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HARDY TYPE AND RELLICH TYPE INEQUALITIES ON THE HEISENBERG GROUP

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Abstract. This paper contains some interesting Hardy type inequalities and Rellich type inequalities for the left invariant vector fields on the Heisenberg group.

1. Introduction

As is well known, Hardy's inequality and Rellich's inequality in Euclidean space R^n (see [7], [9]) and their generalizations played important roles in many areas of mathematics. A natural and interesting question is: Can similar inequalities hold on the nilpotent Lie group, in particular, on the Heisenberg group H_n?

Recently Garofalo and Lanconelli [5] established the following Hardy type inequality:

$$\int_{H_n} \Big(\frac{|z|}{d}\Big)^2 \frac{|\Phi(x,y,t)|^2}{d^2} \le \Big(\frac{2}{Q-2}\Big)^2 \int_{H_n} |\nabla_{H_n}\Phi|^2,\ \forall \Phi \in C_0^\infty(H_n\backslash\{O\})$$

where d denotes the Heisenberg distance: $d(x,y,t) = (|z|^4 + t^2)^{\frac{1}{4}}$, $|z|^2 = x^2 + y^2$, $z = (x,y) \in R^n \times R^n$, $t \in R$, $O = (0,0,0)$, Q the homogeneous dimension, $\nabla_{H_n}\Phi = (X_1\Phi, \cdots, X_n\Phi, Y_1\Phi, \cdots, Y_n\Phi)$, $\{X_j, Y_j\}_{j=1}^n$ the basis of left invariant vector fields on H_n, $X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$. Then they discuss some important topics including the unique continuation of the sub-Laplacian $\Delta_{H_n} = \sum_{j=1}^n (X_j^2 + Y_j^2)$.

In this paper we give a general Hardy type inequality and Rellich type inequality on H_n. The methods here are based on the approach in Allegretto and Huang [3] for the p-Laplacian on R^n.

Theorem 1 (Hardy type inequality). *Let $\Phi \in C_0^\infty(H_n\backslash\{O\})$, $1 < p < Q$. Then it follows that*

$$\int_{H_n} |\nabla_{H_n}\Phi|^p \ge \Big(\frac{Q-p}{p}\Big)^p \int_{H_n} \Big(\frac{|z|}{d}\Big)^p \frac{|\Phi|^p}{d^p} \tag{1}$$

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and

$$\int_{H_n} |\nabla_{H_n}\Phi|^p \geq \Big(\frac{Q-p}{p}\Big)^p \int_{H_n} \Big(\frac{|z|}{d}\Big)^p \frac{|\Phi|^p}{(1+d)^p}. \tag{2}$$

Theorem 2 (Rellich type inequality)**.** *Let $\Phi \in C_0^\infty(H_n\backslash\{O\}), p \geq 1$. Then the inequality*

$$\int_{H_n} |\Delta_{H_n}\Phi|^p + C_0 \int_{H_n} \frac{|z|^{2(p-2)}}{d^{4(p-1)}}|\Phi|^p \geq C_1 \int_{H_n} \Big(\frac{|z|^{2p}}{d^{4p}}\Big)|\Phi|^p \tag{3}$$

holds, where C_0 and C_1 only depend on Q and p.

2. Proof of Theorem 1

We first deduce the Picone type identity for $\{X_j, Y_j\}$ which is especially useful for existence and nonexistence of p-sub-Laplace's equations and systems (for Laplace's equations and systems in R^n, see [3]).

Lemma 2.1 (Picone type identity)**.** *For differentiable functions $v > 0, u \geq 0$ on $\Omega \subset H_n$, where Ω is a bounded or unbounded domain in H_n, or the whole space H_n, it holds that*

$$L(u,v) = R(u,v) \geq 0, \tag{4}$$

where

$$L(u,v) = |\nabla_{H_n}u|^p + (p-1)\frac{u^p}{v^p}|\nabla_{H_n}v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla_{H_n}u|\nabla_{H_n}v|^{p-2}\nabla_{H_n}v,$$

$$R(u,v) = |\nabla_{H_n}u|^p - \nabla_{H_n}\Big(\frac{u^p}{v^{p-1}}\Big)|\nabla_{H_n}v|^{p-2}\nabla_{H_n}v.$$

Moreover, $L(u,v) = 0$ a.e. on Ω iff $\nabla_{H_n}(\frac{u}{v}) = 0$ a.e. on Ω.

