spaces, see [28–30, 35–40].

The following is the Hardy's inequality on Herz-Morrey spaces.

Theorem 3.4. *Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$ and $1 \leq p, q < \infty$. If $\frac{1}{q} + \alpha - \lambda < 1$, then there is a constant $C > 0$ such that for any $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$*

$$\|Hf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}.$$

Proof. Let $K(s,t) = t^{-1}\chi_E(s,t)$ where $E = \{(s,t) : s < t\}$. We find that for any $\lambda > 0$, $K(\lambda s, \lambda t) = \lambda^{-1}K(s,t)$. Moreover, K satisfies

$$\int_0^\infty |K(x,1)|x^{-\frac{1}{q}-\alpha+\lambda}\,dx = \int_0^1 x^{-\frac{1}{q}-\alpha+\lambda}\,dx = \frac{x^{-\frac{1}{q}-\alpha+\lambda+1}}{-\frac{1}{q}-\alpha+\lambda+1}\Bigg|_0^1 < \infty$$

because $\frac{1}{q} + \alpha - \lambda < 1$. Theorem 3.1 gives the Hardy's inequality on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$. □

We also have the corresponding result for the operator $\tilde{H}$.

Theorem 3.5. *Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$ and $1 \leq p, q < \infty$. If $0 < \frac{1}{q} + \alpha - \lambda$, there is a constant $C > 0$ such that for any $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$*

$$\|\tilde{H}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}.$$

Since the proof of the preceding theorem is similar to the proof of Theorem 3.4, for simplicity, we leave the details to the reader.

Next, we establish the Hilbert's inequality on the Herz-Morrey spaces. For any Lebesgue measurable function f, the Hilbert operator is defined as

$$\mathcal{H}f(t) = \int_0^\infty \frac{f(s)}{s+t}\,ds.$$

Theorem 3.6. *Let $\alpha \in \mathbb{R}$, $\lambda \geq 0$ and $1 \leq p, q < \infty$. If $0 < \frac{1}{q} + \alpha - \lambda < 1$, then there is a constant $C > 0$ such that for any $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$*

$$\|\mathcal{H}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}.$$

Proof. We have

$$\mathcal{H}f(t) = \int_0^\infty K(s,t)f(s)\,ds$$

where $K(s,t) = \frac{1}{s+t}$. K obviously satisfies (3.1). Since $0 < \frac{1}{q} + \alpha - \lambda < 1$, we obtain

$$\int_0^\infty |K(x,1)|x^{-\frac{1}{q}-\alpha+\lambda}\,dx = \int_0^\infty (x+1)^{-1}x^{-\frac{1}{q}-\alpha+\lambda}\,dx \leq \int_0^1 x^{-\frac{1}{q}-\alpha+\lambda}\,dx + \int_1^\infty x^{-\frac{1}{q}-\alpha+\lambda-1}\,dx < \infty.$$

Consequently, the boundedness of $\mathcal{H}$ on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$ is assured by Theorem 3.1. □

In particular, we have the following results for central Morrey spaces and Herz spaces.

Corollary 3.7. *Let $0 < \theta < 1$ and $1 \leq q < \infty$.*

1. *There is a constant $C > 0$ such that for any $f \in \dot{B}^{q,\theta}(\mathbb{R})$, we have*

$$\|Hf\|_{\dot{B}^{q,\theta}(\mathbb{R})} \leq C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}.$$

2. *There is a constant* $C > 0$ *such that for any* $f \in \dot{B}^{q,\theta}(\mathbb{R})$,

$$\|\tilde{H}f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \leq C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}.$$

3. *There is a constant* $C > 0$ *such that for any* $f \in \dot{B}^{q,\theta}(\mathbb{R})$,

$$\|\mathcal{H}f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \leq C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}.$$

We have the above results because $0 < \frac{1-\theta}{q} < 1$ is valid when $0 < \theta < 1$ and $1 \leq q < \infty$.

Corollary 3.8. *Let* $\alpha \in \mathbb{R}$ *and* $1 \leq p, q < \infty$.

