HARDY TYPE INEQUALITIES FOR Δ_λ-LAPLACIANS

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ABSTRACT. We derive Hardy type inequalities for a large class of sub-elliptic operators that belong to the class of Δ_λ-Laplacians and find explicit values for the constants involved. Our results generalize previous inequalities obtained for Grushin type operators

$$\Delta_x + |x|^{2\alpha}\Delta_y, \qquad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \ \alpha \geq 0,$$

which were proved to be sharp.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain, where $N \geq 3$. The N-dimensional version of the classical Hardy inequality states that there exists a constant $c > 0$ such that

$$c \int_\Omega \frac{|u(x)|^2}{|x|^2} dx \leq \int_\Omega |\nabla u(x)|^2 dx,$$

for all $u \in H_0^1(\Omega)$. If the origin $\{0\}$ belongs to the set Ω, the optimal constant is $c = \left(\frac{N-2}{2}\right)^2$, but not attained in $H_0^1(\Omega)$. Hardy originally proved this inequality in 1920 for the one-dimensional case. Hardy inequalities are an important tool in the analysis of linear and non-linear PDEs (see, e.g., [1],[2],[3],[4]), and over the years the classical Hardy inequality has been improved and extended in many directions.

After the seminal paper [5] by Garofalo and Lanconelli, where the Hardy inequality for the Kohn Laplacian on the Heisenberg group was proved, a large amount of work has been devoted to Hardy type inequalities in sub-elliptic settings. For a wide bibliography regarding these topics we refer to [2].

Our aim is to derive Hardy type inequalities for a large family of degenerate elliptic operators belonging to the class of Δ_λ-*Laplacians*. In recent years, Δ_λ-Laplacians are attracting increasing attention and their properties have been widely studied ([6],[7], [8],[9], [10], [11], [12], [13],[14], [15], [16]). The class of operators we consider contains Grushin type operators

$$\Delta_x + |x|^{2\alpha}\Delta_y, \qquad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

and, e.g., operators of the form

$$\Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z, \qquad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β and γ are real positive constants. Our results extend the family of Hardy inequalities derived for Grushin type operators in [17]. Improved Hardy inequalities for Grushin type operators were obtained in [18, 19], Hardy inequalities involving the control distance in [20] and Hardy inequalities in half spaces with the degeneracy at the boundary in [21].

The proof of our inequalities is based on an approach introduced by Mitidieri in [22] for the classical Laplacian. Our results coincide for the particular case of Grushin type operators with the inequalities D'Ambrosio obtained in [17], where he proved that the inequalities are sharp. We derive explicit values for the constants in the inequalities, but are currently not able to show its optimality in the general case.

The outline of our paper is as follows: We first introduce the class of operators we consider and formulate several examples. In Section 3 we explain our approach to derive Hardy type inequalities

and give a motivation for the weights appearing in the inequalities. The main results are stated and proved in Section 4. In the appendix we illustrate the relation between the fundamental solution and Hardy inequalities and comment on the difficulties we encounter proving the optimality of the constant in our inequalities.

2. Δ_λ-LAPLACIANS

Here and in the sequel, we use the following notations. We split $\mathbb{R}^N$ into

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k},$$

and write

$$x = (x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^N, \qquad x^{(i)} = (x_1^{(i)}, \ldots, x_{N_i}^{(i)}), \qquad i = 1, \ldots, k.$$

The degenerate elliptic operators we consider are of the form

$$\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \cdots + \lambda_k^2 \Delta_{x^{(k)}},$$

where the functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are pairwise different and $\Delta_{x^{(i)}}$ denotes the classical Laplacian in $\mathbb{R}^{N_i}$. We denote by $|x|$ the euclidean norm of $x \in \mathbb{R}^m$, $m \in \mathbb{N}$, and assume the functions λ_i are of the form

$$\lambda_1(x) = 1,$$
$$\lambda_2(x) = |x^{(1)}|^{\alpha_{21}},$$
$$\lambda_3(x) = |x^{(1)}|^{\alpha_{31}}|x^{(2)}|^{\alpha_{32}},$$
$$\vdots$$
$$\lambda_k(x) = |x^{(1)}|^{\alpha_{k1}}|x^{(2)}|^{\alpha_{k2}} \cdots |x^{(k-1)}|^{\alpha_{kk-1}}, \qquad x \in \mathbb{R}^N,$$

where $\alpha_{ij} \geq 0$ for $i = 2, \ldots, k, j = 1, \ldots, i-1$. Setting $\alpha_{ij} = 0$ for $j \geq i$ we can write

$$(2.1) \qquad \lambda_i(x) = \prod_{j=1}^k |x^{(j)}|^{\alpha_{ij}}, \qquad i = 1, \ldots, k.$$

This implies that there exists a *group of dilations* $(\delta_r)_{r>0}$,

$$\delta_r : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x^{(1)}, \ldots, x^{(k)}) = (r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)}),$$

where $1 = \sigma_1 \leq \sigma_i$ such that λ_i is δ_r-*homogeneous of degree* $\sigma_i - 1$, i.e.,

$$\lambda_i(\delta_r(x)) = r^{\sigma_i - 1} \lambda_i(x), \qquad \forall x \in \mathbb{R}^N, \ r > 0, \ i = 1, \ldots, k,$$

and the operator Δ_λ is δ_r-homogeneous of degree two, i.e.,

$$\Delta_\lambda(u(\delta_r(x))) = r^2 (\Delta_\lambda u)(\delta_r(x)) \qquad \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the *homogeneous dimension* of $\mathbb{R}^N$ with respect to the group of dilations $(\delta_r)_{r>0}$, i.e.,

$$Q := \sigma_1 N_1 + \cdots + \sigma_k N_k.$$

Q will play the same role as the dimension N for the classical Laplacian in our Hardy type inequalities.

For functions λ_i of the form (2.1) we find

$$\sigma_1 = 1,$$
$$\sigma_2 = 1 + \sigma_1 \alpha_{21},$$
$$\sigma_3 = 1 + \sigma_1 \alpha_{31} + \sigma_2 \alpha_{32},$$
$$\vdots$$
$$\sigma_k = 1 + \sigma_1 \alpha_{k1} + \sigma_2 \alpha_{k2} + \cdots + \sigma_{k-1} \alpha_{kk-1}.$$

If the functions λ_i are smooth, i.e., if the exponents α_{ji} are integers, the operator Δ_λ belongs to the general class of operators studied by Hörmander in [23] and it is hypoelliptic (see Remark 1.3, [9]). The simplest example is the operator

$$\partial_{x_1}^2 + |x_1|^{2\alpha}\partial_{x_2}^2, \qquad x = (x_1, x_2) \in \mathbb{R}^2, \ \alpha \in \mathbb{N},$$

where $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}$, $i = 1, 2$, that Grushin studied in [24]. He provided a complete characterization of the hypoellipticity for such operators when lower terms with complex coefficients are added. For real $\alpha > 0$ the operator is commonly called of *Grushin-type*.

Operators Δ_λ with functions λ_i of the form (2.1) belong to the class of Δ_λ-*Laplacians*.

Franchi and Lanconelli introduced operators of Δ_λ-Laplacian type in 1982 and studied their properties in a series of papers. In [25] they defined a metric associated to these operators that plays the same role as the euclidian metric for the standard Laplacian. Using this metric in [26] and [27] they extended the classical De Giorgi theorem and obtained Sobolev type embedding theorems for such operators.

Recently, adding the assumption that the operators are homogeneous of degree two, they were named Δ_λ-Laplacians by Kogoj and Lanconelli in [9], where existence, non-existence and regularity results for solutions of the semilinear Δ_λ-Laplace equation were analyzed. The global well-posedness and longtime behavior of solutions of semilinear degenerate parabolic equations involving Δ_λ-Laplacians were studied in [11], and this result was extended in [12], where also hyperbolic problems were considered. We finally remark that the Δ_λ-Laplacians belong to the more general class of X-*elliptic operators* introduced in [28]. For these operators Hardy inequalities of other kind with weights determined by the control distance were proved by Grillo in [29].

To conclude this section we recall some of the examples in our previous paper [11].