Proof. Since

$$\nabla_{H_n}\Big(\frac{u^p}{v^{p-1}}\Big) = \frac{1}{v^{2p-2}}\Big[pu^{p-1}v^{p-1}\nabla_{H_n}u - (p-1)u^pv^{p-2}\nabla_{H_n}v\Big],$$

$$\nabla_{H_n}\Big(\frac{u^p}{v^{p-1}}\Big)\cdot\nabla_{H_n}v = p\frac{u^{p-1}}{v^{p-1}}\nabla_{H_n}u\cdot\nabla_{H_n}v - (p-1)\frac{u^p}{v^p}|\nabla_{H_n}v|^2,$$

it follows that (4) is obtained. On the other hand, $\frac{1}{p}+\frac{1}{q} = 1$ and Young's inequality yield

$$\begin{aligned} L(u,v) &= |\nabla_{H_n}u|^p + (p-1)\frac{u^p}{v^p}|\nabla_{H_n}v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla_{H_n}u|\cdot|\nabla_{H_n}v|^{p-1} \\ &\quad + p\frac{u^{p-1}}{v^{p-1}}|\nabla_{H_n}v|^{p-2}\Big(|\nabla_{H_n}u|\cdot|\nabla_{H_n}v| - \nabla_{H_n}u\cdot\nabla_{H_n}v\Big) \\ &= p\left[\frac{|\nabla_{H_n}u|^p}{p} + \frac{(\frac{u}{v}|\nabla_{H_n}v|)^{q(p-1)}}{q}\right] - p\frac{u^{p-1}}{v^{p-1}}|\nabla_{H_n}u|\cdot|\nabla_{H_n}v|^{p-1} \\ &\quad + p\frac{u^{p-1}}{v^{p-1}}|\nabla_{H_n}v|^{p-2}\Big(|\nabla_{H_n}u|\cdot|\nabla_{H_n}v| - \nabla_{H_n}u\cdot\nabla_{H_n}v\Big) \\ &\geq p\frac{u^{p-1}}{v^{p-1}}|\nabla_{H_n}v|^{p-2}\Big(|\nabla_{H_n}u|\cdot|\nabla_{H_n}v| - \nabla_{H_n}u\cdot\nabla_{H_n}v\Big) \geq 0 \end{aligned} \tag{5}$$

and the equality holds if and only if $|\nabla_{H_n} u| = \frac{u}{v}|\nabla_{H_n} v|$, $|\nabla_{H_n} u||\nabla_{H_n} v| = \nabla_{H_n} u \cdot \nabla_{H_n} v$. Now suppose $L(u,v)(x_0) = 0$. If $u(x_0) \neq 0$, then $\nabla_{H_n}(\frac{u}{v})(x_0) = 0$. If $u(x_0) = 0$, then $\nabla_{H_n} u = 0$ a.e. on $S = \{x \in \Omega \mid u(x) = 0\}$ and $\nabla_{H_n}(\frac{u}{v}) = 0$ a.e. on S. Therefore the statement is proved. □

Remark 1. If the vector fields $\{X_j, Y_j\}$ are replaced by vector fields satisfying Hörmander's condition, then a similar identity is also valid.

It is clear that $\nabla_{H_n}(\frac{u}{v}) = 0$ implies $u = kv$ for some constant k.