1. *If* $\frac{1}{q} + \alpha < 1$, *then there is a constant* $C > 0$ *such that for any* $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$, *we have*

$$\|Hf\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \leq C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}.$$

2. *If* $0 < \frac{1}{q} + \alpha$, *there is a constant* $C > 0$ *such that for any* $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$,

$$\|\tilde{H}f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \leq C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}.$$

3. *If* $0 < \frac{1}{q} + \alpha < 1$, *then there is a constant* $C > 0$ *such that for any* $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$,

$$\|\mathcal{H}f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \leq C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}.$$

In particular, when $\alpha < 1 - \frac{1}{q}$, we have

$$\|Hf\|_{\dot{K}^{\alpha}_{q,q}(\mathbb{R})} \leq C\|f\|_{\dot{K}^{\alpha}_{q,q}(\mathbb{R})}.$$

Notice that $\dot{K}^{\alpha}_{q,q}(\mathbb{R})$ is the power weighted Lebesgue space $L^q((0,\infty), |x|^{\alpha q})$ [22, Remark 1.1.3]. This result recovers the well known results for the Hardy's inequality on power weighted Lebesgue spaces, see [40–42].

Furthermore, Corollary 3.8 also gives the Hilbert inequality on power weighted Lebesgue spaces $L^q((0,\infty), |x|^{\alpha q})$ when $0 < \frac{1}{q} + \alpha < 1$.

3.2 Hadamard fractional integrals

The Hadamard fractional integrals are the fractional integrals corresponding to the Mellin transform

$$\mathfrak{M}f(s) = \int_0^\infty u^{s-1} f(u)\, du, \quad s = c + it,\ c, t \in \mathbb{R},$$

see [43].

In [43], Butzer, Kilbas and Trujillo introduce and study the following generalizations of Hadamard fractional integrals. They are defined by using the confluent hypergeometric function, which is also named as Kummer function. The confluent hypergeometric function $\Phi[a, c; z]$ is defined for $|z| < 1$, $c > 0$ and $a \neq -j$, $j \in \mathbb{N} \cup \{0\}$ by

$$\Phi[a, c; z] = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k$, $k \in \mathbb{N} \cup \{0\}$, is the Pochhammer symbol [44, Section 6.1] given by

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}.$$

For $\beta > 0$, $\gamma \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{C}$, the generalized Hadamard fractional integrals $\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}f$, $\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma}f$, $\mathcal{I}^{\beta}_{0+,\mu;\gamma,\sigma}f$ and $\mathcal{I}^{\beta}_{-,\mu;\gamma,\sigma}f$ are defined as

$$\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \left(\frac{t}{x}\right)^{\mu} \left(\log\frac{x}{t}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log\frac{x}{t}\right] f(t) \frac{dt}{t},$$

$$\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f(x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \left(\frac{x}{t}\right)^{\mu} \left(\log\frac{t}{x}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log\frac{t}{x}\right] f(t) \frac{dt}{t},$$

$$\mathfrak{I}^{\beta}_{0+,\mu;\gamma,\sigma} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \left(\frac{t}{x}\right)^{\mu} \left(\log\frac{x}{t}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log\frac{x}{t}\right] f(t) \frac{dt}{x},$$

$$\mathfrak{I}^{\beta}_{-,\mu;\gamma,\sigma} f(x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \left(\frac{x}{t}\right)^{\mu} \left(\log\frac{t}{x}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log\frac{t}{x}\right] f(t) \frac{dt}{x},$$

and

$$(\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma} f)(x) = (\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f)(x) = (\mathfrak{I}^{\beta}_{0+,\mu;\gamma,\sigma} f)(x) = (\mathfrak{I}^{\beta}_{-,\mu;\gamma,\sigma} f)(x) = 0, \quad x \le 0,$$

where $\Gamma(\beta)$ is the Gamma function.

Note that $\Phi[a, c; 0] = 1$, when $\sigma = 0$, the above Hadamard fractional integral $\mathcal{J}^{\beta}_{0+,\mu;\gamma,0}$ becomes the Hadamard fractional integral $\mathcal{J}^{\beta}_{0+}$. Additionally, $\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma}$, $\mathfrak{I}^{\beta}_{0+,\mu;\gamma,\sigma}$ and $\mathfrak{I}^{\beta}_{-,\mu;\gamma,\sigma}$ are the Hadamard type fractional integrals introduced and studied in [21]. For the studies of these integrals and their applications on fractional calculus, see [21, 43, 45].