Example 2.1. Let α be a real positive constant and $k = 2$. We consider the Grushin-type operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha}\Delta_{x^{(2)}},$$

where $\lambda = (\lambda_1, \lambda_2)$, with $\lambda_1(x) = 1$ and $\lambda_2(x) = |x^{(1)}|^\alpha$, $x \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Our group of dilations is

$$\delta_r\left(x^{(1)}, x^{(2)}\right) = \left(rx^{(1)}, r^{\alpha+1}x^{(2)}\right),$$

and the homogenous dimension with respect to $(\delta_r)_{r>0}$ is $Q = N_1 + N_2(\alpha + 1)$.

More generally, for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$ with real constants $\alpha_i > 0$, $i = 1, \ldots, k-1$, we consider

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1}\Delta_{x^{(2)}} + \ldots + |x^{(1)}|^{2\alpha_{k-1}}\Delta_{x^{(k)}}.$$

The group of dilations is given by

$$\delta_r\left(x^{(1)}, \ldots, x^{(k)}\right) = \left(rx^{(1)}, r^{1+\alpha_1}x^{(2)}, \ldots, r^{1+\alpha_{k-1}}x^{(k)}\right),$$

and the homogeneous dimension is $Q = N + \alpha_1 N_2 + \alpha_2 N_3 + \cdots + \alpha_{k-1}N_k$.

Example 2.2. For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$ with real constants $\alpha_i > 0$, $i = 1, \ldots, k-1$, we define

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1}\Delta_{x^{(2)}} + |x^{(2)}|^{2\alpha_2}\Delta_{x^{(3)}} + \ldots + |x^{(k-1)}|^{2\alpha_{k-1}}\Delta_{x^{(k)}}.$$

Then, in our notation $\lambda = (\lambda_1, \ldots, \lambda_k)$ with

$$\lambda_1(x) = 1, \quad \lambda_i(x) = |x^{(i-1)}|^{\alpha_{i-1}}, \ i = 2, \ldots, k, \quad x \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k},$$

and the group of dilations is given by

$$\delta_r\left(x^{(1)}, \ldots, x^{(k)}\right) = \left(r^{\sigma_1}x^{(1)}, \ldots, r^{\sigma_k}x^{(k)}\right)$$

with $\sigma_1 = 1$ and $\sigma_i = \alpha_{i-1}\sigma_{i-1} + 1$ for $i = 2, \ldots, k$.

In particular, if $\alpha_1 = \ldots = \alpha_{k-1} = \alpha$, the dilations become

$$\delta_r\left(x^{(1)}, \ldots, x^{(k)}\right) = \left(rx^{(1)}, r^{\alpha+1}x^{(2)}, \ldots, r^{\alpha^{k-1}+\ldots+\alpha+1}x^{(k)}\right).$$

Example 2.3. Let α, β and γ be positive real constants. For the operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha}\Delta_{x^{(2)}} + |x^{(1)}|^{2\beta}|x^{(2)}|^{2\gamma}\Delta_{x^{(3)}},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with

$$\lambda_1(x) = 1, \quad \lambda_2(x) = |x^{(1)}|^\alpha, \quad \lambda_3(x) = |x^{(1)}|^\beta|x^{(2)}|^\gamma, \quad x \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

we find the group of dilations

$$\delta_r\left(x^{(1)}, x^{(2)}, x^{(3)}\right) = \left(rx^{(1)}, r^{\alpha+1}x^{(2)}, r^{\beta+(\alpha+1)\gamma+1}x^{(3)}\right).$$

3. How we derive Hardy-type inequalities

Our Hardy type inequalities are based on the following approach indicated by Mitidieri in [22].

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open subset and $p > 1$. We assume $u \in C_0^1(\Omega)$, and the vector field $h \in C^1(\Omega; \mathbb{R}^N)$ satisfies $\mathrm{div}h > 0$. The divergence theorem implies

$$\int_\Omega |u(x)|^p \mathrm{div}h(x)\, dx = -p \int_\Omega |u(x)|^{p-2}u(x)\nabla u(x) \cdot h(x)\, dx,$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. Taking the absolute value and using Hölder's inequality we obtain

$$\int_\Omega |u(x)|^p \mathrm{div}h(x)dx = -p \int_\Omega |u(x)|^{p-2}u(x)\nabla u(x) \cdot h(x)dx$$

$$\leq p \left(\int_\Omega |u(x)|^p \mathrm{div}h(x)dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{|h(x)|^p}{(\mathrm{div}h(x))^{p-1}}|\nabla u(x)|^p dx\right)^{\frac{1}{p}},$$

and it follows that

$$(3.1) \qquad \int_\Omega |u(x)|^p \mathrm{div}h(x)dx \leq p^p \int_\Omega \frac{|h(x)|^p}{(\mathrm{div}h(x))^{p-1}}|\nabla u(x)|^p dx.$$

If we choose the vector field

$$h_\varepsilon(x) := \frac{x}{(|x|^2 + \varepsilon)^{\frac{p}{2}}},$$

where $\varepsilon > 0$, then

$$\mathrm{div}h_\varepsilon(x) = \frac{N - p\frac{|x|^2}{|x|^2+\varepsilon}}{(|x|^2 + \varepsilon)^{\frac{p}{2}}}, \qquad |h_\varepsilon(x)| = \frac{|x|}{(|x|^2 + \varepsilon)^{\frac{p}{2}}}.$$

Assuming that $N > p$ we have $\mathrm{div}h_\varepsilon > 0$, and from inequality (3.1) we obtain

$$\frac{1}{p^p}\int_\Omega \left(N - p\frac{|x|^2}{|x|^2 + \varepsilon}\right)\frac{|u(x)|^p}{(|x|^2 + \varepsilon)^{\frac{p}{2}}}\, dx \leq \int_\Omega \left(N - p\frac{|x|^2}{|x|^2 + \varepsilon}\right)^{-(p-1)}\frac{|x|^p}{(|x|^2 + \varepsilon)^{\frac{p}{2}}}|\nabla u(x)|^p\, dx.$$

Taking the limit ε tends to zero, the classical Hardy inequality follows from the dominated convergence theorem,

$$\left(\frac{N - p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx \leq \int_\Omega |\nabla u(x)|^p\, dx,$$

and by a density argument it is satisfied for all functions $u \in H_0^1(\Omega)$. If the origin $\{0\}$ belongs to the domain Ω, the constant $\frac{N-p}{p}$ is optimal, but not attained in $H_0^1(\Omega)$.

This approach can be generalized to deduce Hardy type inequalities for degenerate elliptic operators. For the operators Δ_λ with functions λ_i of the form (2.1) and a function u of class $C^1(\Omega)$ we define

$$\nabla_\lambda u := (\lambda_1 \nabla_{x^{(1)}} u, \ldots, \lambda_k \nabla_{x^{(k)}} u), \qquad \lambda_i \nabla_{x^{(i)}} := (\lambda_i \partial_{x_1^{(i)}}, \ldots, \lambda_i \partial_{x_{N_i}^{(i)}}), \quad i = 1, \ldots, k.$$

We will obtain a wide family of Hardy type inequalities, that include as particular cases inequalities of the form

$$(3.2) \qquad \left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]_\lambda^p} dx \le \int_\Omega \psi(x)|\nabla_\lambda u(x)|^p dx,$$

$$(3.3) \qquad \left(\frac{Q-p}{p}\right)^p \int_\Omega \varphi(x)\frac{|u(x)|^p}{[[x]]_\lambda^p} dx \le \int_\Omega |\nabla_\lambda u(x)|^p dx,$$

where Q is the homogeneous dimension, and φ and ψ are suitable weight functions. Moreover, $[[\cdot]]_\lambda$ is a homogeneous norm that replaces the euclidean norm in the classical Hardy inequality.

We introduce the following notation. For a vector field h of class $C^1(\Omega;\mathbb{R}^N)$ we define

$$\operatorname{div}_\lambda h := \sum_{i=1}^k \lambda_i \operatorname{div}_{x^{(i)}} h, \qquad \operatorname{div}_{x^{(i)}} h := \sum_{j=1}^{N_i} \partial_{x_j^{(i)}} h.$$

The subsequent lemma follows from the divergence theorem and can be shown similarly as inequality (3.1). See also Theorem 3.5 in [17] for the particular case of Grushin-type operators.

