Harmonic almost contact structures via the intrinsic torsion

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For an oriented Riemannian manifold M of dimension n, given G a Lie subgroup of $SO(n)$, M is said to be equipped with a G-**structure**, if there exists a subbundle $\mathcal{G}(M)$, with structure group G, of the oriented orthonormal frame bundle $\mathcal{SO}(M)$.

For an oriented Riemannian manifold M of dimension n, given G a Lie subgroup of $SO(n)$, M is said to be equipped with a G-**structure**, if there exists a subbundle $\mathcal{G}(M)$, with structure group G, of the oriented orthonormal frame bundle $\mathcal{SO}(M)$.

G	dim M	name of the G-structure
$U(n)$	$2n$	almost Hermitian
$SU(n)$	$2n$	special almost Hermitian
$U(n) \times 1$	$2n+1$	almost contact metric
$Sp(n)$	$4n$	almost hyperHermitian
$Sp(n)\,Sp(1)$	$4n$	almost quaternion Hermitian
G_2	7	G_2-structure
$Spin(7)$	8	$Spin(7)$-structure

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$$(M, \langle\cdot,\cdot\rangle_M), \qquad (N, \langle\cdot,\cdot\rangle_N)$$

$$f : M \longrightarrow N$$

For M compact and oriented, the **energy** of f is given by:

$$\mathcal{E}(f) = \frac{1}{2}\int_M \|f_*\|^2 dv$$

$$\|f_*\|^2 = \langle f_* e_i, f_* e_i\rangle_N$$

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Tension field

$$\tau(f) = \widetilde{\nabla}_{e_i}(f_* e_i) - f_* \nabla_{e_i} e_i,$$

where $\widetilde{\nabla}$ is the induced connection by ∇^N on f^*TN

the pullback bundle $f^*TN = \{(m, \widetilde{X}), \quad m \in M \text{ and } \widetilde{X} \in \mathrm{T}_{f(m)}N\}$

f **harmonic map** if and only if $\tau(f) = 0$

J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.

$(M, \langle\cdot,\cdot\rangle)$ compact and oriented, $G \subseteq SO(n)$, G closed and connected

$$\mathcal{G}(M) \subseteq \mathcal{SO}(M)$$

The presence of a G-structure is equivalent to the presence of a section

$$\sigma : M \longrightarrow \mathcal{SO}(M)/G$$

$$\langle A, B\rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B\rangle + \langle \phi A, \phi B\rangle.$$

The **energy** of a G-structure σ

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \|\sigma_*\|^2 dv$$

C. M. Wood, Harmonic sections of homogeneous fibre bundles, *Differential Geom. Appl.* 19 (2003), 193-210
J. C. González-Dávila and FMC, Harmonic G-structures, *Math. Proc. Cambridge Philos. Soc.* (to appear). arXiv:math.DG/0706.0116

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}, \qquad \langle \varphi_1, \varphi_2 \rangle = \langle \varphi_1(u_i), \varphi_2(u_i) \rangle$$

$$(g.\varphi)(x) = g\varphi(g^{-1}x), \quad x \in \mathbb{R}^n$$

$$\mathfrak{so}(M) = \mathfrak{g}_\sigma \oplus \mathfrak{m}_\sigma,$$

G-**connection**: $\widetilde{\nabla}$, **torsion**: $\widetilde{\xi}_X = \widetilde{\nabla}_X - \nabla_X \in \mathfrak{so}(M)$

$$\widetilde{\xi}_X = (\widetilde{\xi}_X)_{\mathfrak{g}_\sigma} + (\widetilde{\xi}_X)_{\mathfrak{m}_\sigma}$$

minimal connection of σ: $\nabla^G_X = \widetilde{\nabla}_X - (\tilde{\xi}_X)_{\mathfrak{g}_\sigma}$

intrinsic torsion of σ: $\xi^G_X = (\widetilde{\xi}_X)_{\mathfrak{m}_\sigma} = \nabla^G_X - \nabla_X \in \mathfrak{m}_\sigma$

For a G-structure σ:

minimal connection ∇^G, intrinsic torsion $\xi^G \in T^*M \otimes \mathfrak{m}_\sigma$

$$\nabla^G = \nabla + \xi^G,$$

S. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Math. Series, **201**, Longman (1989).

