HARISH–CHANDRA HOMOMORPHISMS AND SYMPLECTIC REFLECTION ALGEBRAS FOR WREATH-PRODUCTS

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To Joseph Bernstein on the occasion of his 60th birthday

ABSTRACT

The main result of the paper is a natural construction of the spherical subalgebra in a symplectic reflection algebra associated with a wreath-product in terms of quantum hamiltonian reduction of an algebra of differential operators on a representation space of an extended Dynkin quiver. The existence of such a construction has been conjectured in [EG].

We also present a new approach to reflection functors and shift functors for generalized preprojective algebras and symplectic reflection algebras associated with wreath-products.

CONTENTS

1. Introduction

The main result of the paper is the proof of [EG, Conjecture 11.22] that provides a natural construction of the spherical subalgebra in a symplectic reflection algebra associated with a wreath-product in terms of quantum hamiltonian reduction of an algebra of differential operators.

To state the main result we briefly recall a few basic definitions.

1.1. *Quantum Hamiltonian reduction*

We work with associative unital $\mathbf{C}$-algebras and write $\mathrm{Hom} = \mathrm{Hom}_{\mathbf{C}}$, $\otimes = \otimes_{\mathbf{C}}$, etc.

Let A be an associative algebra, that may also be viewed as a Lie algebra with respect to the commutator Lie bracket. Given a Lie algebra $\mathfrak{g}$ and a Lie algebra homomorphism $\rho : \mathfrak{g} \to A$, one has an *adjoint* $\mathfrak{g}$-action on A given by

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$\mathtt{ad}x : a \mapsto \rho(x)\cdot a - a\cdot\rho(x)$, $x \in \mathfrak{g}$, $a \in \mathrm{A}$. The left ideal $\mathrm{A}\cdot\rho(\mathfrak{g})$ is stable under the adjoint action. Furthermore, one shows that multiplication in A induces a well defined associative algebra structure on

$$\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho) := (\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}))^{\mathrm{ad}\mathfrak{g}},$$

the space of $\mathtt{ad}\mathfrak{g}$-invariants in $\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})$. The resulting algebra $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho)$ is called the *quantum Hamiltonian reduction* of A at ρ.

Observe that, if $a \in \mathrm{A}$ is such that the element $a \bmod \mathrm{A}\cdot\rho(\mathfrak{g}) \in \mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})$ is $\mathtt{ad}\mathfrak{g}$-invariant, then the operator of right multiplication by a descends to a well-defined map $\mathrm{R}_a : \mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}) \to \mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})$. Moreover, the assignment $a \mapsto \mathrm{R}_a$ induces an algebra isomorphism $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho) = (\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}))^{\mathrm{ad}\mathfrak{g}} \xrightarrow{\sim} (\mathrm{End}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})))^{\mathrm{op}}$.

If A, viewed as an $\mathtt{ad}\mathfrak{g}$-module, is semisimple, i.e., splits into a (possibly infinite) direct sum of irreducible finite dimensional $\mathfrak{g}$-representations, then the operations of taking $\mathfrak{g}$-invariants and taking the quotient commute, and we may write

$$\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho) = (\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}))^{\mathrm{ad}\mathfrak{g}} = \mathrm{A}^{\mathrm{ad}\mathfrak{g}}/(\mathrm{A}\cdot\rho(\mathfrak{g}))^{\mathrm{ad}\mathfrak{g}}. \tag{1.1.1}$$

Observe that, in this formula, $(\mathrm{A}\cdot\rho(\mathfrak{g}))^{\mathrm{ad}\mathfrak{g}}$ is a *two-sided* ideal of the algebra $\mathrm{A}^{\mathrm{ad}\mathfrak{g}}$.

Any A-module M may be viewed also as a $\mathfrak{g}$-module, via the homomorphism ρ, and we write $\mathrm{M}^{\mathfrak{g}} := \{m \in \mathrm{M} \mid \rho(x)m = 0,\ \forall x \in \mathfrak{g}\}$ for the corresponding space of $\mathfrak{g}$-invariants. Let $(\mathrm{A}, \mathfrak{g})$-mod be the full subcategory of the abelian category of left A-modules whose objects are semisimple as $\mathfrak{g}$-modules. Let $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho)$-mod be the abelian category of left $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho)$-modules.

One defines an exact functor, called *Hamiltonian reduction functor*, as follows

$$\begin{aligned} &\mathbf{H}: \ (\mathrm{A}, \mathfrak{g})\text{-mod} \to \mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho)\text{-mod}, \\ &\mathrm{M} \mapsto \mathbf{H}(\mathrm{M}) := \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}), \mathrm{M}) = \mathrm{M}^{\mathfrak{g}}, \end{aligned} \tag{1.1.2}$$

where the action of $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho)$ on $\mathbf{H}(\mathrm{M})$ comes from the tautological *right* action of $\mathrm{End}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g}))$ on $\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})$ and the above mentioned isomorphism $\mathfrak{A}(\mathrm{A}, \mathfrak{g}, \rho) = (\mathrm{End}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}\cdot\rho(\mathfrak{g})))^{\mathrm{op}}$.

1.2. *Symplectic reflection algebras for wreath-products*

Let n be a positive integer. Let S_n be the permutation group of $[1, n] := \{1, \dots, n\}$, and write $s_{\ell m} \in \mathrm{S}_n$ for the transposition $\ell \leftrightarrow m$. Let L be a 2-dimensional complex vector space, and ω a symplectic form on L.

Let Γ be a finite subgroup of $\mathrm{Sp}(\mathrm{L})$, and let $\mathbf{\Gamma}_n := \mathrm{S}_n \ltimes \Gamma^n$ be a wreath product group acting naturally in L^n. Given $\ell \in [1, n]$ and $\gamma \in \Gamma$, resp. $v \in \mathrm{L}$, we will write $\gamma_{(\ell)} \in \mathbf{\Gamma}_n$ for γ placed in the ℓ-th factor Γ, resp. $v_{(\ell)} \in \mathrm{L}^n$ for v placed in the ℓ-th factor L.

According to [EG], there is a family of associative algebras, called symplectic reflection algebras, attached to the pair $(L^n, \mathbf{\Gamma}_n)$ as above. To define these algebras, write $Z\Gamma$ for the center of the group algebra $\mathbf{C}[\Gamma]$ and let $Z_o\Gamma \subset Z\Gamma$ be a codimension 1 subspace formed by the elements

$$c = \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \cdot \gamma \in Z\Gamma, \quad \forall c_\gamma \in \mathbf{C}. \tag{1.2.1}$$

Given $t, k \in \mathbf{C}$ and $c \in Z_o\Gamma$, the corresponding *symplectic reflection algebra* $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$, with parameters t, k, c, may be defined, cf. [GG, Lemma 3.1.1], as a quotient of the smash product algebra $T(L^n) \rtimes \mathbf{C}[\mathbf{\Gamma}_n]$ by the following relations:

$$[x_{(\ell)}, y_{(\ell)}] = t \cdot 1 + \frac{k}{2} \sum_{m \neq \ell} \sum_{\gamma \in \Gamma} s_{\ell m} \gamma_{(\ell)} \gamma_{(m)}^{-1} + \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma_{(\ell)}, \quad \forall \ell \in [1, n]; \tag{1.2.2}$$

$$[u_{(\ell)}, v_{(m)}] = -\frac{k}{2} \sum_{\gamma \in \Gamma} \omega(\gamma u, v) s_{\ell m} \gamma_{(\ell)} \gamma_{(m)}^{-1}, \quad \forall u, v \in L, \ \ell, m \in [1, n], \ \ell \neq m, \tag{1.2.3}$$

where $\{x, y\}$ is a fixed basis for L with $\omega(x, y) = 1$.

1.3. *Quivers*

Let Q be an extended Dynkin quiver with vertex set I, and let $o \in I$ be an extending vertex of Q.

Definition **1.3.1.** — *The quiver* Q_{CM} *obtained from* Q *by adjoining an additional vertex* s *and an arrow* $b : s \to o$ *is called the* Calogero–Moser quiver *for* Q. *Thus,* $I_{\mathsf{CM}} = I \sqcup \{s\}$ *is the vertex set for* Q_{CM}, *and the vertex* s *is called the* special *vertex.*

Given $\alpha = \{\alpha_i\}_{i \in I_{\mathsf{CM}}} \in \mathbf{Z}^{I_{\mathsf{CM}}}$, a dimension vector for Q_{CM}, write

$$\begin{aligned} \mathsf{Rep}_\alpha(Q_{\mathsf{CM}}) &:= \bigoplus_{\{a: i \to j \ | \ a \in Q_{\mathsf{CM}}\}} \mathrm{Hom}(\mathbf{C}^{\alpha_i}, \mathbf{C}^{\alpha_j}) \\ &= \bigoplus_{\{a: i \to j \ | \ a \in Q_{\mathsf{CM}}\}} \mathrm{Mat}(\alpha_j \times \alpha_i, \mathbf{C}) \end{aligned} \tag{1.3.2}$$

for the space of representations of Q_{CM} of dimension α. Let $\mathscr{D}(Q_{\mathsf{CM}}, \alpha)$ be the algebra of polynomial differential operators on the vector space $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$.

The group $GL(\alpha) := \prod_{i \in I_{\mathsf{CM}}} GL(\mathbf{C}^{\alpha_i})$ acts naturally on $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$, by conjugation. Hence, each element h of the Lie algebra $\mathfrak{gl}(\alpha) := \mathrm{Lie}\, GL(\alpha)$ gives rise to a vector field $\boldsymbol{\xi}_h$ on $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$. This yields a Lie algebra map $\boldsymbol{\xi} : \mathfrak{gl}(\alpha) \to \mathscr{D}(Q_{\mathsf{CM}}, \alpha)$.

The center of the reductive Lie algebra $\mathfrak{gl}(\alpha) = \oplus_{i \in I} \mathfrak{gl}(\alpha_i)$ is clearly isomorphic to $\mathbf{C}^I$. Therefore, associated with any $\chi = \{\chi_i\}_{i \in I} \in \mathbf{C}^I$, one has a Lie algebra homomorphism $\chi : \mathfrak{gl}(\alpha) \to \mathbf{C}$, $x = \oplus_{i\in\mathrm{I}} x_i \mapsto \sum_{i\in\mathrm{I}} \chi_i \cdot \mathrm{Tr}\, x_i$. We will use additive notation for such homomorphisms and write $\boldsymbol{\xi} - \chi : \mathfrak{gl}(\alpha) \to \mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)$ (rather than $\boldsymbol{\xi} \otimes (-\chi)$) for the Lie algebra map $h \mapsto \boldsymbol{\xi}_h - \chi(h) \cdot 1_{\mathscr{D}}$. Let $\mathrm{Im}(\boldsymbol{\xi} - \chi)$ denote the image of the latter map.

We may apply Hamiltonian reduction (1.1.1) to the algebra $\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)$ and to the Lie algebra map $\boldsymbol{\xi} - \chi$. This way, we get the algebra

$$(\mathbf{1.3.3}) \qquad \mathfrak{A}(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha),\ \mathfrak{gl}(\alpha),\ \boldsymbol{\xi} - \chi) = \mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)^{\mathrm{GL}(\alpha)} / \mathbf{J}_\chi,$$

where

$$\mathbf{J}_\chi := (\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \cdot \mathrm{Im}(\boldsymbol{\xi} - \chi))^{\mathrm{GL}(\alpha)}.$$

Let $\mathrm{T}^*\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$ be the cotangent bundle on $\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. The total space of the cotangent bundle comes equipped with the canonical symplectic structure and with a moment map

$$(\mathbf{1.3.4}) \qquad \boldsymbol{\mu} :\ \mathrm{T}^*\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}}) \to \mathfrak{gl}(\alpha)^* \cong \mathfrak{gl}(\alpha).$$

We may apply the *classical* Hamiltonian reduction to $\mathbf{C}[\mathrm{T}^*\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})]$, the Poisson algebra of polynomial functions on $\mathrm{T}^*\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. This way, we get the Poisson algebra $\mathbf{C}[\boldsymbol{\mu}^{-1}(0)]^{\mathrm{GL}(\alpha)}$ of $\mathrm{GL}(\alpha)$-invariant polynomial functions on the zero fiber of the moment map. The algebra in (1.3.3) may be viewed as a quantization of the Poisson algebra $\mathbf{C}[\boldsymbol{\mu}^{-1}(0)]^{\mathrm{GL}(\alpha)}$.

1.4. *Main result*

From now on, we fix $n \in \mathbf{N}$, a 2-dimensional symplectic vector space L and $\Gamma \subset \mathrm{Sp}(\mathrm{L})$, a finite subgroup as in Section 1.2. To (n, L, Γ), we will associate a quiver Q, a dimension vector α, and a character χ as follows.

We let Q be an affine Dynkin quiver associated to Γ via the McKay correspondence. Thus, the set I of vertices of Q is identified with the set of isomorphism classes of irreducible representations of Γ. Let N_i be the irreducible representation of Γ corresponding to the vertex $i \in \mathrm{I}$, and let $\delta_i = \dim \mathrm{N}_i$. The extending vertex $o \in \mathrm{I}$ corresponds to the trivial representation of Γ, so $\delta_o = 1$. The vector $\delta = \{\delta_i\}_{i\in\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$ is the minimal positive imaginary root of the affine root system associated to Q. Motivated by M. Holland [Ho], we put

$$(\mathbf{1.4.1}) \qquad \partial = \{\partial_i\}_{i\in\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}, \quad \partial_i := n(-\delta_i + \sum_{\{a\in\mathrm{Q}_{\mathsf{CM}} \mid t(a)=i\}} \delta_{h(a)}), \quad \forall i \in \mathrm{I}.$$

Given a central element $c \in \mathrm{Z}\Gamma$, write $\mathrm{Tr}(c; \mathrm{N}_i)$ for the trace of c in the simple Γ-module N_i, $i \in \mathrm{I}$. Thus, for any $c \in \mathrm{Z}_o\Gamma$, see (1.2.1), we have $\sum_{i\in\mathrm{I}} \delta_i \cdot \mathrm{Tr}(c; \mathrm{N}_i) = 0$. Associated with any data $n \in \mathbf{N}$, $k \in \mathbf{C}$, and $c \in \mathrm{Z}_o\Gamma$, we introduce three vectors

$$\chi = \{\chi_i\}_{i\in\mathrm{I}_{\mathsf{CM}}}, \quad \chi' = \{\chi'_i\}_{i\in\mathrm{I}_{\mathsf{CM}}} \in \mathbf{C}^{\mathrm{I}_{\mathsf{CM}}},$$

and

$$\lambda(c) = \{\lambda(c)_i\}_{i\in \mathrm{I}} \in \mathbf{C}^{\mathrm{I}}, \quad \text{such that } \delta \cdot \lambda(c) = 1,$$

where we have used standard notation $\delta \cdot \lambda = \sum_i \delta_i \cdot \lambda_i$. These vectors are defined as follows

(**1.4.2**)
$$\begin{aligned}
&\lambda(c)_i := \mathrm{Tr}(c; \mathrm{N}_i) + \delta_i/|\Gamma|, \quad \forall i \in \mathrm{I};\\
&\chi_s := n(k \cdot |\Gamma|/2 - 1) + 1,\\
&\chi_o := \lambda(c)_o - \partial_o - k \cdot |\Gamma|/2,\\
&\chi_i := \lambda(c)_i - \partial_i, \quad \forall i \in \mathrm{I}\smallsetminus\{o\};\\
&\chi'_s := \chi_s - 1 = n(k \cdot |\Gamma|/2 - 1),\\
&\chi'_i = \chi_i, \quad \forall i \in \mathrm{I}.
\end{aligned}$$

We are going to consider representations of the quiver Q_{CM} with dimention vector

(**1.4.3**) $$\alpha = \{\alpha_i\}_{i\in \mathrm{I}_{\mathsf{CM}}} \in \mathbf{Z}^{\mathrm{I}_{\mathsf{CM}}}_{\geq 0}, \quad \text{where } \alpha_s := 1, \text{ and } \alpha_i := n \cdot \delta_i, \quad \forall i \in \mathrm{I}.$$

Let $\chi' \in \mathbf{C}^{\mathrm{I}_{\mathsf{CM}}}$ be as in (1.4.2), and let $\mathbf{J}_{\chi'} = (\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \cdot \mathrm{Im}(\boldsymbol{\xi} - \chi'))^{\mathrm{GL}(\alpha)}$ be the corresponding two-sided ideal in $\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)$, cf. (1.3.3). Write $\mathbf{e} := \frac{1}{|\mathbf{\Gamma}_n|}\sum_{g\in\mathbf{\Gamma}_n} g$ for the 'symmetrizer' idempotent viewed as an element of the symplectic reflection algebra $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$.

We are now in a position to state our main result about deformed Harish–Chandra homomorphisms for symplectic reflection algebras associated with a wreath-product. According to [EG], the importance of the deformed Harish–Chandra homomorphism is due to the fact that this homomorphism provides a description of the *spherical subalgebra* $\mathbf{e}\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)\mathbf{e} \subset \mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$ in terms of quantum Hamiltonian reduction of the ring of polynomial differential operators on the vector space $\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. In the special case of a *cyclic* group $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$, that is, for quivers Q of type $\widetilde{\mathbf{A}}_m$ (equipped with the cyclic orientation), the deformed Harish–Chandra homomorphism has been already constructed in [Ob], see also [Go]. In all other cases, a construction of the deformed Harish–Chandra homomorphism $\Phi_{k,c}$ will be given in the present paper.

Our main result reads

Theorem **1.4.4.** — *Assume that* $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$ *is not a cyclic group of* odd *order (i.e.* Q *is not of type* $\widetilde{\mathbf{A}}_{2m}$*), and put* $t := 1/|\Gamma|$. *Then, for any* $n \in \mathbf{N}$, $k \in \mathbf{C}$, $c \in Z_o\Gamma$, *there is an algebra isomorphism*

$$\begin{aligned}\Phi_{k,c} :\ &\mathfrak{A}(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha), \mathfrak{gl}(\alpha), \boldsymbol{\xi} - \chi')\\ &= \mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)^{\mathrm{GL}(\alpha)}/\mathbf{J}_{\chi'} \xrightarrow{\sim} \mathbf{e}\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)\mathbf{e}.\end{aligned}$$

Furthermore, the map $\Phi_{k,c}$ *is compatible with natural increasing filtrations on the algebras involved and the corresponding associated graded map gives rise to a graded Poisson algebra isomorphism, cf.* (1.3.4)*:*

$$\operatorname{gr}\Phi_{k,c}: \ \mathbf{C}[\boldsymbol{\mu}^{-1}(0)]^{\mathrm{GL}(\alpha)} \xrightarrow{\sim} \operatorname{gr}(\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}).$$

This theorem is a slightly modified and corrected version of [EG, Conjecture 11.22] (in [EG], as well as in the main body of the present paper, everything is stated in terms of the quiver Q rather than in terms of the Calogero–Moser quiver Q_{CM}, see Definition 5.2.1 and Theorem 5.2.4 in Section 5.2 below; however, the two approaches are easily seen to be equivalent). Theorem 1.4.4 is a common generalization of two earlier results. The first one is [GG2, Theorem 6.2.3], cf. also [EG, Corollary 7.4]; it corresponds to the (somewhat degenerate) case of $\Gamma = \{1\}$. The second result, due to M. Holland [Ho], is a special case of Theorem 1.4.4 for $n = 1$, where the symplectic reflection algebra is Morita equivalent to a deformed preprojective algebra of [CBH]. Also, in the special case of a cyclic group $\Gamma = \mathbf{Z}/m\mathbf{Z}$ the isomorphism of Theorem 1.4.4 has been recently constructed in [Go] using the results from [Ob].

A 'classical' counterpart of Theorem 1.4.4 involving classical Hamiltonian reduction (at *generic* values of the moment map (1.3.4)) has been proved in [EG, Theorem 11.16].

Combining Theorem 1.4.4 with (1.1.2), and using the same argument as in the proof of [GG2, Proposition 6.8.1], we deduce

Corollary **1.4.5.** — *There exists an exact functor of Hamiltonian reduction*

$$\mathbf{H}: (\mathscr{D}(Q_{\mathsf{CM}}, \alpha), \mathfrak{gl}(\alpha))\text{-}\mathsf{mod} \to \mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}\text{-}\mathsf{mod}.$$

This functor induces an equivalence

$$(\mathscr{D}(Q_{\mathsf{CM}}, \alpha), \mathfrak{gl}(\alpha))\text{-}\mathsf{mod}/\operatorname{Ker}\mathbf{H} \xrightarrow{\sim} \mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}\text{-}\mathsf{mod}.$$

□

We expect that the Hamiltonian reduction functor induces an equivalence between the subcategory of $(\mathscr{D}(Q_{\mathsf{CM}}, \alpha), \mathfrak{gl}(\alpha))$-mod formed by $\mathscr{D}$-modules whose characteristic variety is contained in the *Nilpotent Lagrangian*, see [Lu1, §12], and the category of finite dimensional $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$-modules.

1.5. *Four homomorphisms*

Our construction of the isomorphism $\Phi_{k,c}$ in Theorem 1.4.4 is rather indirect. It involves four additional algebras and four homomorphisms between those algebras, which are important in their own right.

The first algebra, to be denoted $\Pi'(\mathrm{Q}_{\mathsf{CM}})$, is a slightly renormalized version of the deformed preprojective algebra, with appropriate parameters, cf. [CBH], associated to the Calogero–Moser quiver Q_{CM}. The second algebra, to be denoted B, contains the spherical algebra $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$ as a subalgebra. The algebra B is a 'Calogero–Moser cousin' of *generalized preprojective algebras* introduced by two of us in [GG, (1.2.3)], see also Definition 6.1.3 below.

The third algebra, $\mathfrak{T}_\chi$, is a 'matrix-valued' counterpart of the algebra introduced in (1.3.3). To define this algebra, we introduce the following vector spaces

$$(\mathbf{1.5.1}) \qquad \mathsf{N} = \oplus_{i\in \mathrm{I}_{\mathsf{CM}}} \mathsf{N}_i, \quad \text{where } \mathsf{N}_s := \mathrm{N}_o^* \cong \mathbf{C}, \text{ and } \mathsf{N}_i := \mathrm{N}_i^* \otimes \mathbf{C}^n, \quad \forall i \in \mathrm{I}.$$

Thus, we have $\mathsf{N}_i \cong \mathbf{C}^{\alpha_i}$, so the group $\mathrm{GL}(\alpha)$ acts on N in an obvious way, and this gives the tautological representation $\tau : \mathfrak{gl}(\alpha) \to \operatorname{End}\mathsf{N}$. Following M. Holland [Ho], we apply the quantum Hamiltonian reduction to the algebra $\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes \operatorname{End}\mathsf{N}$ and to the Lie algebra homomorphism

$$\begin{aligned} &\boldsymbol{\xi} - (\chi - \tau) : \mathfrak{gl}(\alpha) \to \mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes \operatorname{End}\mathsf{N}, \\ &h \mapsto \boldsymbol{\xi}_h \otimes \mathtt{Id}_{\mathsf{N}} - 1_{\mathscr{D}} \otimes (\chi(h)\mathtt{Id}_{\mathsf{N}} - \tau(h)), \end{aligned}$$

where $\chi : \mathfrak{gl}(\alpha) \to \mathbf{C}$ is as in (1.4.2). This way, we get an algebra

$$(\mathbf{1.5.2}) \qquad \mathfrak{T}_\chi := \frac{(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes \operatorname{End}\mathsf{N})^{\mathrm{GL}(\alpha)}}{((\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes \operatorname{End}\mathsf{N}) \cdot \operatorname{Im}(\boldsymbol{\xi} - (\chi - \tau)))^{\mathrm{GL}(\alpha)}}.$$

Now, let $\mathbf{P}^1 = (\mathrm{L}\smallsetminus\{0\})/\mathbf{C}^\times$ be the projective line. We will consider an appropriate $\boldsymbol{\Gamma}_n$-equivariant vector bundle of rank $\dim\mathsf{N}$ on $\mathfrak{X}$, where $\mathfrak{X} \subset (\mathbf{P}^1)^n$ is a $\boldsymbol{\Gamma}_n$-stable Zariski open dense subset in the cartesian product of n copies of $\mathbf{P}^1$. Further, we will define a certain algebra $\mathscr{D}(\mathfrak{X}, p, \varrho)$ of *twisted* differential operators acting in that vector bundle, see Section 3.1 for the notation and also (3.6.1).

One has the following diagram of four algebra homomorphisms, all denoted by various Θ's, involving the four algebras introduced above

$$(\mathbf{1.5.3}) \qquad \begin{array}{ccccc} & & \Pi'(\mathrm{Q}_{\mathsf{CM}}) & & \\ & \swarrow^{\Theta^{\mathrm{Holland}}} & & \searrow^{\Theta^{\mathrm{Quiver}}} & \\ \mathfrak{T}_\chi & & & & \mathsf{B} \\ & \searrow_{\Theta^{\mathrm{Radial}}} & & \swarrow_{\Theta^{\mathrm{Dunkl}}} & \\ & & \mathscr{D}(\mathfrak{X}, p, \varrho)^{\boldsymbol{\Gamma}_n} & & \end{array}$$

In this diagram, the map $\Theta^{\mathrm{Holland}}$ is (a slightly renormalized version of) an algebra homomorphism introduced by M. Holland in [Ho]. The map Θ^{Dunkl} is a Γ-analog of the Dunkl representation for rational Cherednik algebras, cf. [EG]. The map Θ^{Radial} is obtained by a 'radial part' type construction with respect to an appropriate transverse slice to generic $\mathrm{GL}(\alpha)$-orbits in $\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. We produce such a slice using a map $\mathrm{L}^{\oplus n} \to \mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$, which is generically injective and is such that its image is generically transverse to $\mathrm{GL}(\alpha)$-orbits in $\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. Our radial part construction associates to a polynomial $\mathrm{GL}(\alpha)$-invariant differential operator $u \in (\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes \mathrm{End}\, \mathsf{N})^{\mathrm{GL}(\alpha)}$ a $\boldsymbol{\Gamma}_n$-invariant twisted differential operator $\Theta^{\text{Radial}}(u) \in \mathscr{D}(\mathfrak{X}, p, \varrho)^{\boldsymbol{\Gamma}_n}$.

The fourth map, Θ^{Quiver}, is new. The main idea behind the construction of this map, as well as the definition of the algebra B, will be outlined in Section 1.7 below and a more rigorous treatment will be given later, in Section 2.2.

Remark **1.5.4.** — In the special case of a cyclic group $\Gamma = \mathbf{Z}/m\mathbf{Z}$, the Dunkl operators that we consider are *not* the same as those introduced earlier by Dunkl-Opdam in [DO].

1.6. *Strategy of the proof of Theorem 1.4.4*

The proof of the main theorem is based on the following key result

Theorem **1.6.1.** — *Diagram* (1.5.3) *commutes, i.e., we have:*

$$\Theta^{\text{Radial}} \circ \Theta^{\text{Holland}} = \Theta^{\text{Dunkl}} \circ \Theta^{\text{Quiver}}.$$

The proof of this theorem is long and messy; it occupies about one half of the paper. In the proof, we explicitly compute both sides of the equation $\Theta^{\text{Radial}} \circ \Theta^{\text{Holland}}(x) = \Theta^{\text{Dunkl}} \circ \Theta^{\text{Quiver}}(x)$, for an appropriate set $\{x,\ x \in \Pi'(\mathrm{Q}_{\mathsf{CM}})\}$ of generators of the algebra $\Pi'(\mathrm{Q}_{\mathsf{CM}})$.

To deduce Theorem 1.4.4 from Theorem 1.6.1, one has to be able to replace in diagram 1.5.3 the algebra $\mathfrak{T}_\chi$, of 'matrix valued' twisted differential operators, by a 'smaller' algebra of *scalar-valued* twisted differential operators of the form $\mathfrak{A}(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha),\ \mathfrak{gl}(\alpha),\ \boldsymbol{\xi} - \chi)$, that appears in Theorem 1.4.4.

To this end, let $\mathsf{p}_s \in \mathrm{End}\, \mathsf{N}$ denote the idempotent corresponding to the projection $\mathsf{N} = \bigoplus_{j \in \mathrm{I}_{\mathsf{CM}}} \mathsf{N}_j \twoheadrightarrow \mathsf{N}_s$. For χ, χ' as in (1.4.2), one proves

$$(\mathbf{1.6.2}) \qquad \mathsf{p}_s \mathfrak{T}_\chi \mathsf{p}_s \cong \mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)^{\mathrm{GL}(\alpha)}/\mathbf{J}_{\chi'} = \mathfrak{A}(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha),\ \mathfrak{gl}(\alpha),\ \boldsymbol{\xi} - \chi') =: \mathfrak{A}_{\chi'}.$$

Write e_i for the idempotent in the algebra $\Pi'(\mathrm{Q}_{\mathsf{CM}})$ corresponding to the trivial path at i. It is easy to see that the map Θ^{Quiver} sends the subalgebra $e_s \Pi'(\mathrm{Q}_{\mathsf{CM}}) e_s \subset \Pi'(\mathrm{Q}_{\mathsf{CM}})$, spanned by paths beginning and ending at the special vertex s, into $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$, a subalgebra in B. Furthermore, restricting diagram (1.5.3) to the subalgebra $e_s\Pi'(\mathrm{Q}_{\mathsf{CM}})e_s$, one obtains four algebra homomorphisms along the perimeter of the following diagram

$$
\textbf{(1.6.3)}\qquad
\begin{array}{ccccc}
 & & e_s\Pi'(\mathrm{Q}_{\mathsf{CM}})e_s & & \\
 & \swarrow^{\Theta^{\mathrm{Holland}}} & & \searrow^{\Theta^{\mathrm{Quiver}}} & \\
\mathscr{D}(\mathrm{Q}_{\mathsf{CM}},\alpha)^{\mathrm{GL}(\alpha)}/\mathbf{J}_{\chi'} & & \overset{\Phi_{k,c}}{\dashrightarrow} & & \mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e} \\
 & \searrow_{\Theta^{\mathrm{Radial}}} & & \swarrow_{\Theta^{\mathrm{Dunkl}}} & \\
 & & \mathscr{D}(\mathfrak{X},p,\varrho_s)^{\mathbf{\Gamma}_n} & &
\end{array}
$$

Here, $\mathscr{D}(\mathfrak{X},p,\varrho_s)^{\mathbf{\Gamma}_n}$ stands for an appropriate ring of *scalar-valued* $\mathbf{\Gamma}_n$-invariant twisted differential operators on $\mathfrak{X}$.

The perimeter of diagram (1.6.3) commutes by Theorem 1.6.1. In addition, one proves

Lemma **1.6.4.** — *In diagram* (1.6.3), *the map* $\Theta^{\mathrm{Holland}}$ *is* surjective *and the map* Θ^{Dunkl} *is* injective.

It is clear that the lemma yields

$$
\begin{aligned}
\mathrm{Ker}\,\Theta^{\mathrm{Holland}} \subset \mathrm{Ker}(\Theta^{\mathrm{Radial}}\circ\Theta^{\mathrm{Holland}}) &= \mathrm{Ker}(\Theta^{\mathrm{Dunkl}}\circ\Theta^{\mathrm{Quiver}}) \\
&= \mathrm{Ker}\,\Theta^{\mathrm{Quiver}}.
\end{aligned}
$$

The resulting inclusion $\mathrm{Ker}\,\Theta^{\mathrm{Holland}}\subset\mathrm{Ker}\,\Theta^{\mathrm{Quiver}}$ implies that we may (and will) define the dashed arrow $\Phi_{k,c}$ in diagram (1.6.3) to be the composite

$$
\frac{\mathscr{D}(\mathrm{Q}_{\mathsf{CM}},\alpha)^{\mathrm{GL}(\alpha)}}{\mathbf{J}_{\chi'}} \xrightarrow[\;]{(\Theta^{\mathrm{Holland}})^{-1}} \frac{e_s\Pi'(\mathrm{Q}_{\mathsf{CM}})e_s}{\mathrm{Ker}\,\Theta^{\mathrm{Holland}}} \xrightarrow{\mathrm{proj}} \frac{e_s\Pi'(\mathrm{Q}_{\mathsf{CM}})e_s}{\mathrm{Ker}\,\Theta^{\mathrm{Quiver}}} \xhookrightarrow{\Theta^{\mathrm{Quiver}}} \mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e}.
$$

To complete the proof of Theorem 1.4.4, one observes that all the objects appearing in diagram (1.6.3) come equipped with natural filtrations, and all the maps in the diagram are filtration preserving. Therefore, to prove that the map $\Phi_{k,c}$ is bijective, it suffices to show a similar statement for $\mathrm{gr}\,\Phi_{k,c}$, the associated graded map. The latter statement follows readily from the results of [CB] and [GG2] concerning the geometry of moment maps arising from representations of affine Dynkin quivers.