Lemma 3.1. *Let $h \in C^1(\Omega;\mathbb{R}^N)$ be such that $\operatorname{div}_\lambda h \ge 0$. Then, for every $p > 1$ and $u \in C_0^1(\Omega)$ such that $\frac{|h|}{(\operatorname{div}_\lambda h)^{\frac{p}{p-1}}}|\nabla_\lambda u| \in L^p(\Omega)$ we have*

$$\int_\Omega |u(x)|^p \operatorname{div}_\lambda h(x)\, dx \le p^p \int_\Omega \frac{|h(x)|^p}{(\operatorname{div}_\lambda h(x))^{p-1}}|\nabla_\lambda u(x)|^p\, dx.$$

Proof. We define

$$\sigma := \begin{pmatrix} I_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k I_k \end{pmatrix},$$

where I_i denotes the identity matrix in $\mathbb{R}^{N_i}$, $i = 1,\dots,k$. The divergence theorem implies

$$0 = \int_{\partial\Omega} |u|^p h \cdot \sigma\nu\, d\zeta = \int_\Omega \operatorname{div}_\lambda(|u|^p h)\, dx = \int_\Omega p|u|^{p-2} u\nabla_\lambda u \cdot h\, dx + \int_\Omega |u|^p \operatorname{div}_\lambda h\, dx,$$

where ν denotes the outward unit normal at $\zeta \in \partial\Omega$.
Applying Hölder's inequality we obtain

$$\int_\Omega |u|^p \operatorname{div}_\lambda h\, dx = -\int_\Omega p|u|^{p-2} u\nabla_\lambda u \cdot h\, dx \le \int_\Omega p|u|^{p-1}|\nabla_\lambda u||h|\, dx$$

$$\le p \left(\int_\Omega |u|^p(\operatorname{div}_\lambda h + \epsilon)\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{|h|^p}{(\operatorname{div}_\lambda h + \epsilon)^{p-1}}|\nabla_\lambda u|^p\, dx\right)^{\frac{1}{p}},$$

and consequently,

$$\frac{\int_\Omega |u|^p \operatorname{div}_\lambda h\, dx}{\left(\int_\Omega |u|^p(\operatorname{div}_\lambda h + \epsilon)\, dx\right)^{\frac{p-1}{p}}} \le p \left(\int_\Omega \frac{|h|^p}{(\operatorname{div}_\lambda h + \epsilon)^{p-1}}|\nabla_\lambda u|^p\, dx\right)^{\frac{1}{p}}.$$

The statement of the lemma now follows from the dominated convergence theorem. $\qquad\square$

To illustrate our approach we first consider Hardy type inequalities of the form (3.2), i.e.,

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]_\lambda^p}\, dx \le \int_\Omega \psi(x)|\nabla_\lambda u(x)|^p\, dx,$$

with a certain weight function ψ and homogeneous norm $[[\cdot]]_\lambda$.

Motivated by Lemma 3.1 we look for a function h satisfying

$$\operatorname{div}_\lambda h(x) = \frac{Q-p}{[[x]]_\lambda^p}.$$

If we choose

$$h(x) = \frac{1}{[[x]]_\lambda^p} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1(x)}, \dots, \frac{\sigma_k x^{(k)}}{\lambda_k(x)} \right),$$

and since λ_i does not depend on $x^{(i)}$ we obtain

$$\text{div}_\lambda h(x) = \frac{Q}{[[x]]_\lambda^p} - p \frac{1}{[[x]]_\lambda^{p+1}} \sum_{i=1}^k \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}(\|x\|_\lambda).$$

Consequently, the homogeneous norm $[[\cdot]]_\lambda$ should fulfill the relation

$$(3.4) \qquad \sum_{i=1}^k \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}([[x]]_\lambda) = [[x]]_\lambda.$$

On the other hand, computing the norm of h we obtain

$$|h(x)|^2 = \frac{1}{[[x]]_\lambda^{2p}} \frac{1}{\prod_{i=1}^k \lambda_i(x)^2} \left(\prod_{j\neq 1} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \dots + \prod_{j\neq k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right),$$

which motivates to consider the homogeneous norm

$$(3.5) \qquad [[x]]_\lambda = \left(\prod_{j\neq 1} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \dots + \prod_{j\neq k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^k (\sigma_i - 1))}}.$$

The exponent is determined by requiring $[[\cdot]]_\lambda$ to be δ_r-homogeneous of degree one. Since the functions λ_i are of the form (2.1), the relation (3.4) is satisfied.

4. Hardy Inequalities for Δ_λ-Laplacians

4.1. **Our homogeneous norms.** We recall that $\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \dots + \lambda_k^2 \Delta_{x^{(k)}}$ with functions λ_i of the form

$$\lambda_i(x) = \prod_{j=1}^k |x^{(j)}|^{\alpha_{ij}}, \qquad i = 1, \dots, k,$$

which are δ_r-homogeneous of degree $\sigma_i - 1$ with respect to a group of dilations

$$\delta_r(x) = (r^{\sigma_1} x^{(1)}, \dots, r^{\sigma_k} x^{(k)}), \qquad x \in \mathbb{R}^N, \ r > 0.$$

Using our previous notations follow the relations

$$\sum_{j=1}^k \alpha_{ij} \sigma_j = \sigma_i - 1, \qquad\qquad \prod_{i=1}^k \lambda_i(x) = \prod_{j=1}^k |x^{(j)}|^{\sum_{i=1}^k \alpha_{ij}}.$$

Definition 4.1. We define the homogenous norm $[[\cdot]]_\lambda$ associated to the Δ_λ-Laplacian by relation (3.5),

$$[[x]]_\lambda := \left(\prod_{i\neq 1} \lambda_i(x)^2 \sigma_1^2 |x^{(1)}|^2 + \dots + \prod_{i\neq k} \lambda_i(x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^k (\sigma_i - 1))}}, \qquad x \in \mathbb{R}^N.$$

Under our hypotheses $[[\cdot]]_\lambda$ can be written as

$$[[x]]_\lambda = \left(\prod_{j=1}^k |x^{(j)}|^{\sum_{i\neq 1} 2\alpha_{ij}} \sigma_1^2 |x^{(1)}|^2 + \dots + \prod_{j=1}^k |x^{(j)}|^{\sum_{i\neq k} 2\alpha_{ij}} \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^k (\sigma_i - 1))}}.$$

We compute the homogeneous norm $[[\cdot]]_\lambda$ for some of the operators in our previous examples.

- For Grushin-type operators

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y, \qquad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where the constant α is non-negative, the definition leads to the same distance from the origin that D'Ambrosio considered in [17],

$$[[(x,y)]]_\lambda = \left(|x|^{2(1+\alpha)} + (1+\alpha)^2|y|^2 \right)^{\frac{1}{2(1+\alpha)}}.$$

- For operators of the form

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}\Delta_z, \qquad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

with non-negative constants α and β, we obtain

$$[[(x,y,z)]]_\lambda = \left(|x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^2 + (1+\beta)^2|x|^{2\alpha}|z|^2 \right)^{\frac{1}{2(1+\alpha+\beta)}}.$$

- For Δ_λ-Laplacians of the form

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z, \qquad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where the constants α, β and γ are non-negative, we get

$$[[(x,y,z)]]_\lambda = \left(|y|^{2\gamma}|x|^{2(1+\alpha+\beta)} + (1+\alpha)^2|x|^{2\beta}|y|^{2(1+\gamma)} + (1+\mu)^2|x|^{2\alpha}|z|^2 \right)^{\frac{1}{2(1+\alpha+\mu)}},$$

where $\mu = \beta + (1+\alpha)\gamma$.