In order to obtain the mapping of the generalized Hadamard fractional integrals, we need to use the following asymptotic behaviours for $\Phi[a, c; x]$

$$\Phi[a, c; x] = \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \left(1 + O\left(\frac{1}{x}\right)\right) \quad \text{as} \quad x \to \infty. \tag{3.3}$$

Moreover, the limit

$$\lim_{k\to\infty} \frac{\frac{(a)_{k+1}}{(c)_{k+1}(k+1)!}}{\frac{(a)_k}{(c)_k k!}} = \lim_{k\to\infty} \frac{a+k}{c+k}\frac{1}{k+1} = 0$$

assures that

$$\Phi[a, c; x] = 1 + O(x) \quad \text{as} \quad x \to 0^+. \tag{3.4}$$

We are now ready to establish the boundedness of the generalized Hadamard fractional integrals on Herz-Morrey spaces.

Theorem 3.9. *Let* $1 \le p, q < \infty$, $\beta > 0$, $\lambda \ge 0$, $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}$ *and* $\mu, \sigma \in \mathbb{C}$.

1. *If* $\operatorname{Re}(\mu - \sigma) > \frac{1}{q} + \alpha - \lambda$, *then there exists a constant* $C > 0$ *such that for any* $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, *we have*
$$\|\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}. \tag{3.5}$$
2. *If* $\operatorname{Re}(\mu - \sigma) > -\frac{1}{q} - \alpha + \lambda$, *then there exists a constant* $C > 0$ *such that for any* $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, *we have*
$$\|\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}. \tag{3.6}$$
3. *If* $\operatorname{Re}(\mu - \sigma) > \frac{1}{q} + \alpha - \lambda - 1$, *then there exists a constant* $C > 0$ *such that for any* $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, *we have*
$$\|\mathfrak{I}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}. \tag{3.7}$$
4. *If* $\operatorname{Re}(\mu - \sigma) > 1 - \frac{1}{q} - \alpha + \lambda$, *then there exists a constant* $C > 0$ *such that for any* $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})$, *we have*
$$\|\mathfrak{I}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R})}. \tag{3.8}$$

Proof. Let $E = \{(u, x) \in (0, \infty) \times (0, \infty) : u < x\}$ and $F = \{(u, x) \in (0, \infty) \times (0, \infty) : x < u\}$. We first consider integral $\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}$. We have

$$\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma} f(x) = \int_0^\infty K_1(u, x) f(u)\, du,$$

where

$$K_1(u, x) = \frac{1}{\Gamma(\beta)} \left(\frac{u}{x}\right)^{\mu} \left(\log \frac{x}{u}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log \frac{x}{u}\right] \frac{1}{u} \chi_E(u, x).$$

For any $\lambda > 0$, we find that

$$\begin{aligned} K_1(\lambda u, \lambda x) &= \frac{1}{\Gamma(\beta)} \left(\frac{\lambda u}{\lambda x}\right)^{\mu} \left(\log \frac{\lambda x}{\lambda u}\right)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log \frac{\lambda x}{\lambda u}\right] \frac{1}{\lambda u} \chi_E(\lambda u, \lambda x) \\ &= \lambda^{-1} K_1(u, x) \end{aligned}$$

since $\chi_E(\lambda u, \lambda x) = \chi_E(u, x)$. Therefore, (3.1) is fulfilled.

Since

$$K_1(u, 1) = \frac{1}{\Gamma(\beta)} u^{\mu-1} (-\log u)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log \frac{1}{u}\right] \chi_{\{u: 0<u<1\}},$$

(3.3) and (3.4) give

$$K_1(u, 1) = \frac{1}{\Gamma(\gamma)} u^{\mu-\sigma-1} (-\log u)^{\beta-1+\gamma-\beta} \sigma^{\gamma-\beta} \left(1 + O\left(\frac{1}{\sigma \log \frac{1}{u}}\right)\right)$$

as $u \to 0^+$ and

$$K_1(u, 1) = \frac{1}{\Gamma(\beta)} u^{\mu-\sigma-1} (-\log u)^{\beta-1} \left(1 + O\left(\sigma \log \frac{1}{u}\right)\right)$$

as $u \to 1$.

By using the substitution $y = -\log u$, we have

$$\begin{aligned} \int_0^\infty |K_1(u, 1)| u^{-\frac{1}{q}-\alpha+\lambda}\, du &= \int_0^1 |K_1(u, 1)| u^{-\frac{1}{q}-\alpha+\lambda}\, du \\ &= \int_0^\infty |K_1(e^{-y}, 1)| e^{-y(-\frac{1}{q}-\alpha+\lambda+1)}\, dy \\ &\le C\left(\int_0^2 e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda)} y^{\beta-1}\, dy + \int_2^\infty e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda)} y^{-1+\gamma}\, dy\right). \end{aligned}$$