Let $S_0^{1,p}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{S^{1,p}} = \Big(\int_\Omega |u|^p + |\nabla_{H_n} u|^p\Big)^{\frac{1}{p}}.$$

Theorem 2.1. *Suppose that for some $\lambda > 0$, $v \in C^\infty(\Omega)$ satisfies*

$$-\Delta_{H_n,p} v \geq \lambda g v^{p-1} \text{ and } v > 0 \text{ in } \Omega,$$

where $\Delta_{H_n,p}$ denotes the p-sub-Laplacian on H_n, i.e.

$$\begin{aligned}\Delta_{H_n,p} = &\sum_{j=1}^n X_j\{[\sum_{j=1}^n(|X_j v|^2 + |Y_j v|^2)]^{\frac{p-2}{2}} X_j v\}\\ &+ \sum_{j=1}^n Y_j\{[\sum_{j=1}^n(|X_j v|^2 + |Y_j v|^2)]^{\frac{p-2}{2}} Y_j v\}.\end{aligned}$$

Then for any u in $S_0^{1,p}$, it holds that

$$\int_\Omega |\nabla_{H_n} u|^p \geq \lambda \int_\Omega g|u|^p. \tag{6}$$

Proof. Let $\Omega_0 \subset \Omega$, Ω_0 be compact. Take $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. By Lemma 2.1, we have

$$\begin{aligned}0 &\leq \int_{\Omega_0} L(\varphi, v) \leq \int_\Omega L(\varphi, v) = \int_\Omega R(\varphi, v)\\ &= \int_\Omega |\nabla_{H_n}\varphi|^p - \nabla_{H_n}\Big(\frac{\varphi^p}{v^{p-1}}\Big)|\nabla_{H_n} v|^{p-2}\nabla_{H_n} v\\ &= \int_\Omega \Big(|\nabla_{H_n}\varphi|^p + \frac{\varphi^p}{v^{p-1}}\Delta_{H_n,p} v\Big) \leq \int_\Omega \Big(|\nabla_{H_n}\varphi|^p - \lambda g\varphi^p\Big).\end{aligned}$$

Let $\varphi \to u$ and (6) is easily obtained. □

Proof of Theorem 1. Set $v = d^{\frac{p-Q}{p}}$. Since

$$\nabla_{H_n} v = \Big(\cdots, \frac{p-Q}{p} d^{-\frac{Q}{p}} X_j d, \cdots, \frac{p-Q}{p} d^{-\frac{Q}{p}} Y_j d, \cdots\Big),$$

then

$$|\nabla_{H_n} v| = \frac{Q-p}{p} d^{-\frac{Q}{p}-1}|z|,\ |\nabla_{H_n} v|^{p-2} = \Big(\frac{Q-p}{p}\Big)^{p-2} d^{-\frac{(p+Q)(p-2)}{p}}|z|^{p-2},$$

where we have used that $|\nabla_{H_n} d| = |z|d^{-1}$. Then

$$\begin{aligned}
-\Delta_{H_n,p} v = &-\sum_{j=1}^{n}\{X_j[(\frac{Q-p}{p})^{p-2} d^{-\frac{(p-2)(p+Q)}{p}} |z|^{p-2} X_j v] \\
&+ Y_j[(\frac{Q-p}{p})^{p-2} d^{-\frac{(p-2)(p+Q)}{p}} |z|^{p-2} Y_j v]\} \\
= &-(\frac{Q-p}{p})^{p-2} \sum_{j=1}^{n}\{-\frac{(p-2)(p+Q)}{p}\cdot\frac{p-Q}{p} d^{-\frac{(p-2)(p+Q)}{p}-1-\frac{Q}{p}} \\
&\cdot |z|^{p-2}(|X_j d|^2 + |Y_j d|^2) \\
&+ (p-2)\frac{p-Q}{p}\cdot d^{-\frac{(p-2)(p+Q)}{p}-\frac{Q}{p}} |z|^{p-4}(x_j X_j d + y_j Y_j d) \\
&+ \frac{p-Q}{p} d^{-\frac{(p-2)(p+Q)}{p}} |z|^{p-2}[-\frac{Q}{p} d^{-\frac{Q}{p}-1}(|X_j d|^2 + |Y_j d|^2) \\
&+ d^{-\frac{Q}{p}}(X_j^2 d + Y_j^2 d)]\}.
\end{aligned}$$