1.7. *The algebra* B *and the map* Θ^{Quiver}

To define the algebra B that appears in diagram (1.5.2), we will first introduce in (2.2.1) certain idempotents $e_{i,n-1}\in\mathbf{C}[\mathbf{\Gamma}_n]$, $i\in\mathrm{I}$. Then, we let

$$
\textbf{(1.7.1)}\qquad \mathbf{M} := \mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)\mathbf{e}\bigoplus(\oplus_{i\in\mathrm{I}}\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)e_{i,n-1}).
$$

Thus, $\mathbf{M}$ is a left $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$-module, and we put $\mathsf{B} := (\mathrm{End}_{\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)} \mathbf{M})^{\mathrm{op}}$. This endomorphism algebra is built out of Hom-spaces between various $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$-modules which appear as direct summands in (1.7.1). The Hom-spaces are easily computed, and we find

$$(\mathbf{1.7.2}) \qquad \mathsf{B} = \bigoplus_{i,j\in \mathrm{I}_{\mathsf{CM}}} \mathsf{B}_{i,j}, \quad \text{where } \mathsf{B}_{s,s} = \mathbf{e}\mathsf{H}\mathbf{e}, \text{ and}$$

$$\mathsf{B}_{s,j} = \mathbf{e}\mathsf{H}e_{j,n-1}, \quad \mathsf{B}_{i,s} = e_{i,n-1}\mathsf{H}\mathbf{e}, \quad \mathsf{B}_{i,j} = e_{i,n-1}\mathsf{H}e_{j,n-1}, \quad \forall i,j \in \mathrm{I}.$$

Each direct summand $\mathsf{B}_{i,j}$ here is a subspace of the algebra $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$, and multiplication in the algebra B is given by 'matrix multiplication' $\mathsf{B}_{i,j} \times \mathsf{B}_{j,k} \to \mathsf{B}_{i,k}$ where, for each $i, j, k \in \mathrm{I}_{\mathsf{CM}}$, the corresponding pairing is induced by the multiplication in $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$.

Our construction of the map Θ^{Quiver} is based on an exact functor

$$(\mathbf{1.7.3}) \qquad \mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)\text{-mod} \to \Pi(Q_{\mathsf{CM}})\text{-mod}, \quad M \mapsto \widetilde{M}.$$

To define this functor, let $L_{(1)}$, resp. $\Gamma_{(1)}$, be a copy (inside the algebra $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$) of our 2-dimensional vector space L, resp. copy of the group Γ, corresponding to the first direct summand in $L^{\oplus n}$. Further, let S_{n-1} be the subgroup of S_n which permutes $[2, n]$, and let $\mathbf{\Gamma}_{n-1} = S_{n-1} \ltimes \Gamma^{n-1} \subset \mathbf{\Gamma}_n$ be the wreath-product subgroup corresponding to the last $n-1$ factors in Γ^n. It is clear from the commutation relations in $T(L^{\oplus n}) \rtimes \mathbf{C}[\mathbf{\Gamma}_n]$ that any element of the subalgebra $\mathsf{H}_{(1)} \subset \mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$, generated by $L_{(1)}$ and $\Gamma_{(1)}$, commutes with $\mathbf{\Gamma}_{n-1}$.

Now, let M be an arbitrary left $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$-module. We deduce that the space $M^{\mathbf{\Gamma}_{n-1}} \subset M$, of $\mathbf{\Gamma}_{n-1}$-invariants, is stable under the action of the subalgebra $\mathsf{H}_{(1)}$. Thus, to each vertex $i \in Q$ we may attach the vector space $M_i := \mathrm{Hom}_{\Gamma_{(1)}}(N_i, M^{\mathbf{\Gamma}_{n-1}})$, the corresponding $\Gamma_{(1)}$-isotypic component. Further, following the strategy of [CBH] and using the McKay correspondence, we see that the action map $L_{(1)} \otimes M^{\mathbf{\Gamma}_{n-1}} \to M^{\mathbf{\Gamma}_{n-1}}$ induces linear maps between various isotypic components M_i. This way, the collection $\{M_i\}_{i\in \mathrm{I}}$ acquires the structure of a representation of the quiver $\overline{Q}$. In addition, the subspace $M_s := M^{\mathbf{\Gamma}_n} \subset M$ is clearly contained in $M_o = \mathrm{Hom}_{\Gamma_{(1)}}(N_o, M^{\mathbf{\Gamma}_{n-1}}) = M^{\mathbf{\Gamma}_{n-1}}$ as a *canonical* direct summand. Therefore the imbedding $b : M_s \to M_o$ and the projection $b^* : M_o \to M_s$ provide additional maps, making the collection $\{M_i\}_{i\in \mathrm{I}_{\mathsf{CM}}}$ a representation of the quiver $\overline{Q_{\mathsf{CM}}}$. One can check that this representation descends to a representation of the algebra $\Pi(Q_{\mathsf{CM}})$, which is a quotient of the path algebra of $\overline{Q_{\mathsf{CM}}}$. Thus, to any $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$-module M we have assigned a $\Pi(Q_{\mathsf{CM}})$-module $\widetilde{M} = \oplus_{i\in \mathrm{I}_{\mathsf{CM}}} M_i$. This gives the desired functor (1.7.3), cf. Section 1.8 below for a generalization.

Finally, we apply the functor $M \mapsto \widetilde{M}$ to $M := \mathbf{M}$, the $\mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$-module in (1.7.1). It is immediate from (1.7.2) that one has a natural bijection $\mathsf{B} \cong \widetilde{\mathbf{M}}$. The bijection gives B the structure of a left $\Pi(Q_{\mathsf{CM}})$-module, moreover, the action of $\Pi(Q_{\mathsf{CM}})$

on B commutes with right multiplication (with respect to the algebra structure) by the elements of B. It follows that the $\Pi(Q_{\mathsf{CM}})$-module structure on B comes, via left multiplication, from an algebra homomorphism $\Pi(Q_{\mathsf{CM}}) \to \mathsf{B}$. The latter homomorphism clearly restricts to a homomorphism $e_s\Pi(Q_{\mathsf{CM}})e_s \to \mathsf{B}_{s,s} = \mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$, denoted Θ^{Radial}.

There is a modification of the above construction, to be explained in Section 2.2, in which the algebra $\Pi(Q_{\mathsf{CM}})$ is replaced by the renormalized algebra $\Pi'(Q_{\mathsf{CM}})$. This way, one obtains similar algebra homomorphisms

$$(\mathbf{1.7.4})\qquad \Theta^{\text{Quiver}} : \Pi'(Q_{\mathsf{CM}}) \to \mathsf{B}, \quad \text{and}$$
$$\Theta^{\text{Quiver}} : e_s\Pi'(Q_{\mathsf{CM}})e_s \to \mathsf{B}_{s,s} = \mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}.$$

1.8. *Applications to reflection functors and shift functors*

In Section 6, we study reflection functors and shift functors for generalized preprojective algebras and symplectic reflection algebras associated with wreath-products, cf. [GG].

More generally, let Q be an arbitrary (not necessarily extended Dynkin) quiver, with vertex set I. Write $\mathfrak{C} = (\mathfrak{C}_{ij})$ for the generalized Cartan matrix of Q and W for the Weyl group W, defined as the group generated by the simple reflections r_i for $i \in I$. The group W acts on $\mathbf{C}^I$ as $r_i : \lambda = \sum_{j\in I} \lambda_j e_j \mapsto \lambda - \sum_{j\in I} \mathfrak{C}_{ij}\lambda_i e_j$.

For any $\lambda \in \mathbf{C}^I$, one has an algebra $\Pi'_\lambda(Q)$, a renormalized version of the corresponding *deformed preprojective algebra* studied in [CBH]. Further, for any integer $n \geq 1$, and complex parameters $\nu \in \mathbf{C}$ and $\lambda \in \mathbf{C}^I$, we have associated in [GG, (1.2.3)], see also Definition 6.1.3 below, a *generalized preprojective algebra* $\mathsf{A}_{n,\lambda,\nu}(Q)$.

For each $i \in I$, there are reflection functors F'_i for the corresponding deformed preprojective algebras $\Pi'_\lambda(Q)$, introduced in [CBH], and also their analogues for generalized preprojective algebras, introduced in [Ga]:

$$(\mathbf{1.8.1})\qquad F_i : \mathsf{A}_{n,\lambda,\nu}(Q)\text{-mod} \to \mathsf{A}_{n,r_i(\lambda),\nu}(Q)\text{-mod}.$$

We will show in Section 6.6 that these functors satisfy standard Coxeter relations:

Proposition **1.8.2.** — *For the reflection functors* F_i *for generalized preprojective algebras, one has:*

(i) *If* $\lambda_i \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $F_i^2 = \mathtt{Id}$.
(ii) *Suppose* $\mathfrak{C}_{ij} = 0$. *If* $\lambda_i \pm p\nu \neq 0$ *and* $\lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $F_iF_j = F_jF_i$.
(iii) *Suppose* $\mathfrak{C}_{ij} = -1$. *If* $\lambda_i \pm p\nu \neq 0$, $\lambda_j \pm p\nu \neq 0$ *and* $\lambda_i + \lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $F_iF_jF_i = F_jF_iF_j$.

Part (i) of the proposition has been already proved in [Ga, Theorem 5.1]; Parts (ii) and (iii) are new. In the special case $n = 1$, the proposition is due to [CBH], [Na], [Lu2], and [Maf]. However, we believe that, even in that special case, our proof appears to be simpler.

Next, given c as in (1.2.1), we put

$$c' := \sum_{\gamma\in\Gamma\smallsetminus\{1\}} (2t - c_\gamma)\cdot\gamma^{-1} \quad\text{and}\quad \mathbf{e}_- := \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\sigma(e_o\otimes\cdots\otimes e_o).$$

Using our main Theorem 1.4.4 and reflection functors, we will deduce

Corollary **1.8.3.** — *For* $t = 1/|\Gamma|$ *and any* c *as in* (1.2.1), *there are algebra isomorphisms*

$$\mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e} \simeq \mathbf{e}_-\mathsf{H}_{t,k-2t,c'}\mathbf{e}_- \simeq \mathbf{e}_-\mathsf{H}_{t,k-2t,c}\mathbf{e}_-.$$

We will prove the first isomorphism above in Section 5.3 and the second in Section 6.7. Using the composite isomorphism in Corollary 1.8.3, we define the *shift functor* to be the functor

(**1.8.4**) $$\mathbf{S} : \mathsf{H}_{t,k,c}\text{-mod} \to \mathsf{H}_{t,k-2t,c}\text{-mod},\quad V \mapsto \mathsf{H}_{t,k-2t,c}\mathbf{e}_- \otimes_{\mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e}} \mathbf{e}V.$$

Finally, we can extend the construction exploited in the definition of the map Θ^{Quiver} to an appropriate, more general, context as follows.

Let T be any nonempty subset of I. Generalizing the definition of Calogero–Moser quiver, let Q_T be a quiver obtained from Q by adjoining a vertex s, called the *special* vertex, and arrows $b_i : s \to i$, one for each $i \in T$. Recall that e_i denotes the idempotent in the path algebra corresponding to a vertex i. Thus, given $\lambda \in \mathbf{C}^I$, we write $\lambda = \sum \lambda_i e_i$, and we also put $e_T := \sum_{i\in T} e_i$.

In Section 6.2, for any $n \geq 1$, $\lambda \in \mathbf{C}^I$, $\nu \in \mathbf{C}$, we introduce an exact functor

(**1.8.5**) $$G' : \mathsf{A}_{n,\lambda,\nu}(Q)\text{-mod} \to \Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod}.$$

The construction of reflection functors for generalized preprojective algebras, see (1.8.1), implies readily that, for any $i \in I$, one has the following commutative diagram

(**1.8.6**) $$\begin{array}{ccc} \mathsf{A}_{n,\lambda,\nu}(Q)\text{-mod} & \xrightarrow{F_i} & \mathsf{A}_{n,r_i(\lambda),\nu}(Q)\text{-mod} \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G'} \\ \Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod} & \xrightarrow{F'_i} & \Pi'_{r_i(\lambda)-\nu e_T+n\nu e_s}(Q_T)\text{-mod}. \end{array}$$

The functor (1.8.5) is a generalization of the functor $M \mapsto \widetilde{M}$ considered in Section 1.7 in the following sense. Let Q be the extended Dynkin quiver associated to a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$. Given a data (n, k, c), as in (1.4.1), put $t = 1/|\Gamma|$ and $\nu = k \cdot |\Gamma|/2$. The generalized preprojective algebra $\mathsf{A}_{n,\lambda,\nu}(\mathrm{Q})$ is *Morita equivalent*, according to [GG], to the symplectic reflection algebra $\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)$, so one has a category equivalence $\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\text{-mod} \xrightarrow{\sim} \mathsf{A}_{n,\lambda,\nu}(\mathrm{Q})\text{-mod}$. Therefore, composing this equivalence with (1.8.5), yields a functor

$$\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\text{-mod} \to \Pi'_{\lambda-\nu e_{\mathrm{T}}+n\nu e_s}(\mathrm{Q}_{\mathrm{T}})\text{-mod}.$$

The latter functor reduces, in the special case of the one point set $\mathrm{T} = \{o\}$, to the functor $\mathrm{M} \mapsto \widetilde{\mathrm{M}}$ considered in Section 1.7.

1.9. *Quantization of the Hilbert scheme of points on the resolution of Kleinian singularity*

The shift functor (1.8.4) is the Γ-analogue of the shift functor introduced in [BEG] in the case of the trivial group Γ. The latter functor has been used by Gordon-Stafford [GS] to construct quantization of the Hilbert scheme of n points of the plane $\mathbf{C}^2$.

Now, let $\mathrm{X} \to \mathrm{L}/\Gamma$ be the minimal resolution of the Kleinian singularity L/Γ and let $\mathrm{Hilb}^n\mathrm{X}$ be the Hilbert scheme of n points in X. It should be possible to use the shift functor (1.8.4) and Theorem 1.4.4 to construct quantizations of $\mathrm{Hilb}^n\mathrm{X}$. This would provide a common generalization to the case of wreath-products $\boldsymbol{\Gamma}_n = \mathrm{S}_n \ltimes \Gamma^n$ of the results of Gordon-Stafford [GS] in the special case $\Gamma = 1$ and $n \geq 1$, and also of the results of Boyarchenko [Bo] in the special case of arbitrary $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$ and $n = 1$, cf. also [Mu] for the case of cyclic group Γ (and $n = 1$).

In a different direction, the construction of the algebra $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$ in terms of Hamiltonian reduction provided by Theorem 1.4.4 gives way to applying the machinery of [BFG] to symplectic reflection algebras over $\Bbbk$, an algebraic closure of the finite field $\mathbf{F}_p$.

In more detail, fix a finite group $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$ and a positive integer n. Then, a routine argument shows that, for all large enough primes $p > n$, each of the schemes X, $\mathrm{Hilb}^n\mathrm{X}$, and $\boldsymbol{\mu}^{-1}(0)$, cf. (1.3.4), has a well defined reduction to a reduced scheme over $\Bbbk$. Further, let $\mathscr{M}_n$ be the irreducible component of $\boldsymbol{\mu}^{-1}(0)$, cf. (1.3.4), as defined in [GG2, Theorem 3.3.3(ii)]. Then, the action of the group $\mathrm{GL}(\alpha)/\mathrm{G}_m$ on $\mathscr{M}_n$ is generically free. Moreover, according to H. Nakajima, there exists a $\mathrm{GL}(\alpha)$-stable Zariski open dense subset $\mathsf{M} \subset \mathscr{M}_n$ of *stable* points, such that one has a smooth universal geometric quotient morphism $\mathsf{M} \to \mathrm{Hilb}^n\mathrm{X}$. Furthermore, in this case all the *Basic assumptions* of [BFG, 4.1.1] hold.

Next, let $\mathbf{Q}[\Gamma]$ be the group algebra of Γ with *rational* coefficients. Write $\mathrm{Z}(\Gamma, \mathbf{Q})$ for the center of $\mathbf{Q}[\Gamma]$, and $\mathrm{Z}_o(\Gamma, \mathbf{Q})$ for the corresponding codimension 1 subspace, cf. (1.2.1). Fix $k \in \mathbf{Q}$ and $c \in \mathrm{Z}_o(\Gamma, \mathbf{Q})$ and let $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n, \mathbf{Q})\mathbf{e}$ be the $\mathbf{Q}$-rational version of the $\mathbf{C}$-algebra $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)\mathbf{e}$. Then, there exists a large enough constant $\mathrm{N}(k,c) > \max(n, |\Gamma|)$ such that for all primes $p > \mathrm{N}(k,c)$ the $\mathbf{Q}$-algebra $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n, \mathbf{Q})\mathbf{e}$ has a well defined reduction to a $\Bbbk$-algebra $\mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n, \Bbbk)\mathbf{e}$.

On the other hand, one can apply a characteristic p version of quantum Hamiltonian reduction, as explained in [BFG, §3], in our present situation. This way, for all large enough primes p, Theorem 4.1.4 from [BFG] provides a construction of a sheaf of Azumaya algebras $\mathscr{A}_{k,c}$ on $(\mathrm{Hilb}^n \mathrm{X})^{(1)}$, the Frobenius twist of the scheme $\mathrm{Hilb}^n \mathrm{X}$.

Mimicing the proof of [BFG, Theorem 7.2.4(i)–(ii)], and using our Theorem 1.4.4, one obtains the following result

Theorem **1.9.1.** — *Fix $k \in \mathbf{Q}$ and $c \in \mathrm{Z}_0(\Gamma, \mathbf{Q})$. Then, there exists a constant $d(k,c) > \max(n, |\Gamma|)$, such that for all primes $p > d(k,c)$ and $t = 1/|\Gamma| \in \Bbbk$, we have*

$$\mathrm{H}^0\big((\mathrm{Hilb}^n \mathrm{X})^{(1)}, \mathscr{A}_{k,c}\big) \cong \mathbf{e}\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n, \Bbbk)\mathbf{e};$$

moreover,

$$\mathrm{H}^i\big((\mathrm{Hilb}^n \mathrm{X})^{(1)}, \mathscr{A}_{k,c}\big) = 0, \quad \forall i > 0.$$

1.10. *Directions of further research*

The map Θ^{Quiver} introduced in this paper turns out to be useful in the theory of deformed double current algebras developed by N. Guay [Gu1, Gu2, Gu3]. Namely, it is possible to view the integer n in the definition of the algebra $\mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e}$ as a parameter and to make an "analytic continuation" of the construction of the map Θ^{Quiver} with respect to that parameter. This way, one obtains a new construction of Γ-deformed double current algebras (for $\mathfrak{gl}(1)$) as appropriate quotients of the algebras $e_s\Pi'(\mathrm{Q}_{\mathsf{CM}})e_s$. This will be discussed in a forthcoming paper [EGR].

We expect that the map Θ^{Dunkl} will be helpful in developing a Borel–Weil–Bott style theory for representations of symplectic reflection algebras for wreath products. Such a theory would provide a geometric realization of finite dimensional representations of these algebras (including those studied in [Mo, Ga]) in the spaces of global sections of appropriate coherent sheaves on $(\mathbf{P}^1)^n$ satisfying appropriate vanishing conditions. First steps in this direction are taken in [E], and forthcoming work of S. Montarani.

1.11. *Acknowledgments*

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2. Calogero–Moser quiver

2.1. *Intertwiners*

Let $\overline{\mathrm{Q}}$ be the double quiver of Q, obtained from Q by adding a reverse edge $a^* : j \to i$ for each edge $a : i \to j$ in Q. For any edge $a : i \to j$ in $\overline{\mathrm{Q}}$, we write its tail $t(a) := i$ and its head $h(a) := j$.

We have an identification $\mathrm{L} \xrightarrow{\sim} \mathrm{L}^* : u \mapsto \omega(u, \cdot)$. Let $a \in \overline{\mathrm{Q}}$ be an edge, then for each intertwiner $\phi_a : \mathrm{L} \otimes \mathrm{N}^*_{t(a)} \to \mathrm{N}^*_{h(a)}$, we have a corresponding intertwiner $\phi'_a : \mathrm{N}^*_{t(a)} \to \mathrm{L} \otimes \mathrm{N}^*_{h(a)}$.

Suppose Q is not of type $\widetilde{\mathrm{A}}_1$. Following [CBH] (cf. also [Me]), we normalize the intertwiners such that for each edge $a \in \mathrm{Q}$, we have $\phi_{a^*}\phi'_a = \delta_{h(a)}\mathtt{Id}_{\mathrm{N}^*_{t(a)}}$, and so $\phi_a\phi'_{a^*} = -\delta_{t(a)}\mathtt{Id}_{\mathrm{N}^*_{h(a)}}$. Thus, $\phi'_a\phi_{a^*}$ is $\delta_{h(a)}$ times the projection of $\mathrm{L} \otimes \mathrm{N}^*_{h(a)}$ to $\mathrm{N}^*_{t(a)}$, and $\phi'_{a^*}\phi_a$ is $-\delta_{t(a)}$ times the projection of $\mathrm{L} \otimes \mathrm{N}^*_{t(a)}$ to $\mathrm{N}^*_{h(a)}$. Hence, for any vertex i,

$$\textbf{(2.1.1)} \qquad \sum_{a \in \mathrm{Q};\, h(a)=i} \phi'_a\phi_{a^*} - \sum_{a \in \mathrm{Q};\, t(a)=i} \phi'_{a^*}\phi_a = \delta_i \mathtt{Id}_{\mathrm{L}\otimes\mathrm{N}^*_i}.$$

Suppose now that Q is of type $\widetilde{\mathrm{A}}_1$. Then Γ is the group with 2 elements $1, \zeta$. Moreover, $\zeta x = -x$ and $\zeta y = -y$. Write the vertices of Q as o and i, where N_o is the trivial representation of Γ and N_i is the sign representation of Γ. We have a decomposition $\mathrm{L} = \mathrm{N}^x_i \oplus \mathrm{N}^y_i$ where N^x_i is spanned by x and N^y_i is spanned by y. Let $\mathrm{pr}^x_i : \mathrm{L}\otimes\mathrm{N}_o \to \mathrm{N}_i$ be the projection map to $\mathrm{N}^x_i \otimes \mathrm{N}_o = \mathrm{N}_i$, and $\mathrm{pr}^y_i : \mathrm{L}\otimes\mathrm{N}_o \to \mathrm{N}_i$ be the projection map to $\mathrm{N}^y_i \otimes \mathrm{N}_o = \mathrm{N}_i$. Let $\mathrm{pr}^x_o : \mathrm{L}\otimes\mathrm{N}_i \to \mathrm{N}_o$ be the projection map to $\mathrm{N}^x_i \otimes \mathrm{N}_i = \mathrm{N}_o$, and $\mathrm{pr}^y_o : \mathrm{L} \otimes \mathrm{N}_i \to \mathrm{N}_o$ be the projection map to $\mathrm{N}^y_i \otimes \mathrm{N}_i = \mathrm{N}_o$. Denote the edges of Q by a_1 and a_2. If $a_1 : o \to i$, then let $\phi_{a_1} = \mathrm{pr}^y_i$ and $\phi_{a_1^*} = \mathrm{pr}^x_o$. If $a_1 : i \to o$, then let $\phi_{a_1} = \mathrm{pr}^x_o$ and $\phi_{a_1^*} = -\mathrm{pr}^y_i$. If $a_2 : o \to i$, then let $\phi_{a_2} = \mathrm{pr}^x_i$ and $\phi_{a_2^*} = -\mathrm{pr}^y_o$. If $a_2 : i \to o$, then let $\phi_{a_2} = \mathrm{pr}^y_o$ and $\phi_{a_2^*} = \mathrm{pr}^x_i$. It is easy to see that with these choices, we again have (2.1.1).

2.2. *Quiver map*

For convenience, we shall fix an isomorphism $\mathrm{N}_i = \mathbf{C}^{\delta_i}$, where $\delta_i = \dim \mathrm{N}_i$. We have $\mathbf{C}\Gamma = \bigoplus_{i\in\mathrm{I}} \mathrm{End}\, \mathrm{N}_i = \bigoplus_{i\in\mathrm{I}} \mathrm{Mat}_{\delta_i}(\mathbf{C})$. Let $e^i_{p,q}$ $(1 \le p, q \le \delta_i)$ be the element of $\mathbf{C}\Gamma$ with 1 in the (p, q)-entry of the matrix for the i-th summand and zero elsewhere. Let e_i be the idempotent $e^i_{1,1}$. In particular, $e_o = \sum_{\gamma\in\Gamma} \gamma/|\Gamma|$, where o is the extending vertex of the affine Dynkin quiver Q. Note that $\mathrm{N}_i = \mathbf{C}[\Gamma]e_i$ and $\phi_a \in e_{h(a)}(\mathrm{L} \otimes \mathbf{C}[\Gamma]e_{t(a)})$. Here, the left action of Γ on $\mathrm{L}\otimes\mathbf{C}[\Gamma]$ is the diagonal one. When Q is not of type $\widetilde{\mathrm{A}}_1$, ϕ_a spans $e_{h(a)}(\mathrm{L}\otimes\mathbf{C}[\Gamma]e_{t(a)})$. When Q is of type $\widetilde{\mathrm{A}}_1$ with vertices o and i, $e_o(\mathrm{L}\otimes\mathbf{C}[\Gamma]e_i)$ and $e_i(\mathrm{L} \otimes \mathbf{C}[\Gamma]e_o)$ are both 2 dimensional and spanned by the intertwiners ϕ_a which they contain.

We put $e_n := \frac{1}{n!}\sum_{\sigma\in S_n}\sigma \in \mathbf{C}[S_n]$. For any $i \in I$, let

$$\textbf{(2.2.1)}\qquad \begin{aligned} e_{i,n-1} &:= e_{n-1}(e_i \otimes e_o \otimes \cdots \otimes e_o) \in \mathbf{C}[\boldsymbol{\Gamma}_n], \quad \text{and} \\ \mathbf{e} &:= e_n(e_o \otimes \cdots \otimes e_o) \in \mathbf{C}[\boldsymbol{\Gamma}_n]. \end{aligned}$$

The idempotent $\mathbf{e}$ is same as the one that appears in Theorem 1.4.4 of the Introduction.

For each vertex i of the Calogero–Moser quiver Q_{CM}, cf. Definition 1.3.1, the idempotent e_i is the trivial path at the vertex i. Let λ_i be the trace of $t \cdot 1 + c$ on N_i, let $\lambda = \sum_{i\in I}\lambda_i e_i$, and let

$$\nu := k|\Gamma|/2.$$

Let $\Pi = \Pi_{\lambda-\nu e_o+n\nu e_s}(Q_{\mathsf{CM}})$, the deformed preprojective algebra of Q_{CM} with parameter $\lambda - \nu e_o + n\nu e_s$ as defined in [CBH]. So Π is the quotient of the path algebra $\mathbf{C}\overline{Q_{\mathsf{CM}}}$ of $\overline{Q_{\mathsf{CM}}}$ by the following relations:

$$\sum_{a\in Q}[a, a^*] + bb^* = \lambda - \nu e_o, \qquad b^*b = -n\nu e_s.$$

We shall define a functor from H-modules to Π-modules. Let M be a H-module. We want to define a Π-module $\widetilde{M}$. For each $i \in I$, let $\widetilde{M}_i := e_{i,n-1}M$. Also, let $\widetilde{M}_s := \mathbf{e}M$. If a is an edge in $\overline{Q}$, then define $a : \widetilde{M}_{t(a)} \to \widetilde{M}_{h(a)}$ to be the map given by the element $\phi_a \otimes e_o \otimes \cdots \otimes e_o \in \mathsf{H}$. Define $b : \widetilde{M}_s \to \widetilde{M}_o$ to be the inclusion map, and define $b^* : \widetilde{M}_o \to \widetilde{M}_s$ to be $-\nu \cdot (1 + s_{12} + \cdots + s_{1n})$.

Lemma **2.2.2.** — *With the above action,* $\widetilde{M}$ *is a* Π*-module.*

Proof. — It is clear that $(1+s_{12}+\cdots+s_{1n})e_{n-1} = ne_n$. On $\widetilde{M}$, we have $b^*b = -n\nu$, and $bb^* = -n\nu e_n = -\nu \cdot (1 + s_{12} + \cdots + s_{1n})$.

By [GG, (3.5.2)], we have an isomorphism $f^{\otimes n}\mathsf{H}f^{\otimes n} = \mathsf{A}_{n,\lambda,\nu}(Q)$ where $f = \sum_{i\in I} e_i$, cf. Definition 6.1.3 below. Now $f^{\otimes n}M$ is a $\mathsf{A}_{n,\lambda,\nu}(Q)$-module, and $e_{i,n-1}M = e_{i,n-1}f^{\otimes n}M$, $\mathbf{e}M = \mathbf{e}f^{\otimes n}M$. The action of the edge $a : \widetilde{M}_{t(a)} \to \widetilde{M}_{h(a)}$ is the action given by the element $a \otimes e_o^{\otimes(n-1)} \in \mathsf{A}_{n,\lambda,\nu}(Q)$.

Now on $\widetilde{M}$, at a vertex $i \neq o, s$, we have

$$\sum_{a\in Q;\, h(a)=i} aa^* - \sum_{a\in Q;\, t(a)=i} a^*a = \lambda_i$$

by the relation (i) in Definition 6.1.3. At the vertex o, we have

$$\sum_{a\in Q;\, h(a)=o} aa^* - \sum_{a\in Q;\, t(a)=o} a^*a = \lambda_o + \nu \cdot (s_{12} + \cdots + s_{1n}) = \lambda_o - \nu - bb^*,$$

using again the relation (i) in Definition 6.1.3. □

It is clear that the assignment $G : M \mapsto \widetilde{M}$ is functorial. We will give a more general construction in Section 6.2.

Applying the functor $M \mapsto \widetilde{M}$ to the H-module $\mathbf{M}$ introduced in (1.7.1) we construct, as has been explained in Section 1.7, the algebra homomorphism $\theta^{\text{Quiver}} : \Pi \to \mathsf{B}$.

Observe that $\theta^{\text{Quiver}}(b^*)$ is 0 when $\nu = 0$. For this reason, we shall need a slight modification of the above constructions.

Define Π' to be the quotient of the path algebra $\mathbf{C}\overline{Q_{\mathsf{CM}}}$ by the following relations:

$$\sum_{a \in Q} [a, a^*] + \nu b b^* = \lambda - \nu e_o, \qquad b^* b = -n e_s.$$

We have a morphism of algebras $\Pi \to \Pi'$ defined on the edges by

$$a \mapsto a \;\text{ for } a \neq b^*, \qquad b^* \mapsto \nu b^*.$$

This is an isomorphism only when $\nu \neq 0$.

Given a H-module M, we define a Π'-module structure on $\widetilde{M}$ as above, except that now, we let $b^* : \widetilde{M}_o \to \widetilde{M}_s$ be $-(1 + s_{12} + \cdots + s_{1n})$. Hence, as above, we obtain a morphism $\Theta^{\text{Quiver}} : \Pi' \to \mathsf{B}$, cf. (1.7.4).

2.3. *Holland's map*

In this subsection, we recall a construction of Holland from [Ho].

Let $\epsilon_i \in \mathbf{Z}^I$ denote the coordinate vector corresponding to the vertex $i \in I$. Let $\delta = \sum_{i \in I} \delta_i \epsilon_i$, the minimal positive imaginary root of Q. Let $\alpha := n\delta + \epsilon_s$, a dimension vector for Q_{CM}. Thus, $\alpha_i = n\delta_i$ for $i \in I$, and $\alpha_s = 1$. *We shall assume that* $\lambda \cdot \delta = 1$, *that is,* $t = 1/|\Gamma|$.

Let $e^a_{p,q}$ and $t^a_{p,q}$ ($a \in Q_{\mathsf{CM}}$, $1 \le p \le \alpha_{h(a)}$, $1 \le q \le \alpha_{t(a)}$) be, respectively, the coordinate vectors and the coordinate functions on $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$. We write $e^a_{q,p}$ for the transpose of $e^a_{p,q}$. Now define a representation of $\overline{Q_{\mathsf{CM}}}$ on $\mathscr{O}(\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})) \otimes N$, the space of N-valued regular functions on $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$, as follows. For any $a \in Q_{\mathsf{CM}}$, we define the following $\operatorname{End} \mathsf{N}$-valued differential operators of order 0 and 1, respectively

$$\hat{a} := -\sum_{p,q} e^a_{p,q} \otimes t^a_{p,q}, \quad \text{resp.,} \quad \hat{a}^* := \sum_{p,q} e^a_{q,p} \otimes \frac{\partial}{\partial t^a_{p,q}}.$$

The assignment $a \mapsto \hat{a}$, $a^* \mapsto \hat{a}^*$ extends by multiplicativity to an algebra homomorphism

$$(\mathbf{2.3.1}) \qquad \widetilde{\theta}^{\text{Holland}} : \mathbf{C}\overline{Q_{\mathsf{CM}}} \to (\mathscr{D}(Q_{\mathsf{CM}}, \alpha) \otimes \operatorname{End} \mathsf{N})^{GL(\alpha)},$$

where $\mathbf{C}\overline{\mathrm{Q}_{\mathsf{CM}}}$ denotes the path algebra of the double quiver $\overline{\mathrm{Q}_{\mathsf{CM}}}$. By [Ho, Theorem 3.14] and [Ho, Lemma 3.16], $\widetilde{\theta}^{\mathrm{Holland}}$ descends to homomorphisms, cf. notation in (1.4.2):

$$\theta^{\mathrm{Holland}} : \Pi \to \mathfrak{T}_\chi \quad \text{and} \quad \theta^{\mathrm{Holland}} : e_s\Pi e_s \to \mathfrak{A}_{\chi'}.$$

We remind that $\mathfrak{A}_{\chi'}$ is the algebra in (1.6.2).