Since $\mathrm{Re}(\mu - \sigma) > \frac{1}{q} + \alpha - \lambda$, we have a constant $C > 0$ such that

$$\int_0^2 e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda)} y^{\beta-1}\, dy \le C \int_0^2 y^{\beta-1}\, dy < C.$$

Furthermore, we also have an $\epsilon > 0$ such that $\mathrm{Re}(\mu - \sigma) > \frac{1}{q} + \alpha - \lambda + \epsilon$ and

$$\int_2^\infty e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda)} y^{-1+\gamma}\, dy < \int_2^\infty e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda-\epsilon)}\, dy < C.$$

Consequently, (3.2) is fulfilled and Theorem 3.1 guarantees (3.5).

Next, we consider $\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma}$. We have

$$\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f(x) = \int_0^\infty K_2(u, x) f(u)\, du,$$

where

$$K_2(u,x) = \frac{1}{\Gamma(\beta)}\left(\frac{x}{u}\right)^{\mu}\left(\log\frac{u}{x}\right)^{\beta-1}\Phi\left[\gamma,\beta;\sigma\log\frac{u}{x}\right]\frac{1}{u}\chi_F(u,x).$$

Obviously, for any $\lambda > 0$, $K_2(\lambda u, \lambda x) = \lambda^{-1}K_2(u,x)$ and

$$K_2(u,1) = \frac{1}{\Gamma(\beta)}u^{-\mu}(\log u)^{\beta-1}\Phi[\gamma,\beta;\sigma\log u]\,u^{-1}\chi_{\{u:1<u\}}.$$

By using the substitution $y = \log u$, we find that

$$\int_0^\infty |K_2(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du = \int_1^\infty |K_2(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du = \int_0^\infty |K_2(e^y,1)|e^{y(-\frac{1}{q}-\alpha+\lambda+1)}\,dy.$$

Consequently, (3.3) and (3.4) yield

$$\int_0^\infty |K_2(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du \le C\left(\int_0^2 y^{\beta-1}dy + \int_2^\infty e^{y(-\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda)}y^{-1+\gamma}\,dy\right) < \infty$$

because $\mathrm{Re}(\mu-\sigma) > -\frac{1}{q}-\alpha+\lambda$. Therefore, Theorem 3.1 yields (3.6).

We consider the operator $\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}$. We have

$$\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}f(x) = \int_0^\infty K_3(u,x)f(u)\,du,$$

where

$$K_3(u,x) = \frac{1}{\Gamma(\beta)}\left(\frac{u}{x}\right)^{\mu}\left(\log\frac{x}{u}\right)^{\beta-1}\Phi\left[\gamma,\beta;\sigma\log\frac{x}{u}\right]\frac{1}{x}\chi_E(u,x).$$

The function K_3 fulfills (3.1) and

$$K_3(u,1) = \frac{1}{\Gamma(\beta)}u^{\mu}(-\log u)^{\beta-1}\Phi\left[\gamma,\beta;\sigma\log\frac{1}{u}\right]\chi_{\{u:0<u<1\}}.$$

As $\mathrm{Re}(\mu-\sigma) > \frac{1}{q}+\alpha-\lambda-1$, by using the substitution $y = -\log u$, we get

$$\begin{aligned}\int_0^\infty |K_3(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du &= \int_0^1 |K_3(u,1)|u^{-\frac{1}{q}-\alpha+\lambda}\,du\\ &= \int_0^\infty |K_3(e^{-y},1)|e^{-y(-\frac{1}{q}-\alpha+\lambda+1)}\,dy\\ &\le C\left(\int_0^2 y^{\beta-1}\,dy + \int_2^\infty e^{-y(\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda+1)}y^{-1+\gamma}\,dy\right) < \infty\end{aligned}$$

because $\mathrm{Re}(\mu-\sigma) > \frac{1}{q}+\alpha-\lambda-1$. Theorem 3.1 gives the boundedness of $\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma}$.