Note that

$$\sum_{j=1}^{n}(x_j X_j d + y_j Y_j d) = |z|^4 d^{-3},\ \Delta_{H_n} d = \sum_{j=1}^{n}(X_j^2 d + Y_j^2 d) = (Q-1)|z|^2 d^{-3}$$

and it follows that

$$\begin{aligned}
-\Delta_{H_n,p} v = &\Big(\frac{Q-p}{p}\Big)^{p-1}\Big[-\frac{(p-2)(p+Q)}{p} + (p-2) - \frac{Q}{p} + (Q-1)\Big] \\
&\cdot |z|^p d^{\frac{-p^2-pQ+Q-p}{p}} \\
= &\Big(\frac{Q-p}{p}\Big)^p \frac{|z|^p}{d^p} d^{\frac{(p-Q)(p-1)}{p}-p} = \Big(\frac{Q-p}{p}\Big)^p \frac{|z|^p}{d^p}\frac{v^{p-1}}{d^p}.
\end{aligned}$$

Inequality (1) is established by using Theorem 2.1 and inequality (2) is a consequence of (1). □

Corollary 1 (Uncertainty principle). *Suppose $u \in C_0^\infty(H_n \setminus \{O\})$. Then*

$$\frac{Q-p}{p}\int_{H_n}\frac{|z|^2}{d^2}|u|^2 \le \Big(\int_{H_n}|\nabla_{H_n} u|^p\Big)^{\frac{1}{p}}\Big(\int_{H_n}|z|^q|u|^q\Big)^{\frac{1}{q}}. \tag{7}$$

Proof. By (1) and Hölder's inequality, we get

$$\begin{aligned}
\int_{H_n}\frac{|z|^2}{d^2}|u|^2 &= \int_{H_n}\frac{|z||u|}{d^2}\cdot|z||u| \le \Big(\int_{H_n}\frac{|z|^p|u|^p}{d^{2p}}\Big)^{\frac{1}{p}}\Big(\int_{H_n}|z|^q|u|^q\Big)^{\frac{1}{q}} \\
&\le \frac{p}{Q-p}\Big(\int_{H_n}|\nabla_{H_n} u|^p\Big)^{\frac{1}{p}}\Big(\int_{H_n}|z|^q|u|^q\Big)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 2. *Suppose $u \in C_0^\infty(H_n)$. Then*

$$\int_{H_n}\frac{|z|^p|u|^p}{(1+d)^{2p}} \le \Big(\frac{p}{Q-p}\Big)^p\int_{H_n}|\nabla_{H_n} u|^p. \tag{8}$$

Remark 2. Hardy type inequalities allow us to study the following eigenvalue problem in H_n (if $p = 2$, linear; if $p \neq 2$, nonlinear) with indefinite weights

$$\begin{aligned} -\Delta_{H_n,p} u &= \lambda g |u|^{p-2} u, && \text{in } H_n, \\ u &\to 0, && \text{as } d(x,y,t) \to \infty \end{aligned}$$

(see Allegretto [1], Allegretto and Huang [2], and Huang [8] for the Laplacian or p-Laplacian case in R^n).

Remark 3. The above inequalities are also applied to the study of unique continuation for $\Delta_{H_n,p}$; see [6], [5].