Later, we will define a homomorphism $\Theta^{\mathrm{Holland}} : e_s\Pi' e_s \to \mathfrak{A}_{\chi'}$.

3. Radial part map

From now on, we assume that Q is not of type $\widetilde{\mathrm{A}}_n$ where n is even. Equivalently, that means that Γ has the (necessarily unique) *central element of order* 2, to be denoted $\zeta \in \Gamma$.

3.1. *Twisted differential operators*

Let $\mathbf{T} \cong (\mathbf{C}^\times)^m$ be a torus with Lie algebra $\mathfrak{t} := \mathrm{Lie}\,\mathbf{T}$, and $p : \mathrm{X} \to \mathfrak{X}$ a principal $\mathbf{T}$-bundle. For any $h \in \mathrm{Lie}\,\mathbf{T}$, the infinitesimal h-action yields a vector field $\boldsymbol{\xi}_h$ on X. Let $\mathscr{D}_\mathrm{X}$ be the sheaf of algebraic differential operators of X. The action of $\mathbf{T}$ on X makes $\mathscr{D}_\mathrm{X}$ a $\mathbf{T}$-equivariant sheaf of algebras, and we write $\Gamma(\mathrm{X}, \mathscr{D}_\mathrm{X})^\mathbf{T}$ for the algebra of $\mathbf{T}$-invariant global differential operators on X. The assignment $h \mapsto \boldsymbol{\xi}_h$ gives a Lie algebra homomorphism $\mathfrak{t} \to \Gamma(\mathrm{X}, \mathscr{D}_\mathrm{X})^\mathbf{T}$.

Let $\rho : \mathfrak{t} \to \mathrm{End}\,\mathrm{F}$ be a finite dimensional representation. We form $\mathscr{D}_{\mathrm{X},\mathrm{F}} := \mathscr{D}_\mathrm{X} \otimes \mathrm{End}\,\mathrm{F}$, a sheaf of associative algebras on X. Let $\boldsymbol{\xi} - \rho : \mathfrak{t} \to \mathscr{D}_{\mathrm{X},\mathrm{F}} = \mathscr{D}_\mathrm{X} \otimes \mathrm{End}\,\mathrm{F}$, $h \mapsto \boldsymbol{\xi}_h \otimes \mathtt{Id}_\mathrm{F} - \rho(h)$ be the diagonal Lie algebra homomorphism. We write $\mathrm{Im}(\boldsymbol{\xi} - \rho)$ for the image of this homomorphism, and $(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}}$ for the subsheaf of those sections of the push-forward sheaf $p_*\mathscr{D}_{\mathrm{X},\mathrm{F}}$, on $\mathfrak{X}$, which commute with $\mathrm{Im}(\boldsymbol{\xi} - \rho)$. Thus, $\mathrm{Im}(\boldsymbol{\xi} - \rho)$ is a *central* subspace of $(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}}$, a sheaf of associative algebras on $\mathfrak{X}$. We write $(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}} \cdot \mathrm{Im}(\boldsymbol{\xi} - \rho)$ for the (automatically two-sided) ideal in $(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}}$ generated by the image of $\boldsymbol{\xi} - \rho$. Thus, the quotient $(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}}/(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}} \cdot \mathrm{Im}(\boldsymbol{\xi} - \rho)$ is a well-defined sheaf of associative algebras on $\mathfrak{X}$. Let

$$(\mathbf{3.1.1}) \qquad \mathscr{D}(\mathfrak{X}, p, \rho) := \Gamma\big(\mathfrak{X}, (p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}}/(p_*\mathscr{D}_{\mathrm{X},\mathrm{F}})^{\mathsf{adt}} \cdot \mathrm{Im}(\boldsymbol{\xi} - \rho)\big)$$

be the algebra of its global sections, to be referred to as the algebra of *twisted differential operators on* $\mathfrak{X}$ associated with the principal $\mathbf{T}$-bundle $p : \mathrm{X} \to \mathfrak{X}$ and representation ρ.

For any open set U (in the ordinary, Hausdorff topology), we write $\mathscr{H}ol(\mathrm{U}, \mathrm{F})$ for the vector space of all holomorphic maps $\mathrm{U} \to \mathrm{F}$. Given such an open subset

$U \subset X$, put

$$\mathscr{H}ol_\rho(U) := \{f \in \mathscr{H}ol(U, F) \quad | \quad \boldsymbol{\xi}_h f = \rho(h) f, \quad \forall h \in \mathfrak{t}\}. \tag{3.1.2}$$

There is a natural action of the algebra $\mathscr{D}(\mathfrak{X}, p, \rho)$ on the vector space $\mathscr{H}ol_\rho(U)$ given, in local coordinates, by differential operators with End F-valued coefficients.

Given a decomposition $F = F_1 \oplus \cdots \oplus F_r$ into a direct sum of $\mathfrak{t}$-subrepresentations, we have $\operatorname{End} F = \bigoplus_{1 \le m, l \le n} \operatorname{Hom}(F_m, F_l)$. This gives the induced direct sum decomposition

$$\mathscr{D}(\mathfrak{X}, p, \rho) = \bigoplus_{1 \le m, l \le n} \mathscr{D}(\mathfrak{X}, p, \rho, F_m \to F_l).$$

Thus, for each (m, l), the direct summand $\mathscr{D}(\mathfrak{X}, p, \rho, F_m \to F_l)$ has a natural structure of left $\mathscr{D}(\mathfrak{X}, p, \rho|_{F_l})$-module and of right $\mathscr{D}(\mathfrak{X}, p, \rho|_{F_m})$-module.

3.2. *The radial part construction*

Let G be a linear algebraic group and Y a smooth G-variety. Assume in addition that we have a smooth subvariety $X \subset Y$ which is stable under the action of a torus $\mathbf{T} \subset G$, and we also have a smooth morphism $p : X \to \mathfrak{X}$, which is a principal $\mathbf{T}$-bundle with respect to the induced $\mathbf{T}$-action on X. Thus, we have the following diagram

$$\mathfrak{X} \xleftarrow[p]{} X \xhookrightarrow[\jmath]{x \mapsto 1 \times x} G \times_{\mathbf{T}} X \xrightarrow[\text{act}]{g \times x \mapsto g(x)} Y. \tag{3.2.1}$$

Let $\mathfrak{g} := \operatorname{Lie} G$ and let $\rho : \mathfrak{g} \to \operatorname{End} F$ be a finite dimensional representation. For any open subset $U_Y \subset Y$, we may consider the vector space $\mathscr{H}ol_\rho(U_Y)$ defined similarly to (3.1.2) but with respect to the $\mathfrak{g}$-representation ρ. Write $\rho_\mathfrak{t} = \rho|_\mathfrak{t}$ for the restriction of ρ to the Lie algebra $\mathfrak{t} = \operatorname{Lie} \mathbf{T}$. Restriction of functions gives the map

$$\mathsf{Res} : \mathscr{H}ol_\rho(U_Y) \to \mathscr{H}ol_{\rho_\mathfrak{t}}(X \cap U_Y), \quad f \mapsto \mathsf{Res} f := \jmath^* f.$$

Let $\mathscr{D}(Y, F) = \Gamma(Y, \mathscr{D}_{Y,F}) = \Gamma(Y, \mathscr{D}_Y) \otimes \operatorname{End} F$ be the algebra of End F-valued differential operators on Y. As above, we have the Lie algebra map $\boldsymbol{\xi} - \rho : \mathfrak{g} \to \mathscr{D}(Y, F)$ and the subalgebra $\mathscr{D}(Y, F)^{\operatorname{ad}\mathfrak{g}}$, of the operators commuting with the image of that map.

Let $K \subset G$ be a finite subgroup that preserves X and normalizes the torus $\mathbf{T}$. The action of K on X, resp. on $\mathscr{D}_Y$, induces a natural K-action on $\mathfrak{X}$, resp. on the algebra $\mathscr{D}(\mathfrak{X}, p, \rho_\mathfrak{t})$, of twisted differential operators on $\mathfrak{X}$. We write $\mathscr{D}(\mathfrak{X}, p, \rho_\mathfrak{t})^K$ for the subalgebra of K-invariants.

One has the following standard result.

Proposition **3.2.2** (***Radial part map***). — *Assume that* G *is connected and the differential of the map* act, *in* (3.2.1), *is an isomorphism at every point* $\jmath(x)$, $x \in \mathrm{X}$. *Then,*

(i) *There is a natural radial part homomorphism* $\theta^{\mathrm{Radial}} : \mathscr{D}(\mathrm{Y}, \mathrm{F})^{\mathrm{ad}\mathfrak{g}} \to \mathscr{D}(\mathfrak{X}, p, \rho_{\mathfrak{t}})^{\mathrm{K}}$ *such that, for any open (in the Hausdorff topology) subset* $\mathrm{U}_{\mathrm{Y}} \subset \mathrm{Y}$, *we have*

$$\theta^{\mathrm{Radial}}(\mathrm{D}) \cdot (\mathsf{Res} f) = \mathsf{Res}(\mathrm{D}f), \quad \forall \mathrm{D} \in \mathscr{D}(\mathrm{Y}, \mathrm{F})^{\mathrm{ad}\mathfrak{g}}, f \in \mathscr{H}ol_{\rho}(\mathrm{U}_{\mathrm{Y}}).$$

(ii) *The two-sided ideal* $(\mathscr{D}(\mathrm{Y}, \mathrm{F})\,\mathrm{Im}(\boldsymbol{\xi} - \rho))^{\mathrm{ad}\mathfrak{g}}$ *is contained in the kernel of the radial part map* θ^{Radial}.

(iii) *Assume, in addition, that* $\mathfrak{X}$ *is affine and the restriction of* ρ *to* $\mathfrak{t}$ *is diagonalizable. Then, there are canonical algebra isomorphisms, cf.* (1.1.1),

$$\begin{aligned}&\mathfrak{A}(\mathscr{D}(\mathrm{X}, \mathrm{F}), \mathfrak{t}, \boldsymbol{\xi} - \rho)\\ &\qquad \cong \mathscr{D}(\mathrm{X}, \mathrm{F})^{\mathrm{ad}\mathfrak{t}}/(\mathscr{D}(\mathrm{X}, \mathrm{F})\,\mathrm{Im}(\boldsymbol{\xi} - \rho))^{\mathrm{ad}\mathfrak{t}} \xrightarrow{\sim} \mathscr{D}(\mathfrak{X}, p, \rho_{\mathfrak{t}}).\end{aligned}$$

□

3.3. *A slice in* $\mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$

We choose the following orientation on Q: the vertex o is a sink, and any other vertex is a source or a sink. Thus, the second order element ζ acts by 1 at sinks and by -1 at sources. Note also that, see (1.4.2)

$$\partial_i = -n\,\mathrm{Tr}\,|_{\mathrm{N}_i}(\zeta), \quad i \in \mathrm{I}.$$

The collection of intertwiners $\phi = (\phi_a)_{a \in \mathrm{Q}}$ introduced in Section 2.1 gives rise to a linear map

$$\phi : \mathrm{L} \to \mathsf{Rep}_\delta(\mathrm{Q}), \ \text{ where } \ \phi_a : \mathrm{L} \to \mathrm{Hom}(\mathrm{N}^*_{t(a)}, \mathrm{N}^*_{h(a)}), \ \ u \mapsto \phi_a(u).$$

We also define a linear map $\jmath : \mathrm{L}^n \to \mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$ by

$$\begin{aligned}&\jmath(u_1, \dots, u_n)_b = (1, 1, \dots, 1), \ \text{ and}\\ &\jmath(u_1, \dots, u_n)_a = (\phi_a(u_1), \dots, \phi_a(u_n)), \ \ \forall a \in \mathrm{Q}.\end{aligned}$$

Lemma **3.3.1.** — *Let* $u, w \in \mathrm{L}$. *Suppose there are* $\beta_i \in \mathrm{End}(\mathrm{N}_i^*)$ *for* $i \in \mathrm{I}$ *such that* $\phi_a(u)\beta_{t(a)} = \beta_{h(a)}\phi_a(w)$ *for all edges* $a \in \mathrm{Q}$. *If the* β_i *are not all zero, then* u *is proportional to* γw *for some* $\gamma \in \Gamma$.

The lemma will be proved later, at the end of Section 8.2. From this lemma, we deduce

Corollary **3.3.2.** — *There exists a* $\boldsymbol{\Gamma}_n$*-stable Zariski-open dense subset* $L^n_{reg} \subset L^n$ *such that* $\jmath(L^n_{reg})$ *is contained in the set of generic representations of* Q *and, moreover,* $\jmath(L^n_{reg})$ *meets generic* $GL(\alpha)$*-orbits in* $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$ *transversely.*

Proof. — First, we claim that the affine space $\jmath(L^n)$ meets generic $GL(\alpha)$-orbits in $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})$.

Recall that generic representations of Q with dimension vector $n\delta$ are direct sums of n representations with dimension vector δ, it suffices to show that the subspace consisting of the representations $\{\phi_a(u)\}_{a\in Q}$ for all $u \in L$ intersects generic $GL(\delta)$-orbits in $\mathsf{Rep}_\delta(Q)$. By the preceding lemma, the (rational) map $\mathsf{Rep}_\delta(Q) \to \mathbf{P}^1$ defined in [Ri, Theorem 6.2] (which parametrizes generic orbits) is nonconstant on L.

The corollary now follows from the standard Bertini–Sard theorem, cf. e.g. [Ve]. □

3.4. *Discriminant function*

Put $L^\times := L\smallsetminus\{0\}$. The multiplicative group $\mathbf{C}^\times$ is imbedded in GL(L) as scalar matrices, and we have the standard $\mathbf{C}^\times$-bundle $L^\times \to \mathbf{P}_L := L^\times/\mathbf{C}^\times \cong \mathbf{P}^1$ (projective line). The group $S := \mathbf{C}^\times \cap \Gamma$ consists of two elements, $S = \{1, \zeta\}$, where $\zeta \in \Gamma$ is our second order element.

Given $\ell \in \mathbf{P}_L$, write $\Gamma^\ell \subset \Gamma$ for the isotropy group of the line $\ell \subset L$. Clearly, one has $S \subset \Gamma^\ell$. Therefore, $|\Gamma^\ell|/2 = |\Gamma^\ell/S|$ is a positive integer, and we put $\boldsymbol{\kappa}_\ell := |\Gamma^\ell/S| - 1$. Thus, we have $\boldsymbol{\kappa}_\ell > 0$ if and only if the group $\Gamma^\ell \subset \Gamma$, is *strictly larger* than S. The lines ℓ with this property form a *finite* set $\mathbf{P}_L^{\text{sing}} \subset \mathbf{P}_L$. For each $\ell \in \mathbf{P}_L^{\text{sing}}$, we choose and fix a vector representative $v_\ell \in \ell\smallsetminus\{0\} \subset L$.

We introduce the following function

$$\Delta := \prod_{\ell\in\mathbf{P}_L^{\text{sing}}} \omega(v_\ell, -)^{\boldsymbol{\kappa}_\ell} \in \mathbf{C}[L],$$

which is uniquely defined up to a nonzero constant factor depending on the choice of representatives v_ℓ, $\ell \in \mathbf{P}_L^{\text{sing}}$. Further, we introduce a *discriminant* function on L^n_{reg}, defined by

$$(\mathbf{3.4.1}) \qquad \Delta_n(u_1, \ldots, u_n) := \prod_{k=1}^{n} \frac{1}{\Delta(u_k)} \prod_{m\neq l}\prod_{\gamma\in\Gamma} \frac{1}{\omega(u_m, \gamma u_l)}, \quad \text{for } u_1, \ldots, u_n \in L.$$

Let L^n_{reg} be a Zariski open set as in Corollary 3.3.2. Shrinking this set if necessary, from now on we assume in addition that L^n_{reg} is an *affine* $\mathbf{T}$-stable subset contained in $(L^\times)^n$ and, moreover, that the denominator of the function Δ_n does not vanish on L^n_{reg}. Thus, the set L^n_{reg} is $\boldsymbol{\Gamma}_n \ltimes \mathbf{T}$-stable, and we have $\Delta_n \in \mathbf{C}[L^n_{\text{reg}}]$.

The natural action of the torus $\mathbf{T}$ on L^n_{reg} induces an action of the Lie algebra $\mathfrak{t} = \mathrm{Lie}\,\mathbf{T}$ on the coordinate ring $\mathbf{C}[L^n_{reg}]$. Given $h = (h_1, ..., h_n) \in \mathfrak{t} = \mathbf{C}^n$, we write the action map as $h : \mathbf{C}[L^n_{reg}] \to \mathbf{C}[L^n_{reg}]$, $f \mapsto \boldsymbol{\xi}_h f$, and also put $\mathrm{Tr}(h) = h_1 + ... + h_n$.

Lemma **3.4.2.** — *The discriminant* $\Delta_n \in \mathbf{C}[L^n_{reg}]$ *is a weight vector for the* $\mathfrak{t}$*-action, specifically, we have*

$$\boldsymbol{\xi}_h \Delta_n = (n|\Gamma| - 2)\,\mathrm{Tr}(h) \cdot \Delta_n, \quad \forall h \in \mathfrak{t}.$$

Proof. — Note that $\sum_{\ell \in \mathbf{P}^{\mathrm{sing}}_{L}} \boldsymbol{\kappa}_\ell = |\Gamma| - 2$. Hence, Δ is a homogeneous polynomial of degree $|\Gamma| - 2$. We see that any vector u_m appears on the RHS of (3.4.1) with degree $-(|\Gamma| - 2)$, in the factor $\prod \Delta(u_k)^{-1}$, and with degree $-(n-1)|\Gamma|$, in the factor $1/\prod\prod \omega(u_m, \gamma u_l)$. □

3.5. *Compatibility with group actions*

Let $\mathbf{T} := (\mathbf{C}^\times)^n$ be the torus, and form the wreath product $\boldsymbol{\Gamma}_n \ltimes \mathbf{T} = S_n \ltimes (\Gamma \times \mathbf{C}^\times)^n$. We are going to define a group imbedding

(**3.5.1**) $$\jmath_{\mathrm{Lie}} : \boldsymbol{\Gamma}_n \ltimes \mathbf{T} \to GL(\alpha).$$

To this end, we recall the direct sum decomposition (1.5.1), so $\dim \mathsf{N}_i = \alpha_i$ and one may identify the group $GL(\alpha_i)$ with $GL(\mathsf{N}_i)$, for any $i \in I_{\mathsf{CM}}$. Now, by the structure of group algebras, we have the canonical algebra isomorphism $\mathbf{C}[\Gamma] \xrightarrow{\sim} \oplus_{i\in I} \mathrm{End}(N_i^*)$. Thus, we have a group imbedding $\Gamma \hookrightarrow GL(\delta)$ and, therefore, an imbedding $\Gamma^n \hookrightarrow GL(\delta) \times \cdots \times GL(\delta) \hookrightarrow \prod_{i\in I} GL(\alpha_i)$. Further, we define a homomorphism $S_n \to GL(\mathsf{N}_i)$ by $\sigma \mapsto \mathtt{Id}_{N_i^*} \otimes \sigma_{\mathbf{C}^n}$, where $\sigma_{\mathbf{C}^n} \in GL(\mathbf{C}^n)$ stands for the permutation matrix corresponding to $\sigma \in S_n$. This way, combining together the above defined imbeddings of Γ^n and S_n, we obtain a group imbedding $\jmath_{\mathrm{Lie}} : \boldsymbol{\Gamma}_n \to GL(\alpha)$, such that its component $\boldsymbol{\Gamma}_n \to GL(\alpha_s)$, at the special vertex s, sends every element of $\boldsymbol{\Gamma}_n$ to 1.

It remains to define the torus imbedding $\jmath_{\mathrm{Lie}} : \mathbf{T} \to GL(\alpha)$, $\mathbf{t} \mapsto g(\mathbf{t}) = \{g_i(\mathbf{t}) \in GL(\alpha_i)\}_{i\in I_{\mathsf{CM}}}$. The latter is given as follows. We put $g_s(\mathbf{t}) = 1$, $\forall \mathbf{t} \in \mathbf{T}$, and, for any $i \in I$, let

$$g_i(\mathbf{t}) := \mathbf{t}^{-1} \otimes \mathtt{Id}_{N_i^*} \quad \text{if } i \text{ is a source in } Q,$$
$$g_i(\mathbf{t}) := \mathtt{Id}_{\mathsf{N}_i} \quad \text{if } i \text{ is a sink in } Q,$$

where, for $\mathbf{t} = (t_1, ..., t_n)$ we let $\mathbf{t}^{-1} \in GL(\mathbf{C}^n)$ denote the diagonal matrix with diagonal entries $t_1^{-1}, ..., t_n^{-1}$. We note that the image of $\mathbf{T}$ under the above imbedding is *not* contained in the center of the group $GL(\alpha)$.

The torus $\mathbf{T} := (\mathbf{C}^\times)^n$ acts naturally on L^n; the element $\mathbf{t} = (t_1, ..., t_n) \in \mathbf{T}$ sends $(u_1, ..., u_n) \in L^n$ to $(t_1 u_1, ..., t_n u_n)$. This action of $\mathbf{T}$ commutes with that of the group Γ^n. Thus, we get an action of the group $\mathbf{\Gamma}_n \ltimes \mathbf{T}$ on L^n. It is easy to show that the group imbedding $\jmath_{\mathrm{Lie}} : \mathbf{\Gamma}_n \ltimes \mathbf{T} \hookrightarrow \mathrm{GL}(\alpha)$ agrees with the slice imbedding $\jmath : \mathrm{L}^n \hookrightarrow \mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}})$. Specifically, one has

$$(\mathbf{3.5.2})\qquad \jmath(g(u)) = \jmath_{\mathrm{Lie}}(g)(\jmath(u)), \quad \forall u \in \mathrm{L}^n,\ g \in \mathbf{\Gamma}_n \ltimes \mathbf{T}.$$

3.6. *The homomorphisms* θ^{Radial} *and* Θ^{Radial}

The element $\chi \in \mathbf{C}^{\mathrm{I}_{\mathsf{CM}}}$, in (1.4.2), gives a Lie algebra homomorphism $\chi : \mathfrak{gl}(\alpha) \to \mathbf{C}$. We also have the tautological representation $\tau : \mathfrak{gl}(\alpha) \to \operatorname{End}\mathsf{N}$, see (1.5.1), and we let $\chi - \tau : \mathfrak{gl}(\alpha) \to \operatorname{End}\mathsf{N}$, $h \mapsto \chi(h)\cdot \mathtt{Id}_{\mathsf{N}} - \tau(h)$, $\forall h \in \mathfrak{gl}(\alpha)$.

The group imbedding (3.5.1) induces the corresponding Lie algebra imbedding $\jmath_{\mathrm{Lie}} : \mathfrak{t} = \operatorname{Lie}\mathbf{T} \hookrightarrow \mathfrak{gl}(\alpha)$. We let $\rho := (\chi - \tau)\circ \jmath_{\mathrm{Lie}}$ be the pull-back of the representation $\chi - \tau$ to the Lie algebra $\mathfrak{t}$ via the imbedding $\mathfrak{t} \hookrightarrow \mathfrak{gl}(\alpha)$.

We are now in a position to apply the general radial part construction of Section 3.2 in our special case. Specifically, the n-th cartesian power of the $\mathbf{C}^\times$-bundle $\mathrm{L}^\times \to \mathbf{P}_{\mathrm{L}}$ gives a principal $\mathbf{T}$-bundle $(\mathrm{L}^\times)^n \to (\mathbf{P}_{\mathrm{L}})^n$. We set $\mathrm{X} := \mathrm{L}^n_{\mathrm{reg}} \subset (\mathrm{L}^\times)^n$, and let $\mathfrak{X} \subset (\mathbf{P}_{\mathrm{L}})^n$ be the image of X. Write $p : \mathrm{X} \to \mathfrak{X}$ for the restriction of the bundle projection to X. Thus, $\mathfrak{X}$ is a $\mathbf{\Gamma}_n$-stable Zariski open dense subset of $(\mathbf{P}_{\mathrm{L}})^n$, and $p : \mathrm{X} \to \mathfrak{X}$ is a principal $\mathbf{T}$-bundle.

We apply Proposition 3.2.2 to

$$\mathrm{G} = \mathrm{GL}(\alpha), \quad \mathbf{T} = (\mathbf{C}^\times)^n, \quad \mathrm{K} = \mathbf{\Gamma}_n, \quad \mathrm{Y} = \mathsf{Rep}_\alpha(\mathrm{Q}_{\mathsf{CM}}),$$
$$p : \mathrm{X} = \mathrm{L}^n_{\mathrm{reg}} \to \mathfrak{X} = \mathrm{L}^n_{\mathrm{reg}}/\mathbf{T}.$$

Thus, we obtain an algebra homomorphism, cf. (1.5.2):

$$(\mathbf{3.6.1})\qquad \theta^{\mathrm{Radial}} : \mathfrak{T}_\chi = \frac{(\mathscr{D}(\mathrm{Q}_{\mathsf{CM}},\alpha)\otimes \operatorname{End}\mathsf{N})^{\mathrm{GL}(\alpha)}}{((\mathscr{D}(\mathrm{Q}_{\mathsf{CM}},\alpha)\otimes \operatorname{End}\mathsf{N})\operatorname{Im}(\boldsymbol{\xi} - (\chi-\tau)))^{\mathrm{GL}(\alpha)}} \to \mathscr{D}(\mathfrak{X}, p, \rho)^{\mathbf{\Gamma}_n}.$$

Further, we introduce another representation $\varrho : \mathfrak{t} \to \operatorname{End}\mathsf{N}$, $h \mapsto \varrho(h)$ by the formula $\varrho(h) := \rho(h) + \frac{1}{2}(n|\Gamma| - 2)\operatorname{Tr}(h)\mathtt{Id}_{\mathsf{N}}$.

It is easy to see that each of the direct summands in the decomposition $\mathsf{N} = \oplus_{i\in \mathrm{Q}_{\mathsf{CM}}}\mathsf{N}_i$, cf. (1.5.1), is stable under the $\mathfrak{t}$-action via either representation ρ or ϱ. Thus we can write $\rho = \oplus_{i\in \mathrm{I}_{\mathsf{CM}}}\rho_i$, and $\varrho = \oplus_{i\in\mathrm{I}_{\mathsf{CM}}}\rho_i$. To describe these representations more explicitly, let c_ζ be the coefficient in (1.2.1) corresponding to our second order element $\zeta \in \Gamma$, and put

$$(\mathbf{3.6.2})\qquad \mu := -\left(\frac{c_\zeta\cdot|\Gamma|}{2} + 1\right), \quad \text{and} \quad \psi := \sum_{\{i\in\mathrm{I}\ |\ i \text{ is a source in } \mathrm{Q}\}} \delta_i\cdot\chi_i \in \mathbf{C}.$$

One finds that the representations ρ_i and ϱ_i are given by the following explicit formulas:

$$\rho_i(h) = \psi \cdot \mathrm{Tr}(h) \cdot \mathtt{Id}_{\mathsf{N}_i}, \quad \varrho_i(h) = \left(\mu + \tfrac{1}{2}\right) \cdot \mathrm{Tr}(h) \cdot \mathtt{Id}_{\mathsf{N}_i}$$

if $i = s$ or i is a sink in Q; and

$$\rho_i(h) = \psi \cdot \mathrm{Tr}(h) \cdot \mathtt{Id}_{\mathsf{N}_i} - h \otimes \mathtt{Id}_{\mathrm{N}_i^*},$$
$$\varrho_i(h) = \left(\mu + \tfrac{1}{2}\right) \cdot \mathrm{Tr}(h) \cdot \mathtt{Id}_{\mathsf{N}_i} - h \otimes \mathtt{Id}_{\mathrm{N}_i^*}$$

if i is a source in Q, where, for any $h = (h_1, \dots, h_n) \in \mathfrak{t} = \mathbf{C}^n$, we write $\mathrm{Tr}(h) := h_1 + \dots + h_n$, and where the tensor factor h, in $h \otimes \mathtt{Id}_{\mathrm{N}_i^*}$, stands for the map $\mathbf{C}^n \to \mathbf{C}^n$ given by the diagonal matrix with diagonal entries $h_1, \dots, h_n$.

Next, according to Lemma 3.7.6 below, we have

$$(\mathbf{3.6.3}) \qquad 2(\mu - \psi) = -2\left(\frac{c_\zeta \cdot |\Gamma|}{2} + 1\right) + 1 - 2\psi$$
$$= (|\Gamma| - 2) + (n-1)|\Gamma| = n|\Gamma| - 2.$$

Hence, Lemma 3.4.2 shows that $\varrho - \rho = \frac{1}{2}(n|\Gamma| - 2) \cdot \mathrm{Tr}(-)$ is nothing but the weight of $\sqrt{\Delta_n}$, the square root of the discriminant function. Thus, given $\mathrm{D} \in \mathscr{D}(\mathfrak{X}, p, \rho)$, we may conjugate D by the operator of multiplication by the function $\sqrt{\Delta_n}$ to obtain a twisted differential operator $\frac{1}{\sqrt{\Delta_n}} \circ \mathrm{D} \circ \sqrt{\Delta_n} \in \mathscr{D}(\mathfrak{X}, p, \varrho)$, such that for any open set $\mathrm{U} \subset \mathrm{L}^n_{\mathrm{reg}}$ the induced action on functions is given by the map

$$\Gamma(\mathrm{U}, \varrho) \to \Gamma(\mathrm{U}, \varrho), \quad f \mapsto (1/\sqrt{\Delta_n}) \cdot \theta^{\mathrm{Radial}}(\mathrm{D})(\sqrt{\Delta_n} \cdot f).$$

We note that although the square root of the discriminant function Δ_n is ill defined as a function, conjugation by the operator of multiplication by such a function is an unambiguously defined operation on twisted differential operators. Furthermore, the result of conjugation by $\sqrt{\Delta_n}$ is clearly independent of the choice of nonzero constant factor involved in the definition of Δ_n, cf. Section 3.4. Thus, we have a canonically defined algebra homomorphism

$$\Theta^{\mathrm{Radial}} : \mathfrak{T}_\chi \to \mathscr{D}(\mathfrak{X}, p, \varrho)^{\mathbf{\Gamma}_n},$$
$$u \mapsto \Theta^{\mathrm{Radial}}(u) := \frac{1}{\sqrt{\Delta_n}} \circ \theta^{\mathrm{Radial}}(u) \circ \sqrt{\Delta_n}.$$

3.7. *Formulas for the map* $\theta^{\mathrm{Radial}} \circ \theta^{\mathrm{Holland}}$

We introduce some notation. Given a map $f : \mathrm{L} \to \mathrm{U}$ and any $1 \le l \le n$, we write f_l for the map $\mathrm{L}^n \to \mathrm{U}$, $(u_1, \dots, u_n) \mapsto f(u_l)$. Thus, given $\gamma \in \Gamma$, we have the composite $\mathrm{L} \xrightarrow{\gamma} \mathrm{L} \xrightarrow{f} \mathrm{U}$, and the corresponding map $(f \circ \gamma)_l : \mathrm{L}^n \to \mathrm{U}$.

Let ω denote the symplectic form on L. For any vector $v \in \mathrm{L}$, we have the linear function $v^\vee : u \mapsto \omega(v, u)$. Thus, for any $v \in \mathrm{L}$ and $\gamma \in \Gamma$, we may consider the following functions

$$
\begin{aligned}
(\mathbf{3.7.1}) \qquad & (\gamma v)_l^\vee := (v^\vee \circ \gamma^{-1})_l : \ \mathrm{L}^n \to \mathbf{C}, \\
& (u_1, \dots, u_n) \mapsto \omega(\gamma v, u_l) = \omega\big(v, \gamma^{-1} u_l\big) \quad \text{and} \\
& \omega(\gamma; m, l) : \ \mathrm{L}^n \to \mathbf{C}, \quad (u_1, \dots, u_n) \mapsto \omega(u_m, \gamma u_l), \quad \forall 1 \le m \neq l \le n.
\end{aligned}
$$

The definition of the open subset $\mathrm{L}^n_{\text{reg}} \subset \mathrm{L}^n$ insures, see Section 3.4, that none of the functions $\omega(\gamma; m, l)$ vanishes on $\mathrm{L}^n_{\text{reg}}$. Hence, we have $1/\omega(\gamma; m, l) \in \mathbf{C}[\mathrm{L}^n_{\text{reg}}]$.