Proposition 4.2. *Our homogeneous norm $[[\cdot]]_\lambda$ satisfies the following properties:*

(1) *It is δ_r-homogeneous of degree one, i.e.,*

$$[[\delta_r(x)]]_\lambda = r[[x]]_\lambda.$$

(2) *It fulfills the relation*

$$\sum_{i=1}^{k} \sigma_i \left(x^{(i)} \cdot \nabla_{x^{(i)}} \right) [[x]]_\lambda = [[x]]_\lambda.$$

Proof. (1) Let $x \in \mathbb{R}^N$. The homogeneity of the functions λ_i implies that

$$[[\delta_r(x)]]_\lambda$$

$$= \left(\prod_{i \neq 1}(\lambda_i(\delta_r(x)))^2 \sigma_1^2 |r^{\sigma_1}x^{(1)}|^2 + \cdots + \prod_{i \neq k}(\lambda_i(\delta_r(x)))^2 \sigma_k^2 |r^{\sigma_k}x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}}$$

$$= \left(\prod_{i \neq 1}r^{2\sigma_1}r^{2(\sigma_i-1)}(\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k}r^{2\sigma_k}r^{2(\sigma_i-1)}(\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}}$$

$$= \left(r^{2+\sum_{i=1}^{k}2(\sigma_i-1)} \prod_{i \neq 1}(\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k}(\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}} = r[[x]]_\lambda.$$

(2) We observe

$$x^{(l)} \cdot \nabla_{x^{(l)}}[[x]]_\lambda$$

$$= \frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))} \left(\prod_{i \neq 1}(\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k}(\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}-1}$$

$$\left((2\sum_{j \neq 1}\alpha_{jl}) \prod_{i \neq 1}(\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + (2\sum_{j \neq k}\alpha_{jl}) \prod_{i \neq k}(\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 + 2 \prod_{i \neq l}(\lambda_i(x))^2 \sigma_l^2 |x^{(l)}|^2 \right),$$

and using the relation $\sum_{l=1}^{k} \sigma_l \alpha_{jl} = \sigma_j - 1$ it follows that

$$\sum_{l=1}^{k} \sigma_l \left(x^{(l)} \cdot \nabla_{x^{(l)}} \right) [[x]]_\lambda$$

$$= \frac{1}{2(1 + \sum_{i=1}^{k}(\sigma_i - 1))} \left(\prod_{i \neq 1} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))} - 1}$$

$$2 \left(((\sum_{j \neq 1}\sum_{l=1}^{k} \sigma_l \alpha_{jl}) + \sigma_1) \prod_{i \neq 1} (\lambda_i(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + ((\sum_{j \neq k}\sum_{l=1}^{k} \sigma_l \alpha_{jl}) + \sigma_k) \prod_{i \neq k} (\lambda_i(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)$$

$$= [[x]]_\lambda.$$

$\square$

4.2. **Main results.** We denote by $\mathring{W}_\lambda^{1,p}(\Omega)$ the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{\mathring{W}_\lambda^{1,p}(\Omega)} := \left(\int_\Omega |\nabla_\lambda u(x)|^p dx \right)^{\frac{1}{p}},$$

and for $\psi \in L^1_{loc}(\Omega)$ such that $\psi > 0$ a.e. in Ω we define the space $\mathring{W}_\lambda^{1,p}(\Omega, \psi)$ as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{\mathring{W}_\lambda^{1,p}(\Omega;\psi)} := \left(\int_\Omega |\nabla_\lambda u(x)|^p \psi(x) dx \right)^{\frac{1}{p}}.$$

Theorem 4.3. *Let $p > 1$ and $\mu_1, \ldots, \mu_k$, $s \in \mathbb{R}$ be such that $s < N_1 + \mu_1$ and*

(4.1) $-p \min\{\alpha_{1i}, \ldots \alpha_{ki}, 1\} + s < N_i + \mu_i \qquad i = 1, \ldots, k.$

Then, for every $u \in \mathring{W}_\lambda^{1,p}(\Omega, \psi)$, we have

$$\left(\frac{Q - s + \sum_{i=1}^{k} \sigma_i \mu_i}{p} \right)^p \int_\Omega \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{[[x]]_\lambda^s} |u(x)|^p \ dx \leq \int_\Omega \psi(x) |\nabla_\lambda u(x)|^p \ dx,$$

where $\psi(x) = \frac{[[x]]_\lambda^{p(1+\sum_{i=1}^{k}(\sigma_i-1))-s}}{\prod_{i=1}^{k} |x^{(i)}|^{p(\sum_{j=1}^{k} \alpha_{ji})-\mu_i}}.$

In particular, for $s = p$ and $\mu_1 = \cdots = \mu_k = 0$ we get

$$\left(\frac{Q - p}{p} \right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]_\lambda^p} \ dx \leq \int_\Omega \frac{[[x]]_\lambda^{\sum_{i=1}^{k}(\sigma_i-1)}}{\prod_{i=1}^{k} \lambda_i(x)^p} |\nabla_\lambda u(x)|^p \ dx,$$

and choosing $s = p(1 + \sum_{i=1}^{k}(\sigma_i - 1))$ and $\mu_i = p \sum_{j=1}^{k} \alpha_{ji}$ we obtain

$$\left(\frac{Q - p}{p} \right)^p \int_\Omega \frac{\prod_{i=1}^{k} \lambda_i(x)^p}{[[x]]_\lambda^{p(1+\sum_{i=1}^{k}(\sigma_i-1))}} |u(x)|^p \ dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \ dx.$$

Proof. We deduce the inequalities from Lemma 3.1. To this end for $\varepsilon > 0$ we define

$$\lambda^\varepsilon := (\lambda_1^\varepsilon, \ldots, \lambda_k^\varepsilon), \qquad \lambda_i^\varepsilon(x) := \prod_{j=1}^{k} \left(|x^{(j)}|^2 + \varepsilon \right)^{\frac{\alpha_{ij}}{2}}, \quad i = 1, \ldots, k,$$

$$[[x]]_{\varepsilon,\lambda} := \left(\sum_{j=1}^{k} \left(\prod_{i \neq j} \lambda_i^\varepsilon(x)^2 \sigma_j^2 |x^{(j)}|^2 \right) \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}}$$

and consider the function

$$h_\varepsilon(x) := \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1^\varepsilon(x)}, \ldots, \frac{\sigma_k x^{(k)}}{\lambda_k^\varepsilon(x)} \right).$$

We obtain

$$
\begin{aligned}
\mathrm{div}_\lambda h_\varepsilon(x) &= \sum_{i=1}^{k} \frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)} \nabla_{x^{(i)}} \cdot \left(\frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \sigma_i x^{(i)} \right) \\
&= \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \left(\sum_{i=1}^{k} \frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)} \left(N_i \sigma_i + \sigma_i \mu_i - s \frac{1}{[[x]]_{\varepsilon,\lambda}} \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}([[x]]_{\varepsilon,\lambda}) \right) \right) \\
&= \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} c_\varepsilon(x),
\end{aligned}
$$

where

$$
c_\varepsilon(x) := \sum_{i=1}^{k} \frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)} \left(N_i \sigma_i + \sigma_i \mu_i - s \frac{1}{[[x]]_{\varepsilon,\lambda}} \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}([[x]]_{\varepsilon,\lambda}) \right).
$$

Using Proposition 4.2 we observe that

$$
\lim_{\varepsilon \to 0} c_\varepsilon(x) = \sum_{i=1}^{k} (N_i \sigma_i + \sigma_i \mu_i - s) = Q - s + \sum_{i=1}^{k} \sigma_i \mu_i,
$$

which is positive by our hypothesis. Moreover, there exist positive constants α_1 and α_2 such that

(4.2)
$$
0 < \alpha_1 \le c_\varepsilon(x) \le \alpha_2 < \infty \qquad \forall x \in \Omega.
$$