R. L. Bryant, Metrics with expceptional holonomy, *Ann. of Math.* **126** (1987), 525–576.

R. Cleyton and A. F. Swann, Einstein metrics via intrinsic or parallel torsion, *Math. Z.* 247 no. 3(2004), 513–528.

$\mathcal{SO}(M)/G$ as a Riemannian manifold

$$\mathcal{SO}(M) \xrightarrow{\pi_{SO(n)}} M \qquad\qquad \mathcal{SO}(M) \xrightarrow{\pi_G} \mathcal{SO}(M)/G$$

$$\mathrm{T}\mathcal{SO}(M)/G = \mathcal{V} \oplus \mathcal{H}$$

$$\mathcal{V} = \pi_{G*}(\ker \pi_{SO(n)*}) \qquad \mathcal{H} = \pi_{G*}(\ker \omega)$$

$\omega : \mathrm{T}\mathcal{SO}(M) \to \mathfrak{so}(n)$ is the connection one-form of the Levi Civita connection ∇

$$\mathcal{SO}(M)/G \xrightarrow{\pi} M$$

$$\pi^*\mathfrak{so}(M) = \mathcal{SO}(M) \times_G \mathfrak{so}(n) = \mathfrak{g}_{\mathcal{SO}(M)} \oplus \mathfrak{m}_{\mathcal{SO}(M)},$$

where $\mathfrak{g}_{\mathcal{SO}(M)} = \mathcal{SO}(M) \times_G \mathfrak{g}$ and $\mathfrak{m}_{\mathcal{SO}(M)} = \mathcal{SO}(M) \times_G \mathfrak{m}$.

A fibred metric on $\pi^*\mathfrak{so}(M)$ is defined by

$$\langle (pG, \varphi_m), (pG, \psi_m) \rangle = \langle \varphi_m, \psi_m \rangle$$

$$\phi_{|\mathcal{V}_{pG}} : \mathcal{V}_{pG} \longrightarrow \left(\mathfrak{m}_{\mathcal{SO}(M)}\right)_{pG}$$

Any vector in $\mathcal{V}_{pG}$ is given by $\pi_{G*p}(a^*_p)$, for some $a = (a_{ji}) \in \mathfrak{m}$

$$\phi_{|\mathcal{V}_{pG}}(\pi_{G*p}(a^*_p)) = (pG, a_{ji}\, p(u_i)^\flat \otimes p(u_j))$$

p is an orthonormal frame on $m \in M$, $p : \mathbb{R}^n \to \mathrm{T}_m M$, and $u_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$.
Extending ϕ to $T_{pG}\mathcal{SO}(M)/G$, by saying $\phi_{|\mathcal{H}_{pG}} = 0$, one can define

$$\langle A, B\rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B\rangle + \langle \phi A, \phi B\rangle$$

$\pi : \mathcal{SO}(M)/G \to M$ is a Riemannian submersion with totally geodesic fibres [Vilms] (Besse's book)

$$\langle A, B\rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B\rangle + \langle \phi A, \phi B\rangle$$

$\pi : \mathcal{SO}(M)/G \to M$ is a Riemannian submersion with totally geodesic fibres

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \|\sigma_*\|^2 dv = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\phi\,\sigma_*\|^2 dv$$

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$$\phi\,\sigma_* = -\xi^G$$

Total bending

$$B(\sigma) = \frac{1}{2}\int_M \|\phi\,\sigma_*\|^2 dv = \frac{1}{2}\int_M \|\xi^G\|^2 dv$$