In Section 2.1, for each edge $a \in \overline{\mathrm{Q}}$, we have introduced intertwiners $\phi_a : \mathrm{L} \otimes \mathrm{N}^*_{t(a)} \to \mathrm{N}^*_{h(a)}$ and $\phi'_a : \mathrm{N}^*_{t(a)} \to \mathrm{L} \otimes \mathrm{N}^*_{h(a)}$. Below, we shall view ϕ_a as a $\mathrm{Hom}(\mathrm{N}^*_{t(a)}, \mathrm{N}^*_{h(a)})$-valued linear function on L, written as $u \mapsto \phi_a(u)$. The n-th cartesian power of this function gives a $\boldsymbol{\Gamma}_n$-equivariant linear map

$$
\begin{aligned}
& \phi^n_a : \mathrm{L}^n \to \mathrm{Hom}(\mathsf{N}_{t(a)}, \mathsf{N}_{h(a)}) = \mathrm{Hom}(\mathrm{N}^*_{t(a)}, \mathrm{N}^*_{h(a)}) \otimes \mathrm{End}\, \mathbf{C}^n, \\
& \qquad (u_1, \dots, u_n) \mapsto \sum_{1 \le l \le n} \phi_a(u_l) \otimes \mathrm{E}_{ll},
\end{aligned}
$$

where E_{ll} stands for the $n \times n$-matrix unit with the only nonzero entry at the place (l, l).

Similarly, we shall view ϕ'_a as a $\mathrm{Hom}(\mathrm{N}^*_{t(a)}, \mathrm{N}^*_{h(a)})$-valued constant vector field on L whose value at each point $u \in \mathrm{L}$ is equal to ϕ'_a. Thus, given $m \in [1, n]$, we shall write $\frac{\partial}{(\partial \phi_{a^*})_m}$ for the $\mathrm{Hom}(\mathrm{N}^*_{h(a)}, \mathrm{N}^*_{t(a)})$-valued first order differential operator on L^n corresponding to the constant vector field $\phi'_{a^*} \in \mathrm{Hom}(\mathrm{N}^*_{h(a)}, \mathrm{N}^*_{t(a)}) \otimes \mathrm{L}$ along the m-th direct factor L in L^n.

Next, recall the map θ^{Holland}, introduced in Section 2.3. The composite $\theta^{\text{Radial}} \circ \theta^{\text{Holland}}$ associates to every edge $a \in \overline{\mathrm{Q}}$ a twisted differential operator from the algebra $\mathscr{D}(\mathfrak{X}, p, \rho)$. By definition, such an operator is a coset modulo the ideal $\mathscr{D}(\mathrm{L}^n_{\text{reg}}, \mathsf{N})^{\text{adt}} \cdot \mathrm{Im}(\boldsymbol{\xi} - \rho)$, see Proposition 3.2.2(iii), of an element

$$
\theta^{\text{Radial}} \circ \theta^{\text{Holland}}(a) \in \mathscr{D}\big(\mathrm{L}^n_{\text{reg}}\big) \otimes \mathrm{Hom}(\mathsf{N}_{t(a)}, \mathsf{N}_{h(a)}) \subset \mathscr{D}\big(\mathrm{L}^n_{\text{reg}}, \mathsf{N}\big).
$$

We will write such an element D as an $n \times n$-matrix with entries in $\mathrm{Hom}(\mathrm{N}^*_{t(a)}, \mathrm{N}^*_{h(a)})$, and write D_{ml} for (m, l)-th entry of that matrix.

Proposition **3.7.2.** — *Let $a \in \mathrm{Q}$. Then*

(i) $\theta^{\text{Radial}} \circ \theta^{\text{Holland}}(a)$ *is a zero-order differential operator on L^n_{reg} given by multiplication by the function ϕ^n_a.*

(ii) *For $l \neq m$, the (m, l)-entry of $\theta^{\text{Radial}} \circ \theta^{\text{Holland}}(a^*)$ is is a zero-order differential operator on L^n_{reg} given by multiplication by the function*

$$-k/2 \sum_{\gamma} \frac{(\phi_{a^*} \circ \gamma)_l}{\omega(\gamma; m, l)} \gamma, \tag{3.7.3}$$

and the (m, m)-entry of $\theta^{\text{Radial}} \circ \theta^{\text{Holland}}(a^*)$ is a first order differential operator

$$\frac{1}{|\Gamma|}\Bigg(\frac{\partial}{\partial(\phi_{a^*})_m} + \sum_{\gamma \neq 1, \zeta} \frac{(\phi_{a^*} \circ (\gamma^{-1} + \mathtt{Id}))_{mm}}{\omega(\gamma; m, m)}\big(-1 + |\Gamma| c_\gamma \gamma^{-1}\big) + \frac{1}{|\Gamma|} \sum_{\ell \neq m} \sum_{\gamma} \frac{(\phi_{a^*} \circ \gamma)_\ell}{\omega(\gamma; m, \ell)}\Bigg). \tag{3.7.4}$$

(iii) *For the edge $b : s \to o$ we have*

$$\begin{aligned} \theta^{\text{Radial}} \circ \theta^{\text{Holland}}(b) &= -\sum_{p} e^b_{p,1}, \\ \theta^{\text{Radial}} \circ \theta^{\text{Holland}}(b^*) &= (1 - \sum_{j \in \mathrm{I}} \delta_j \chi_j) \sum_p e^b_{1,p} = \nu \sum_p e^b_{1,p}. \end{aligned} \tag{3.7.5}$$

The proof of Proposition 3.7.2 will be given later, in Section 8. We will use

Lemma **3.7.6.** — *We have $c_\zeta + n = (1 - 2\psi)/|\Gamma|$, and $k = 2(1 - \sum_j \delta_j \chi_j)/|\Gamma|$. Furthermore,*

$$c_\gamma = \big(1 - \sum_j \chi_j(\delta_j - \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma))\big)/|\Gamma| \quad \textit{for } \gamma \neq \zeta.$$

Proof. — Since $\lambda_i = \mathrm{Tr}\,|_{\mathrm{N}_i}(t \cdot 1 + c)$, we obtain by orthogonality relations that $c_\gamma = 1/|\Gamma| \sum_{j \in \mathrm{I}} \lambda_j \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma)$. Hence, for ∂ as in (1.4.2), we compute

$$\begin{aligned} &\sum_j \chi_j(\delta_j - \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma)) \\ &\quad = \lambda \cdot \delta - \nu - \partial \cdot \delta - \sum_j \lambda_j \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma) + \nu - n \sum_j \mathrm{Tr}\,|_{\mathrm{N}_j}(\zeta)\, \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma) \\ &\quad = 1 - |\Gamma| c_\gamma - n \sum_j \mathrm{Tr}\,|_{\mathrm{N}_j}(\zeta)\, \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\gamma). \end{aligned}$$

If $\gamma \neq \zeta$, then this is equal to $1 - |\Gamma| c_\gamma$. If $\gamma = \zeta$, then it is equal to $1 - |\Gamma| c_\zeta - n|\Gamma|$. Moreover,

$$\sum_j \chi_j(\delta_j - \mathrm{Tr}\,|_{\mathrm{N}^*_j}(\zeta)) = 2 \sum_{\{j \in \mathrm{Q} \mid j \text{ is a source}\}} \chi_j \delta_j = 2\psi.$$

We also have $\sum_j \chi_j \delta_j = 1 - \nu = 1 - k|\Gamma|/2$. □

4. Dunkl representation

4.1. *Dunkl operators*

Recall the principal $\mathbf{T}$-bundle $p : \mathrm{X} = \mathrm{L}^n_{\mathrm{reg}} \to \mathfrak{X} = \mathrm{L}^n_{\mathrm{reg}}/\mathbf{T}$, and other notation introduced in Section 3.6. We are going to define a certain representation η of the Lie algebra $\mathfrak{t} = \operatorname{Lie}\mathbf{T}$ which is normalized by the natural $\mathbf{\Gamma}_n$-action on $\mathfrak{t}$. Thus, there is an action of $\mathbf{\Gamma}_n$ on $\mathscr{D}(\mathfrak{X}, p, \eta)$, the corresponding algebra of twisted differential operators.

Our goal is to define certain elements in $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes \mathbf{\Gamma}_n$ which may be thought of as Γ-analogues of *Dunkl operators*. The construction of these operators will be given in five steps.

Step 1. — Write $\mathrm{L}_{\mathrm{reg}}$ for the preimage of $\mathbf{P}_{\mathrm{L}} \smallsetminus \mathbf{P}^{\mathrm{sing}}_{\mathrm{L}}$ under the projection $\mathrm{L}^\times \twoheadrightarrow \mathbf{P}_{\mathrm{L}}$. Thus $\mathrm{L}_{\mathrm{reg}} \subset \mathrm{L}$ is an open dense subset formed by the points $v \in \mathrm{L}$ such that, for any $\gamma \in \Gamma \smallsetminus \{1, \zeta\}$, we have $\gamma(v) \notin \mathbf{C}v$.

For any $\gamma \in \Gamma$, we have a quadratic function $\omega^\gamma : \mathrm{L} \to \mathbf{C}$, $u \mapsto \omega(u, \gamma u)$. This function does not vanish on $\mathrm{L}_{\mathrm{reg}}$, thus, we have $1/\omega^\gamma \in \mathbf{C}[\mathrm{L}_{\mathrm{reg}}]$. Given $v \in \mathrm{L}$, we also have the linear function $v^\vee : u \mapsto \omega(v, u)$, on L.

Recall the coefficients $c_\gamma \in \mathbf{C}$ given by (1.2.1). To each $v \in \mathrm{L}$ we assign the following element

$$\text{(4.1.1)} \qquad \mathrm{D}^v := 2|\Gamma|^{-1}\frac{\partial}{\partial v} + \sum_{\gamma \neq 1, \zeta} c_\gamma \frac{(\gamma v + v)^\vee}{\omega^\gamma}\gamma \in \mathscr{D}(\mathrm{L}_{\mathrm{reg}}) \rtimes \Gamma.$$

Step 2. — Let $\mathrm{F} = \mathbf{C}^2$ be a 2-dimensional vector space with fixed basis (f^+, f^-), and identify $\operatorname{End}\mathrm{F}$ with the algebra of 2×2-matrices. We set $\mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) = \mathscr{D}(\mathrm{L}_{\mathrm{reg}}) \otimes \operatorname{End}\mathrm{F}$, and form the smash product $\mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) \rtimes \Gamma = (\mathscr{D}(\mathrm{L}_{\mathrm{reg}}) \otimes \operatorname{End}\mathrm{F}) \rtimes \Gamma$, where Γ acts trivially on F and on $\operatorname{End}\mathrm{F}$.

For each $v \in \mathrm{L}$, we introduce the following element written as a matrix with entries in $\mathscr{D}(\mathrm{L}_{\mathrm{reg}}) \rtimes \Gamma$:

$$\text{(4.1.2)} \qquad \mathrm{D}^v_{\mathrm{F}} := \begin{pmatrix} 0 & -v^\vee \\ \mathrm{D}^v & 0 \end{pmatrix} = \mathrm{D}^v \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - v^\vee \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) \rtimes \Gamma.$$

Step 3. — For any affine variety $\mathscr{Y}$ and $n \geq 1$, one has the standard algebra isomorphism $\mathscr{D}(\mathscr{Y}^n) \cong \mathscr{D}(\mathscr{Y})^{\otimes n}$. Since $\operatorname{End}(\mathrm{F}^{\otimes n}) \cong (\operatorname{End}\mathrm{F})^{\otimes n}$, we deduce an algebra isomorphism $\mathscr{D}(\mathscr{Y}^n, \mathrm{F}^{\otimes n}) \cong \mathscr{D}(\mathscr{Y}, \mathrm{F})^{\otimes n}$. The symmetric group S_n acts naturally on $\mathscr{Y}^n$ and on $(\operatorname{End}\mathrm{F})^{\otimes n}$, hence, also on the tensor product $\mathscr{D}(\mathscr{Y}^n, \mathrm{F}^{\otimes n}) \cong \mathscr{D}(\mathscr{Y}, \mathrm{F})^{\otimes n}$.

We take $\mathscr{Y} = \mathrm{L}_{\mathrm{reg}}$ and put $\mathrm{X} := \mathrm{L}^n_{\mathrm{reg}}$, cf. Section 3.4. Thus, X is a $\mathbf{\Gamma}_n$-stable affine open dense subset of $(\mathrm{L}_{\mathrm{reg}})^n$, and we have a chain of natural algebra imbeddings

$$(\mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) \rtimes \Gamma)^{\otimes n} = \mathscr{D}\big((\mathrm{L}_{\mathrm{reg}})^n, \mathrm{F}^{\otimes n}\big) \rtimes \Gamma^n \hookrightarrow \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}) \rtimes \Gamma^n \hookrightarrow \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n,$$

where the middle imbedding is given by restriction from $(\mathrm{L}_{\mathrm{reg}})^n$ to X.

For any $v \in \mathrm{L}$ and $l = 1, \ldots, n$, let $\mathrm{D}^{v,n}_{\mathrm{F},l} \in \mathscr{D}(\mathrm{L}^n_{\mathrm{reg}}, \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n$ denote the image of the element $1^{\otimes(l-1)} \otimes \mathrm{D}^v_{\mathrm{F}} \otimes 1^{\otimes(n-l)}$, cf. (4.1.2), under this imbedding.

Step 4. — For any $l = 1, \ldots, n$, and $\gamma \in \Gamma$, let $\gamma_l \in \Gamma^n$ denote a copy of the element γ placed in the l-th factor of Γ^n. In particular, given any $1 \leq m \neq l \leq n$, and $\gamma \in \Gamma$, we have the corresponding transposition $s_{ml} = \{m \leftrightarrow l\} \in \mathrm{S}_n$ and the element $s_{ml}\gamma_m\gamma_l^{-1} \in \mathbf{\Gamma}_n$. Given $\gamma \in \Gamma$, $v \in \mathrm{L}$ and any $1 \leq m \neq l \leq n$, we also have regular functions $v_l^\vee$ and $1/\omega(\gamma; m, l)$ on $\mathrm{L}^n_{\mathrm{reg}}$, see (3.7.1).

With this notation, for any $v \in \mathrm{L}$ and $1 \leq m \neq l \leq n$, we will now define an element

$$\mathrm{R}^v_{ml} \in (\mathbf{C}[\mathrm{X}] \otimes \mathrm{End}\,\mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n = \mathrm{Hom}(\mathrm{F}^{\otimes n}, (\mathbf{C}[\mathrm{X}] \otimes \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n). \tag{4.1.3}$$

To this end, write

$$\left\{\vec{f} = f_1 \otimes \ldots \otimes f_n \mid f_i = f^\pm,\ i = 1, \ldots, n\right\} \tag{4.1.4}$$

for the standard basis of the vector space $\mathrm{F}^{\otimes n}$. Given such a basis element $\vec{f} = f_1 \otimes \ldots \otimes f_n$ and $1 \leq m \leq n$, let $\vec{f}^-_m := f_1 \otimes \ldots \otimes f_{m-1} \otimes f^- \otimes f_{m+1} \otimes \ldots \otimes f_n$. Now, view each R^v_{ml} in (4.1.3) as a linear map $\mathrm{F}^{\otimes n} \to (\mathbf{C}[\mathrm{X}] \otimes \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n$, which we define as follows

$$\begin{aligned} &\mathrm{R}^v_{ml}(\vec{f}) = \frac{1}{2}\sum_{\gamma\in\Gamma}\left(\frac{(\gamma v_l)^\vee}{\omega(\gamma; m, l)} \otimes \vec{f}^-_m\right) \cdot s_{ml}\gamma_m\gamma_l^{-1}, \quad \text{if } f_m = f^+, f_l = f^+, \\ &\mathrm{R}^v_{ml}(\vec{f}) = \frac{1}{2}\sum_{\gamma\in\Gamma}\left(\frac{v_m^\vee}{\omega(\gamma; m, l)} \otimes \vec{f}^-_m\right) \cdot s_{ml}\gamma_m\gamma_l^{-1}, \quad \text{if } f_m = f^+, f_l = f^-, \\ &\mathrm{R}^v_{ml}(\vec{f}) = 0, \quad \text{if } f_m = f^-. \end{aligned} \tag{4.1.5}$$

We identify the algebra $\mathbf{C}[\mathrm{X}] \otimes \mathrm{End}\,\mathrm{F}^{\otimes n}$ with the subalgebra of $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})$ formed by zero order differential operators. Therefore, we may view the elements R^v_{ml}, in (4.1.5), as being elements of $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n$, which have zero order as differential operators. Given $k \in \mathbf{C}$ and $v \in \mathrm{L}$, we define first order differential operators

$$\mathrm{Dunkl}^v_l := \mathrm{D}^{v,n}_{\mathrm{F},l} + k\sum_{l\neq m}\mathrm{R}^v_{lm} \in \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}) \rtimes \mathbf{\Gamma}_n, \quad \forall 1 \leq l \leq n. \tag{4.1.6}$$

Step 5. — Let $\mu \in \mathbf{C}$ be the constant defined in (3.6.2). We introduce a representation of the 1-dimensional Lie algebra $\mathbf{C}$ on the vector space F. Specifically, we let the generator $\mathbf{1} \in \mathbf{C}$ act, in the basis $\{f^+, f^-\}$, via the diagonal matrix $\mathrm{diag}(\mu + \frac{1}{2}, \mu - \frac{1}{2})$. The n-th cartesian power of this 2-dimensional representation gives a Lie algebra representation $\eta : \mathfrak{t} = \mathbf{C}^n \to \mathrm{End}\, \mathrm{F}^{\otimes n}$.

We consider the Lie algebra homomorphism

$$\boldsymbol{\xi} - \eta : \mathfrak{t} \to \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}), \quad h \mapsto \boldsymbol{\xi}_h \otimes \mathtt{Id}_{\mathrm{F}^{\otimes n}} - \mathtt{Id}_{\mathscr{D}} \otimes \eta(h).$$

The group $\boldsymbol{\Gamma}_n$ acts naturally both on the Lie algebra $\mathfrak{t}$ and on $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})$, and it is clear that the map $\boldsymbol{\xi} - \eta$ is $\boldsymbol{\Gamma}_n$-equivariant. It follows in particular that $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathrm{ad}\mathfrak{t}}$, the centralizer of the image of the map $\boldsymbol{\xi} - \eta$ in $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})$, is a $\boldsymbol{\Gamma}_n$-stable subalgebra.

Now, we apply the general construction of algebras of twisted differential operators given in Section 3.1 to the torus $\mathbf{T} = (\mathbf{C}^\times)^n$ acting on $\mathrm{X} = \mathrm{L}^n_{\mathrm{reg}}$ and to the representation η defined above. This way, with the notation of Section 3.6, we get an algebra $\mathscr{D}(\mathfrak{X}, p, \eta)$. By construction, the algebra $\mathscr{D}(\mathfrak{X}, p, \eta)$ is a quotient of $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathrm{ad}\mathfrak{t}}$, and this quotient inherits a natural $\boldsymbol{\Gamma}_n$-action. Thus, we may form the corresponding algebra $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes \boldsymbol{\Gamma}_n$.

It is straightforward to verify, counting homogeneity degrees of the coefficients, that for any $v \in \mathrm{L}$ the operator in (4.1.6) is $\mathrm{ad}\mathfrak{t}$-invariant. That is, for each $l = 1, \ldots, n$, we have $\mathbf{Dunkl}^v_l \in \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathrm{ad}\mathfrak{t}} \rtimes \boldsymbol{\Gamma}_n$. Therefore, the element $\mathbf{Dunkl}^v_l$ has a well defined image in $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes \boldsymbol{\Gamma}_n$, to be denoted by the same symbol $\mathbf{Dunkl}^v_l$ and to be called the l-th *Dunkl operator* associated with $v \in \mathrm{L}$.

4.2. *Equalizer construction*

Recall from Section 3.4, the group $\mathrm{S} = \{1, \zeta\} \cong \mathbf{Z}/2$. Thus, $\mathrm{S} = \mathbf{C}^\times \cap \Gamma \subset GL(\mathrm{L})$ may be (and will be) viewed either as a subgroup of $\mathbf{C}^\times$ or as a subgroup of Γ. We put $\mathbf{S} := \mathrm{S}^n$. This group comes equipped with a natural group imbedding $\boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}} : \mathbf{S} \hookrightarrow \Gamma^n \subset \boldsymbol{\Gamma}_n$, such that the image of $\mathbf{S}$ is a *normal* subgroup in $\boldsymbol{\Gamma}_n$, and also with a natural imbedding $\mathbf{S} \hookrightarrow \mathbf{T}$.

In general, let A be an associative algebra equipped with a $\boldsymbol{\Gamma}_n$-action $\boldsymbol{\Gamma}_n \ni g : a \mapsto a^g$, by algebra automorphisms, and also with a homomorphism $\mathsf{a} : \mathbf{S} \to \mathrm{A}$, that is, with a map such that $\mathsf{a}(1) = 1$, and such that $\mathsf{a}(ss') = \mathsf{a}(s) \cdot \mathsf{a}(s')$, $\forall s, s' \in \mathbf{S}$. Assume in addition that the following identities hold (the one on the left says that a is a $\boldsymbol{\Gamma}_n$-*equivariant* map):

$$(\mathbf{4.2.1}) \qquad \mathsf{a}(s)^g = \mathsf{a}(gsg^{-1}), \quad \text{and} \quad \mathsf{a}(s) \cdot a \cdot \mathsf{a}(s^{-1}) = a^{\boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}}(s)}, \quad \forall s \in \mathbf{S},\ g \in \boldsymbol{\Gamma}_n,\ a \in \mathrm{A}.$$

We form the smash product $\mathrm{A} \rtimes \boldsymbol{\Gamma}_n$ and introduce the following two homomorphisms

$$\Upsilon_1, \Upsilon_2 : \mathbf{S} \to \mathrm{A} \rtimes \boldsymbol{\Gamma}_n, \quad \text{where} \quad \Upsilon_1 : s \mapsto \mathsf{a}(s) \rtimes 1, \quad \Upsilon_2 : s \mapsto 1 \rtimes \boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}}(s).$$

It is straightforward to verify that equations (4.2.1) imply that the left ideal in the algebra $A \rtimes \mathbf{\Gamma}_n$ generated by the set $\{\Upsilon_1(s) - \Upsilon_2(s),\ s \in \mathbf{S}\}$ is in effect a two-sided ideal. We let $A \rtimes_{\mathbf{S}} \mathbf{\Gamma}_n$ be the quotient of $A \rtimes \mathbf{\Gamma}_n$ by that two-sided ideal, to be called *equalizer smash product* algebra.

4.3. *The homomorphism* θ^{Dunkl}

Let

$$\mathbf{I} = \{(\epsilon_1, ..., \epsilon_n) \mid \epsilon_m = 0 \text{ or } 1 \text{ for all } m\} \subset \mathbf{Z}^n.$$

Let $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n) \in \mathbf{I}$, and write $F_{\boldsymbol{\epsilon}}$ for the one dimensional subspace of $F^{\otimes n}$ spanned by $f_1 \otimes \cdots \otimes f_n$, where $f_m = f^+$ if $\epsilon_m = 0$, and $f_m = f^-$ if $\epsilon_m = 1$ $(m = 1, ..., n)$. The representation η of $\mathfrak{t}$ on the vector space $F^{\otimes n}$ induces an adjoint action of $\mathfrak{t}$ on $\mathrm{End}(F^{\otimes n})$. We have a decomposition

$$\mathrm{End}(F^{\otimes n}) = \bigoplus_{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in \mathbf{I}} F_{\boldsymbol{\epsilon}'} \otimes (F_{\boldsymbol{\epsilon}})^*,$$

where each component in the direct sum is stable under the action of $\mathfrak{t}$. Moreover, $\mathfrak{t}$ acts on $F_{\boldsymbol{\epsilon}'} \otimes (F_{\boldsymbol{\epsilon}})^*$ by the character $\boldsymbol{\epsilon}' - \boldsymbol{\epsilon}$. Therefore, the $\mathfrak{t}$-action on $F_{\boldsymbol{\epsilon}'} \otimes (F_{\boldsymbol{\epsilon}})^*$ exponentiates to a $\mathbf{T}$-action. Taking the direct sum over various pairs $(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}')$, we obtain a well defined *adjoint* $\mathbf{T}$-action on $\mathrm{End}\, F^{\otimes n} = F^{\otimes n} \otimes (F^{\otimes n})^*$. Thus, for any $\mathbf{t} \in \mathbf{T}$, the adjoint action of $\mathbf{t}$ gives an algebra automorphism $\mathtt{Ad}_F(\mathbf{t}) : \mathrm{End}\, F^{\otimes n} \to \mathrm{End}\, F^{\otimes n}$.

The torus $\mathbf{T}$ also acts naturally on $X = L^n_{\mathrm{reg}}$. The tensor product of the induced $\mathbf{T}$-action on $\mathscr{D}(X)$ with the adjoint $\mathbf{T}$-action on $\mathrm{End}\, F^{\otimes n}$ gives a $\mathbf{T}$ action on $\mathscr{D}(X, F^{\otimes n}) = \mathscr{D}(X) \otimes \mathrm{End}\, F^{\otimes n}$, to be called the *adjoint* action $\mathtt{Ad}_{\mathscr{D}\otimes F} : \mathbf{T} \to \mathrm{Aut}(\mathscr{D}(X, F^{\otimes n}))$. The map $\mathtt{Ad}_{\mathscr{D}\otimes F}$ is clearly $\mathbf{\Gamma}_n$-equivariant. It is also clear from the construction that the differential of the $\mathtt{Ad}_{\mathscr{D}\otimes F}$-action of $\mathbf{T}$ is nothing but the adjoint action of the Lie algebra $\mathfrak{t}$. In particular, we have $\mathscr{D}(X, F^{\otimes n})^{\mathrm{ad}\mathfrak{t}} = \mathscr{D}(X, F^{\otimes n})^{\mathtt{Ad}_{\mathscr{D}\otimes F}\mathbf{T}}$.

Next, we are going to apply the general construction of Section 4.2 in the following special case. Let $S \to \mathscr{D}(L_{\mathrm{reg}}, F) = \mathscr{D}(L_{\mathrm{reg}}) \otimes \mathrm{End}\, F$ be a homomorphism given by the assignment

$$1 \mapsto 1_{\mathscr{D}} \otimes \mathtt{Id}_F, \quad \zeta \mapsto 1_{\mathscr{D}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a homomorphism $\mathsf{a}_F : \mathbf{S} \to \mathscr{D}(X, F^{\otimes n})$ to be the composite of the n-th cartesian power of the above homomorphism $S \to \mathscr{D}(L_{\mathrm{reg}}, F)$, followed by the natural imbedding $\mathscr{D}((L_{\mathrm{reg}})^n, F^{\otimes n}) \hookrightarrow \mathscr{D}(X, F^{\otimes n})$. This homomorphism is clearly $\mathbf{\Gamma}_n$-equivariant and the image of a_F is contained in $\mathscr{D}(X, F^{\otimes n})^{\mathrm{ad}\mathfrak{t}}$.

Write $a \mapsto a^g$ for the action of an element $g \in \mathbf{\Gamma}_n$ on $a \in \mathscr{D}(X, F^{\otimes n})$. One checks by direct computation that the map a_F is related to the two natural imbeddings

$\boldsymbol{\varepsilon}_{\mathbf{T}} : \mathbf{S} \hookrightarrow \mathbf{T}$ and $\boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}} : \mathbf{S} \hookrightarrow \boldsymbol{\Gamma}_n$ via the following identity

$$(\mathbf{4.3.1}) \qquad (\mathtt{Ad}_{\mathscr{D}\otimes\mathrm{F}} \circ \boldsymbol{\varepsilon}_{\mathbf{T}}(s))(a) = \mathsf{a}_{\mathrm{F}}(s^{-1}) \cdot a^{\boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}}(s)} \cdot \mathsf{a}_{\mathrm{F}}(s), \quad \forall s \in \mathbf{S},\ a \in \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n}).$$

It follows from (4.3.1) that, for any $a \in \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathtt{adt}} = \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathtt{Ad}_{\mathscr{D}\otimes\mathrm{F}}\mathbf{T}}$ and $s \in \mathbf{S}$, one has $a^{\boldsymbol{\varepsilon}_{\boldsymbol{\Gamma}}(s)} = \mathsf{a}_{\mathrm{F}}(s) \cdot a \cdot \mathsf{a}_{\mathrm{F}}(s^{-1})$. We conclude that both conditions in (4.2.1) hold for the thus obtained homomorphism $\mathsf{a} : \mathbf{S} \to \mathrm{A} := \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathtt{adt}}$.

Further, we have the algebra projection $\mathrm{pr} : \mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})^{\mathtt{adt}} \twoheadrightarrow \mathscr{D}(\mathfrak{X}, p, \eta)$, and we let $\mathrm{pr} \circ \mathsf{a} : \mathbf{S} \to \mathscr{D}(\mathfrak{X}, p, \eta)$ be the composite homomorphism. The $\mathtt{Ad}_{\mathscr{D}\otimes\mathrm{F}}$-action of $\mathbf{T}$ on $\mathscr{D}(\mathrm{X}, \mathrm{F}^{\otimes n})$ clearly descends to an $\mathtt{Ad}_{\mathscr{D}\otimes\mathrm{F}}$-action on $\mathscr{D}(\mathfrak{X}, p, \eta)$. It follows that conditions (4.2.1) hold for the map $\mathrm{pr} \circ \mathsf{a}$ as well. Thus, we are in a position to form $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n$, the corresponding equalizer smash product. We let $\mathrm{Dunkl}(v, l)$ denote the image in $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n$ of the element $\mathrm{Dunkl}_l^v \in \mathscr{D}(\mathfrak{X}, p, \eta) \rtimes \boldsymbol{\Gamma}_n$.

The main result of this section reads

Theorem **4.3.2.** — *Put* $t = 1/|\Gamma|$. *The assignment, given on the generators* $g \in \Gamma$, $v_l \in \mathrm{L}_l$, $l = 1, \ldots, n$ *(where* L_l *stands for the* l*-th direct summand in* L^n*), of the algebra* $\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)$ *by the formulas below extends uniquely to a well defined and injective algebra homomorphism*

$$\theta^{\mathrm{Dunkl}} :\ \mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n) \to \mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n, \quad g \mapsto g,\ v_l \mapsto \mathrm{Dunkl}(v, l).$$

The injectivity statement in the theorem is not difficult; it follows easily from the PBW theorem for the algebra $\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)$, by considering principal symbols of differential operators. The difficult part is to verify that the assignment of the theorem does define an algebra homomorphism. The proof of this is quite long and involves a lot of explicit computations. That proof will be given later, in Section 9. In the special case $n = 1$, the proof is less technical and is presented below.

4.4. *Proof of Theorem 4.3.2 in the special case:* $n = 1$

Let u_1, u_2 denote the coordinates in the symplectic vector space (L, ω).

For $n = 1$, the assignment of Theorem 4.3.2 reduces to the map $\mathrm{L} \to \mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) \rtimes_{\mathbf{S}} \Gamma$ that reads

$$\theta^{\mathrm{Dunkl}} : v \mapsto \begin{pmatrix} 0 & -v^{\vee} \\ \mathrm{D}^v & 0 \end{pmatrix}, \quad \text{where } \mathrm{D}^v = \frac{2}{|\Gamma|}\frac{\partial}{\partial v} + \sum_{\gamma \neq 1, \zeta} c_\gamma \frac{(\gamma v + v)^{\vee}}{\omega^{\gamma}} \gamma.$$

For any $v, w \in \mathrm{L}$ we are going to compute all 4 entries of the 2×2-matrix representing the operator $[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)] \in \mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{F}) \rtimes \Gamma$. First of all, it is easy to see that $[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)]_{12} = [\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)]_{21} = 0$.