Indeed, we compute

$$
\begin{aligned}
&x^{(l)} \cdot \nabla_{x^{(l)}}[[x]]_{\varepsilon,\lambda} \\
&= \frac{1}{2(1 + \sum_{i=1}^{k}(\sigma_i - 1))} \left(\prod_{i \neq 1}(\lambda_i^\varepsilon(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k}(\lambda_i^\varepsilon(x))^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i-1))}-1} \\
&= \left\{ \frac{|x^{(l)}|^2}{|x^{(l)}|^2 + \varepsilon} \left[\left(2 \sum_{j \neq 1} \alpha_{jl}\right) \prod_{i \neq 1}(\lambda_i^\varepsilon(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \left(2 \sum_{j \neq k} \alpha_{jl}\right) \prod_{i \neq k}(\lambda_i^\varepsilon(x))^2 \sigma_k^2 |x^{(k)}|^2 \right] \right. \\
&\quad \left. + 2 \prod_{i \neq l}(\lambda_i^\varepsilon(x))^2 \sigma_l^2 |x^{(l)}|^2 \right\},
\end{aligned}
$$

and consequently, using the relation $\sum_{l=1}^{k} \sigma_l \alpha_{jl} = \sigma_j - 1$ it follows that

$$
\begin{aligned}
c_\varepsilon(x) &= \sum_{l=1}^{k} \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} (N_l \sigma_l + \sigma_l \mu_l) - s \frac{1}{[[x]]_{\varepsilon,\lambda}} \sum_{l=1}^{k} \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} \sigma_l x^{(l)} \cdot \nabla_{x^{(l)}}([[x]]_{\varepsilon,\lambda}) \\
&\quad \sum_{l=1}^{k} \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} (N_l \sigma_l + \sigma_l \mu_l) - s \frac{1}{[[x]]_{\varepsilon,\lambda}} [[x]]_{\varepsilon,\lambda}^{1-2(1+\sum_{i=1}^{k}(\sigma_i-1))} \sum_{l=1}^{k} \sigma_l x^{(l)} \cdot \nabla_{x^{(l)}}([[x]]_{\varepsilon,\lambda}) \\
&\ge \sum_{l=1}^{k} \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} (N_l \sigma_l + \sigma_l \mu_l) - s \frac{[[x]]_{\varepsilon,\lambda}^{-2(1+\sum_{i=1}^{k}(\sigma_i-1))}}{(1 + \sum_{i=1}^{k}(\sigma_i - 1))} \cdot \\
&\quad \left\{ \left(\left(\sum_{j \neq 1} \sum_{l=1}^{k} \sigma_l \alpha_{jl}\right) + \sigma_1 \right) \prod_{i \neq 1}(\lambda_i^\varepsilon(x))^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \left(\left(\sum_{j \neq k} \sum_{l=1}^{k} \sigma_l \alpha_{jl}\right) + \sigma_k \right) \prod_{i \neq k}(\lambda_i^\varepsilon(x))^2 \sigma_k^2 |x^{(k)}|^2 \right\} \\
&= \sum_{l=1}^{k} \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} (N_l \sigma_l + \sigma_l \mu_l) - s \ge N_1 + \mu_1 - s > 0.
\end{aligned}
$$

On the other hand,

$$c_\varepsilon(x) = \sum_{l=1}^k \frac{\lambda_l(x)}{\lambda_l^\varepsilon(x)} \left(N_l \sigma_l + \sigma_l \mu_l - s \frac{1}{[[x]]_{\varepsilon,\lambda}} \sigma_l x^{(l)} \cdot \nabla_{x^{(l)}} ([[x]]_{\varepsilon,\lambda}) \right)$$

$$\leq \sum_{l=1}^k \left(N_l \sigma_l + \sigma_l \mu_l \right) < \infty,$$

which concludes the proof of property (4.2).

Moreover, we compute

$$|h_\varepsilon(x)| = \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \left(\sum_{i=1}^k \frac{\sigma_i^2 |x^{(i)}|^2}{\lambda_i^\varepsilon(x)^2} \right)^{\frac{1}{2}}$$

$$= \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \frac{\left(\sum_{i=1}^k \prod_{j\neq i} \lambda_j^\varepsilon(x)^2 \sigma_i^2 |x^{(i)}|^2 \right)^{\frac{1}{2}}}{\prod_{i=1}^k \lambda_i^\varepsilon(x)}$$

$$= \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i} [[x]]_{\varepsilon,\lambda}^{(1+\sum_{i=1}^k (\sigma_i - 1)) - s}}{\prod_{i=1}^k \lambda_i^\varepsilon(x)},$$

and Lemma 3.1 applied to h_ε yields

$$\frac{1}{p^p} \int_\Omega c_\varepsilon(x) \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} |u(x)|^p dx$$

$$\leq \int_\Omega \frac{1}{c_\varepsilon(x)^{(p-1)}} \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_{\varepsilon,\lambda}^s} \left(\sum_{i=1}^k \frac{\sigma_i^2 |x^{(i)}|^2}{\lambda_i^\varepsilon(x)^2} \right)^{\frac{p}{2}} |\nabla_\lambda u(x)|^p \ dx,$$

$$\leq \frac{1}{\alpha_1^{(p-1)}} \int_\Omega \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_\lambda^s} \left(\sum_{i=1}^k \frac{\sigma_i^2 |x^{(i)}|^2}{\lambda_i(x)^2} \right)^{\frac{p}{2}} |\nabla_\lambda u(x)|^p \ dx$$

$$= \frac{1}{\alpha_1^{(p-1)}} \int_\Omega \psi(x) |\nabla_\lambda u(x)|^p \ dx.$$

Since

$$\lim_{\varepsilon \to 0} c_\varepsilon(x) = Q + \sum_{i=1}^k \sigma_i \mu_i - s,$$

the theorem now follows from the dominated convergence theorem by taking the limit ε tends to zero. $\square$

Remark 4.4. The first condition on the exponents in Theorem 4.3 allows to derive the uniform estimates for $c_\epsilon(x)$ in the proof, while the condition (4.1) ensures that ψ belongs to $L_{loc}^1(\Omega)$.

We formulated a very general family of Hardy-type inequalities, the parameters allow to adjust the weights and to move them from one side of the inequality to the other. Particular choices lead to inequalities of the form (3.2) or (3.3).

Remark 4.5. For Grushin-type operators $\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y$, $\alpha \geq 0, (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, we recover the Hardy inequalities of Theorem 3.1 in [17], where it was proved that the constants are optimal.

For the convenience of the reader we first formulated Hardy type inequalities for the particular case of our homogeneous norms $[[\cdot]]_\lambda$. We now generalize Theorem 4.3 and consider homogeneous distances from the origin $\| \cdot \|_\lambda$ that satisfy the relation

$$\sum_{j=1}^k \sigma_j \left(x^{(j)} \cdot \nabla_{x^{(j)}} \right) \|x\|_\lambda = \|x\|_\lambda, \qquad x \in \mathbb{R}^N.$$

For instance, we could choose

$$(4.3) \qquad \|x\|_\lambda := \left(\sum_{j=1}^{k} |x^{(j)}|^2 \prod_{i \neq j} \sigma_i \right)^{\frac{1}{2\prod_{i=1}^{k} \sigma_i}}, \qquad x \in \mathbb{R}^N,$$

$$(4.4) \qquad \text{or} \qquad \|x\|_\lambda := \left(\sum_{j=1}^{k} (\sigma_j |x^{(j)}|)^2 \prod_{i \neq j} \sigma_i \right)^{\frac{1}{2\prod_{i=1}^{k} \sigma_i}}, \qquad x \in \mathbb{R}^N.$$

Remark 4.6. For Grushin-type operators the second distance $\|\cdot\|_\lambda$ coincides with our homogeneous norm $[[\cdot]]_\lambda$ and with the distance considered by D'Ambrosio in [17].

We compute the first of the homogeneous distances for our previous examples.

- For operators of the form

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} \Delta_z, \qquad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

with non-negative constants α and β, we obtain

$$\|(x, y, z)\|_\lambda = \left(|x|^{2(1+\alpha)(1+\beta)} + |y|^{2(1+\beta)} + |z|^{2(1+\alpha)} \right)^{\frac{1}{2(1+\alpha)(1+\beta)}}.$$

- For Δ_λ-Laplacians of the form

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z, \qquad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where the constants α, β and γ are non-negative, we get

$$\|(x, y, z)\|_\lambda = \left(|x|^{2(1+\alpha)(1+\mu)} + |y|^{2(1+\mu)} + |z|^{2(1+\alpha)} \right)^{\frac{1}{2(1+\alpha)(1+\mu)}},$$

where $\mu = \beta + (1+\alpha)\gamma$.