3. Proof of Theorem 2

Lemma 3.1. *If v is a smooth function satisfying*

$$v > 0, \Delta_{H_n} v < 0, \ \text{in } \Omega,$$

and $u \geq 0$, $p > 1$, then it follows that

$$L_1(u,v) = R_1(u,v) \geq 0, \tag{9}$$

where

$$\begin{aligned} L_1(u,v) &= |\Delta_{H_n} u|^p - p \frac{u^{p-1}}{v^{p-1}} \Delta_{H_n} u \cdot \Delta_{H_n} v |\Delta_{H_n} v|^{p-2} + (p-1) \frac{u^p}{v^p} |\Delta_{H_n} v|^p \\ &\quad - p(p-1) \frac{u^{p-2}}{v^{p-1}} |\Delta_{H_n} v|^{p-2} \Delta_{H_n} v \\ &\quad \cdot (|\nabla_{H_n} u|^2 - 2 \frac{u}{v} \nabla_{H_n} u \cdot \nabla_{H_n} v + \frac{u^2}{v^2} |\nabla_{H_n} v|^2), \\ R_1(u,v) &= |\Delta_{H_n} u|^p - \Delta_{H_n} \big(\frac{u^p}{v^{p-1}} \big) |\Delta_{H_n} v|^{p-2} \Delta_{H_n} v. \end{aligned}$$

Proof. Note that

$$\begin{aligned} \Delta_{H_n} (\frac{u^p}{v^{p-1}}) &= \frac{1}{v^{2p-2}} [p(p-1) u^{p-2} |\nabla_{H_n} u|^2 v^{p-1} + p u^{p-1} v^{p-1} \Delta_{H_n} u \\ &\quad - 2p(p-1) u^{p-1} v^{p-2} \nabla_{H_n} u \cdot \nabla_{H_n} v - (p-1)(p-2) u^p v^{p-3} |\nabla_{H_n} v|^2 \\ &\quad - (p-1) u^p v^{p-2} \Delta_{H_n} v] \\ &\quad + \frac{2(p-1)^2}{v^{p+1}} u^p |\nabla_{H_n} v|^2 \end{aligned}$$

and (9) is evidently obtained.

Since $\Delta_{H_n} v < 0$ and

$$\frac{u^{p-1}}{v^{p-1}} \Delta_{H_n} u \Delta_{H_n} v |\Delta_{H_n} v|^{p-2} \leq \frac{|\Delta_{H_n} u|^p}{p} + \frac{1}{q} \frac{u^p}{v^p} |\Delta_{H_n} v|^p, \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$
\begin{aligned}
L_1(u,v) \geq{} & |\Delta_{H_n} u|^p + (p-1)\frac{u^p}{v^p}|\Delta_{H_n} v|^p - p\Big(\frac{|\Delta_{H_n} u|^p}{p} + \frac{1}{q}\frac{u^p}{v^p}|\Delta_{H_n} v|^p\Big) \\
& - p(p-1)\frac{u^{p-2}}{v^{p-1}}|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v|\nabla_{H_n} u - \frac{u}{v}\nabla_{H_n} v|^2 \\
={} & \Big(p-1-\frac{p}{q}\Big)\frac{u^p}{v^p}|\Delta_{H_n} v|^p - p(p-1)\frac{u^{p-2}}{v^{p-1}}|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v \\
& \cdot |\nabla_{H_n} u - \frac{u}{v}\nabla_{H_n} v|^2 \\
\geq{} & 0.
\end{aligned}
$$

□

Theorem 3.1. *Let $v \in C^\infty(\Omega)$, $v > 0$, satisfying*

$$\Delta_{H_n}(|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v) \geq -\lambda g_1 v^{p-1} + \mu g_2 v^{p-1}$$

for some constant $\lambda, \mu > 0$, $\Delta_{H_n} v < 0$. Then for any $u \in S_0^{2,p}(\Omega)$, it holds that

$$\int_\Omega |\Delta_{H_n} u|^p + \lambda \int_\Omega g_1 |u|^p \geq \mu \int_\Omega g_2 |u|^p \tag{10}$$

where $S_0^{2,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Folland-Stein space $S^{2,p}(\Omega)$ (see [4]*).*