Next, write $\mathsf{eu} = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}$ for the Euler operator. We compute

$$
\begin{aligned}
[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)]_{11} &= w^{\vee} \mathrm{D}^{v} - v^{\vee} \mathrm{D}^{w} \\
&= \frac{2}{|\Gamma|} \left(w^{\vee} \frac{\partial}{\partial v} - v^{\vee} \frac{\partial}{\partial w} \right) + \sum_{\gamma \neq 1, \zeta} \frac{c_\gamma}{\omega^\gamma} (w^{\vee}(\gamma v + v)^{\vee} - v^{\vee}(\gamma w + w)^{\vee}) \gamma \\
&= -\frac{2\omega(v, w)}{|\Gamma|} \mathsf{eu} + \sum_{\gamma \neq 1, \zeta} \frac{c_\gamma}{\omega^\gamma} (w^{\vee}(\gamma v)^{\vee} - (\gamma w)^{\vee} v^{\vee})) \gamma \\
&= t\omega(v, w) \Big(-\frac{2}{|\Gamma|} \mathsf{eu} + \sum_{\gamma \neq 1, \zeta} c_\gamma \gamma \Big).
\end{aligned}
$$

One proves similarly that $[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)]_{22} = \omega(v, w)(\frac{2}{|\Gamma|}(\mathsf{eu} + 2) + \sum_{\gamma \neq 1, \zeta} c_\gamma \gamma)$. Thus, we find

$$
\begin{aligned}
&[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)] \\
&\quad = \frac{2\omega(v, w)}{|\Gamma|} \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathsf{eu} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) + \omega(v, w) \sum_{\gamma \neq 1, \zeta} c_\gamma \gamma.
\end{aligned}
$$

Now, in the 1-dimensional Lie algebra $\mathfrak{t} = \mathbf{C}$, we have the generator $\mathbf{1}$ which acts in F via the matrix $\mathrm{diag}(\mu + \frac{1}{2}, \mu - \frac{1}{2})$. By definition of twisted differential operators, in the algebra $\mathscr{D}(\mathfrak{X}, p, \eta)$, we have $\mathsf{eu} = \mathbf{1} = \mathrm{diag}(\mu + \frac{1}{2}, \mu - \frac{1}{2})$. Therefore, in the algebra $\mathscr{D}(\mathfrak{X}, p, \eta)$, we get

$$
\begin{aligned}
&[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)] \\
&\quad = \frac{2\omega(v, w)}{|\Gamma|} \left(\begin{pmatrix} -\mu - \frac{1}{2} & 0 \\ 0 & \mu - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) + \omega(v, w) \sum_{\gamma \neq 1, \zeta} c_\gamma \gamma.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{2}{|\Gamma|} \left(\begin{pmatrix} -\mu - \frac{1}{2} & 0 \\ 0 & \mu - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) \\
&\quad = \frac{2}{|\Gamma|} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - (\mu + 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\
&\quad = \frac{1}{|\Gamma|} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_\zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

where in the last equality we have used the definition of μ from (3.6.2). We find

$$
(\mathbf{4.4.1}) \qquad [\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)] = \omega(v, w) \Big(|\Gamma|^{-1} + c_\zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{\gamma \neq 1, \zeta} c_\gamma \gamma \Big).
$$

In these formulas, the matrix $\operatorname{diag}(1,-1) \in \operatorname{End} F$ is viewed as an element of $\mathscr{D}(L_{\mathrm{reg}}, F)$. The image of this element in $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \Gamma$, the equalizer smash product algebra, equals $1 \rtimes \boldsymbol{\varepsilon}_{\Gamma}(\zeta)$, by definition. Hence, from (4.4.1) we deduce

$$[\theta^{\mathrm{Dunkl}}(v), \theta^{\mathrm{Dunkl}}(w)] = \omega(v, w)(1/|\Gamma| + \sum_{\gamma \neq 1} c_\gamma \gamma).$$

This completes the proof of Theorem 4.3.2 in the special case $n = 1$. □

4.5. *The map* Θ^{Dunkl}

In Section 4.3, for any $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ and $\mathbf{t} \in \mathbf{T}$, we have defined the *adjoint* action $\mathtt{Ad}_F(\mathbf{t}) : \mathrm{Hom}(F_{\boldsymbol{\epsilon}}, F_{\boldsymbol{\epsilon}'}) \to \mathrm{Hom}(F_{\boldsymbol{\epsilon}}, F_{\boldsymbol{\epsilon}'})$. This $\mathtt{Ad}_F$-action of $\mathbf{T}$ descends to an action on $\mathscr{D}(\mathfrak{X}, p, \eta, F_{\boldsymbol{\epsilon}} \to F_{\boldsymbol{\epsilon}'})$, the corresponding quotient space. The reader should be alerted that the resulting $\mathtt{Ad}_F$-action on $\mathscr{D}(\mathfrak{X}, p, \eta, F_{\boldsymbol{\epsilon}} \to F_{\boldsymbol{\epsilon}'})$ that we are considering at the moment is *different* from the $\mathtt{Ad}_{\mathscr{D}\otimes F}$-action of $\mathbf{T}$ considered in Section 4.3: the latter action, comes from the action of $\mathbf{T}$ on *both* factors of the tensor product $\mathscr{D}(X, F^{\otimes n}) = \mathscr{D}(X) \otimes \operatorname{End} F^{\otimes n}$, while the former comes from the action of $\mathbf{T}$ on the *second* tensor factor, $\operatorname{End} F^{\otimes n}$, only.

Lemma **4.5.1.** — *Let* $i, j \in I$. *For the algebra* $\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n$, *as defined in Section 4.3, one has:*

$$e_{i,n-1}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n)e_{j,n-1} = \mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_j \to \mathsf{N}_i)^{\boldsymbol{\Gamma}_n},$$
$$e_{i,n-1}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n)\mathbf{e} = \mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_s \to \mathsf{N}_i)^{\boldsymbol{\Gamma}_n},$$
$$\mathbf{e}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n)e_{j,n-1} = \mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_j \to \mathsf{N}_s)^{\boldsymbol{\Gamma}_n},$$
$$\mathbf{e}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n)\mathbf{e} = \mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_s \to \mathsf{N}_s)^{\boldsymbol{\Gamma}_n}.$$

Proof. — We prove the first equality; the rest are similar.
Note that $\mathbf{C}\boldsymbol{\Gamma}_n e_{j,n-1} = \bigoplus_{l=1}^n N_o^{\otimes(l-1)} \otimes N_j \otimes N_o^{\otimes(n-l)} = \mathsf{N}_j^*$, so

$$(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \boldsymbol{\Gamma}_n)e_{j,n-1} = \bigoplus_{1 \le l \le n} \mathscr{D}(\mathfrak{X}, p, \eta) \otimes_{\mathbf{S}} \big(N_o^{\otimes(l-1)} \otimes N_j \otimes N_o^{\otimes(n-l)}\big),$$

where on the right hand side, $s \in \mathbf{S}$ acts on $\mathscr{D}(\mathfrak{X}, p, \eta)$ by right multiplication by $\mathsf{a}(s)$.

For any $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \mathbf{I}$, we write $s^{\boldsymbol{\epsilon}}$ for the character of $\mathbf{S}$ whose value at $\zeta_{(l)}$ is $(-1)^{\epsilon_l}$.

Suppose $\mathbf{S}$ acts on $N_o^{\otimes(l-1)} \otimes N_j \otimes N_o^{\otimes(n-l)}$ by the character $s^{\boldsymbol{\epsilon}(l)}$, where $\boldsymbol{\epsilon}(l) \in \mathbf{I}$. If j is a sink in Q, then $\boldsymbol{\epsilon}(l) = (0, \ldots, 0)$, while if j is a source in Q, then $\boldsymbol{\epsilon}(l) = (0, \ldots, 1, \ldots, 0)$ (where the 1 is in the l-th position). Under the above right action of $\mathbf{S}$ on $\mathscr{D}(\mathfrak{X}, p, \eta)$ via a, the $s^{\boldsymbol{\epsilon}(l)}$-isotypic component of $\mathscr{D}(\mathfrak{X}, p, \eta)$ is

$\oplus_{\epsilon\in\mathbf{I}}\mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon})$, so

$$\mathscr{D}(\mathfrak{X},p,\eta) \otimes_{\mathbf{S}} \big(\mathrm{N}_o^{\otimes(l-1)} \otimes \mathrm{N}_j \otimes \mathrm{N}_o^{\otimes(n-l)}\big) = \bigoplus_{\epsilon\in\mathbf{I}} \mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon}) \otimes \mathrm{N}_j.$$

Now, the space $e_{i,n-1}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \mathbf{\Gamma}_n)e_{j,n-1}$ can be written as

$$\Big(\bigoplus_{1\le m,l\le n} \big(\mathrm{N}_o^{\otimes(m-1)} \otimes \mathrm{N}_i^* \otimes \mathrm{N}_o^{\otimes(n-m)}\big) \bigotimes (\oplus_{\epsilon\in\mathbf{I}}\mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon}) \otimes \mathrm{N}_j)\Big)^{\mathbf{\Gamma}_n}.$$

The subgroup $\mathbf{S} \subset \mathbf{\Gamma}_n$ acts on $\mathrm{N}_o^{\otimes(m-1)} \otimes \mathrm{N}_i^* \otimes \mathrm{N}_o^{\otimes(n-m)}$ by the character $s^{\epsilon'(m)}$, where $\epsilon'(m)$ is $(0, ..., 0)$ if i is a sink, or $(0, ..., 1, ..., 0)$ (where the 1 is in m-th position) if i is a source.

Recall the Ad_{F}-action of $\mathbf{T}$ on $\mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon})$ described before this lemma. We have the group imbedding $\boldsymbol{\varepsilon}_{\mathbf{T}} : \mathbf{S} \hookrightarrow \mathbf{T}$, see Sections 4.2, 4.3. Equation (4.3.1) implies that the restriction, via $\boldsymbol{\varepsilon}_{\mathbf{T}}$, of the Ad_{F}-action of $\mathbf{T}$ to the subgroup $\mathbf{S}$ coincides with the restriction, via $\boldsymbol{\varepsilon}_{\mathbf{\Gamma}}$, of the action of $\mathbf{\Gamma}_n$ on $\mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon})$ to its subgroup $\mathbf{S}$. We have

$$\Big(\big(\mathrm{N}_o^{\otimes(m-1)} \otimes \mathrm{N}_i^* \otimes \mathrm{N}_o^{\otimes(n-m)}\big) \bigotimes \big(\oplus_{\epsilon\in\mathbf{I}}\, \mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon}) \otimes \mathrm{N}_j\big)\Big)^{\mathbf{S}} = \mathrm{N}_i^* \otimes \mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon'(m)}) \otimes \mathrm{N}_j.$$

We conclude that

$$e_{i,n-1}(\mathscr{D}(\mathfrak{X}, p, \eta) \rtimes_{\mathbf{S}} \mathbf{\Gamma}_n)e_{j,n-1} = \Big(\bigoplus_{1\le m,l\le n} \mathrm{N}_i^* \otimes \mathscr{D}(\mathfrak{X}, p, \eta, \mathrm{F}_{\epsilon(l)} \to \mathrm{F}_{\epsilon'(m)}) \otimes \mathrm{N}_j\Big)^{\mathbf{\Gamma}_n}.$$

The last expression is equal to $\mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_j \to \mathsf{N}_i)^{\mathbf{\Gamma}_n}$. □

Recall the homomorphism θ^{Dunkl} of Theorem 4.3.2. For any $i,j \in \mathrm{I}_{\mathsf{CM}}$, recall the subspace $\mathsf{B}_{i,j}$ of H defined in (1.7.2). Using Lemma 4.5.1, we obtain by restricting θ^{Dunkl} to $\mathsf{B}_{i,j}$, a homomorphism

$$\Theta^{\mathrm{Dunkl}} : \mathsf{B}_{i,j} \to \mathscr{D}(\mathfrak{X}, p, \varrho, \mathsf{N}_j \to \mathsf{N}_i)^{\mathbf{\Gamma}_n} \subset \mathscr{D}(\mathfrak{X}, p, \varrho)^{\mathbf{\Gamma}_n}.$$

We define the following algebra homomorphism

$$\Theta^{\mathrm{Dunkl}} : \mathsf{B} \to \mathscr{D}(\mathfrak{X}, p, \varrho)^{\mathbf{\Gamma}_n},\ \sum_{i,j} u_{i,j} \mapsto \sum_{i,j} \Theta^{\mathrm{Dunkl}}(u_{i,j}), \qquad \forall u_{i,j} \in \mathsf{B}_{i,j},\ i,j \in \mathrm{I}_{\mathsf{CM}}.$$

4.6. *Computation of* $\Theta^{\mathrm{Dunkl}} \circ \theta^{\mathrm{Quiver}}$

Let a be an edge of $\overline{Q_{CM}}$, viewed as an element of the algebra Π'. We would like to compute $D^a := \Theta^{\mathrm{Dunkl}} \circ \theta^{\mathrm{Quiver}}(a) \in \mathscr{D}(\mathfrak{X}, p, \varrho)^{\boldsymbol{\Gamma}_n}$, the image of that element under the composite map $\Theta^{\mathrm{Dunkl}} \circ \theta^{\mathrm{Quiver}}$.

We will freely use the notation from Section 3.7.

Proposition **4.6.1.** — *If* $a \in Q$ *then* $D^a = -\phi_a^n$, *and* D^{a^*} *is an* $n \times n$*-matrix with the entries*

$$(D^{a^*})_{mm} = 2|\Gamma|^{-1}\frac{\partial}{(\partial\phi_{a^*})_m} + \sum_{\gamma\neq 1,\zeta} c_\gamma \frac{(\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id}))_m}{\omega(\gamma;m,m)}\gamma^{-1}, \quad \textit{and}$$

$$(D^{a^*})_{ml} = -\frac{k}{2}\sum_{\gamma\in\Gamma}\frac{(\phi_{a^*}\circ\gamma)_l}{\omega(\gamma;m,l)}\gamma, \quad \textit{for } l\neq m.$$

Proof of Proposition 4.6.1 for $n = 1$. — In this special case, we have $N_i = \mathbf{C}\Gamma e_i$ and $N_i^* = e_i\mathbf{C}\Gamma$, with the pairing defined by

$$(e_i\gamma, \gamma' e_i) = e_i\gamma\gamma' e_i \in \mathbf{C}.$$

Thus, for $n = 1$ the formulas of Proposition 4.6.1 read

$$D^a = -\phi_a, \;\; D^{a^*} = 2|\Gamma|^{-1}\frac{\partial}{\partial\phi_{a^*}} + \sum_{\gamma\neq 1,\zeta} c_\gamma \frac{\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id})}{\omega^\gamma}\gamma^{-1}, \;\; \forall a\in Q.$$

To verify these formulas, we write the Dunkl map in the form $\Theta^{\mathrm{Dunkl}}(v) = \sum_{\gamma\in\Gamma} d_\gamma(v)\gamma$, where $v \in L$ and $d_\gamma(v) \in \mathscr{D}(L_{\mathrm{reg}}, F)$. We prove the formula for D^{a^*} because the formula for D^a is easier to prove.

We recall the construction of the map θ^{Quiver}. Let

$$\tilde{\phi}_{a^*} \in \mathrm{Hom}_\Gamma(N_{h(a^*)}, N_{t(a^*)}\otimes L)$$

be the element corresponding to $\phi_{a^*} \in \mathrm{Hom}_\Gamma(N^*_{t(a^*)}, N^*_{h(a^*)}\otimes L)$. Then the map

$$\begin{aligned}&\mathrm{Hom}_\Gamma(N_{h(a^*)}, N_{t(a^*)}\otimes L)\\ &\qquad\simeq \mathrm{Hom}_\Gamma(N^*_{t(a^*)}, N^*_{h(a^*)}\otimes L) \to e_{h(a^*)}\mathbf{C}[\Gamma]\otimes L e_{t(a^*)}\end{aligned}$$

from the construction of Θ^{Quiver} is defined by $\tilde{\phi}_{a^*} \mapsto \tilde{\phi}_{a^*}(1\cdot e_{h(a^*)})$.

Choose a basis v_1, v_2 in L. Then $\tilde{\phi}_{a^*} = \sum_{s=1}^2 \tilde{\phi}^s_{a^*} \otimes v_s$ where $\tilde{\phi}^s_{a^*} \in \mathrm{Hom}(N_{h(a^*)}, N_{t(a^*)})$. We consider D^{a^*} as element of $\mathscr{D}(L_{\mathrm{reg}}, N_{h(a^*)} \to N_{t(a^*)})^\Gamma$. From the construction of θ^{Quiver} we find

$$\begin{aligned}\mathrm{D}^{a^*} &= \sum_{s=1}^{2} \Theta^{\mathrm{Dunkl}}(v_s)\tilde{\phi}^s_{a^*} = \sum_{s=1}^{2}\sum_{\gamma\in\Gamma} d_\gamma(v_s)\big(\gamma\circ\tilde{\phi}^s_{a^*}\big)\\ &= 2|\Gamma|^{-1}\frac{\partial}{\partial\tilde{\phi}_{a^*}} + \sum_{s=1}^{2}\sum_{\gamma\neq 1,\zeta} c_\gamma \frac{(\gamma v_s + v_s)^\vee}{\omega^\gamma}\big(\gamma\circ\tilde{\phi}^s_{a^*}\big)\\ &= 2|\Gamma|^{-1}\frac{\partial}{\partial\tilde{\phi}_{a^*}} + \sum_{\gamma\neq 1,\zeta} c_\gamma \frac{\gamma\circ\tilde{\phi}_{a^*}\circ(\gamma^{-1}+\mathtt{Id})}{\omega^\gamma}.\end{aligned}$$

We have natural isomorphisms

$$\mathrm{Hom}(\mathrm{N}_{h(a^*)}, \mathrm{N}_{t(a^*)}) \cong \mathrm{N}^*_{h(a^*)}\otimes\mathrm{N}_{t(a^*)} \cong \mathrm{Hom}(\mathrm{N}^*_{t(a^*)}, \mathrm{N}^*_{h(a^*)}).$$

We deduce

$$\mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{N}_{h(a^*)}\to\mathrm{N}_{t(a^*)})^\Gamma \simeq \mathscr{D}(\mathrm{L}_{\mathrm{reg}}, \mathrm{N}^*_{t(a^*)}\to\mathrm{N}^*_{h(a^*)})^\Gamma.$$

Under this isomorphism, the element $\gamma\tilde{\phi}_{a^*}\circ(\gamma^{-1}+\mathtt{Id})$ corresponds to $\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id})\circ\gamma^{-1}$ and $\frac{\partial}{\partial\tilde{\phi}_{a^*}}$ corresponds to $\frac{\partial}{\partial\phi_{a^*}}$. This completes the proof. □

We omit the proof of Proposition 4.6.1 for $n > 1$; it is similar to the above computation in the case $n = 1$.

It is easy to see that for the edge $b\colon s\to o$, we have

$$\mathrm{D}^b := \Theta^{\mathrm{Dunkl}}\circ\theta^{\mathrm{Quiver}}(b) = -(1, \ldots, 1)^t$$

and

$$\mathrm{D}^{b^*} := \Theta^{\mathrm{Dunkl}}\circ\theta^{\mathrm{Quiver}}(b^*) = \nu\cdot(1, \ldots, 1).$$

Thus, for all $a\in\overline{\mathrm{Q}_{\mathsf{CM}}}$, we have computed the operators $\mathrm{D}^a := \Theta^{\mathrm{Dunkl}}\circ\theta^{\mathrm{Quiver}}(a)$ where $\mathrm{D}^a\in\mathscr{D}(\mathfrak{X}, p, \varrho)$.

Theorem **4.6.2.** — *For all values of* c, k, *we have* $\Theta^{\mathrm{Radial}}\circ\theta^{\mathrm{Holland}} = \Theta^{\mathrm{Dunkl}}\circ\theta^{\mathrm{Quiver}}$. □

In the special case $n = 1$, for any edge $a\in\mathrm{Q}$, we have

$$\begin{aligned}\theta^{\mathrm{Radial}}\theta^{\mathrm{Holland}}(a^*) = 2|\Gamma|^{-1}\frac{\partial}{\partial\phi_{a^*}} &+ \sum_{\gamma\neq 1,\zeta} c_\gamma\frac{\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id})}{\omega^\gamma}\gamma^{-1}\\ &- |\Gamma|^{-1}\sum_{\gamma\neq 1,\zeta}\frac{\phi_{a^*}\circ\gamma^{-1}}{\omega^\gamma}.\end{aligned}$$

Therefore, replacing here the map θ^{Radial} by Θ^{Radial}, we find

$$\Theta^{\mathrm{Radial}}\theta^{\mathrm{Holland}}(a^*) = 2|\Gamma|^{-1}\frac{\partial}{\partial\phi_{a^*}} + \sum_{\gamma\neq 1,\zeta} c_\gamma \frac{\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id})}{\omega^\gamma}\gamma^{-1}.$$

When $n > 1$, it is completely similar.

5. Harish–Chandra homomorphism

Recall that we assume $\lambda\cdot\delta = 1$, i.e. $t = |\Gamma|^{-1}$. We shall write $\mathsf{H}_{k,c}$ for $\mathsf{H}_{t,k,c}(\boldsymbol{\Gamma}_n)$.

5.1. *Modified Holland's map*

In this subsection, we define a map $\Theta^{\mathrm{Holland}} : e_s\Pi' e_s \to \mathfrak{A}_{\chi'}$, cf. (1.6.2). To this end, assume for the moment that ν is a formal variable, and the algebras Π, Π', $\mathfrak{T}_\chi$, $\mathfrak{A}_{\chi'}$ are all defined over $\mathbf{C}[\nu]$.

Lemma **5.1.1.** — *The map* $\mathrm{gr}(\theta^{\mathrm{Radial}}) : \mathrm{gr}\,\mathfrak{A}_{\chi'} \to \mathrm{gr}(\mathscr{D}(\mathfrak{X}, p, \rho_s)^{\boldsymbol{\Gamma}_n} \otimes \mathbf{C}[\nu])$ *is injective.*

Proof. — This follows from Proposition 7.2.2, Theorem 7.2.3 and Proposition 7.2.5. □

The preceding lemma implies that $\mathrm{gr}\,\mathfrak{A}_{\chi'}$ and $\mathfrak{A}_{\chi'}$ are free $\mathbf{C}[\nu]$-modules.

We define a homomorphism

$$\tilde{\Theta}^{\mathrm{Holland}} : \mathbf{C}\overline{\mathrm{Q}_{\mathsf{CM}}} \otimes \mathbf{C}[\nu] \to (\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha) \otimes (\mathsf{End}\,\mathsf{N}) \otimes \mathbf{C}[\nu, \nu^{-1}])^{\mathrm{GL}(\alpha)}$$

by

$$\begin{aligned} \tilde{\Theta}^{\mathrm{Holland}}(e_j) &= \tilde{\theta}^{\mathrm{Holland}}(e_j) \quad \text{for all vertices } j,\\ \tilde{\Theta}^{\mathrm{Holland}}(a) &= \tilde{\theta}^{\mathrm{Holland}}(a) \quad \text{for any edge } a\neq b^*,\\ \tilde{\Theta}^{\mathrm{Holland}}(b^*) &= \nu^{-1}\tilde{\theta}^{\mathrm{Holland}}(b^*). \end{aligned}$$

It is easy to see that since $\tilde{\theta}^{\mathrm{Holland}}$ descends to a homomorphism $\theta^{\mathrm{Holland}} : \Pi \to \mathfrak{T}_\chi$, the homomorphism $\tilde{\Theta}^{\mathrm{Holland}}$ descends to a homomorphism

$$\Theta^{\mathrm{Holland}} : \Pi' \to \mathfrak{T}_\chi[\nu^{-1}], \quad \text{such that} \quad \Theta^{\mathrm{Holland}} : e_s\Pi' e_s \to \mathfrak{A}_{\chi'}[\nu^{-1}].$$

Suppose that $\mathrm{X} \in \mathbf{C}\overline{\mathrm{Q}}$. By (3.7.5), $\theta^{\mathrm{Radial}}\theta^{\mathrm{Holland}}(b^*\mathrm{X}b)$ vanishes if we set $\nu = 0$, so by Lemma 5.1.1,

$$\tilde{\theta}^{\mathrm{Holland}}(b^*\mathrm{X}b) \in (\mathscr{D}(\mathrm{Q}_{\mathsf{CM}}, \alpha)(\boldsymbol{\xi} - (\lambda - \partial - ne_s))(\mathfrak{gl}(\alpha)))^{\mathrm{GL}(\alpha)}.$$

Since $\chi' = (\lambda - \partial - ne_s) - \nu e_o + n\nu e_s$, we have that $\tilde{\theta}^{\text{Holland}}(b^*Xb)$ belongs to

$$\mathscr{D}(Q_{\mathsf{CM}}, \alpha)(\boldsymbol{\xi} - \chi')(\mathfrak{gl}(\alpha)) + \nu \cdot \mathscr{D}(Q_{\mathsf{CM}}, \alpha)(e_o - ne_s)(\mathfrak{gl}(\alpha)).$$

Since $\mathrm{GL}(\alpha)$ is a reductive group, we have a projection map

$$\mathrm{pr} : \mathscr{D}(Q_{\mathsf{CM}}, \alpha) \to \mathscr{D}(Q_{\mathsf{CM}}, \alpha)^{\mathrm{GL}(\alpha)}$$

such that

$$\mathrm{pr}(\mathscr{D}(Q_{\mathsf{CM}}, \alpha)(\boldsymbol{\xi} - \chi')(\mathfrak{gl}(\alpha))) \subset (\mathscr{D}(Q_{\mathsf{CM}}, \alpha)(\boldsymbol{\xi} - \chi')(\mathfrak{gl}(\alpha)))^{\mathrm{GL}(\alpha)}.$$

Thus, the element $\tilde{\theta}^{\text{Holland}}(b^*Xb) = \mathrm{pr}(\tilde{\theta}^{\text{Holland}}(b^*Xb))$ belongs to

$$(\mathscr{D}(Q_{\mathsf{CM}}, \alpha)(\boldsymbol{\xi} - \chi')(\mathfrak{gl}(\alpha)))^{\mathrm{GL}(\alpha)} + \nu \cdot \mathscr{D}(Q_{\mathsf{CM}}, \alpha)^{\mathrm{GL}(\alpha)}.$$

Therefore, $\Theta^{\text{Holland}}(e_s\Pi' e_s) \subseteq \mathfrak{A}_{\chi'}$. Thus, for any $\nu \in \mathbf{C}$, we have a homomorphism

$$\Theta^{\text{Holland}} : e_s\Pi' e_s \to \mathfrak{A}_{\chi'}.$$

Theorem **5.1.2.** — *The following diagram commutes:*

$$\begin{array}{ccc} e_s\Pi' e_s & \xrightarrow{\Theta^{\text{Quiver}}} & \mathbf{e}\mathsf{H}_{k,c}\mathbf{e} \\ \Big\downarrow{\scriptstyle \Theta^{\text{Holland}}} & & \Big\downarrow{\scriptstyle \Theta^{\text{Dunkl}}} \\ \mathfrak{A}_{\chi'} & \xrightarrow{\Theta^{\text{Radial}}} & \mathscr{D}(\mathfrak{X}, p, \varrho_s)^{\boldsymbol{\Gamma}_n}. \end{array}$$

Proof. — This follows from Theorem 4.6.2. □

Proposition **5.1.3.** — *The map* $\Theta^{\text{Holland}} : e_s\Pi' e_s \to \mathfrak{A}_{\chi'}$ *is surjective.*

Proof. — The algebra Π' has a filtration with $\deg(a) = 1$ for edges $a \neq b, b^*$, and $\deg(b) = \deg(b^*) = 0$. It suffices to show that the associated graded map

$$\mathrm{gr}(\Theta^{\text{Holland}}) : e_s\,\mathrm{gr}(\Pi')e_s \to \mathrm{gr}(\mathfrak{A}_{\chi'})$$

is surjective.

Now $e_s\,\mathrm{gr}(\Pi')e_s$ is generated by e_s and $b^*(e_o\Pi(Q)e_o)b$ where $\Pi(Q)$ is the preprojective algebra of Q. By [CB, Theorem 3.4], we have

$$\mathbf{C}[\mathsf{Rep}_{n\delta}(\Pi(Q))]^{\mathrm{GL}(n\delta)} = \left(\left(\mathbf{C}[\mathsf{Rep}_{\delta}(\Pi(Q))]^{\mathrm{GL}(\delta)}\right)^{\otimes n}\right)^{S_n}.$$

Denote by

$$\Theta^{\mathrm{CBH}} : \Pi(Q) \to \mathbf{C}\Big[\mathsf{Rep}_\delta(\Pi(Q)) \otimes \mathrm{End}\left(\oplus_{i\in I}\mathbf{C}^{\delta_i}\right)\Big]^{\mathrm{GL}(\delta)}$$

the natural morphism defined in [CBH], which gives a morphism

$$\Theta^{\mathrm{CBH}} : e_o\Pi(Q)e_o \to \mathbf{C}[\mathsf{Rep}_\delta(\Pi(Q))]^{\mathrm{GL}(\delta)}.$$

This latter morphism is an isomorphism by [CBH, Theorem 8.10].

Now, given an element $X \in e_o\Pi(Q)e_o$, we claim that

$$\mathrm{gr}(\Theta^{\mathrm{Holland}})(b^*Xb) = -\sum_{p=1}^{n} 1^{\otimes(p-1)} \otimes \Theta^{\mathrm{CBH}}(X) \otimes 1^{\otimes(n-p)}. \tag{5.1.4}$$

Indeed, we have

$$\begin{aligned}
&\mathrm{gr}(\theta^{\mathrm{Radial}})\,\mathrm{gr}(\Theta^{\mathrm{Holland}})(b^*Xb)\\
&= \mathrm{gr}(\Theta^{\mathrm{Dunkl}})\,\mathrm{gr}(\Theta^{\mathrm{Quiver}})(b^*Xb) \quad \text{(by Theorem 5.1.2)}\\
&= \mathrm{gr}(\theta^{\mathrm{Radial}})\Big(-\sum_{p=1}^{n} 1^{\otimes(p-1)} \otimes \Theta^{\mathrm{CBH}}(X) \otimes 1^{\otimes(n-p)}\Big) \quad \text{(by (6.3.3) below).}
\end{aligned}$$

Hence, (5.1.4) follows from injectivity of $\mathrm{gr}(\theta^{\mathrm{Radial}})$. It follows from (5.1.4) and Lemma 6.3.4 below that $\mathrm{gr}(\Theta^{\mathrm{Holland}})$ is surjective. □

5.2. — It follows from Theorem 5.1.2, Proposition 5.1.3, and the injectivity of Θ^{Dunkl} that we have a homomorphism

$$(\Theta^{\mathrm{Dunkl}})^{-1} \circ \Theta^{\mathrm{Radial}} : \mathfrak{A}_{\chi'} \to \mathbf{e}\mathsf{H}_{k,c}\mathbf{e}.$$

Since $\mathsf{Rep}_\alpha(Q_{\mathsf{CM}}) = \mathsf{Rep}_{n\delta}(Q) \oplus \mathbf{C}^n$, we have an obvious embedding

$$\varpi : \mathscr{D}(Q, n\delta) \to \mathscr{D}(Q_{\mathsf{CM}}, \alpha).$$

Definition **5.2.1.** — *The Harish–Chandra homomorphism* $\Phi_{k,c}$ *is defined to be the composition*

$$\mathscr{D}(Q, n\delta)^{\mathrm{GL}(n\delta)} \xrightarrow{\varpi} \mathfrak{A}_{\chi'} \xrightarrow{(\Theta^{\mathrm{Dunkl}})^{-1}\circ\Theta^{\mathrm{Radial}}} \mathbf{e}\mathsf{H}_{k,c}\mathbf{e}. \tag{5.2.2}$$

Following [EG], we define a 1-parameter space of representations V_d of $\mathfrak{gl}_n$ as follows. As a vector space, V_d is spanned by expressions $(x_1 \cdots x_n)^d \cdot P$, where P is a Laurent polynomial in $x_1, \ldots, x_n$ of total degree 0. The Lie algebra $\mathfrak{gl}_n$ has an action on V_d by formal differentiation, where $e_{p,q}$ acts by $x_p \frac{\partial}{\partial x_q}$. We restrict this to an $\mathfrak{sl}_n$ action. The desired $\mathfrak{gl}_n$ action on V_d is obtained by pulling back the $\mathfrak{sl}_n$ action via the natural Lie algebra projection $\mathfrak{gl}_n \to \mathfrak{sl}_n$, so that the center of $\mathfrak{gl}_n$ acts trivially.

Let Fun( ) denote the vector space of functions on a formal neighborhood of a point of the slice L^n_{reg}. Recall that $\partial = -n \sum_{i\in I} \operatorname{Tr}|_{N_i}(\zeta) e_i$. We have

$$(\mathbf{5.2.3}) \qquad \begin{aligned}&(\mathsf{Fun}(\mathsf{Rep}_\alpha(Q_{\mathsf{CM}})) \otimes \mathbf{C}_{-\chi'})^{\mathfrak{gl}(\alpha)}\\ &\quad = (\mathsf{Fun}(\mathsf{Rep}_{n\delta}(Q))\widehat{\otimes} V_{\nu-1} \otimes \mathbf{C}_{-\lambda+e_o+\partial})^{\mathfrak{gl}(n\delta)}.\end{aligned}$$

The $GL(n\delta)$ action on $\mathsf{Rep}_{n\delta}(Q)$ induces a Lie algebra map $\mathtt{ad} : \mathfrak{gl}(n\delta) \to \mathscr{D}(Q, n\delta)$. Let $\mathtt{ad} : U\mathfrak{gl}(n\delta) \to \mathscr{D}(Q, n\delta)$ be the induced map on the universal enveloping algebra of $\mathfrak{gl}(n\delta)$. Define the left ideal

$$J_{k,c} := \mathscr{D}(Q, n\delta) \cdot \mathtt{ad}(\mathtt{Ann}\,(V_{\nu-1} \otimes \mathbf{C}_{-\lambda+e_o+\partial})) \subset \mathscr{D}(Q, n\delta).$$

By (5.2.3), the ideal $J_{k,c}^{GL(n\delta)}$ is in the kernel of the map ϖ in (5.2.2).