Theorem 4.7. *Let $p > 1$ and $\mu_1, \ldots, \mu_k, s, t \in \mathbb{R}$ be such that $s + t < N_1 + \mu_1$ and*

$$(4.5) \qquad -p \min\{\alpha_{1i}, \ldots \alpha_{ki}, 1\} + s + \frac{t}{\sigma_i} < N_i + \mu_i \qquad i = 1, \ldots, k.$$

Then, for every $u \in \mathring{W}_\lambda^{1,p}(\Omega, \psi)$ we have

$$\left(\frac{Q - s - t + \sum_{i=1}^{k} \sigma_i \mu_i}{p} \right)^p \int_\Omega \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{\|x\|_\lambda^t [[x]]_\lambda^s} |u(x)|^p \, dx \leq \int_\Omega \psi(x) |\nabla_\lambda u(x)|^p \, dx,$$

where $\psi(x) = \dfrac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i - p \sum_{j=1}^{k} \alpha_{ji}}}{\|x\|_\lambda^t [[x]]_\lambda^{s - p(1 + \sum_{i=1}^{k} (\sigma_i - 1))}}$, and $\|\cdot\|_\lambda$ denotes the homogeneous norm (4.3) or (4.4).

In particular, for $s = 0$, $\mu_i = 0$ and $t = p$ we obtain

$$\left(\frac{Q - p}{p} \right)^p \int_\Omega \frac{|u(x)|^p}{\|x\|_\lambda^p} \, dx \leq \int_\Omega \frac{[[x]]_\lambda^{p(1 + \sum_{i=1}^{k} (\sigma_i - 1))}}{\|x\|_\lambda^p \prod_{j=1}^{k} \lambda_j(x)^p} |\nabla_\lambda u(x)|^p \, dx.$$

For $t = 0$ we recover the Hardy inequalities in Theorem 4.3 with our homogeneous norms $[[\cdot]]_\lambda$.

Proof. We prove the statement for the homogeneous norm (4.3). The result for the distance (4.4) follows analogously. We deduce the inequalities from Lemma 3.1. To this end we define the function

$$h_\varepsilon(x) := \frac{\prod_{i=1}^{k} |x^{(i)}|^{\mu_i}}{\|x\|_{\varepsilon,\lambda}^t [[x]]_{\varepsilon,\lambda}^s} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1^\varepsilon(x)}, \ldots, \frac{\sigma_k x^{(k)}}{\lambda_k^\varepsilon(x)} \right),$$

where $\|\cdot\|_{\varepsilon,\lambda}$ is a smooth approximation of $\|\cdot\|_\lambda$,

$$\|x\|_{\varepsilon,\lambda} = \left(\sum_{j=1}^{k} (|x^{(j)}|^2 + \varepsilon)^{\prod_{i \neq j} \sigma_i} \right)^{\frac{1}{2\prod_{i=1}^{k} \sigma_i}}.$$

We obtain

$$|h_\varepsilon(x)| = \frac{\prod_{i=1}^{k}|x^{(i)}|^{\mu_i}\,[[x]]_{\varepsilon,\lambda}^{(1+\sum_{i=1}^{k}(\sigma_i-1))-s}}{\prod_{i=1}^{k}\lambda_i^\varepsilon(x)\,\|x\|_{\varepsilon,\lambda}^t},$$

$$\text{div}_\lambda h_\varepsilon(x) = \frac{\prod_{i=1}^{k}|x^{(i)}|^{\mu_i}}{\|x\|_{\varepsilon,\lambda}^t[[x]]_{\varepsilon,\lambda}^s}\left(c_\varepsilon(x) - t\frac{1}{\|x\|_{\varepsilon,\lambda}}\sum_{i=1}^{k}\frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)}\sigma_i x^{(i)}\cdot\nabla_{x^{(i)}}(\|x\|_{\varepsilon,\lambda})\right)$$

$$= \frac{\prod_{i=1}^{k}|x^{(i)}|^{\mu_i}}{\|x\|_{\varepsilon,\lambda}^t[[x]]_{\varepsilon,\lambda}^s}\left(c_\varepsilon(x) - \eta_\varepsilon(x)\right),$$

where c_ε was defined in the proof of Theorem 4.3 and

$$\eta_\varepsilon(x) := t\frac{1}{\|x\|_{\varepsilon,\lambda}}\sum_{i=1}^{k}\frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)}\sigma_i x^{(i)}\cdot\nabla_{x^{(i)}}(\|x\|_{\varepsilon,\lambda}).$$

We observe that

$$0 \le \eta_\varepsilon(x) = t\frac{1}{\|x\|_{\varepsilon,\lambda}}\left(\sum_{i=1}^{k}\frac{\lambda_i(x)}{\lambda_i^\varepsilon(x)}\frac{|x^{(i)}|^2}{|x^{(i)}|^2+\varepsilon}(|x^{(i)}|^2+\varepsilon)^{\Pi_{j\ne i}\sigma_j}\right)\|x\|_{\varepsilon,\lambda}^{1-2\prod_{j=1}^{k}\sigma_k}$$

$$\le t\frac{1}{\|x\|_{\varepsilon,\lambda}}\left(\sum_{i=1}^{k}(|x^{(i)}|^2+\varepsilon)^{\Pi_{j\ne i}\sigma_j}\right)\|x\|_{\varepsilon,\lambda}^{1-2\prod_{j=1}^{k}\sigma_k} = t$$

and consequently, it follows from the proof of Theorem 4.3 that

$$c_\varepsilon(x) - \eta_\varepsilon(x) \ge N_1 + \mu_1 - s - t > 0.$$

Moreover, we have

$$\lim_{\varepsilon\to 0}(c_\varepsilon(x) - \eta_\varepsilon(x)) = Q + \sum_{i=1}^{k}\sigma_i\mu_i - s - t.$$

By our assumptions $Q > s+t-\sum_{i=1}^{k}\sigma_i\mu_i$, which implies that $\text{div}_\lambda h_\varepsilon > 0$ for all sufficiently small $\varepsilon > 0$. Lemma 3.1 applied to the function h_ε leads to the inequality

$$\frac{1}{p^p}\int_\Omega(c_\varepsilon(x) - \eta_\varepsilon(x))\frac{\prod_{i=1}^{k}(|x^{(i)}|^{\mu_i}}{\|x\|_{\varepsilon,\lambda}^t[[x]]_{\varepsilon,\lambda}^s}|u(x)|^p\,dx$$

$$\le \int_\Omega\frac{1}{(c_\varepsilon(x)-\eta_\varepsilon(x))^{(p-1)}}\psi_\varepsilon(x)\,|\nabla_\lambda u(x)|^p\,dx,$$

where

$$\psi_\varepsilon(x) = \frac{\prod_{i=1}^{k}|x^{(i)}|^{\mu_i}}{\|x\|_{\varepsilon,\lambda}^t[[x]]_{\varepsilon,\lambda}^s}\left(\sum_{i=1}^{k}\frac{\sigma_i^2|x^{(i)}|^2}{\lambda_i^\varepsilon(x)^2}\right)^{\frac{p}{2}}$$

$$\le \frac{\prod_{i=1}^{k}|x^{(i)}|^{\mu_i}}{\|x\|_\lambda^t[[x]]_\lambda^s}\left(\sum_{i=1}^{k}\frac{\sigma_i^2|x^{(i)}|^2}{\lambda_i(x)^2}\right)^{\frac{p}{2}} = \psi(x).$$

By taking the limit ε tends to zero the statement of the theorem follows from the dominated convergence theorem. $\qquad\square$

Remark 4.8. The first condition on the exponents in Theorem 4.7 allows to derive the uniform estimates for $\eta_\epsilon(x)$ in the proof, while the condition (4.5) ensures that ψ belongs to $L_{loc}^1(\Omega)$.

Finally, we formulate Hardy type inequalities without weights.