$t \to \sigma_t \in \Gamma^\infty(\mathcal{SO}(M)/G)$ variation such that $\sigma_0 = \sigma$
variation field $m \to \varphi(m) = \frac{d}{dt}_{|t=0}\sigma_t(m)$, $\varphi \in \Gamma^\infty(\sigma^*\mathcal{V})$.
Therefore, $\Gamma^\infty(\sigma^*\mathcal{V}) \cong \mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$

$$\sigma^*\mathcal{V} \cong \sigma^*\mathfrak{m}_{\mathcal{SO}(M)} \cong \mathfrak{m}_\sigma$$

Then $\mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G) \cong \Gamma^\infty(\mathfrak{m}_\sigma)$

The first variation formula

If σ is a G-structure, then, for all
$\varphi \in \Gamma^\infty(\mathfrak{m}_\sigma) \cong \mathrm{T}_\sigma\Gamma^\infty(\mathcal{SO}(M)/G)$, we have

$$d\mathcal{E}_\sigma(\varphi) = -\int_M \langle \xi^G, \nabla\varphi\rangle dv = -\int_M \langle d^*\xi^G, \varphi\rangle dv,$$

where ξ^G is the intrinsic torsion of σ.

Harmonic G-structures

The coderivative $d^*\xi^G$ is a global section of $\mathfrak{m}_\sigma$ and is given by

$$d^*\xi^G = -(\nabla_{e_i}\xi^G)_{e_i} = -(\nabla^G_{e_i}\xi^G)_{e_i} - \xi^G_{\xi^G_{e_i}e_i}.$$

the following conditions are equivalent:

(i) σ is a critical point for the energy functional on $\Gamma^\infty(\mathcal{SO}(M)/G)$.

(ii) $d^*\xi^G = 0$.

(iii) $(\nabla^G_{e_i}\xi^G)_{e_i} = -\xi^G_{\xi^G_{e_i}e_i}$.

$$(\nabla_X \sigma_*)(Y) = \nabla^q_X \sigma_* Y - \sigma_*(\nabla_X Y),$$

If σ is a G-structure on $(M, \langle\cdot,\cdot\rangle)$, then:

(a) $\phi(\nabla_X \sigma_*)Y = -\frac{1}{2}\left((\nabla_X \xi^G)_Y + (\nabla_Y \xi^G)_X\right)$.

(b) $2\langle \pi_*(\nabla_X \sigma_*)Y, Z\rangle = \langle \xi^G_X, R(Y,Z)\rangle + \langle \xi^G_Y, R(X,Z)\rangle$.

$$\tau(\sigma) = (\nabla_{e_i}\sigma_*)(e_i), \quad \pi_*\tau(\sigma) = \langle \xi^G_{e_i}, R(e_i,\cdot)\rangle^\sharp, \quad \phi\tau(\sigma) = d^*\xi^G$$

A G-structure σ on a closed and oriented Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is harmonic as a map if and only if σ is a harmonic G-structure such that

$$\langle \xi^G_{e_i}, R(e_i,\cdot)\rangle = 0.$$

$$\nabla^*\nabla\Psi = -\left(\nabla^2\Psi\right)_{e_i,e_i}, \qquad (\nabla^2\Psi)_{X,Y} = \nabla_X(\nabla_Y\Psi) - \nabla_{\nabla_X Y}\Psi$$

Let $(M, \langle\cdot,\cdot\rangle)$ be an oriented Riemannian n-manifold equipped with a G-structure, where the Lie group G is closed, connected and $G \subseteq SO(n)$. If Ψ is a (r,s)-tensor field on M which is stabilised under the action of G, then

$$\nabla^*\nabla\Psi = (\nabla^G_{e_i}\xi^G)_{e_i}\Psi + \xi^G_{\xi^G_{e_i}e_i}\Psi - \xi^G_{e_i}(\xi^G_{e_i}\Psi).$$