Theorem **5.2.4.** — *The Harish–Chandra homomorphism induces an algebra isomorphism*

$$\Phi_{k,c} : \mathscr{D}(Q, n\delta)^{GL(n\delta)}/J_{k,c}^{GL(n\delta)} \xrightarrow{\sim} \mathbf{e}\mathsf{H}_{k,c}\mathbf{e}.$$

Proof. — By Theorem 7.2.3 and [EG, Theorem 1.3], the associated graded map, $\operatorname{gr}\Phi_{k,c}$, is the isomorphism in (7.2.4), hence $\Phi_{k,c}$ is itself an isomorphism. □

5.3. *Proof of Corollary 1.8.3*

Given any $C = \sum_{\gamma\in\Gamma} C_\gamma \gamma \in \mathbf{C}[\Gamma]$, we let $\overline{C} = \sum_{\gamma\in\Gamma} C_\gamma \gamma^{-1}$. Correspondingly, if $\lambda = \sum_{i\in I} \operatorname{Tr}|_{N_i}(C) e_i$, then let $\overline{\lambda} = \sum_{i\in I} \operatorname{Tr}|_{N_i}(\overline{C}) e_i$. We have an anti-isomorphism

$$(\mathbf{5.3.1}) \qquad \mathsf{H}_{k,c} \xrightarrow{\sim} \mathsf{H}_{k,\bar{c}}, \quad g \mapsto g^{-1}, \quad u \mapsto \sqrt{-1}u, \qquad \forall g \in \boldsymbol{\Gamma}_n,\ u \in L^n.$$

We also have an isomorphism

$$(\mathbf{5.3.2}) \qquad \begin{aligned}&\mathsf{H}_{k,c} \xrightarrow{\sim} \mathsf{H}_{-k,c}, \quad \sigma \mapsto (-1)^\sigma \sigma, \quad g \mapsto g, \quad u \mapsto u,\\ &\qquad\qquad\qquad \forall \sigma \in S_n,\ g \in \Gamma^n,\ u \in L^n.\end{aligned}$$

The isomorphism in (5.3.2) sends $\mathbf{e}$ to $\mathbf{e}_-$.

Now, for any $i \in \mathrm{I}$ set $\lambda_i^\dagger := \mathrm{Tr}\,|_{\mathrm{N}_i}(t \cdot 1 + c^\dagger)$. We put

$$c^\dagger := -c + 2|\Gamma|^{-1} \sum_{\gamma \neq 1} \gamma, \quad \text{and} \quad \lambda^\dagger := \sum_{i \in \mathrm{I}} \lambda_i^\dagger e_i = -\lambda + 2e_o.$$

The group $\mathrm{GL}(n\delta)$ acts on $\det(\mathsf{Rep}_{n\delta}(\mathrm{Q})^*)$ by the character 2∂. We have

$$\begin{aligned}(\mathrm{V}_{\nu-1} \otimes \mathbf{C}_{-\lambda+e_o+\partial})^* \otimes \det(\mathsf{Rep}_{n\delta}(\mathrm{Q})^*) &\simeq \mathrm{V}_{-\nu} \otimes \mathbf{C}_{\lambda-e_o-\partial} \otimes \mathbf{C}_{2\partial} \\ &= \mathrm{V}_{-\nu} \otimes \mathbf{C}_{-\lambda^\dagger+e_o+\partial}.\end{aligned}$$

Let $\mathsf{i} : \mathscr{D}(\mathrm{Q}, n\delta) \to \mathscr{D}(\mathrm{Q}, n\delta)$ be the anti-isomorphism sending a differential operator to its adjoint. Then for any $\mathrm{GL}(n\delta)$-module V,

$$\mathsf{i}(\mathtt{ad}(\mathtt{Ann}\,(\mathrm{V}))) = \mathtt{ad}((\mathtt{Ann}\,(\mathrm{V}^* \otimes \det(\mathsf{Rep}_{n\delta}(\mathrm{Q})^*)))).$$

The proof of the first isomorphism in Corollary 1.8.3 is now completed by the following isomorphisms

$$\begin{aligned}\mathbf{e}\mathsf{H}_{k,c}\mathbf{e} &\simeq (\mathbf{e}\mathsf{H}_{2|\Gamma|^{-1}-k,c^\dagger}\mathbf{e})^{op} && \text{using Theorem 5.2.4} \\ &\simeq \mathbf{e}\mathsf{H}_{2|\Gamma|^{-1}-k,\overline{c^\dagger}}\mathbf{e} && \text{using (5.3.1)} \\ &\simeq \mathbf{e}_-\mathsf{H}_{k-2|\Gamma|^{-1},\overline{c^\dagger}}\mathbf{e}_- && \text{using (5.3.2).}\end{aligned}$$

We will prove the second isomorphism in Corollary 1.8.3 later in Section 6.7.

6. Reflection isomorphisms

Except for Section 6.7, this section is independent of the earlier sections.

6.1. — Let Q be an arbitrary quiver (not necessarily of type $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{D}}$, or $\widetilde{\mathrm{E}}$). Denote by I the set of vertices of Q. Let $\mathrm{R} = \bigoplus_{i \in \mathrm{I}} \mathbf{C}$, and E the vector space with basis formed by the set of edges $\{a \in \overline{\mathrm{Q}}\}$. Thus, E is naturally a R-bimodule. The path algebra of $\overline{\mathrm{Q}}$ is $\mathbf{C}\overline{\mathrm{Q}} := \mathrm{T_R E} = \bigoplus_{n \geq 0} \mathrm{T}^n_{\mathrm{R}}\mathrm{E}$, where $\mathrm{T}^n_{\mathrm{R}}\mathrm{E} = \mathrm{E} \otimes_{\mathrm{R}} \cdots \otimes_{\mathrm{R}} \mathrm{E}$ is the n-fold tensor product. The trivial path for the vertex i is denoted by e_i, an idempotent in R.

Fix a positive integer n. Let $\mathsf{R} := \mathrm{R}^{\otimes n}$. For any $\ell \in [1, n]$, define the E-bimodules

$$\mathsf{E}_\ell := \mathrm{R}^{\otimes(\ell-1)} \otimes \mathrm{E} \otimes \mathrm{R}^{\otimes(n-\ell)} \qquad \text{and} \qquad \mathsf{E} := \bigoplus_{1 \leq \ell \leq n} \mathsf{E}_\ell.$$

The natural inclusion $\mathsf{E}_\ell \hookrightarrow \mathrm{R}^{\otimes(\ell-1)} \otimes \mathrm{T_R E} \otimes \mathrm{R}^{\otimes(n-\ell)}$ induces a canonical identification $\mathrm{T}_{\mathsf{R}}\mathsf{E}_\ell = \mathrm{R}^{\otimes(\ell-1)} \otimes \mathrm{T_R E} \otimes \mathrm{R}^{\otimes(n-\ell)}$. Given two elements $\varepsilon \in \mathsf{E}_\ell$ and $\varepsilon' \in \mathsf{E}_m$ of the form

(**6.1.1**) $\qquad \varepsilon = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n},$

(**6.1.2**) $\qquad \varepsilon' = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n},$

where $\ell \neq m$, $a, b \in \overline{Q}$ and $i_1, \ldots, i_n \in I$, we define

$$\begin{aligned}\lfloor \varepsilon, \varepsilon' \rfloor := &(e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n})\\ &\times (e_{i_1} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n})\\ &- (e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n})\\ &\times (e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes t(b) \otimes \cdots \otimes e_{i_n}).\end{aligned}$$

Note that $\lfloor \varepsilon, \varepsilon' \rfloor$ is an element in $T^2_R E$.

Definition **6.1.3** ([GG, Def. 1.2.3]). — *For any* $\lambda = \sum_{i \in I} \lambda_i e_i$ *where* $\lambda_i \in \mathbf{C}$, *and* $\nu \in \mathbf{C}$, *define the algebra* $\mathsf{A}_{n,\lambda,\nu}(Q)$ *to be the quotient of* $T_R E \rtimes \mathbf{C}[S_n]$ *by the following relations.*

(i) *For any* $i_1, \ldots, i_n \in I$ *and* $\ell \in [1, n]$*:*

$$\begin{aligned}&e_{i_1} \otimes \cdots \otimes \Big(\sum_{\{a \in Q \mid h(a) = i_\ell\}} a \cdot a^* - \sum_{\{a \in Q \mid t(a) = i_\ell\}} a^* \cdot a - \lambda_{i_\ell} e_{i_\ell} \Big) \otimes \cdots \otimes e_{i_n}\\ &\quad = \nu \sum_{\{j \neq \ell \mid i_j = i_\ell\}} (e_{i_1} \otimes \cdots \otimes e_{i_\ell} \otimes \cdots \otimes e_{i_n}) s_{j\ell}.\end{aligned}$$

(ii) *For any* $\varepsilon, \varepsilon'$ *of the form* (6.1.1)–(6.1.2)*:*

$$\lfloor \varepsilon, \varepsilon' \rfloor = \begin{cases} \nu \cdot (e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } b \in Q,\ a = b^*, \\ -\nu \cdot (e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } a \in Q,\ b = a^*, \\ 0 & \text{else.} \end{cases}$$

When $n = 1$, there is no parameter ν, and $\mathsf{A}_{n,\lambda,\nu}(Q)$ is the deformed preprojective algebra $\Pi_\lambda(Q)$ defined in [CBH].

6.2. *Quiver functors*

The goal of this section is to put the construction of the functor $M \to \widetilde{M}$ exploited in Section 2.2 into an appropriate, more general, context.

Let T be a nonempty subset of I, and let $e_T := \sum_{i \in T} e_i$. In particular, $e_I = 1$. Let Q_T be a quiver obtained from Q by adjoining a vertex s, and arrows $b_i : s \to i$ for $i \in T$. We call s the *special* vertex. We shall define a functor G from $\mathsf{A}_{n,\lambda,\nu}(Q)$-modules to $\Pi_{\lambda - \nu e_T + n\nu e_s}(Q_T)$-modules.

Let M be an $\mathsf{A}_{n,\lambda,\nu}(Q)$-module. We want to define a $\Pi_{\lambda - \nu e_T + n\nu e_s}(Q_T)$-module G(M). For each $i \in I$, let $G(M)_i := e_{n-1}(e_i \otimes e_T^{\otimes(n-1)})M$. Also, let $G(M)_s := e_n e_T^{\otimes n} M$.

If a is an edge in $\overline{Q}$, then define $a : G(M)_{t(a)} \to G(M)_{h(a)}$ to be the map given by the element $a \otimes e_T^{\otimes(n-1)} \in \mathsf{A}_{n,\lambda,\nu}(Q)$. We have an inclusion $G(M)_s \subset e_{n-1}e_T^{\otimes n}M = \bigoplus_{j\in T} G(M)_j$. For $i \in T$, we have a projection map $\mathsf{pr}_i : \bigoplus_{j\in T} G(M)_j \to G(M)_i$. Define $b_i : G(M)_s \to G(M)_i$ to be the restriction of pr_i to $G(M)_s$. Define $b_i^* : G(M)_i \to G(M)_s$ to be $-\nu \cdot (1 + s_{12} + \cdots + s_{1n})$.

The following lemma is a generalization of Lemma 2.2.2.

Lemma **6.2.1.** — *With the above actions,* $G(M)$ *is a* $\Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T)$*-module.*

Proof. — It is clear that $(1 + s_{12} + \cdots + s_{1n})e_{n-1} = ne_n$.
On $G(M)$, at the special vertex s, we have $\sum_{i\in T} b_i^* b_i = -n\nu$.
At a vertex $i \in I$, $i \notin T$, we have

$$\sum_{a\in Q;\, h(a)=i} aa^* - \sum_{a\in Q;\, t(a)=i} a^*a = \lambda_i$$

by the relation (i) in Definition 6.1.3. At a vertex $i \in T$, we have

$$\sum_{a\in Q;\, h(a)=i} aa^* - \sum_{a\in Q;\, t(a)=i} a^*a = \lambda_i + \nu \cdot \mathsf{pr}_i(s_{12} + \cdots + s_{1n}) = \lambda_i - \nu - b_i b_i^*,$$

using again the relation (i) in Definition 6.1.3. □

It is clear that the assigment $M \mapsto G(M)$ is functorial. We have constructed a functor

$$(\textbf{6.2.2}) \qquad G : \mathsf{A}_{n,\lambda,\nu}(Q)\text{-mod} \to \Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod}.$$

Recall the symmetrizer $e_n := \frac{1}{n!}\sum_{s\in S_n} s$.

Definition **6.2.3.** — *Let* $\mathsf{U}_{n,\lambda,\nu}(Q) := e_n\mathsf{A}_{n,\lambda,\nu}(Q)e_n$ *be the* spherical subalgebra *in* $\mathsf{A}_{n,\lambda,\nu}(Q)$.

The idempotents e_n and $\mathsf{e}_T^n := e_T^{\otimes n}$ commute. For $M := \mathsf{A}_{n,\lambda,\nu}(Q)e_n\mathsf{e}_T^n$, we get

$$G(M)_s = \mathsf{e}_T^n \mathsf{U}_{n,\lambda,\nu}\mathsf{e}_T^n.$$

In this case, $G(M)_s$ is an algebra, and the action of $e_s\Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T)e_s$ on $G(M)_s$ commutes with right multiplication by the elements of $G(M)_s$. Thus, our construction yields an algebra homomorphism

$$(\textbf{6.2.4}) \qquad \widehat{G} : e_s\Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T)e_s \to \mathsf{e}_T^n \mathsf{U}_{n,\lambda,\nu}\mathsf{e}_T^n.$$

6.3. *Modified version*

The map $\widehat{G}$ is 0 on nonconstant paths when $\nu = 0$. For this reason, we shall need a slight modification of the constructions in the previous subsection.

Define $\Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)$ to be the quotient of the path algebra $\mathbf{C}\overline{Q_T}$ by the following relations:

$$\sum_{a\in Q}[a, a^*] + \nu \sum_{i\in T} b_i b_i^* = \lambda - \nu e_T, \qquad \sum_{i\in T} b_i^* b_i = -ne_s.$$

We have an algebra morphism $\Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T) \to \Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)$ defined on the edges by

$$a \mapsto a \quad \text{for } a \neq b_i^*, \qquad b_i^* \mapsto \nu b_i^*.$$

This is an isomorphism only when $\nu \neq 0$.

Given a $\mathsf{A}_{n,\lambda,\nu}(Q)$-module M, we construct a $\Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)$-module $G'(M)$ analogous to $G(M)$ in the previous subsection, the only difference is that now, we let $b_i^* : G'(M)_i \to G'(M)_s$ be $-(1+s_{12}+\cdots+s_{1n})$. Hence, as above, we obtain a functor

$$(\mathbf{6.3.1}) \qquad G' : \mathsf{A}_{n,\lambda,\nu}(Q)\text{-mod} \to \Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod}$$

as well as a morphism

$$(\mathbf{6.3.2}) \qquad \widehat{G}' : e_s\Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)e_s \to \mathsf{e}_T^n \mathsf{U}_{n,\lambda,\nu}\mathsf{e}_T^n.$$

The algebra $\Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)$ has a filtration with $\deg(a) = 1$ for $a \neq b_i, b_i^*$, and $\deg(b_i) = \deg(b_i^*) = 0$ for $i \in T$. Also, $e_s\,\mathrm{gr}(\Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T))e_s$ is generated by e_s and $(\bigoplus_{i,j\in T} b_j^*\Pi_0(Q)b_i)$.

We shall assume that Q *is a connected quiver without edge-loops, and* Q *is not a finite Dynkin quiver.* Then, by [GG, Theorem 2.2.1] and [GG, Remark 2.2.6], $\mathrm{gr}\,\mathsf{A}_{n,\lambda,\nu}(Q) = \Pi_0(Q)^{\otimes n} \rtimes \mathbf{C}[S_n]$. Thus,

$$\mathrm{gr}\left(\mathsf{e}_T^n\mathsf{U}_{n,\lambda,\nu}\mathsf{e}_T^n\right) = \left((e_T\Pi_0(Q)e_T)^{\otimes n}\right)^{S_n}.$$

Now given an element $X \in e_j\Pi_0(Q)e_i$ where $i, j \in T$, we have

$$(\mathbf{6.3.3}) \qquad \mathrm{gr}(\widehat{G}')(b_j^* X b_i) = -\sum_{p=1}^{n} e_T^{\otimes(p-1)} \otimes X \otimes e_T^{\otimes(n-p)}.$$

Lemma **6.3.4.** — *Let* A *be any associative algebra with unit* $1 \in A$. *Then* $(A^{\otimes n})^{S_n}$ *is generated as an algebra by elements of the form*

$$\sum_{p=1}^{n} 1^{\otimes(p-1)} \otimes X \otimes 1^{\otimes(n-p)}, \qquad X \in A.$$

Proof. — Since $(A^{\otimes n})^{S_n}$ is spanned by elements of the form $a^{\otimes n}$ where $a \in A$, it suffices to show that the lemma is true for $A = \mathbf{C}[a]$, but this follows from the main theorem on symmetric functions. □

Proposition **6.3.5.** — *The map* $\widehat{G}'$ *in* (6.3.2) *is surjective.*

Proof. — It suffices to show that $\mathrm{gr}(\widehat{G}')$ is surjective. This follows from (6.3.3) and the preceding lemma. □

6.4. *Reflection functors*

Recall the setting of reflection functors as in (1.8.1). In particular, we have the Weyl group W generated by the simple reflections r_i for $i \in I$. We also have a non-empty subset $T \subset I$ and we fix a vertex $i \notin T$.

Let us apply the reflection functor F_i to the $\mathsf{A}_{n,\lambda,\nu}(Q)$-module $\mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T$. By construction, we have

$$\mathsf{e}^n_T F_i\big(\mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T\big) = \mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T$$

and the left action of $\mathsf{e}^n_T\mathsf{A}_{n,r_i(\lambda),\nu}(Q)\mathsf{e}^n_T$ on $\mathsf{e}^n_T F_i(\mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T)$ commutes with the right multiplication by $\mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T$. Hence, for $i \notin T$, we obtain a homomorphism

$$\widehat{F}_i : \mathsf{e}^n_T\mathsf{A}_{n,r_i(\lambda),\nu}(Q)\mathsf{e}^n_T \to \mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T. \qquad (\mathbf{6.4.1})$$

Note that $\widehat{F}_i(\mathsf{e}^n_T\mathsf{U}_{n,r_i(\lambda),\nu}(Q)\mathsf{e}^n_T) \subset \mathsf{e}^n_T\mathsf{U}_{n,\lambda,\nu}\mathsf{e}^n_T$.

In the special case when $n = 1$, reflection functors were constructed in [CBH]; let us recall their definition. Since $\Pi_\lambda(Q)$ does not depend on the orientation of Q, we may assume without loss of generality that i is a sink in Q. Let M be a $\Pi_\lambda(Q)$-module, and $M_j = e_j M$ for each $j \in I$. For each edge $a \in Q$ such that $h(a) = i$, write

$$\pi_a : \bigoplus_{\xi\in Q;\, h(\xi)=i} M_{t(\xi)} \to M_{t(a)}, \quad \mu_a : M_{t(a)} \to \bigoplus_{\xi\in Q;\, h(\xi)=i} M_{t(\xi)}$$

for the projection map and inclusion map, respectively. Define

$$\pi : \bigoplus_{a\in Q;\, h(a)=i} M_{t(a)} \to M_i, \quad \pi := \sum_{a\in Q;\, h(a)=i} a \circ \pi_a,$$

and

$$\mu : M_i \to \bigoplus_{a\in Q;\, h(a)=i} M_{t(a)}, \quad \mu := \sum_{a\in Q;\, h(a)=i} \mu_a \circ a^*.$$

Observe that $\pi\mu = \lambda_i$. Let $(F_i(M))_j := M_j$ if $j \neq i$, and let $(F_i(M))_i := \mathrm{Ker}(\pi)$. If $a \in \overline{Q}$ and $h(a), t(a) \neq i$, then let $a : (F_i(M))_{t(a)} \to (F_i(M))_{h(a)}$ be the same map as $a : M_{t(a)} \to M_{h(a)}$. If $a \in Q$ and $h(a) = i$, then let $a : (F_i(M))_{t(a)} \to (F_i(M))_i$ be the map $(-\lambda_i + \mu\pi)\mu_a$, and let $a^* : (F_i(M))_i \to (F_i(M))_{t(a)}$ be the map π_a restricted to $(F_i(M))_i$. Letting

$$F_i(M) = \oplus_{j\in I}(F_i(M))_j,$$

we have defined the functor

$$F_i : \Pi_\lambda(Q)\text{-mod} \to \Pi_{r_i(\lambda)}(Q)\text{-mod} \quad \text{for any } i \in I.$$

In particular, for the quiver Q_T, and for $i \in I$ but $i \notin T$, we have

$$F_i : \Pi_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod} \to \Pi_{r_i(\lambda)-\nu e_T+n\nu e_s}(Q_T)\text{-mod} \tag{6.4.2}$$

Let $i \in I$ but $i \notin T$. We define a functor

$$F'_i : \Pi'_{\lambda-\nu e_T+n\nu e_s}(Q_T)\text{-mod} \to \Pi'_{r_i(\lambda)-\nu e_T+n\nu e_s}(Q_T)\text{-mod}$$

in exactly the same way as F_i in (6.4.2). It is easy to see from definitions that the diagram (1.8.6) commutes.

6.5. *Relations in rank 1*

In this subsection, the rank n is equal to 1. Let $\mathfrak{C} = (\mathfrak{C}_{ij})$ be the generalized Cartan matrix of Q.

Proposition **6.5.1.** — *For all $\lambda \in R$, we have the following.*

(i) *The map*

$$\widehat{F}_i : (1 - e_i)\Pi_\lambda(Q)(1 - e_i) \to (1 - e_i)\Pi_{r_i(\lambda)}(Q)(1 - e_i)$$

is an isomorphism, and $\widehat{F}_i^2 = \mathtt{Id}$.

(ii) *If* $\mathfrak{C}_{ij} = 0$, *then*

$$\widehat{F}_i \circ \widehat{F}_j = \widehat{F}_j \circ \widehat{F}_i : (1 - e_i - e_j)\Pi_\lambda(Q)(1 - e_i - e_j) \to (1 - e_i - e_j)\Pi_{r_ir_j(\lambda)}(Q)(1 - e_i - e_j).$$

(iii) *If* $\mathfrak{C}_{ij} = -1$, *then*

$$\widehat{F}_i \circ \widehat{F}_j \circ \widehat{F}_i = \widehat{F}_j \circ \widehat{F}_i \circ \widehat{F}_j : (1 - e_i - e_j)\Pi_\lambda(Q)(1 - e_i - e_j) \to (1 - e_i - e_j)\Pi_{r_ir_jr_i(\lambda)}(Q)(1 - e_i - e_j).$$

Proof. — (i) The algebra $(1-e_i)\Pi_\lambda(Q)(1-e_i)$ is generated by edges $a \in \overline{Q}$ with $h(a), t(a) \neq i$, and paths of length two: a_2a_1 with $h(a_2), t(a_1) \neq i$ and $t(a_2) = h(a_1) = i$.

If $a \in \overline{Q}$ and $h(a), t(a) \neq i$, then $\widehat{F}_i(a) = a$.

Now let a_2a_1 be a path with $h(a_2), t(a_1) \neq i$ and $t(a_2) = h(a_1) = i$. If $a_2 \neq a_1^*$ or $a_1 \neq a_2^*$, then $\widehat{F}_i(a_2a_1) = a_2a_1$. If $a_2 = a_1^*$, then $\widehat{F}_i(a_2a_1) = -\lambda_i e_{t(a_1)} + a_2a_1$, and so $\widehat{F}_i(\widehat{F}_i(a_2a_1)) = -\lambda_i e_{t(a_1)} + \lambda_i e_{t(a_1)} + a_2a_1 = a_2a_1$.

(ii) When $\mathfrak{C}_{ij} = 0$, there is no edge joining i and j. In this case, it is clear that $F_iF_j = F_jF_i$, so $\widehat{F}_i\widehat{F}_j = \widehat{F}_j\widehat{F}_i$.

(iii) When $\mathfrak{C}_{ij} = -1$, there is precisely one edge in Q joining i and j, say $a : i \to j$. The algebra $(1-e_i-e_j)\Pi_\lambda(Q)(1-e_i-e_j)$ is generated by:

- edges $a_1 \in \overline{Q}$ with $h(a_1), t(a_1) \neq i, j$;
- paths a_2a_1 with $t(a_2) = h(a_1) = i$ and $h(a_2), t(a_1) \neq i, j$;
- paths a_2a_1 with $t(a_2) = h(a_1) = j$ and $h(a_2), t(a_1) \neq i, j$;
- paths $a_3a_2a_1$ with $a_2 = a$, $t(a_3) = j$, $h(a_1) = i$ and $h(a_3), t(a_1) \neq i, j$;
- paths $a_3a_2a_1$ with $a_2 = a^*$, $t(a_3) = i$, $h(a_1) = j$ and $h(a_3), t(a_1) \neq i, j$.

In the first case above, we have

$$\widehat{F}_i\widehat{F}_j\widehat{F}_i(a_1) = a_1 = \widehat{F}_j\widehat{F}_i\widehat{F}_j(a_1).$$

In the second case above, when $a_2 \neq a_1^*$ or $a_1 \neq a_2^*$, we have

$$\widehat{F}_i\widehat{F}_j\widehat{F}_i(a_2a_1) = a_2a_1 = \widehat{F}_j\widehat{F}_i\widehat{F}_j(a_2a_1).$$

When $a_2 = a_1^*$, we have

$$\begin{aligned}\widehat{F}_i\widehat{F}_j\widehat{F}_i(a_2a_1) &= \widehat{F}_i\widehat{F}_j(-\lambda_i + a_2a_1) = \widehat{F}_i(-\lambda_i + a_2a_1)\\ &= -\lambda_i - \lambda_j + a_2a_1,\end{aligned}$$

since $r_j(r_i(\lambda))e_i = \lambda_j$; and on the other hand, since $r_j(\lambda)e_i = \lambda_i + \lambda_j$, we find

$$\widehat{F}_j\widehat{F}_i\widehat{F}_j(a_2a_1) = \widehat{F}_j\widehat{F}_i(a_2a_1) = \widehat{F}_j(-\lambda_i - \lambda_j + a_2a_1) = -\lambda_i - \lambda_j + a_2a_1.$$

The third case above is similar to the second case.

In the fourth and fifth cases above, note that no two of the edges a_1, a_2, a_3 are reverse of the other, so we have

$$\widehat{F}_i\widehat{F}_j\widehat{F}_i(a_3a_2a_1) = a_3a_2a_1 = \widehat{F}_j\widehat{F}_i\widehat{F}_j(a_3a_2a_1).$$

□

Lemma **6.5.2.** — (i) *If* $\lambda_i \neq 0$, *then*

$$\Pi_\lambda(Q) = \Pi_\lambda(Q)(1-e_i)\Pi_\lambda(Q).$$

(ii) *If* $\mathfrak{C}_{ij} = -1$, *and* $\lambda_i \neq 0$, $\lambda_j \neq 0$, $\lambda_i + \lambda_j \neq 0$, *then*

$$\Pi_\lambda(Q) = \Pi_\lambda(Q)(1 - e_i - e_j)\Pi_\lambda(Q).$$

Proof. — (i) As a $\Pi_\lambda(Q)$-module, $\frac{\Pi_\lambda(Q)}{\Pi_\lambda(Q)(1-e_i)\Pi_\lambda(Q)}$ is zero at all vertices not equal to i, so all edges of $\overline{Q}$ must act by 0. But then it must also be zero at the vertex i since $\lambda_i e_i = \sum_{a\in Q;h(a)=i} aa^* - \sum_{a\in Q;t(a)=i} a^*a$.

(ii) There is only one edge in Q joining i and j, say $a : i \to j$. Let V be the $\Pi_\lambda(Q)$-module $\frac{\Pi_\lambda(Q)}{\Pi_\lambda(Q)(1-e_i-e_j)\Pi_\lambda(Q)}$. Now V is zero at all vertices not equal to i or j, so $V = V_i \oplus V_j$. Suppose $V \neq 0$, say $V_j \neq 0$. Then $aa^* = \lambda_j e_j$ on V_j implies that a, a^* are nonzero maps, and a has a right inverse $\lambda_j^{-1}a^*$. But then $a^*a = -\lambda_i e_i$ on V_i implies that a has a left inverse $-\lambda_i^{-1}a^*$. Hence, $\lambda_j = -\lambda_i$, a contradiction. □

Using Proposition 6.5.1(i), $\Pi_{r_i(\lambda)}(Q)(1 - e_i)$ is a right $(1 - e_i)\Pi_\lambda(Q)(1 - e_i)$-module, and $\Pi_{r_i(\lambda)}(Q)(1 - e_i - e_j)$ is a right $(1 - e_i - e_j)\Pi_\lambda(Q)(1 - e_i - e_j)$-module.

Corollary **6.5.3.** — (i) *If* $\lambda_i \neq 0$, *then*

$$F_i(M) = \Pi_{r_i(\lambda)}(Q)(1 - e_i) \otimes_{(1-e_i)\Pi_\lambda(Q)(1-e_i)} (1 - e_i)M$$

for any $M \in \Pi_\lambda(Q)$-mod.

(ii) *If* $\mathfrak{C}_{ij} = -1$, *and* $\lambda_i \neq 0$, $\lambda_j \neq 0$, $\lambda_i + \lambda_j \neq 0$, *then*

$$F_i(M) = \Pi_{r_i(\lambda)}(Q)(1 - e_i - e_j) \otimes_{(1-e_i-e_j)\Pi_\lambda(Q)(1-e_i-e_j)} (1 - e_i - e_j)M$$

for any $M \in \Pi_\lambda(Q)$-mod.

Proof. — (i) Let $M \in \Pi_\lambda(Q) - \text{mod}$. By Lemma 6.5.2(i),

$$\begin{aligned} F_i(M) &= \Pi_{r_i(\lambda)}(Q)(1 - e_i) \otimes_{(1-e_i)\Pi_{r_i(\lambda)}(Q)(1-e_i)} (1 - e_i)F_i(M) \\ &= \Pi_{r_i(\lambda)}(Q)(1 - e_i) \otimes_{(1-e_i)\Pi_\lambda(Q)(1-e_i)} (1 - e_i)M. \end{aligned}$$

The proof of (ii) is similar, using Lemma 6.5.2(ii). □

Corollary **6.5.4.** — (i) *If* $\lambda_i \neq 0$, *then* $F_i^2 = \mathtt{Id}$.

(ii) *If* $\mathfrak{C}_{ij} = 0$, *then* $F_iF_j = F_jF_i$.

(iii) *If* $\mathfrak{C}_{ij} = -1$ *and* $\lambda_i \neq 0$, $\lambda_j \neq 0$, $\lambda_i + \lambda_j \neq 0$, *then* $F_iF_jF_i = F_jF_iF_j$.

Proof. — (ii) is trivial, while (i) and (iii) are immediate from Proposition 6.5.1 and Corollary 6.5.3. □

Our proof of Corollary 6.5.4 appears to be simpler than earlier proofs, see [CBH, Theorem 5.1] (for (i)), [Na, Remark 3.20], [Na, Theorem 3.4], [Lu2], and [Maf].

6.6. *Relations in higher rank*

In this subsection, n is an integer greater than 1. We shall show that the reflection functors F_i of (1.8.1) satisfy the Weyl group relations when the parameters are generic.

We omit the proof of the following proposition since it is completely similar to the proof of Proposition 6.5.1.

Proposition **6.6.1.** — *Let* $i,j \in I$. *The homomorphisms* $\widehat{F}_i$ *of* (6.4.1) *satisfy the following for any* $\lambda \in R$ *and* $\nu \in \mathbf{C}$*:*

(i) *Let* $T = I \setminus \{i\}$. *Then the map*

$$\widehat{F}_i : \mathsf{e}^n_T \mathsf{A}_{n,r_i(\lambda),\nu}(Q)\mathsf{e}^n_T \to \mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T$$

is an isomorphism, and $\widehat{F}_i \circ \widehat{F}_i = \mathtt{Id}$.

(ii) *Let* $T = I \setminus \{i,j\}$. *If* $\mathfrak{C}_{ij} = 0$, *then*

$$\widehat{F}_i \circ \widehat{F}_j = \widehat{F}_j \circ \widehat{F}_i : \mathsf{e}^n_T \mathsf{A}_{n,r_i r_j(\lambda),\nu}(Q)\mathsf{e}^n_T \to \mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T.$$

(iii) *Let* $T = I \setminus \{i,j\}$. *If* $\mathfrak{C}_{ij} = -1$, *then*

$$\widehat{F}_i \circ \widehat{F}_j \circ \widehat{F}_i = \widehat{F}_j \circ \widehat{F}_i \circ \widehat{F}_j : \mathsf{e}^n_T \mathsf{A}_{n,r_i r_j r_i(\lambda),\nu}(Q)\mathsf{e}^n_T \to \mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}(Q)\mathsf{e}^n_T.$$ □

Next, we have the following generalization of Lemma 6.5.2.

Lemma **6.6.2.** — (i) *Let* $T = I \setminus \{i\}$. *If* $\lambda_i \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $\mathsf{A}_{n,\lambda,\nu} = \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}$.

(ii) *Let* $T = I \setminus \{i,j\}$ *and suppose* $\mathfrak{C}_{ij} = 0$. *If* $\lambda_i \pm p\nu \neq 0$ *and* $\lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $\mathsf{A}_{n,\lambda,\nu} = \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}$.

(iii) *Let* $T = I \setminus \{i,j\}$ *and suppose* $\mathfrak{C}_{ij} = -1$. *If* $\lambda_i \pm p\nu \neq 0$, $\lambda_j \pm p\nu \neq 0$ *and* $\lambda_i + \lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then* $\mathsf{A}_{n,\lambda,\nu} = \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}$.

Proof. — The proof is similar to the proof of Lemma 6.5.2.