Theorem 4.9. *Let $N_1 > p > 1$. Then, for every $u \in \mathring{W}_\lambda^{1,p}(\Omega)$ we have*

$$\left(\frac{N_1 - p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x^{(1)}|^p} \, dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \, dx,$$

$$\left(\frac{N_1 - p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{\|x\|_\lambda^p} \, dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \, dx.$$

Proof. It suffices to prove the first inequality. The second inequality is an immediate consequence of the first, since the norms satisfy $\|x\|_\lambda \geq |x^{(1)}|$, $x \in \mathbb{R}^N$. We define the function

$$h_\varepsilon(x) := \frac{1}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} \left(x^{(1)}, 0, \ldots, 0\right)$$

and compute

$$\operatorname{div}_\lambda h_\varepsilon(x) = \frac{N_1 - p\frac{|x^{(1)}|^2}{|x^{(1)}|^2 + \varepsilon}}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} > 0,$$

$$|h_\varepsilon(x)| = \frac{|x^{(1)}|}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}}.$$

Since $N_1 > p$ we have $\operatorname{div}_\lambda h_\varepsilon > 0$, and Lemma 3.1 applied to h_ε yields the inequality

$$\frac{1}{p^p} \int_\Omega \left(N_1 - p\frac{|x^{(1)}|^2}{|x^{(1)}|^2 + \varepsilon}\right) \frac{|u(x)|^p}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} \, dx$$

$$\leq \int_\Omega \left(N_1 - p\frac{|x^{(1)}|^2}{|x^{(1)}|^2 + \varepsilon}\right)^{-(p-1)} \frac{|x^{(1)}|^p}{(|x^{(1)}|^2 + \varepsilon)^{\frac{p}{2}}} |\nabla_\lambda u(x)|^p \, dx.$$

The first inequality of the theorem now follows from the dominated convergence theorem by taking the limit ε tends to zero. $\square$

Appendix A. Some Remarks on the Optimality of the Constant

For the particular case of Grushin type operators D'Ambrosio proved in [17] that the constants in the inequalities in Theorem 4.3 are optimal. The optimality was shown similarly to the classical case using the explicit form of the function for which the Hardy inequality becomes an equality. This function does not belong to the Sobolev space $H_0^1(\Omega)$, but an approximating sequence in $H_0^1(\Omega)$ is used in the proof. Moreover, the function is strongly related to the fundamental solution at the origin. For more general Δ_λ-Laplacians this function as well as the fundamental solution are unknown, and at present we are not able to prove that our Hardy type inequalities are sharp.

Using the fundamental solution at the origin the following observations yield a simple proof for Hardy inequalities. We will only consider the case $p = 2$ here.

Let λ be of the form (2.1), $\Omega \subset \mathbb{R}^N$ be a domain, $N \geq 3$, and Φ be the fundamental solution at the origin of $-\Delta_\lambda$ on Ω, i.e.,

$$-\Delta_\lambda \Phi = c\delta_0,$$

$$\Phi > 0,$$

for some constant $c > 0$, where δ_0 denotes the Dirac delta function. Moreover, let $u \in C_0^1(\Omega)$ and $v := u\Phi^{-\frac{1}{2}}$. Then, the following identities follow from integration by parts and the properties of the fundamental solution (see [30] for the case of the classical Laplacian),

(A.1)
$$\int_\Omega |\nabla_\lambda u|^2 dx = \frac{1}{4} \int_\Omega \frac{|\nabla_\lambda \Phi|^2}{|\Phi|^2} u^2 dx + \frac{1}{2} \int_\Omega \nabla_\lambda \Phi \nabla_\lambda (v^2) dx + \int_\Omega |\nabla_\lambda v|^2 \Phi dx$$

$$= \frac{1}{4} \int_\Omega \frac{|\nabla_\lambda \Phi|^2}{|\Phi|^2} u^2 dx + \frac{1}{2} cv^2(0) + \int_\Omega |\nabla_\lambda v|^2 \Phi dx$$

$$= \frac{1}{4} \int_\Omega \frac{|\nabla_\lambda \Phi|^2}{|\Phi|^2} u^2 dx + \int_\Omega |\nabla_\lambda v|^2 \Phi dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla_\lambda \Phi|^2}{|\Phi|^2} u^2 dx,$$

where we used that $v(0) = u(0)\Phi(0)^{-\frac{1}{2}} = 0$.

The fundamental solution at the origin for the Grushin-type operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y, \qquad \alpha \geq 0, \ z = (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is of the form

$$\Phi(x,y) = \frac{c}{[[(x,y)]]_\lambda^{Q-2}},$$

for some constant $c \geq 0$ (see [31]). The estimate (A.1) implies the weighted Hardy type inequality

$$\int_\Omega |\nabla_\lambda u(z)|^2 dz \geq \frac{1}{4}\int_\Omega \frac{|\nabla_\lambda \Phi(z)|^2}{|\Phi(z)|^2} u(z)^2 \, dz = \frac{(Q-2)^2}{4}\int_\Omega \frac{|x|^{2\alpha}}{[[(x,y)]]_\lambda^{2(1+\alpha)}} u(z)^2 dz,$$

which is a particular case of the inequalities in Theorem 4.3. To show the optimality of the constant we consider the identity

$$\int_\Omega |\nabla_\lambda u(z) - \varphi(z)u(z)|^2 \, dz = \int_\Omega |\nabla_\lambda u(z)|^2 + |u(z)|^2 \left(|\varphi(z)|^2 + \operatorname{div}_\lambda \varphi(z)\right) dz$$

and observe that the function

$$\varphi(x,y) = -\frac{Q-2}{2}\frac{|x|^{2\alpha}}{[[(x,y)]]_\lambda^{2(1+\alpha)}}\left(x, \frac{(1+\alpha)y}{|x|^\alpha}\right),$$

which we applied in the proof of Theorem 4.3, satisfies

$$|\varphi(x,y)|^2 + \operatorname{div}_\lambda \varphi(x,y) = -\left(\frac{Q-2}{2}\right)^2 \frac{|x|^{2\alpha}}{[[(x,y)]]_\lambda^{2(1+\alpha)}}.$$

A solution of the equation

$$\nabla_\lambda u(x,y) = -\frac{Q-2}{2}\frac{|x|^{2\alpha}}{[[(x,y)]]_\lambda^{2(1+\alpha)}}\left(x, \frac{(1+\alpha)y}{|x|^\alpha}\right) u(x,y)$$

is the function

$$u(x,y) = \frac{1}{[[(x,y)]]_\lambda^{\frac{Q-2}{2}}},$$

which was used in [17] to prove the optimality of the constant. It transforms the Hardy inequality into an equality, but does not belong to the class $\mathring{W}_\lambda^{1,2}(\Omega)$ if the domain Ω contains the origin (see [17], p.728).

The fundamental solution for general Δ_λ-Laplacians is unknown. Assuming that there exists a homogeneous distance from the origin d_λ such that the fundamental solution is given by $\Phi = d_\lambda^{2-Q}$ we obtain

$$\frac{|\nabla_\lambda \Phi(x)|^2}{|\Phi(x)|^2} = (Q-2)^2 \frac{|\nabla_\lambda d_\lambda(x)|^2}{|d_\lambda(x)|^2},$$

and (A.1) implies the Hardy type inequality

$$\int_\Omega |\nabla_\lambda u(x)|^2 dx \geq \frac{(Q-2)^2}{4}\int_\Omega \frac{|\nabla_\lambda d_\lambda(x)|^2}{|d_\lambda(x)|^2}|u(x)|^2 \, dx.$$

Consequently, if the fundamental solution was known we could define the distance $d_\lambda := \Phi^{\frac{1}{2-Q}}$ and compute explicit, weighted Hardy inequalities.