Proof. Suppose $\Omega_0 \subset \Omega$, Ω_0 is compact. Take $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. It follows that

$$
\begin{aligned}
0 &\leq \int_{\Omega_0} L_1(\varphi, v) \leq \int_\Omega L_1(\varphi, v) = \int_\Omega R_1(\varphi, v) \\
&= \int_\Omega |\Delta_{H_n}\varphi|^p - \int_\Omega \Delta_{H_n}\Big(\frac{\varphi^p}{v^{p-1}}\Big)|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v \\
&= \int_\Omega |\Delta_{H_n}\varphi|^p - \int_\Omega \frac{\varphi^p}{v^{p-1}}\Delta_{H_n}\Big(|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v\Big) \\
&\leq \int_\Omega |\Delta_{H_n}\varphi|^p + \lambda\int_\Omega g_1\varphi^p - \mu\int_\Omega g_2\varphi^p.
\end{aligned}
$$

Letting $\varphi \to u$, we prove the result. □

Proof of Theorem 2. Set $v = d^\beta$, where $\beta < 0$ will be determined later. It is clear that

$$\Delta_{H_n} v = \beta(\beta + Q - 2)|z|^2 d^{\beta-4}.$$

Letting $\beta > 2 - Q$ yields $\Delta_{H_n} v < 0$. A direct calculation gives

$$\begin{aligned}
\Delta_{H_n}\left(|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v\right) &= \beta|\beta|^{p-2}(\beta+Q-2)^{p-1}\Delta_{H_n}[|z|^{2(p-1)}d^{(\beta-4)(p-1)}] \\
&= \beta|\beta|^{p-2}(\beta+Q-2)^{p-1}\{4n(p-1)|z|^{2(p-2)}d^{(\beta-4)(p-1)} \\
&\quad + 4(p-1)(p-2)|z|^{2(p-2)}d^{(\beta-4)(p-1)} \\
&\quad + 4(p-1)^2(\beta-4)|z|^{2(p-2)}d^{(\beta-4)(p-1)-1} \\
&\qquad \cdot \sum_{j=1}^{n}(x_j X_j d + y_j Y_j d) \\
&\quad + (p-1)(\beta-4)[(\beta-4)(p-1)-1] \\
&\qquad \cdot |z|^{2(p-1)}d^{(\beta-4)(p-1)-2}|\nabla_{H_n} d|^2 \\
&\quad + (p-1)(\beta-4)|z|^{2(p-1)}d^{(\beta-4)(p-1)-1}\Delta_{H_n} d\} \\
&= \beta|\beta|^{p-2}(\beta+Q-2)^{p-1}(p-1)\{4(n+p-2) \\
&\qquad \cdot |z|^{2(p-2)}d^{(\beta-4)(p-1)} \\
&\quad + (\beta-4)[4(p-1)+(\beta-4)(p-1)+Q-2] \\
&\qquad \cdot |z|^{2p}d^{(\beta-4)(p-1)-4}\}
\end{aligned}$$

where we have used the identities $\sum_{j=1}^{n}(x_j X_j d + y_j Y_j d) = |z|^4 d^{-3}$, $|\nabla_{H_n} d|^2 = |z|^2 d^{-2}$, $\Delta_{H_n} d = (Q-1)|z|^2 d^{-3}$.

We take β satisfying $\max(\frac{2-Q}{p-1}, \frac{-(n-2p+2)-\sqrt{n^2+4p-4}}{p-1}) < \beta < 0$ and denote

$$C_0 = -4\beta|\beta|^{p-2}(\beta+Q-2)^{p-1}(p-1)(n+p-2) > 0,$$

$$C_1 = \beta|\beta|^{p-2}(\beta+Q-2)^{p-1}(p-1)(\beta-4)\big[\beta(p-1)+Q-2\big] > 0.$$

It follows that $C_1 > C_0$ and

$$\Delta_{H_n}\left(|\Delta_{H_n} v|^{p-2}\Delta_{H_n} v\right) \geq -C_0\frac{|z|^{2(p-2)}}{d^{4(p-1)}}v^{p-1} + C_1\frac{|z|^{2p}}{d^{4p}}v^{p-1}.$$

By Theorem 3.1, (3) is deduced. ∎

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