Finally, we consider $\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma}$. We have

$$\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma}f(x) = \int_0^\infty K_4(u,x)f(u)\,du,$$

where

$$K_4(u,x) = \frac{1}{\Gamma(\beta)}\left(\frac{x}{u}\right)^{\mu}\left(\log\frac{u}{x}\right)^{\beta-1}\Phi\left[\gamma,\beta;\sigma\log\frac{u}{x}\right]\frac{1}{x}\chi_F(u,x).$$

We see that K_4 satisfies (3.1) and

$$K_4(u,1) = \frac{1}{\Gamma(\beta)} u^{-\mu} (\log u)^{\beta-1} \Phi\left[\gamma, \beta; \sigma \log u\right] \chi_{\{u:1<u\}}.$$

Since $\mathrm{Re}(\mu - \sigma) > 1 - \frac{1}{q} - \alpha + \lambda$, by using the substitution $y = \log u$, we obtain

$$\begin{aligned}
\int_0^\infty |K_4(u,1)| u^{-\frac{1}{q}-\alpha+\lambda}\, du &= \int_1^\infty |K_4(u,1)| u^{-\frac{1}{q}-\alpha+\lambda}\, du \\
&= \int_0^\infty |K_4(e^{-y},1)| e^{y(-\frac{1}{q}-\alpha+\lambda+1)}\, dy \\
&\le C \left(\int_0^2 y^{\beta-1}\, dy + \int_2^\infty e^{y(-\mathrm{Re}(\mu-\sigma)-\frac{1}{q}-\alpha+\lambda+1)} y^{-1+\gamma}\, dy \right) < \infty
\end{aligned}$$

Theorem 3.1 guarantees the validity of (3.8). □

Since the central Morrey space $\dot{B}^{q,\theta}(\mathbb{R})$ and the Herz space $\dot{K}^{\alpha}_{p,q}(\mathbb{R})$ are members of Her-Morrey spaces, as special cases of Theorem 3.9, we have the mapping properties of the Hadamard fractional integrals on central Morrey spaces and Herz spaces.

Corollary 3.10. *Let $1 \le q < \infty$, $\beta > 0$, $0 < \theta < 1$, $\gamma \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{C}$.*

1. *If $\mathrm{Re}(\mu - \sigma) > \frac{1-\theta}{q}$, then there exists a constant $C > 0$ such that for any $f \in \dot{B}^{q,\theta}(\mathbb{R})$, we have*

$$\|\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \le C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}. \tag{3.9}$$

2. *If $\mathrm{Re}(\mu - \sigma) > -\frac{1-\theta}{q}$, then there exists a constant $C > 0$ such that for any $f \in \dot{B}^{q,\theta}(\mathbb{R})$, we have*

$$\|\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \le C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}. \tag{3.10}$$

3. *If $\mathrm{Re}(\mu - \sigma) > \frac{1-\theta}{q} - 1$, then there exists a constant $C > 0$ such that for any $f \in \dot{B}^{q,\theta}(\mathbb{R})$, we have*

$$\|\mathcal{I}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \le C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}. \tag{3.11}$$

4. *If $\mathrm{Re}(\mu - \sigma) > 1 - \frac{1-\theta}{q}$, then there exists a constant $C > 0$ such that for any $f \in \dot{B}^{q,\theta}(\mathbb{R})$, we have*

$$\|\mathcal{I}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{\dot{B}^{q,\theta}(\mathbb{R})} \le C\|f\|_{\dot{B}^{q,\theta}(\mathbb{R})}. \tag{3.12}$$

Corollary 3.11. *Let $1 \le p, q < \infty$, $\beta > 0$, $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{C}$.*

1. *If $\mathrm{Re}(\mu - \sigma) > \frac{1}{q} + \alpha$, then there exists a constant $C > 0$ such that for any $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$, we have*

$$\|\mathcal{J}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \le C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}. \tag{3.13}$$

2. *If $\mathrm{Re}(\mu - \sigma) > -\frac{1}{q} - \alpha$, then there exists a constant $C > 0$ such that for any $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$, we have*

$$\|\mathcal{J}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \le C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}. \tag{3.14}$$

3. *If $\mathrm{Re}(\mu - \sigma) > \frac{1}{q} + \alpha - 1$, then there exists a constant $C > 0$ such that for any $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$, we have*

$$\|\mathcal{I}^{\beta}_{0+,\mu;\gamma,\sigma} f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \le C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}. \tag{3.15}$$

4. *If $\mathrm{Re}(\mu - \sigma) > 1 - \frac{1}{q} - \alpha$, then there exists a constant $C > 0$ such that for any $f \in \dot{K}^{\alpha}_{p,q}(\mathbb{R})$, we have*

$$\|\mathcal{I}^{\beta}_{-,\mu;\gamma,\sigma} f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})} \le C\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R})}. \tag{3.16}$$

For the studies of fractional Hadamard integrals on other function spaces such as amalgam spaces, BMO and modular spaces, see [27, 30, 46].

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