Moreover, if the G-structure is harmonic, then

$$\nabla^*\nabla\Psi = -\xi^G_{e_i}(\xi^G_{e_i}\Psi).$$

An almost contact metric manifold $(M^{2n+1}, \langle\cdot,\cdot\rangle, \varphi, \zeta)$

$$\langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)$$
$$\varphi^2 = -I + \eta\otimes\zeta, \qquad \zeta^\flat = \eta$$

$$G = U(n)\times 1 \subseteq SO(2n+1), \qquad T^*_m M = \mathbb{R}\eta + \eta^\perp$$

$$\mathfrak{so}(2n+1) \cong \Lambda^2 T^*M \cong \Lambda^2\eta^\perp + \eta^\perp\wedge\mathbb{R}\eta = \mathfrak{u}(n) + \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp\wedge\mathbb{R}\eta$$

$$\mathfrak{u}(n)^\perp = \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp\wedge\mathbb{R}\eta$$

$$T^*M\otimes\mathfrak{u}(n)^\perp = \eta^\perp\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp\otimes\eta^\perp\wedge\eta + \eta\otimes\eta^\perp\wedge\eta$$

$$T^*M \otimes \mathfrak{u}(n)^\perp = \eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \otimes \eta^\perp \wedge \eta + \eta \otimes \eta^\perp \wedge \eta$$

$$\begin{aligned} \eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \text{ (Gray\&Hervella's modules)} \\ \eta^\perp \otimes \eta^\perp \wedge \eta &= \mathcal{C}_5 + \mathcal{C}_8 + \mathcal{C}_9 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_{10} \\ \eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_{11} \\ \eta \otimes \eta^\perp \wedge \eta &= \mathcal{C}_{12} \end{aligned}$$

Fundamental two-form, $F = \langle \cdot, \varphi \cdot \rangle$

$$\xi^{U(n)} \to -\xi^{U(n)}F = \nabla^{U(n)}F - \xi^{U(n)}F = \nabla F,$$

D. Chinea and J. C. González-Dávila, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.* (4) 156 (1990), 15–36.

1. if $n = 1$, $\xi^{U(1)} \in \mathrm{T}^*M \otimes \mathfrak{u}(1)^\perp = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{12}$;
2. if $n = 2$, $\xi^{U(2)} \in \mathrm{T}^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \cdots \oplus \mathcal{C}_{12}$;
3. if $n \geqslant 3$, $\xi^{U(n)} \in \mathrm{T}^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{12}$.

$$\begin{aligned} \xi_X^{U(n)} &= -\tfrac{1}{2}\varphi \circ \nabla_X \varphi + \nabla_X \eta \otimes \zeta - \tfrac{1}{2}\eta \otimes \nabla_X \zeta \\ &= \tfrac{1}{2}(\nabla_X \varphi) \circ \varphi + \tfrac{1}{2}\nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta \end{aligned}$$

if the almost contact structure is of type $\mathcal{C}_5 \oplus \cdots \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$, then

$$\xi_X^{U(n)} = \nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta$$

$$\mathrm{Ric}^{\mathrm{ac}}(X, Y) = \langle R_{e_i, X} \varphi e_i, \varphi Y \rangle$$

$$\mathrm{Ric}^{\mathrm{ac}}(\varphi X, \varphi Y) = \mathrm{Ric}^{\mathrm{ac}}(Y_{\zeta^\perp}, X_{\zeta^\perp}), \qquad \mathrm{Ric}^{\mathrm{ac}}(X, \zeta) = 0$$

$$\mathrm{Ric}^{ac}(X,Y) = \langle R_{e_i,X}\varphi e_i, \varphi Y\rangle$$

Lemma

If $(M,\langle\cdot,\cdot\rangle,\varphi,\zeta)$ is an almost contact metric $2n+1$-manifold, then the almost contact Ricci curvature satisfies

$$\mathrm{Ric}^{ac}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = \langle (\nabla^{U(n)}_{e_i}\xi)_{\varphi e_i}\varphi X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle + \langle \xi_{\xi_{e_i}\varphi e_i}\varphi X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle,$$
$$\mathrm{Ric}^{ac}(\zeta, X) = ((\nabla^{U(n)}_{e_i}\xi)_{\varphi e_i}\eta)(\varphi X) + (\xi_{\xi_{e_i}\varphi e_i}\eta)(\varphi X),$$

for all $X, Y \in \mathfrak{X}(M)$. Furthermore, we have:

(i) *The restriction $\mathrm{Ric}^{ac}_{\mathrm{alt}|\zeta^\perp}$ of $\mathrm{Ric}^{ac}_{\mathrm{alt}}$ to the space $\zeta^\perp$ is in $\mathfrak{u}(n)^\perp_{|\zeta^\perp}$ and determines a $U(n)$-component of the Weyl curvature tensor W.*

(ii) *The one-form $\mathrm{Ric}^{ac}(\zeta,\cdot)$ is in $\eta^\perp$ and determines another $U(n)$-component of W.*

As a consequence, if an almost contact metric $2n+1$-manifold with $n>1$ is conformally flat, i.e. $W=0$, then $\mathrm{Ric}^{ac}_{\mathrm{alt}|\zeta^\perp}=0$ and $\mathrm{Ric}^{ac}(\zeta,\cdot)=0$, or equivalently, $\mathrm{Ric}^{ac}_{\mathrm{alt}}=0$.

$$d^2F = 0, \qquad (\nabla F)_{(i)} = (-\xi F)_{(i)} \to \xi_{(i)}$$

Lemma

For almost contact metric structures of type $\mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_{10}$, the following identity is satisfied

$$\begin{aligned}
0 = \quad & 3\langle(\nabla^{U(n)}_{e_i}\xi_{(1)})_{e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle - \langle(\nabla^{U(n)}_{e_i}\xi_{(3)})_{e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle \\
& +(n-2)\langle(\nabla^{U(n)}_{e_i}\xi_{(4)})_{e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle + \langle\xi_{(3)X_{\zeta^\perp}}e_i, \xi_{(1)e_i}Y_{\zeta^\perp}\rangle \\
& -\langle\xi_{(3)Y_{\zeta^\perp}}e_i, \xi_{(1)e_i}X_{\zeta^\perp}\rangle + \langle\xi_{(3)X_{\zeta^\perp}}e_i, \xi_{(2)e_i}Y_{\zeta^\perp}\rangle \\
& -\langle\xi_{(3)Y_{\zeta^\perp}}e_i, \xi_{(2)e_i}X_{\zeta^\perp}\rangle - \tfrac{n-5}{n-1}\langle\xi_{(1)\xi_{(4)e_i}e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle \\
& -\tfrac{n-2}{n-1}\langle\xi_{(2)\xi_{(4)e_i}e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle + \langle\xi_{(3)\xi_{(4)e_i}e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle \\
& +(n-2)(\xi_{(5)\,e_i}\eta)\wedge(\xi_{(10)\,e_i}\eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) - 2(\xi_{(8)\,e_i}\eta)\wedge(\xi_{(10)\,e_i}\eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) \\
& +(n-2)(\xi_{(6)\,e_i}\eta)\wedge(\xi_{(10)\,e_i}\eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) - 2(\xi_{(7)\,e_i}\eta)\wedge(\xi_{(10)\,e_i}\eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}).
\end{aligned}$$

[A. Swann and FMC]