To prove (i), let V be the $\mathsf{A}_{n,\lambda,\nu}$-module $\frac{\mathsf{A}_{n,\lambda,\nu}}{\mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T\mathsf{A}_{n,\lambda,\nu}}$ where $T = I \setminus \{i\}$. For any n-tuple of vertices $i_1, \dots, i_n$, we let $V_{i_1,\dots,i_n} := (e_{i_1} \otimes \cdots \otimes e_{i_n})V$, so $V = \bigoplus_{i_1,\dots,i_n \in I} V_{i_1,\dots,i_n}$. Since $\mathsf{e}^n_T V = 0$, we have $V_{i_1,\dots,i_n} = 0$ when none of $i_1, \dots, i_n$ is i. Suppose now that i appears m times in $i_1, \dots, i_n$. We shall prove by induction on m that $V_{i_1,\dots,i_n} = 0$, so we assume that the statement is true whenever i appears less than m times. Without loss of generality, say $i_1 = \cdots = i_m = i$. Then by the relation (i) in Definition 6.1.3 and the induction hypothesis, we have

$$(\lambda_i + \nu \sum_{\ell=2}^{m} s_{1\ell})V_{i_1,\dots,i_n} = 0.$$

By [Ga, Prop. 5.12], the element $\lambda_i + \nu \sum_{\ell=2}^{m} s_{1\ell}$ is invertible in the group algebra $\mathbf{C}[S_n]$. Hence, $V_{i_1,\dots,i_n} = 0$, and (i) follows by induction.

The proofs of (ii) and (iii) are similar, using induction. □

As in the previous subsection, we obtain

Corollary **6.6.3.** — (i) *Let* $T = I \setminus \{i\}$. *If* $\lambda_i \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then*

$$F_i(M) = \mathsf{A}_{n,r_i(\lambda),\nu}\mathsf{e}^n_T \otimes_{\mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T} \mathsf{e}^n_T M, \quad \forall M \in \mathsf{A}_{n,\lambda,\nu}\text{-mod}.$$

(ii) *Let* $T = I \setminus \{i,j\}$ *and suppose* $\mathfrak{C}_{ij} = 0$. *If* $\lambda_i \pm p\nu \neq 0$ *and* $\lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then*

$$F_i(M) = \mathsf{A}_{n,r_i(\lambda),\nu}\mathsf{e}^n_T \otimes_{\mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T} \mathsf{e}^n_T M, \quad \forall M \in \mathsf{A}_{n,\lambda,\nu}\text{-mod}.$$

(iii) *Let* $T = I \setminus \{i,j\}$ *and suppose* $\mathfrak{C}_{ij} = -1$. *If* $\lambda_i \pm p\nu \neq 0$, $\lambda_j \pm p\nu \neq 0$ *and* $\lambda_i + \lambda_j \pm p\nu \neq 0$ *for* $p = 0, 1, \dots, n-1$, *then*

$$F_i(M) = \mathsf{A}_{n,r_i(\lambda),\nu}\mathsf{e}^n_T \otimes_{\mathsf{e}^n_T \mathsf{A}_{n,\lambda,\nu}\mathsf{e}^n_T} \mathsf{e}^n_T M, \quad \forall M \in \mathsf{A}_{n,\lambda,\nu}\text{-mod}.$$

□

Proposition 1.8.2 is immediate from Proposition 6.6.1 and Corollary 6.6.3.

6.7. *Shift functors*

In this subsection, we return to the case when Q is the affine Dynkin quiver associated to Γ.

Let $\mathfrak{C} = (\mathfrak{C}_{ij})$ be the generalized Cartan matrix of Q. The affine Weyl group $\widetilde{W}$ is generated by the simple reflections r_i for $i \in I$. It acts on $\mathbf{C}^I$ by $r_i : \mathbf{C}^I \to \mathbf{C}^I$, where $r_i(\lambda) = \lambda - \sum_{j\in I} \mathfrak{C}_{ij}\lambda_i e_j$.

Let Q' be the finite Dynkin quiver obtained from Q by deleting the vertex o. The Weyl group W of Q' is the subgroup of $\widetilde{W}$ generated by the r_i for $i \neq o$. Let $\mathfrak{C}' = (\mathfrak{C}'_{ij})$ be the Cartan matrix of Q'. Then W acts on $\oplus_{i\neq o}\mathbf{C}e_i$ by $r_i(\lambda) = \lambda - \sum_{j\neq o} \mathfrak{C}'_{ij}\lambda_i e_j$. Denote by $w_0 \in W$ the longest element of W.

If $i \in I$, then let $i^* \in I$ be the vertex such that $N_{i^*} = N_i^*$. Recall that if $\lambda = \sum_{i\in I} \lambda_i e_i$, then $\overline{\lambda} = \sum_{i\in I} \lambda_{i^*} e_i$.

Lemma **6.7.1.** — *For any* $\lambda \in \mathbf{C}^I$ *with* $\lambda \cdot \delta = 1$, *we have* $w_0(\lambda) = -\overline{\lambda} + 2e_0$.

Proof. — The projection $\mathbf{C}^I \to \oplus_{i\neq o}\mathbf{C}e_i$ is W-equivariant with kernel $\mathbf{C}e_o$. We write $\lambda = \lambda_o e_o + \lambda'$ where $\lambda' \in \oplus_{i\neq o}\mathbf{C}e_i$. Now $w_0(\lambda) - w_0(\lambda') \in \mathbf{C}e_0$ and $w_0(\lambda') = -\overline{\lambda'}$. It follows that $w_0(\lambda) = -\overline{\lambda} + 2(\lambda \cdot \delta)e_0$. □

We will now prove the second isomorphism in Corollary 1.8.3. For each vertex $i \neq o$, we have, from (6.4.1), the homomorphism

$$\widehat{\mathrm{F}}_i : \mathbf{e}_- \mathsf{A}_{n, r_i(\lambda), \nu-1}(\mathrm{Q})\mathbf{e}_- \to \mathbf{e}_- \mathsf{A}_{n,\lambda,\nu-1}(\mathrm{Q})\mathbf{e}_-$$

which is an isomorphism by Proposition 6.6.1(i). By writing w_0 as a product of simple reflections, we get an isomorphism

$$(\mathbf{6.7.2}) \qquad \widehat{\mathrm{F}}_{w_0} : \mathbf{e}_- \mathsf{A}_{n, w_0(\lambda), \nu-1}(\mathrm{Q})\mathbf{e}_- \xrightarrow{\sim} \mathbf{e}_- \mathsf{A}_{n,\lambda,\nu-1}(\mathrm{Q})\mathbf{e}_-.$$

Proposition 6.6.1 implies that this isomorphism does not depend on the choice of presentation of w_0 as a product of simple reflections.

Write $\mathsf{H}_{t,k,c} = \mathsf{H}_{t,k,c}(\mathbf{\Gamma}_n)$. By [GG, (3.5.2)], there is an isomorphism $f^{\otimes n}\mathsf{H}_{t,k,c}f^{\otimes n} = \mathsf{A}_{n,\lambda,\nu}(\mathrm{Q})$ where $f = \sum_{i\in \mathrm{I}} e_i$. In particular, $\mathbf{e}_-\mathsf{H}_{t,k-2t,c'}\mathbf{e}_- = \mathbf{e}_-\mathsf{A}_{n,\overline{\lambda^\dagger},\nu-1}(\mathrm{Q})\mathbf{e}_-$, and $\mathbf{e}_-\mathsf{H}_{t,k-2t,c}\mathbf{e}_- = \mathbf{e}_-\mathsf{A}_{n,\lambda,\nu-1}\mathbf{e}_-$. By Lemma 6.7.1, $\overline{\lambda^\dagger} = w_0(\lambda)$, so by (6.7.2) we have the isomorphism

$$\widehat{\mathrm{F}}_{w_0} : \mathbf{e}_-\mathsf{H}_{t,k-2t,c'}\mathbf{e}_- \xrightarrow{\sim} \mathbf{e}_-\mathsf{H}_{t,k-2t,c}\mathbf{e}_-.$$

This completes the proof of Corollary 1.8.3.

Using the isomorphism $\mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e} \simeq \mathbf{e}_-\mathsf{H}_{t,k-2t,c}\mathbf{e}_-$ of Corollary 1.8.3, we can consider $\mathsf{H}_{t,k-2t,c}\mathbf{e}_-$ as a $(\mathsf{H}_{t,k-2t,c}, \mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e})$-bimodule.

Definition **6.7.3.** — *The shift functor is defined to be*

$$\mathbf{S} : \mathsf{H}_{t,k,c}\text{-mod} \to \mathsf{H}_{t,k-2t,c}\text{-mod}, \quad \mathrm{V} \mapsto \mathsf{H}_{t,k-2t,c}\mathbf{e}_- \otimes_{\mathbf{e}\mathsf{H}_{t,k,c}\mathbf{e}} \mathbf{e}\mathrm{V}.$$

7. Extended Dynkin quiver

7.1. Γ-*analogue of commuting variety*

In this subsection, we will prove a generalization of [EG, Theorem 12.1].

Let $\mathrm{R}(\Gamma, n)$ be the space of extensions of the representation $\mathbf{C}\Gamma \otimes \mathbf{C}^n$ of Γ to a representation of $\mathrm{T(L)} \rtimes \mathbf{C}\Gamma$, i.e.,

$$\mathrm{R}(\Gamma, n) := \mathrm{Hom}_\Gamma\big(\mathrm{L}, \mathrm{End}_{\mathbf{C}}(\mathbf{C}\Gamma \otimes \mathbf{C}^n)\big).$$

Let $\mathscr{X} = \mathscr{X}(\Gamma, n)$ be the (not necessarily reduced) subscheme of $\mathrm{R}(\Gamma, n)$ consisting of those representations ρ such that $\rho([\mathrm{X}, \mathrm{Y}]) = 0$ for all $\mathrm{X}, \mathrm{Y} \in \mathrm{L}$.

Now $\mathbf{C}\Gamma = \oplus_{i\in\mathrm{I}}\,\mathrm{End}(\mathrm{N}_i)$. Let $p_i \in \mathbf{C}\Gamma$ be the idempotent element corresponding to the identity element of $\mathrm{End}(\mathrm{N}_i)$. Define $\mathscr{X}_1 = \mathscr{X}_1(\Gamma, n)$ to be the (not necessarily reduced) subscheme of $\mathrm{R}(\Gamma, n)$ consisting of those representations ρ such that, for all

$X, Y \in L$, we have $\rho([X,Y]p_i) = 0$ for $i \neq o$, and $\wedge^2\rho([X,Y]p_o) = 0$. We remark that $\rho([X,Y]p_o)$ is a $n \times n$-matrix and $\wedge^2\rho([X,Y]p_o) = 0$ means that all 2×2 minors of this matrix vanish (so its rank is at most 1).

We shall denote by J and J_1 the defining ideals of $\mathscr{Z}(\Gamma, n)$ and $\mathscr{Z}_1(\Gamma, n)$, respectively. Thus,

$$\mathscr{Z}(\Gamma, n) = \operatorname{Spec} \mathbf{C}[R(\Gamma, n)]/J$$

and

$$\mathscr{Z}_1(\Gamma, n) = \operatorname{Spec} \mathbf{C}[R(\Gamma, n)]/J_1.$$

Let $G := \mathtt{Aut}_\Gamma(\mathbf{C}\Gamma \otimes \mathbf{C}^n)$. Observe that the group G acts on $R(\Gamma, n)$, $\mathscr{Z}(\Gamma, n)$, and $\mathscr{Z}_1(\Gamma, n)$.

Theorem **7.1.1.** — *One has:* $J^G = J_1^G$.

It is clear that $J \supset J_1$, so $J^G \supset J_1^G$. To prove Theorem 7.1.1, we have to show that $J^G \subset J_1^G$. We need the following lemmas. First, let us fix a basis X, Y for L.

Lemma **7.1.2.** — *The ideal* J^G *is generated in* $\mathbf{C}[R(\Gamma, n)]^G$ *by functions of the form* $\rho \mapsto \operatorname{Tr}(\rho(Q[X,Y]))$, *where* $Q \in T(L) \rtimes \mathbf{C}\Gamma$.

Proof. — This follows from Weyl's fundamental theorem of invariant theory. □

Therefore, it suffices to show that $\operatorname{Tr}(\rho(Q[X,Y]p_i)) = 0 \pmod{J_1}$ for all $\rho \in R(\Gamma, n)$, $Q \in T(L) \rtimes \mathbf{C}\Gamma$, and $i \in I$. This is obvious for $i \neq o$ from the definition of J_1. For $i = o$, we shall prove it by induction on the degree of Q. The case $\deg Q = 0$ is clear, so let $d > 0$ and assume that $\operatorname{Tr}(\rho(Q[X,Y]p_o)) = 0 \pmod{J_1}$ whenever $\deg Q < d$.

Lemma **7.1.3.** — *Let* $\deg Q = d$. *If* $Q = Q_1[X,Y]Q_2$ *for some* $Q_1, Q_2 \in T(L) \rtimes \mathbf{C}\Gamma$, *then* $\operatorname{Tr}(\rho(Q[X,Y]p_o)) = 0 \pmod{J_1}$.

Proof. — We may replace Q, Q_1, Q_2 by $p_oQp_o, p_oQ_1p_o, p_oQ_2p_o$ respectively. Modulo J_1, and writing in terms of matrix elements, we have

$$\begin{aligned}
\operatorname{Tr}(\rho(Q[X,Y]p_o)) &= \operatorname{Tr}(\rho(Q_1[X,Y]p_oQ_2[X,Y]p_o)) \\
&= \sum \rho(Q_1)_{lm}\rho([X,Y]p_o)_{mq}\rho(Q_2)_{qr}\rho([X,Y]p_o)_{rl} \\
&= \sum \rho(Q_1)_{lm}\rho([X,Y]p_o)_{ml}\rho(Q_2)_{qr}\rho([X,Y]p_o)_{rq} \\
&\qquad\qquad (\text{since } \wedge^2\rho([X,Y]p_o) = 0) \\
&= \operatorname{Tr}(\rho(Q_1[X,Y]p_o))\operatorname{Tr}(\rho(Q_2[X,Y]p_o)).
\end{aligned}$$

This is equal to zero by induction hypothesis. □

Let $\varphi : \mathrm{T(L)} \rtimes \mathbf{C}\Gamma \to \mathrm{S(L)} \rtimes \mathbf{C}\Gamma$ be the quotient map, where S(L) denotes the symmetric algebra on L. The preceding lemma implies the following corollary.

Corollary **7.1.4.** — *If* $\deg \mathrm{Q} \le d$ *and* $\mathrm{Q} \in \operatorname{Ker}\varphi$, *then* $\mathrm{Tr}(\rho(\mathrm{Q}[\mathrm{X},\mathrm{Y}]p_o)) = 0$ $(\mathrm{mod}\, \mathrm{J}_1)$. □

Note that elements of the form $(a\mathrm{X}+b\mathrm{Y})^m$ (where $a, b \in \mathbf{C}$) span a set of representatives of S(L) in T(L). Thus, it remains to show that $\mathrm{Tr}(\rho((a\mathrm{X}+b\mathrm{Y})^m[\mathrm{X},\mathrm{Y}]p_o))$ $= 0$ $(\mathrm{mod}\, \mathrm{J}_1)$ for any $a, b \in \mathbf{C}$ and $m \le d$. This is equivalent to showing that $\mathrm{Tr}(\rho((a\mathrm{X}+b\mathrm{Y})^m[\mathrm{X},\mathrm{Y}])) = 0$ $(\mathrm{mod}\, \mathrm{J}_1)$. But we have

$$\mathrm{Tr}(\rho((a\mathrm{X}+b\mathrm{Y})^m[\mathrm{X},\mathrm{Y}])) = \frac{1}{a(m+1)}\mathrm{Tr}(\rho([(a\mathrm{X}+b\mathrm{Y})^{m+1},\mathrm{Y}])) = 0.$$

This completes the proof of Theorem 7.1.1.

7.2. — Let $\mu_{\mathsf{CM}} : \mathsf{Rep}_\alpha(\overline{\mathrm{Q}_{\mathsf{CM}}}) \to \mathfrak{gl}(\alpha)$ be the moment map, and $\mathscr{Z}_{\mathsf{CM}} = \mu_{\mathsf{CM}}^{-1}(0)$ the scheme theoretic inverse image of the point 0. It was proved in [GG2, Theorem 1.3.1] that $\mathscr{Z}_{\mathsf{CM}}$ is a reduced scheme. Now, there are natural algebra morphisms

$$(\mathbf{7.2.1}) \qquad \mathbf{C}[\mathscr{Z}]^{\mathrm{G}} \xleftarrow{f} \mathbf{C}[\mathscr{Z}_1]^{\mathrm{G}} \xrightarrow{g} \mathbf{C}[\mathscr{Z}_{\mathsf{CM}}]^{\mathrm{G}}.$$

By Theorem 7.1.1, f is an isomorphism. The following proposition and its proof is a straightforward generalization of [GG2, Proposition 2.8.2], given our Theorem 7.1.1.

Proposition **7.2.2.** — *The morphism* g *in* (7.2.1) *is an isomorphism.* □

From Proposition 7.2.2 and [GG2, Theorem 1.3.1], we have the following generalization of [GG2, Theorem 1.2.1].

Theorem **7.2.3.** — *One has:* $\mathrm{J}^{\mathrm{G}} = \sqrt{\mathrm{J}}^{\mathrm{G}}$. □

Let $\mathscr{Z}^{red} := \operatorname{Spec} \mathbf{C}[\mathsf{Rep}_{n\delta}(\overline{\mathrm{Q}})]/\sqrt{\mathrm{J}}$, a closed subvariety of $\mathsf{Rep}_{n\delta}(\overline{\mathrm{Q}})$. Define an embedding $\jmath : \mathrm{L}^n_{\mathrm{reg}} \to \mathsf{Rep}_{n\delta}(\overline{\mathrm{Q}})$ by $\jmath(u_1, \ldots, u_n)_a = (\phi_a(u_1), \ldots, \phi_a(u_n))$ for any $a \in \overline{\mathrm{Q}}$. Using formulas (8.2.1) from Section 8.2 below, we deduce that the image of $\jmath$ lies in $\mathscr{Z}^{red}$. Pullback of functions gives a morphism

$$(\mathbf{7.2.4}) \qquad \jmath^* : \mathbf{C}[\mathscr{Z}^{red}]^{\mathrm{G}} \to \mathbf{C}[\mathrm{L}^n_{\mathrm{reg}}]^{\Gamma_n}.$$

By [CB, Theorem 3.4] and [Kr, Corollary 3.2], we have the following proposition.

Proposition **7.2.5.** — *The map* $\jmath^*$ *in* (7.2.4) *is an isomorphism.* □

8. Proof of Proposition 3.7.2

8.1. — The formula of Proposition 3.7.2(i) is clear. Next, we have

$$\theta^{\mathrm{Holland}}(a^*) = \sum_{p,q} e^a_{q,p} \otimes \frac{\partial}{\partial t^a_{p,q}}.$$

To compute the restriction of $e^a_{q,p} \otimes \frac{\partial}{\partial t^a_{p,q}}$ to $\jmath(\mathrm{L}^n_{\mathrm{reg}})$ at a point $\mathbf{u} = (u_1, \ldots, u_n) \in \mathrm{L}^n_{\mathrm{reg}}$, let $g_{p,q}(\varepsilon) = \mathtt{Id} + \varepsilon \mathbf{B}_{p,q}$ be an element of $\mathrm{GL}(\alpha)$ such that

(**8.1.1**) $$g_{p,q}(\varepsilon) \cdot \big(\jmath(\mathbf{u}) + \varepsilon e^a_{p,q}\big) = \jmath(\mathbf{u}) + \varepsilon \jmath(\mathbf{w}), \quad \mathbf{w} \in \mathrm{L}^n_{\mathrm{reg}},$$

where we omit terms of higher order in ε. Then for a function $f \in \mathscr{O}(\chi, \mathrm{N})$, we have

(**8.1.2**) $$\begin{aligned} e^a_{q,p} \otimes \frac{\partial}{\partial \varepsilon} f\big(\jmath(\mathbf{u}) + \varepsilon e^a_{p,q}\big) &= e^a_{q,p} \otimes \frac{\partial}{\partial \varepsilon} f\big(g_{p,q}(\varepsilon)^{-1} \cdot (\jmath(\mathbf{u}) + \varepsilon \jmath(\mathbf{w}))\big) \\ &= e^a_{q,p} \otimes \frac{\partial}{\partial \varepsilon} \chi(g_{p,q}(\varepsilon)) g_{p,q}(\varepsilon)^{-1} f(\jmath(\mathbf{u}) + \varepsilon \jmath(\mathbf{w})) \\ &= e^a_{q,p} \otimes \frac{\partial}{\partial \mathbf{w}} f(\jmath(\mathbf{u})) + e^a_{q,p} \otimes \Big(\sum_{j \in \mathrm{I}} \chi_j \operatorname{Tr}\big(\mathbf{B}^{(j)}_{p,q}\big) \mathtt{Id} - \mathbf{B}_{p,q}\Big) f(\jmath(\mathbf{u})) \end{aligned}$$

where $\mathbf{B}^{(j)}_{p,q}$ is the component of $\mathbf{B}_{p,q}$ in $\mathfrak{gl}(\alpha_j)$. We shall write $\mathbf{B}^{(j)}_{p,q}$ for $j \in \mathrm{I}$ as a $n \times n$ block matrix $\oplus_{1 \le \ell, m \le n} \mathbf{B}^{(j)}_{p,q}(\ell, m)$ where $\mathbf{B}^{(j)}_{p,q}(\ell, m) \in \mathfrak{gl}(\delta_j)$ is the (ℓ, m)-th block. Similarly, we write $e^a_{p,q}$ as $\oplus_{1 \le \ell, m \le n} e^a_{p,q}(\ell, m)$. By (8.1.1), we need to solve the equations:

$$\mathbf{B}^{(h(a))}_{p,q}(\ell, m) \phi_a(u_m) - \phi_a(u_\ell) \mathbf{B}^{(t(a))}_{p,q}(\ell, m) + e^a_{p,q}(\ell, m) = \begin{cases} 0 & \text{if } \ell \neq m \\ \phi_a(w_\ell) & \text{if } \ell = m \end{cases}$$

$$\sum_{m=1}^{n} \mathbf{B}^{(o)}_{p,q}(\ell, m) - \mathbf{B}^{(s)}_{p,q} = 0.$$

where $1 \le \ell, m \le n$. We shall set $\mathbf{B}^{(s)}_{p,q} = 0$.

Suppose $(\ell - 1)\delta_{h(a)} < p \le \ell \delta_{h(a)}$ and $(m-1)\delta_{t(a)} < q \le m \delta_{t(a)}$ where $\ell, m \in [1, n]$. If $\ell \neq m$, then we set $\mathbf{B}^{(j)}_{p,q}(\ell', m') = 0$ whenever $\ell' \neq m'$ and $(\ell', m') \neq (\ell, m)$. If $\ell = m$, then we set $\mathbf{B}^{(j)}_{p,q}(\ell', m') = 0$ whenever $\ell' \neq m'$.

8.2. *Proof of* (3.7.3)

First of all, it is immediate from (2.1.1) that

(**8.2.1**) $$\sum_{a \in \mathrm{Q};\, t(a) = i} \phi_{a^*}(u) \phi_a(w) = \delta_i \omega(w, u) \mathtt{Id}_{\mathrm{N}^*_i}, \quad \text{for each source } i \text{ in } \mathrm{Q};$$

$$\sum_{a \in \mathrm{Q};\, h(a) = j} \phi_a(w) \phi_{a^*}(u) = \delta_j \omega(w, u) \mathtt{Id}_{\mathrm{N}^*_j}, \quad \text{for each sink } j \text{ in } \mathrm{Q}.$$

Next, we find a collection of operators $\beta_i \in \mathrm{End}(\mathrm{N}_i^*)$ such that

(**8.2.2**) $$\phi_a(u)\beta_i - \beta_j\phi_a(w) = f_a$$

where $i = t(a), j = h(a)$, and $f_a : \mathrm{N}_i^* \to \mathrm{N}_j^*$ are given operators. We write the collection β_i as an element $\sum \beta_\gamma \gamma$ of $\mathbf{C}[\Gamma]$. Since $\gamma\phi_a(w) = \phi_a(\gamma w)\gamma$, we get

$$\sum \beta_\gamma \phi_a(u - \gamma w)\gamma = f_a, \qquad \sum \beta_\gamma \gamma \phi_a(\gamma^{-1}u - w) = f_a.$$

We multiply the first equation above by $\phi_{a^*}(u)$ on the left and add over all edges going out from i. Similarly, let us multiply the second equation above by $\phi_{a^*}(w)$ on the right and add over all edges going into j.

Using formulas (8.2.1), we obtain:

$$\delta_i \sum \beta_\gamma \omega(u, \gamma w)\gamma|_{\mathrm{N}_i^*} = \sum_{a\in \mathrm{Q};\, t(a)=i} \phi_{a^*}(u) f_a, \quad \text{for sources } i;$$

$$\delta_j \sum \beta_\gamma \omega(u, \gamma w)\gamma|_{\mathrm{N}_j^*} = \sum_{a\in \mathrm{Q};\, h(a)=j} f_a \phi_{a^*}(w), \quad \text{for sinks } j.$$

This implies that

(**8.2.3**) $$\beta_\gamma = \omega(u, \gamma w)^{-1}|\Gamma|^{-1}\sum_{a\in \mathrm{Q}} \Big(\mathrm{Tr}\,|_{\mathrm{N}^*_{h(a)}} \big(f_a\phi_{a^*}(w)\gamma^{-1}\big) + \mathrm{Tr}\,|_{\mathrm{N}^*_{t(a)}} \big(\phi_{a^*}(u) f_a \gamma^{-1}\big)\Big).$$

Hence, if $\ell \neq m$, for $\oplus_{j\in \mathrm{I}} \mathbf{B}^{(j)}_{p,q}(\ell, m)$ we get the expression

(**8.2.4**) $$\sum_{\gamma\in\Gamma} \omega(u_\ell, \gamma u_m)^{-1}|\Gamma|^{-1}\Big(\mathrm{Tr}\,|_{\mathrm{N}^*_{h(a)}} \big(e^a_{p,q}\phi_{a^*}(u_m)\gamma^{-1}\big) + \mathrm{Tr}\,|_{\mathrm{N}^*_{t(a)}} (\phi_{a^*}(u_\ell) e^a_{p,q}\gamma^{-1})\Big)\gamma$$

and so, for $\mathbf{B}^{(o)}_{p,q}(\ell, \ell) = -\mathbf{B}^{(o)}_{p,q}(\ell, m)$ we obtain the expression

$$-\sum_{\gamma\in\Gamma} \omega(u_\ell, \gamma u_m)^{-1}|\Gamma|^{-1}\Big(\mathrm{Tr}\,|_{\mathrm{N}^*_{h(a)}} \big(e^a_{p,q}\phi_{a^*}(u_m)\gamma^{-1}\big) + \mathrm{Tr}\,|_{\mathrm{N}^*_{t(a)}} \big(\phi_{a^*}(u_\ell) e^a_{p,q}\gamma^{-1}\big)\Big).$$

Thus, for all $j \in \mathrm{I}$, for $\mathbf{B}^{(j)}_{p,q}(\ell, \ell)$ we obtain the expression

$$-\sum_{\gamma\in\Gamma} \omega(u_\ell, \gamma u_m)^{-1}|\Gamma|^{-1}\Big(\mathrm{Tr}\,|_{\mathrm{N}^*_{h(a)}} \big(e^a_{p,q}\phi_{a^*}(u_m)\gamma^{-1}\big) + \mathrm{Tr}\,|_{\mathrm{N}^*_{t(a)}} \big(\phi_{a^*}(u_\ell) e^a_{p,q}\gamma^{-1}\big)\Big)\mathtt{Id}_{\delta_i\times\delta_i}.$$

It follows from the last formula and from (8.1.2) that for $\ell \neq m$, the (m, ℓ)-entry of the radial part of $\theta^{\mathrm{Holland}}(a^*)$ is

$$\sum_{p=(\ell-1)\delta_{h(a)}+1}^{\ell\delta_{h(a)}} \sum_{q=(m-1)\delta_{t(a)}+1}^{m\delta_{t(a)}} e^a_{q,p}(m,\ell)\Big(\sum_{j\in \mathrm{I}} \chi_j \operatorname{Tr}\big(\mathbf{B}^{(j)}_{p,q}(\ell,\ell)\big) - \sum_{j\in \mathrm{I}} \mathbf{B}^{(j)}_{p,q}(\ell,\ell)\Big)$$
$$= |\Gamma|^{-1} \sum_{\gamma} \frac{(\phi_{a^*})_m \cdot \gamma^{-1} + \gamma^{-1}(\phi_{a^*})_\ell}{\omega(\gamma;\ell,m)} (1 - \sum_j \chi_j \delta_j)$$
$$= \frac{k}{2} \sum_{\gamma} \frac{(\phi_{a^*} \circ (\mathtt{Id} + \gamma^{-1}))_{m,\ell}}{\omega(\gamma;\ell,m)} \gamma^{-1}.$$

Note that, since ζ acts by -1 on L and by 1 on $\mathrm{N}^*_{h(a)}$,

$$\frac{(\phi_{a^*})_m}{\omega(\gamma\zeta;\ell,m)} (\gamma\zeta)^{-1} = -\frac{(\phi_{a^*})_m}{\omega(\gamma;\ell,m)} \gamma^{-1}$$

and so

$$\sum_{\gamma} \frac{(\phi_{a^*})_m}{\omega(\gamma;\ell,m)} \gamma^{-1} = 0.$$

Hence, the (m, ℓ)-entry of the radial part of $\theta^{\mathrm{Holland}}(a^*)$ is equal to

$$-\frac{k}{2} \sum_{\gamma} \frac{(\phi_{a^*} \circ \gamma)_\ell}{\omega(\gamma;m,\ell)} \gamma.$$

Proof of Lemma 3.3.1. — We set $f_a = 0$ in (8.2.2). Then from (8.2.3), we have $\omega(u, \gamma w)\beta_\gamma = 0$ for all $\gamma \in \Gamma$. Since not all β_γ are zero, we must have $\omega(u, \gamma w) = 0$ for some γ. □

8.3. *Proof of* (3.7.4)

We need to solve $\phi_a(u)\beta_i - \beta_j\phi_a(u) = f_a - \phi_a(w)$.

As above, for $\gamma \neq 1, \zeta$, we obtain

$$\beta_\gamma = \omega(u, \gamma u)^{-1} |\Gamma|^{-1} \sum_{a\in \mathrm{Q}} \Big(\operatorname{Tr}|_{\mathrm{N}^*_{h(a)}} \big(f_a \phi_{a^*}(u)\gamma^{-1}\big) + \operatorname{Tr}|_{\mathrm{N}^*_{t(a)}} \big(\phi_{a^*}(u) f_a \gamma^{-1}\big)\Big).$$

Moreover, multiplying on the right by $\phi_{a^*}(v)$ and summing over all incoming edges $a \in \mathrm{Q}$ at the vertex j, we get

$$\delta_j \sum \beta_\gamma \omega(\gamma^{-1}u - u, v)\gamma|_{\mathrm{N}^*_j} = \sum_{a\in \mathrm{Q}: h(a)=j} f_a \phi_{a^*}(v) - \delta_j \omega(w, v).$$

Take the trace of both sides this equation and sum up over all sinks j. We have $\oplus_{\mathrm{sink}\, j} (\mathrm{N}^*_j)^{\oplus \delta_j} = \mathbf{C}[\Gamma/\mathrm{S}]$, where $\mathrm{S} = \{1, \zeta\}$. It follows that the trace of γ in the last sum vanishes if $\gamma \neq 1, \zeta$. Let $\beta_\zeta = 0$. Then

$$(\mathbf{8.3.1})\qquad w = 2|\Gamma|^{-1}\sum_{a\in Q}\operatorname{Tr}(f_a\phi_{a^*}).$$

Hence, for $\ell = m$, we get

$$(\mathbf{8.3.2})\qquad \oplus_{j\in I}\mathbf{B}^{(j)}_{p,q}(m,m) = \sum_{\gamma\neq 1,\zeta}\omega(u_m,\gamma u_m)^{-1}|\Gamma|^{-1}\Big(\operatorname{Tr}|_{N^*_{h(a)}}\big(e^a_{p,q}(\phi_{a^*})_m\gamma^{-1}\big) + \operatorname{Tr}|_{N^*_{t(a)}}\big((\phi_{a^*})_m e^a_{p,q}\gamma^{-1}\big)\Big)(\gamma-1).$$

It follows from (8.1.2), (8.3.1), (8.3.2) and (8.2.4) that the (m,m)-entry of the radial part of $\theta^{\mathrm{Holland}}(a^*)$ is

$$(\mathbf{8.3.3})\qquad \frac{2}{|\Gamma|}\frac{\partial}{\partial(\phi_{a^*})_m} + \sum_{p=(m-1)\delta_{h(a)}+1}^{m\delta_{h(a)}}\ \sum_{q=(m-1)\delta_{t(a)}+1}^{m\delta_{t(a)}} e^a_{q,p}(m,m)\Big(\sum_{j\in I}\chi_j\operatorname{Tr}(\mathbf{B}^{(j)}_{p,q}(m,m))$$
$$-\sum_{j\in I}\mathbf{B}^{(j)}_{p,q}(m,m)\Big) - \sum_{\ell\neq m}\ \sum_{p=(\ell-1)\delta_{h(a)}+1}^{\ell\delta_{h(a)}}\ \sum_{q=(m-1)\delta_{t(a)}+1}^{m\delta_{t(a)}} e^a_{q,p}(m,\ell)\sum_{j\in I}\mathbf{B}^{(j)}_{p,q}(\ell,m)$$
$$= \frac{2}{|\Gamma|}\frac{\partial}{\partial(\phi_{a^*})_m} + \frac{1}{|\Gamma|}\sum_{\gamma\neq 1,\zeta}\frac{\gamma^{-1}\cdot(\phi_{a^*})_m + (\phi_{a^*})_m\cdot\gamma^{-1}}{\omega(\gamma;m,m)}\Big(-\gamma+1$$
$$-\sum_j\chi_j(\delta_j - \operatorname{Tr}|_{N^*_j}(\gamma))\Big) - \frac{1}{|\Gamma|}\sum_{\ell\neq m}\sum_\gamma\frac{(\phi_{a^*}\circ(\mathtt{Id}+\gamma^{-1}))_{m,\ell}}{\omega(\gamma;\ell,m)}$$
$$= \frac{2}{|\Gamma|}\frac{\partial}{\partial(\phi_{a^*})_m} + \frac{1}{|\Gamma|}\sum_{\gamma\neq 1,\zeta}\frac{(\phi_{a^*}\circ(\gamma^{-1}+\mathtt{Id}))_{mm}}{\omega(\gamma;m,m)}\big(-1+|\Gamma|c_\gamma\gamma^{-1}\big)$$
$$+\frac{1}{|\Gamma|}\sum_{\ell\neq m}\sum_\gamma\frac{(\phi_{a^*}\circ\gamma)_\ell}{\omega(\gamma;m,\ell)}.$$

The last term in (8.3.3) comes from (8.2.4).