On the other hand, suitable to analyze the optimality of the constants in our family of Hardy type inequalities is the relation

$$(A.2) \quad \int_\Omega \left|\frac{\varphi(x)}{\psi(x)}u(x) - \psi(x)\nabla_\lambda u(x)\right|^2 dx = \int_\Omega \psi(x)^2 |\nabla_\lambda u(x)|^2 + u(x)^2\left(\frac{|\varphi(x)|^2}{\psi(x)^2} + \operatorname{div}_\lambda \varphi(x)\right) dx,$$

which follows from integration by parts, where $\varphi : \mathbb{R} \to \mathbb{R}^N$ is a vector field and $\psi : \mathbb{R} \to \mathbb{R}$ a scalar function. Comparing with the first inequality in Theorem 4.3 we choose

$$\psi(x)^2 = \frac{[[x]]_\lambda^{2(1+\sum_{i=1}^k (\sigma_i - 1)) - s}}{\prod_{i=1}^k |x^{(i)}|^{2\sum_{j=1}^k \alpha_{ji} - \mu_i}},$$

and observe that the function

$$\varphi(x) = -\frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_\lambda^s} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1(x)}, \dots, \frac{\sigma_k x^{(k)}}{\lambda_k(x)} \right),$$

which we used to prove the theorem, satisfies

$$\left(\frac{|\varphi(x)|^2}{\psi(x)^2} + \operatorname{div}_\lambda \varphi(x) \right) = -\left(\frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \right)^2 \frac{\prod_{i=1}^k |x^{(i)}|^{\mu_i}}{[[x]]_\lambda^s}.$$

Consequently, the Hardy type inequality in Theorem 4.3 is an equality if u is a solution of the equation

$$\nabla_\lambda u(x) = \frac{\varphi(x)}{\psi(x)^2} u(x),$$

i.e.,

$$(A.3) \qquad \nabla_{x^{(i)}} u(x) = -\frac{Q - s + \sum_{i=1}^k \sigma_i \mu_i}{2} \frac{\prod_{j \neq i} \lambda_j(x)^2}{\|x\|_\lambda^{2(1+\sum_{i=1}^k (\sigma_i - 1))}} \sigma_i x^{(i)} u(x), \qquad i = 1, \dots, k.$$

Except for Grushin type operators we are unable to solve this equation, not even for the particular Δ_λ-Laplacians in Examples 2.1 to 2.3.

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References

[1] Brezis H, Vázquez JL. Blow-up solutions of some nonlinear elliptic problems. Rev Mat Univ Complut Madrid. 1997;10(2):443–469.

[2] D'Ambrosio L. Hardy-type inequalities related to degenerate elliptic differential operators. Ann Sc Norm Super Pisa Cl Sci (5). 2005;4(3):451–486.

[3] Vazquez JL, Zuazua E. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J Funct Anal. 2000;173(1):103–153; Available from: `http://dx.doi.org/10.1006/jfan.1999.3556`.

[4] Baras P, Goldstein JA. The heat equation with a singular potential. Trans Amer Math Soc. 1984;284(1):121–139; Available from: `http://dx.doi.org/10.2307/1999277`.

[5] Garofalo N, Lanconelli E. Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation. Ann Inst Fourier (Grenoble). 1990;40(2):313–356; Available from: `http://www.numdam.org/item?id=AIF_1990__40_2_313_0`.

[6] Thuy NTC, Tri NM. Some existence and nonexistence results for boundary value problems for semilinear elliptic degenerate operators. Russ J Math Phys. 2002;9(3):365–370.

[7] Monti R, Morbidelli D. John domains for the control distance of diagonal vector fields. J Anal Math. 2004; 92:259–284; Available from: `http://dx.doi.org/10.1007/BF02787764`.

[8] Thuy PT, Tri NM. Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations. NoDEA Nonlinear Differential Equations Appl. 2012;19(3):279–298; Available from: `http://dx.doi.org/10.1007/s00030-011-0128-z`.

[9] Kogoj AE, Lanconelli E. On semilinear Δ_λ-Laplace equation. Nonlinear Anal. 2012;75(12):4637–4649; Available from: `http://dx.doi.org/10.1016/j.na.2011.10.007`.

[10] Thuy PT, Tri NM. Long time behavior of solutions to semilinear parabolic equations involving strongly degenerate elliptic differential operators. NoDEA Nonlinear Differential Equations Appl. 2013;20(3):1213–1224; Available from: `http://dx.doi.org/10.1007/s00030-012-0205-y`.

[11] Kogoj AE, Sonner S. Attractors for a class of semi-linear degenerate parabolic equations. J Evol Equ. 2013; 13(3):675–691; Available from: `http://dx.doi.org/10.1007/s00028-013-0196-0`.

[12] Kogoj AE, Sonner S. Attractors met X-elliptic operators. J Math Anal Appl. 2014;420(1):407–434; Available from: `http://dx.doi.org/10.1016/j.jmaa.2014.05.070`.

[13] Anh CT. Global attractor for a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$. NoDEA Nonlinear Differential Equations Appl. 2014;21(5):663–678; Available from: `http://dx.doi.org/10.1007/s00030-013-0261-y`.

[14] Tri NM, Luyen DT. Existence of solutions to boundary-value problems for semilinear Δ_γ differential equations. Math Notes. 2015;97(1):73–84.

[15] Thao M. On the global attractor for a semilinear strongly degenerate parabolic equation. Acta Mathematica Vietnamica. 2015;:1–15Available from: `http://dx.doi.org/10.1007/s40306-015-0133-0`.

[16] Wu JM. Geometry of grushin spaces. Illinois J Math, to appear.

[17] D'Ambrosio L. Hardy inequalities related to Grushin type operators. Proc Amer Math Soc. 2004;132(3):725–734; Available from: `http://dx.doi.org/10.1090/S0002-9939-03-07232-0`.

[18] Su D, Yang QH. Improved Hardy inequalities in the Grushin plane. J Math Anal Appl. 2012;393(2):509–516; Available from: `http://dx.doi.org/10.1016/j.jmaa.2012.04.010`.

[19] Yang Q, Su D, Kong Y. Improved Hardy inequalities for Grushin operators. J Math Anal Appl. 2015;424(1):321–343; Available from: `http://dx.doi.org/10.1016/j.jmaa.2014.11.010`.

[20] Xiao YX. Some Hardy inequalities on half spaces for Grushin type operators. Math Inequal Appl. 2013;16(3):793–807; Available from: `http://dx.doi.org/10.7153/mia-16-60`.

[21] Zhu L. A Hardy inequality for the Grushin type operators. Math Inequal Appl. 2012;15(4):923–930; Available from: `http://dx.doi.org/10.7153/mia-15-78`.

[22] Mitidieri E. A simple approach to Hardy inequalities. Mat Zametki. 2000;67(4):563–572; Available from: `http://dx.doi.org/10.1007/BF02676404`.

[23] Hörmander L. Hypoelliptic second order differential equations. Acta Math. 1967;119:147–171.

[24] Grušin VV. A certain class of hypoelliptic operators. Mat Sb (NS). 1970;83 (125):456–473.

[25] Franchi B, Lanconelli E. Une métrique associée à une classe d'opérateurs elliptiques dégénérés. Rend Sem Mat Univ Politec Torino. 1983;(Special Issue):105–114 (1984); conference on linear partial and pseudodifferential operators (Torino, 1982).

[26] Franchi B, Lanconelli E. An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality. Comm Partial Differential Equations. 1984;9(13):1237–1264; Available from: `http://dx.doi.org/10.1080/03605308408820362`.

[27] Franchi B, Lanconelli E. Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients. Ann Scuola Norm Sup Pisa Cl Sci (4). 1983;10(4):523–541; Available from: `http://www.numdam.org/item?id=ASNSP_1983_4_10_4_523_0`.

[28] Lanconelli E, Kogoj AE. X-elliptic operators and X-control distances. Ricerche Mat. 2000;49(suppl.):223–243; contributions in honor of the memory of Ennio De Giorgi (Italian).

[29] Grillo G. Hardy and Rellich-type inequalities for metrics defined by vector fields. Potential Anal. 2003;18(3):187–217; Available from: `http://dx.doi.org/10.1023/A:1020963702912`.

[30] Adimurthi, Sekar A. Role of the fundamental solution in Hardy-Sobolev-type inequalities. Proc Roy Soc Edinburgh Sect A. 2006;136(6):1111–1130; Available from: `http://dx.doi.org/10.1017/S030821050000490X`.

[31] D'Ambrosio L, Lucente S. Nonlinear Liouville theorems for Grushin and Tricomi operators. J Differential Equations. 2003;193(2):511–541; Available from: `http://dx.doi.org/10.1016/S0022-0396(03)00138-4`.

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