It is even easier to compute the radial part for the edge $b: s\to o$. We omit this computation. This completes the proof of Proposition 3.7.2.

9. Proof of Theorem 4.3.2

9.1. — It easy to check that the operators R^v_{ml} have the following $\boldsymbol{\Gamma}_n$-equivariance properties:

$$\gamma_m R^v_{ml} = R^{\gamma(v)}_{ml}\gamma_m,\qquad \gamma_l R^v_{ml} = R^v_{ml}\gamma_l,$$
$$s_{ml}R^v_{ml} = R^v_{lm}s_{ml},\qquad s_{mj}R^v_{ml} = R^v_{jl}s_{mj},\qquad s_{lj}R^v_{ml} = R^v_{mj}s_{lj},$$

where $j\neq m, l$. It implies that $g\Theta^{\mathrm{Dunkl}}(v) = \Theta^{\mathrm{Dunkl}}g(v)$, for any $g\in\boldsymbol{\Gamma}_n$ and $v\in L^n_{\mathrm{reg}}$.

Next, we prove that

$$\textbf{(9.1.1)}\qquad \big[\Theta^{\mathrm{Dunkl}}(w_i),\,\Theta^{\mathrm{Dunkl}}(v_i)\big] = \omega(w,v)\Big(t\cdot 1+\frac{k}{2}\sum_{j\neq i}\sum_{\gamma\in\Gamma}s_{ij}\gamma_i\gamma_j^{-1}+\sum_{\gamma\in\Gamma\smallsetminus\{1\}}c_\gamma\gamma_i\Big),$$

where $1\le i\le n$.

First we prove

$$\textbf{(9.1.2)}\qquad \big[\mathrm{R}_{ij}^{w},\,\Theta_i^{\mathrm{Dunkl}}(v)\big]+\big[\Theta_i^{\mathrm{Dunkl}}(w),\,\mathrm{R}_{ij}^{v}\big]=\frac{1}{2}\omega(w,v)\sum_{j\neq i}\sum_{\gamma\in\Gamma}s_{ij}\gamma_i\gamma_j^{-1},\qquad 1\le i\neq j\le n.$$

We prove that (9.1.2) holds if we apply both sides to $\vec{f}$, a basis vector in $\mathrm{F}^{\otimes n}$ such that $f_i, f_j = f^+$, cf. (4.1.4). Indeed, in that case we have

$$\big[\mathrm{R}_{ij}^{w},\,\Theta_i^{\mathrm{Dunkl}}(v)\big](\vec{f})=\Big(\mathrm{R}_{ij}^{w}(\mathrm{D}^{v})_i+\Theta_i^{\mathrm{Dunkl}}(v)\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{(\gamma w)_j^{\vee}}{\omega(1;i,j)}\Big)(\vec{f})=\mathrm{A}\vec{f}$$

$$\big[\Theta_i^{\mathrm{Dunkl}}(w),\,\mathrm{R}_{ij}^{v}\big](\vec{f})=\Big(-\frac{1}{2}\Theta_i^{\mathrm{Dunkl}}(w)\sum_{\gamma\in\Gamma}\frac{(\gamma v)_j^{\vee}}{\omega(\gamma^{-1};i,j)}-\mathrm{R}_{ij}^{w}(\mathrm{D}^{w})_i\Big)(\vec{f}) = \mathrm{B}\vec{f},$$

where

$$\mathrm{A}=-\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{v_i^{\vee}(\gamma w)_j^{\vee}}{\omega(\gamma^{-1};i,j)}s_{ij}\gamma_i\gamma_j^{-1},\quad\text{resp.,}\quad \mathrm{B}=\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{w_i^{\vee}(\gamma v)_j^{\vee}}{\omega(\gamma^{-1};i,j)}s_{ij}\gamma_i\gamma_j^{-1}.$$

These formulas yield

$$\begin{aligned}\Big(\big[\mathrm{R}_{ij}^{w},\Theta_i^{\mathrm{Dunkl}}(v)\big]+\big[\Theta_i^{\mathrm{Dunkl}}(w),\,\mathrm{R}_{ij}^{v}\big]\Big)\vec{f}&=-\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{v_i^{\vee}(\gamma w)_j^{\vee}-w_i^{\vee}(\gamma v)_j^{\vee}}{\omega(\gamma^{-1};i,j)}s_{ij}\gamma_i\gamma_j^{-1}\vec{f}\\&=\frac{1}{2}\omega(w,v)\sum_{\gamma\in\Gamma}s_{ij}\gamma_i\gamma_j^{-1}\vec{f}.\end{aligned}$$

We consider the case $f_i=f^-, f_j=f^+$. Then we have:

$$\big[\mathrm{R}_{ij}^{w},\,\Theta_i^{\mathrm{Dunkl}}(v)\big]\vec{f}=-\mathrm{R}_{ij}^{w}v_i^{\vee}\vec{f}=\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{(\gamma w)_j^{\vee}}{\omega(\gamma^{-1};i,j)}v_i^{\vee}s_{ij}\gamma_i\gamma_j^{-1}\vec{f}=0,$$

because in the sum the terms corresponding γ and $\zeta\gamma$ mutually cancel. Analogously,

$$\big[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{ij}^{w}\big]\vec{f} = \mathrm{R}_{ij}^{v} w_i^{\vee}\vec{f} = -\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{(\gamma v)_j^{\vee}}{\omega(\gamma^{-1}; i,j)} s_{ij}\gamma_i\gamma_j^{-1} w_i^{\vee}\vec{f} = 0.$$

For the same reason we also have $\sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1}\vec{f} = 0$.

We consider the case $f_i = f^+, f_j = f^-$. A similar argument yields

$$\big[\mathrm{R}_{ij}^{w}, \Theta_i^{\mathrm{Dunkl}}(v)\big]\vec{f} = \big[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{ij}^{w}\big]\vec{f} = \sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1}\vec{f} = 0.$$

The case $f_i, f_j = f^-$ is analogous to the first case and we have

$$\big[\mathrm{R}_{ij}^{w}, \Theta_i^{\mathrm{Dunkl}}(v)\big]\vec{f} = \frac{1}{2}\sum_{\gamma\in\Gamma}\frac{w_i^{\vee}(\gamma v)_j^{\vee}}{\omega(\gamma^{-1}; i,j)}\vec{f},$$

$$\big[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{ij}^{v}\big]\vec{f} = -\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{v_i^{\vee}(\gamma w)_j^{\vee}}{\omega(\gamma^{-1}; i,j)} s_{ij}\gamma_i\gamma_j^{-1}\vec{f},$$

and

$$\begin{aligned}\Big(\big[\mathrm{R}_{ij}^{w}, \Theta_i^{\mathrm{Dunkl}}(v)\big] + \big[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{ij}^{v}\big]\Big)\vec{f} &\\ = -\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{w_i^{\vee}(\gamma v)_j^{\vee} - v_i^{\vee}(\gamma w)_j^{\vee}}{\omega(\gamma^{-1}; i,j)} s_{ij}\gamma_i\gamma_j^{-1}\vec{f} &\\ = \frac{1}{2}\omega(w,v)\sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1}\vec{f}.&\end{aligned}$$

We remark that for any $1 \le j \ne i \ne k \le n$ and $w, v \in \mathrm{L}$ we have $\mathrm{R}_{ij}^{w}\mathrm{R}_{ik}^{v} = \mathrm{R}_{ik}^{v}\mathrm{R}_{ij}^{w} = 0$. Now, (9.1.1) follows from this equation, the $n = 1$ case of Theorem 4.3.2, and (9.1.2).

9.2. — Next we prove:

(**9.2.1**) $$\big[\Theta^{\mathrm{Dunkl}}(w_i), \Theta^{\mathrm{Dunkl}}(v_m)\big] = -\frac{k}{2}\sum_{\gamma\in\Gamma}\omega_{\mathrm{L}}(\gamma u, v) s_{im}\gamma_i\gamma_m^{-1}, \quad 1 \le i \ne m \le n.$$

To this end, we rewrite the RHS of (9.1.1) as follows

$$\begin{aligned}\big[\Theta^{\mathrm{Dunkl}}(w_i), \Theta^{\mathrm{Dunkl}}(v_m)\big] &= k\Big(\big[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{mi}^{v}\big] + \big[\mathrm{R}_{im}^{w}\Theta_m^{\mathrm{Dunkl}}(v)\big]\Big)\\ &\quad + k^2\Big(\big[\mathrm{R}_{im}^{w}, \mathrm{R}_{mi}^{v}\big] + \sum_{j\ne m,i}\big[\mathrm{R}_{im}^{w}, \mathrm{R}_{mj}^{v}\big] + \big[\mathrm{R}_{ij}^{w}, \mathrm{R}_{mj}^{v}\big]\\ &\quad + \big[\mathrm{R}_{ij}^{w}, \mathrm{R}_{mi}^{v}\big]\Big).\end{aligned}$$

We first prove that

$$\left[\mathrm{R}^w_{im}, \mathrm{R}^v_{mj}\right] + \left[\mathrm{R}^w_{ij}, \mathrm{R}^v_{mj}\right] + \left[\mathrm{R}^w_{ij}, \mathrm{R}^v_{mi}\right] = 0, \quad j \neq m, i.$$

For that it is enough to show that

$$(\mathbf{9.2.2}) \qquad \mathrm{R}^w_{im}\mathrm{R}^v_{mj} - \mathrm{R}^v_{mj}\mathrm{R}^w_{ij} + \mathrm{R}^w_{ij}\mathrm{R}^v_{mi} = 0, \quad \text{and} \quad -\mathrm{R}^v_{mj}\mathrm{R}^w_{im} + \mathrm{R}^w_{ij}\mathrm{R}^v_{mj} - \mathrm{R}^v_{mi}\mathrm{R}^w_{ij} = 0.$$

We prove the first equation, the second is proved similarly. Let $\vec{f}$ be a basis vector in $\mathrm{F}^{\otimes n}$ such that $f_i, f_j, f_m = f^+$, cf. (4.1.4). We compute

$$\begin{aligned}
&4\big(\mathrm{R}^w_{im}\mathrm{R}^v_{mj} - \mathrm{R}^v_{mj}\mathrm{R}^w_{ij} + \mathrm{R}^w_{ij}\mathrm{R}^v_{mi}\big)(\vec{f})\\
&= \Bigg(\sum_{\beta,\gamma\in\Gamma} \frac{(w)^\vee_i(\gamma v)^\vee_j}{\omega(\beta^{-1}; i, m)\omega((\gamma\beta)^{-1}; i, j)} s_{im}s_{mj}\beta_i\gamma_m(\gamma\beta)^{-1}_j\\
&\quad - \sum_{\gamma,\beta\in\Gamma} \frac{(\beta v)^\vee_j(\beta^{-1}\gamma w)^\vee_m}{\omega(\beta^{-1}; m, j)\omega(\beta; i, m)} s_{mj}s_{ij}(\beta^{-1}\gamma)_i\beta_m\gamma^{-1}_j\\
&\quad + \sum_{\gamma,\beta\in\Gamma} \frac{(\beta w)^\vee_j(\beta\gamma v)^\vee_j}{\omega(\beta^{-1}; i, j)\omega((\beta\gamma)^{-1}; m, j)} s_{ij}s_{mi}\gamma^{-1}_i(\beta\gamma)_m\beta^{-1}_j \Bigg)\vec{f}.
\end{aligned}$$

We change summation indices at the first term as $\gamma \to \gamma\beta$, $\beta \to \gamma$ and at the third term as $\gamma \to \beta^{-1}$, $\beta \to \gamma\beta$. We get

$$\begin{aligned}
&\sum_{\gamma,\beta\in\Gamma} \Bigg(\frac{w^\vee_i(\gamma v)^\vee_j}{\omega(\beta^{-1}; i, m)\omega(\beta^{-1}; i, j)} - \frac{(\gamma v)^\vee_j(\beta w)^\vee_m}{\omega(\gamma^{-1}; m, j)\omega(\beta^{-1}; i, m)}\\
&\qquad + \frac{(\gamma\beta w)^\vee_j\gamma v^\vee_j}{\omega(\beta^{-1}; i, j)\omega(\gamma^{-1}; m, j)} \Bigg) \cdot s_{im}s_{mj}\beta_i\gamma_m(\beta^{-1}\gamma^{-1})_j\vec{f}\\
&= 4\Bigg(\sum_{\gamma,\beta\in\Gamma} \frac{\gamma v^\vee_j}{\omega(\beta^{-1}; i, m)\omega(\beta^{-1}; i, j)\omega(\gamma^{-1}; m, j)} \cdot \mathrm{Y}_{\beta,\gamma,m,j} \Bigg)\vec{f} = 0,
\end{aligned}$$

where $\mathrm{Y}_{\beta,\gamma,m,j}$ is given by the following expression

$$\begin{aligned}
\mathrm{Y}_{\beta,\gamma,m,j} = &\big(w^\vee_i\omega(\gamma^{-1}; m, j) - (\beta w)^\vee_m\omega(\beta^{-1}; i, j)\\
&+ (\gamma\beta w)^\vee_j\omega(\beta^{-1}; i, m)\big) \cdot s_{im}s_{mj}\beta_i\gamma_m(\beta^{-1}\gamma^{-1})_j.
\end{aligned}$$

We consider the case $f_i, f_m = f^+, f_j = f^-$:

$$4\left(\mathrm{R}^w_{im}\mathrm{R}^v_{mj} - \mathrm{R}^v_{mj}\mathrm{R}^w_{ij} + \mathrm{R}^w_{ij}\mathrm{R}^v_{mi}\right)(\vec{f})$$
$$= \Bigg(\sum_{\beta,\gamma\in\Gamma} \frac{(w)^\vee_i \beta^{-1} v^\vee_i}{\omega(\beta^{-1};\, i,\, m)\omega((\gamma\beta)^{-1};\, i, j)} s_{im}s_{mj}\beta_i\gamma_m(\gamma\beta)^{-1}_j$$
$$- \sum_{\gamma,\beta\in\Gamma} \frac{v^\vee_m w^\vee_i}{\omega(\beta^{-1};\, m, j)\omega(\gamma^{-1}\beta;\, i,\, m)} s_{mj}s_{ij}(\beta^{-1}\gamma)_i\beta_m\gamma_j^{-1}$$
$$+ \sum_{\gamma,\beta\in\Gamma} \frac{w^\vee_i \beta\gamma v^\vee_j}{\omega(\beta^{-1};\, i, j)\omega((\beta\gamma)^{-1};\, m, j)} s_{ij}s_{mi}\gamma_i^{-1}(\beta\gamma)_m\beta_j^{-1}\Bigg)\vec{f}.$$

Performing the same change of summation indices as in the previous paragraph, we deduce that the following sum vanishes

$$\sum_{\gamma,\beta\in\Gamma} \frac{w^\vee_i(\beta^{-1}v^\vee_i\omega(\gamma^{-1};\, m, j) - v^\vee_m\omega((\gamma\beta)^{-1};\, i, j) + \gamma v^\vee_j\omega(\beta^{-1};\, i,\, m))}{\omega(\beta^{-1};\, i,\, m)\omega((\gamma\beta)^{-1};\, i, j)\omega(\gamma^{-1};\, m, j)}$$
$$\times s_{im}s_{mj}\beta_i\gamma_m(\beta^{-1}\gamma^{-1})_j\vec{f}.$$

It easy to see that if $f_i = f^-$ or $f_m = f^-$ then $\mathrm{R}^w_{im}\mathrm{R}^v_{mj}\vec{f} = \mathrm{R}^v_{mj}\mathrm{R}^w_{ij}\vec{f} = \mathrm{R}^w_{ij}\mathrm{R}^v_{mi}\vec{f} = 0$. Thus, we have proved (9.2.2).

Now we prove that

$$\left[\mathrm{R}^w_{im}, \mathrm{R}^v_{mi}\right] = 0.$$

It is easy to see that $[\mathrm{R}^w_{im}, \mathrm{R}^v_{mi}]\vec{f} = 0$ when $f_i = f^-$ or $f_m = f^-$. If $f_i, f_m = f^+$ then $[\mathrm{R}^w_{im}, \mathrm{R}^v_{mi}]\vec{f} = 0$ is equivalent to:

$$\textbf{(9.2.3)} \qquad \sum_{\gamma,\beta\in\Gamma} \frac{w^\vee_i\beta\gamma v^\vee_m - v^\vee_m\beta^{-1}\gamma^{-1}w^\vee_i}{\omega(\beta^{-1};\, i,\, m)\omega((\beta\gamma\beta)^{-1};\, i,\, m)}(\beta\gamma)_m(\gamma\beta)^{-1}_i\vec{f} = 0.$$

Fix some $\gamma, \beta \in \Gamma$. Let $\alpha := \beta\gamma$, $\beta := \gamma\beta$. It is easy to see that the coefficient in front of $\alpha_m\beta_i^{-1}\vec{f}$ in (9.2.3) is equal to

$$\textbf{(9.2.4)} \qquad \left(w^\vee_i\alpha v^\vee_m - v^\vee_m\beta^{-1}w^\vee_i\right)\sum_{\delta\in \mathrm{Z}_\beta} \frac{1}{\omega(\delta\beta;\, i,\, m)\omega(\beta^{-1}\delta\beta;\, i,\, m)}$$

where Z_β is the notation for the centaralizer of $\beta \in \Gamma$. We notice that α is conjugate to β, hence if $\beta = 1$ or $\beta = \zeta$ then $\alpha = \beta = \beta^{-1}$ and (9.2.4) is zero.

Thus we can assume that $\beta \neq 1, \zeta$. In this case Z_β is a cyclic group. We denote the order of Z_β by l and let ρ be a generator of Z_β. Then we can assume that $\beta = \rho^q$ for some q, $0 < q < l$.

Let $a, b \in L$ be the basis in L such that $\rho a = \epsilon a$, $\rho b = \epsilon^{-1} b$ where ϵ is some primitive lth root of 1. Let $x = a^*$ and $y = b^*$. We make change of variables $\beta u_m \to u_m$ in (9.2.4). Let $z_{mi} = x_i y_m/(x_m y_i)$. Then, (9.2.4) is proportional to

$$\sum_{p=0}^{l-1} \frac{1}{\omega(u_i, \delta^p u_m)\omega(\delta^q u_i, \delta^p u_m)} = \frac{1}{(x_m y_i)^2} \sum_{p=0}^{l-1} \frac{1}{(\epsilon^{-p} - \epsilon^p z_{mi})(\epsilon^{-p+q} - \epsilon^{p-q} z_{mi})}$$

$$= \frac{1}{(\epsilon^q - \epsilon^{-q})(x_m x_i)^2} \sum_{p=0}^{l-1} \frac{1}{z_{mi} - \epsilon^{2q-2p}} - \frac{1}{z_{mi} - \epsilon^{-2p}} = 0.$$

Finally we show that

$$\left[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{mi}^v\right] + \left[\mathrm{R}_{im}^w \Theta_m^{\mathrm{Dunkl}}(v)\right] = -\frac{1}{2} \sum_{\gamma \in \Gamma} \omega_{\mathrm{L}}(\gamma w, v) s_{im} \gamma_i \gamma_m^{-1}.$$

If $f_i, f_m = f^{\pm}$ then the terms in the LHS of the sum below corresponding to γ and $\gamma\zeta$ mutually cancel out, and we deduce

$$-\frac{1}{2}\Big(\sum_{\gamma \in \Gamma} \omega_{\mathrm{L}}(\gamma w, v) s_{im} \gamma_i \gamma_m^{-1}\Big) \vec{f} = 0.$$

In the case $f_i, f_m = f^-$ we know that

$$\left[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{mi}^v\right]\vec{f} = \left[\mathrm{R}_{im}^w, \Theta_m^{\mathrm{Dunkl}}(v)\right]\vec{f} = 0.$$

In the case $f_i = f_m = f^+$ we have

$$\left[\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{mi}^v\right]\vec{f} = \frac{1}{2}\Bigg(\sum_{\gamma \in \Gamma} -(\mathrm{D}^w)_i \frac{\gamma v_i^\vee}{\omega(\gamma^{-1}; m, i)} s_{mi} \gamma_m \gamma_i^{-1} + \frac{v_m^\vee}{\omega(\gamma^{-1}; m, i)} (\mathrm{D}^{\gamma^{-1} w})_m s_{mi} \gamma_m \gamma_i^{-1}\Bigg)\vec{f},$$

$$\left[\mathrm{R}_{im}^w, \Theta_m^{\mathrm{Dunkl}}(v)\right]\vec{f} = \frac{1}{2}\Bigg(\sum_{\gamma \in \Gamma} -\frac{w_i^\vee}{\omega(\gamma^{-1}; i, m)} (\mathrm{D}^{\gamma^{-1} v})_i s_{im} \gamma_i \gamma_m^{-1} + (\mathrm{D}^v)_m \frac{\gamma w_m^\vee}{\omega(\gamma^{-1}; i, m)} s_{im} \gamma_i \gamma_m^{-1}\Bigg)\vec{f}.$$

Fix a conjugacy class $C \subset \Gamma$. Then the coefficient in front of $-\frac{1}{2} c_\beta$, $\beta \in C$ at $([\Theta_i^{\mathrm{Dunkl}}(w), \mathrm{R}_{mi}^v] + [\mathrm{R}_{im}^w \Theta_m^{\mathrm{Dunkl}}(v)])\vec{f}$ is equal to

$$\Bigg(\sum_{\gamma\in\Gamma,\beta\in\mathrm{C}}\frac{(\beta w_i^\vee+w_i^\vee)\beta\gamma v_i^\vee}{\omega(\beta;\,i,\,i)\omega(\gamma^{-1}\beta^{-1};\,m,\,i)}s_{mi}(\beta\gamma)_m(\gamma^{-1})_i$$
$$-\frac{v_m^\vee\left(\beta\gamma^{-1}w_m^\vee+\gamma^{-1}w_m^\vee\right)}{\omega(\gamma^{-1};\,m,\,i)\omega(\beta;\,m,\,m)}s_{mi}\gamma_m(\beta\gamma^{-1})_i$$
$$-\frac{(\beta v_m^\vee+v_m^\vee)\beta\gamma w_m^\vee}{\omega(\beta;\,m,\,m)\omega(\gamma^{-1}\beta^{-1};\,i,\,m)}s_{im}(\beta\gamma)_i(\gamma^{-1})_m$$
$$+\frac{w_i^\vee\left(\beta\gamma^{-1}v_i^\vee+\gamma^{-1}v_i^\vee\right)}{\omega(\gamma^{-1};\,i,\,m)\omega(\beta;\,i,\,i)}s_{im}\gamma_i(\beta\gamma^{-1})_m\Bigg)\vec{f}.$$

Fix $\gamma\in\Gamma$, $\beta\in\mathrm{C}$. Then the coefficient in front of $s_{im}(\beta\gamma)_m\gamma_i^{-1}$ is equal to

$$\mathrm{F}=\frac{(\beta w_i^\vee+w_i^\vee)\beta\gamma v_i^\vee}{\omega(\beta;\,i,\,i)\omega(\beta\gamma;\,m,\,i)}-\frac{v_m^\vee\left(\gamma^{-1}w_m^\vee+\gamma^{-1}\beta^{-1}w_m^\vee\right)}{\omega(\beta\gamma;\,m,\,i)\omega(\gamma^{-1}\beta\gamma;\,m,\,m)}$$
$$-\frac{\left(\gamma^{-1}\beta\gamma v_m^\vee+v_m^\vee\right)\gamma^{-1}w_m^\vee}{\omega(\gamma^{-1}\beta\gamma;\,m,\,m)\omega(\gamma;\,i,\,m)}+\frac{w_i^\vee(\beta\gamma v_i^\vee+\gamma v_i^\vee)}{\omega(\gamma^{-1};\,i,\,m)\omega(\beta;\,i,\,i)}.$$

We see that $\mathrm{F}=\mathrm{F}(u_i,u_m)$ is a homogeneous function in two variables, of bidegree $(-1,-1)$, that is $\mathrm{F}\in\mathrm{H}^0(\mathbf{P}\times\mathbf{P},\mathscr{O}(-1)\boxtimes\mathscr{O}(-1))$. It could have simple poles along the divisors $u_i\sim\beta u_i$, $u_m\sim\gamma^{-1}\beta\gamma u_m$, $u_i\sim\beta\gamma u_m$, $u_i\sim\gamma u_m$ where $\sim$ stands for being proportional. But is easy to check that the residues actually vanish. We deduce that $\mathrm{F}=0$.

The coefficient in front of $s_{mi}\gamma_m\gamma_i^{-1}\vec{f}$ in the part of $([\Theta_i^{\mathrm{Dunkl}}(w),\mathrm{R}_{mi}^v]+[\mathrm{R}_{im}^w\Theta_m^{\mathrm{Dunkl}}(v)])\vec{f}$ that does not contain coefficients c_γ, $\gamma\in\Gamma$, equals

$$-\frac{\partial}{\partial w_i}\frac{\gamma v_i^\vee}{\omega(\gamma^{-1};\,m,\,i)}+\frac{v_m^\vee}{\omega(\gamma^{-1};\,m,\,i)}\frac{\partial}{\partial(\gamma^{-1}w)_m}$$
$$-\frac{w_i^\vee}{\omega(\gamma^{-1};\,i,\,m)}\frac{\partial}{\partial(\gamma v)_i}+\frac{\partial}{\partial v_m}\frac{\gamma^{-1}w_m^\vee}{\omega(\gamma^{-1};\,i,\,m)}.$$

It is easy to show that this expression vanishes since we have

$$w_i^\vee\frac{\partial}{\partial(\gamma v)_i^\vee}-\gamma v_i^\vee\frac{\partial}{\partial w_i}=\gamma^{-1}w_m^\vee\frac{\partial}{\partial(\gamma v)_m^\vee}-v_m^\vee\frac{\partial}{\partial(\gamma^{-1}w)_m}=\omega(w,v)\mathsf{eu}.$$

We consider the case $f_i=f^-, f_m=f^+$. Then we have $[\mathrm{R}_{im}^w,\Theta_m(v)]\vec{f}=0$ and

$$\left[\Theta_i^{\mathrm{Dunkl}}(w),\mathrm{R}_{mi}^v\right]\vec{f}=\frac{1}{2}\sum_{\gamma\in\Gamma}\frac{w_i^\vee v_m^\vee-(\gamma v)_i^\vee(\gamma^{-1}w)_m^\vee}{\omega(\gamma^{-1};\,m,\,i)}s_{mi}\gamma_m\gamma_i^{-1}\vec{f}$$
$$=-\sum_{\gamma\in\Gamma}\omega(\gamma w,v)s_{mi}\gamma_i\gamma_m^{-1}\vec{f}.$$

The analysis of the case $f_i=f^+, f_m=f^-$ is similar.

REFERENCES

[BB] A. Beilinson and J. Bernstein, A proof of Jantzen conjectures, *I. M. Gelfand Seminar*, Adv. Soviet Math., vol. 16, part 1, pp. 1–50, Amer. Math. Soc., Providence, RI, 1993.

[BEG] Y. Berest, P. Etingof and V. Ginzburg, Cherednik algebras and differential operators on quasi-invariants, *Duke Math. J.*, **118** (2003), 279–337. math.QA/0111005.

[BFG] R. Bezrukavnikov, M. Finkelberg and V. Ginzburg, with an Appendix by P. Etingof, Cherednik algebras and Hilbert schemes in characteristic *p*, *Represent. Theory*, **10** (2006), 254–298. math.RT/0312474.

[Bo] M. Boyarchenko, Quantization of minimal resolutions of Kleinian singularities, *Adv. Math.*, **211** (2007), 244–265.

[CB] W. Crawley-Boevey, Decomposition of Marsden–Weinstein reductions for representations of quivers, *Compos. Math.*, **130** (2002), 225–239. math.AG/0007191.

[CBH] W. Crawley-Boevey and M. P. Holland, Noncommutative deformations of Kleinian singularities, *Duke Math. J.*, **92** (1998), 605–635.

[DO] C. Dunkl and E. Opdam, Dunkl operators for complex reflection groups, *Proc. London Math. Soc.*, **86** (2003), 70–108.

[E] P. Etingof, Cherednik and Hecke algebras of varieties with a finite group action, preprint. math.QA/0406499.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism, *Invent. Math.*, **147** (2002), 243–348. math.AG/0011114.

[EGR] P. Etingof, V. Ginzburg and E. Rains, in preparation.

[Ga] W. L. Gan, Reflection functors and symplectic reflection algebras for wreath products, *Adv. Math.*, **205** (2006), 599–630. math.RT/0502035.

[GG] W. L. Gan and V. Ginzburg, Deformed preprojective algebras and symplectic reflection algebras for wreath products, *J. Algebra*, **283** (2005), 350–363. math.QA/0401038.

[GG2] W. L. Gan and V. Ginzburg, Almost-commuting variety, D-modules, and Cherednik algebras, *IMPR, Int. Math. Res. Pap.*, **2006** (2006), Article ID 26439. math.RT/0409262.

[Go] I. Gordon, A remark on rational Cherednik algebras and differential operators on the cyclic quiver, *Glasg. Math. J.*, **48** (2006), 145–160. math.RT/0507413.

[GS] I. Gordon and J. T. Stafford, Rational Cherednik algebras and Hilbert schemes I, II, *Adv. Math.*, **198** (2005), 222–274 and *Duke Math. J.*, **132** (2006), 73–135. math.RA/0407516 and math.RT/0410293.

[Gu1] N. Guay, Cherednik algebras and Yangians, *Int. Math. Res. Not.*, **2005** (2005), 3551–3593.

[Gu2] N. Guay, Affine Yangians and deformed double current algebras in type A, *Adv. Math.*, **211** (2007), 436–484.

[Gu3] N. Guay, Quantum algebras and symplectic reflection algebras for wreath products, preprint.

[Ho] M. P. Holland, Quantization of the Marsden–Weinstein reduction for extended Dynkin quivers, *Ann. Sci. Éc. Norm. Supér., IV. Sér.*, **32** (1999), 813–834.

[Kr] P. B. Kronheimer, The construction of ALE spaces as hyper-Kahler quotients, *J. Differ. Geom.*, **29** (1989), 665–683.

[Lu1] G. Lusztig, Quivers, Perverse sheaves, and quantized enveloping algebras, *J. Amer. Math. Soc.*, **4** (1991), 365–421.

[Lu2] G. Lusztig, Quiver varieties and Weyl group actions, *Ann. Inst. Fourier*, **50** (2000), 461–489.

[Maf] A. Maffei, A remark on quiver varieties and Weyl groups, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (5), **1** (2002), 649–686. math.AG/0003159.

[Me] A. Mellit, Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation, *Algebra Discrete Math.*, **2004** (2004), 89–110.

[Mo] S. Montarani, Finite dimensional representations of symplectic reflection algebras associated to wreath products II, preprint. math.RT/0501156.

[Mu] I. Musson, Hilbert schemes and noncommutative deformations of type A Kleinian singularities, *J. Algebra*, **293** (2005), 102–129. math.RT/0504543.

[Na] H. Nakajima, Reflection functors for quiver varieties and Weyl group actions, *Math. Ann.*, **327** (2003), 671–721.

[Ob] A. Oblomkov, Deformed Harish–Chandra homomorphism for the cyclic quiver, preprint. math.RT/0504395.

[Ri] C. Ringel, The rational invariants of the tame quivers, *Invent. Math.*, **58** (1980), 217–239.

[Ve] J.-L. Verdier, Stratifications de Whitney et théorème de Bertini–Sard, *Invent. Math.*, **36** (1976), 295–312.

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