Lemma

For almost contact metric manifolds of type $\mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_{10}$, the following identity is satisfied

$$\begin{aligned} 0 = \quad & -\left((\nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i}\eta\right)(Y_{\zeta_\perp}) - 3\left((\nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i}\eta\right)(Y_{\zeta_\perp}) \\ & -3\left((\nabla^{U(n)}_{e_i}\xi_{(7)})_{e_i}\eta\right)(Y_{\zeta_\perp}) - \left((\nabla^{U(n)}_{e_i}\xi_{(8)})_{e_i}\eta\right)(Y_{\zeta_\perp}) \\ & +3\left((\nabla^{U(n)}_{e_i}\xi_{(9)})_{e_i}\eta\right)(Y_{\zeta_\perp}) + \left((\nabla^{U(n)}_{e_i}\xi_{(10)})_{e_i}\eta\right)(Y_{\zeta_\perp}) \\ & -(\xi_{(6)e_i}\eta)(\xi_{(1)e_i}Y_{\zeta_\perp}) - (\xi_{(7)e_i}\eta)(\xi_{(1)e_i}Y_{\zeta_\perp}) - (\xi_{(10)e_i}\eta)(\xi_{(1)e_i}Y_{\zeta_\perp}) \\ & -(\xi_{(6)e_i}\eta)(\xi_{(2)e_i}Y_{\zeta_\perp}) - (\xi_{(7)e_i}\eta)(\xi_{(2)e_i}Y_{\zeta_\perp}) + (\xi_{(10)e_i}\eta)(\xi_{(2)e_i}Y_{\zeta_\perp}) \\ & +(\xi_{(5)e_i}\eta)(\xi_{(3)e_i}Y_{\zeta_\perp}) + (\xi_{(8)e_i}\eta)(\xi_{(3)e_i}Y_{\zeta_\perp}) + (\xi_{(9)e_i}\eta)(\xi_{(3)e_i}Y_{\zeta_\perp}) \\ & +\tfrac{n}{n-1}(\xi_{(8)\xi_{(4)e_i}e_i}\eta)(Y_{\zeta_\perp}) - (\xi_{(10)\xi_{(4)e_i}e_i}\eta)(Y_{\zeta_\perp}). \end{aligned}$$

In particular, if the almost contact manifold is of type $\mathcal{C}_5 \oplus \cdots \oplus \mathcal{C}_{10}$, then

$$\begin{aligned} 0 &= (n-1)(\nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i} + 2(\nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i} + 2(\nabla^{U(n)}_{e_i}\xi_{(7)})_{e_i} \\ &\quad -(\nabla^{U(n)}_{e_i}\xi_{(8)})_{e_i} - 2(\nabla^{U(n)}_{e_i}\xi_{(9)})_{e_i} + (\nabla^{U(n)}_{e_i}\xi_{(10)})_{e_i}, \\ 0 &= (n-2)(\nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i} - (\nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i} - (\nabla^{U(n)}_{e_i}\xi_{(7)})_{e_i} \\ &\quad -2(\nabla^{U(n)}_{e_i}\xi_{(8)})_{e_i} + (\nabla^{U(n)}_{e_i}\xi_{(9)})_{e_i} + 2(\nabla^{U(n)}_{e_i}\xi_{(10)})_{e_i}. \end{aligned}$$

Example: $\mathcal{C}_5 \oplus \mathcal{C}_6$ (trans-Sasakian)

$$\left.\begin{aligned} 0 &= (n-1)(\nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i} + 2(\nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i}, \\ 0 &= (n-2)(\nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i} - (\nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i}, \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \nabla^{U(n)}_{e_i}\xi_{(5)})_{e_i} &= 0, \\ \nabla^{U(n)}_{e_i}\xi_{(6)})_{e_i} &= 0 \end{aligned}$$

$$\xi_{e_i}e_i = b\zeta, \qquad \xi_{\xi_{e_i}e_i} = b\xi_\zeta = 0$$

If $(M, \langle\cdot,\cdot\rangle, \varphi, \zeta)$ is an almost contact metric $2n+1$-manifold with fundamental two-form F, then the following conditions are equivalent:

1. The almost contact metric structure is harmonic.
2. $\nabla^*\nabla\varphi \in \mathfrak{u}(n) + \zeta_c^\perp$, where $\zeta_c^\perp = \{a \otimes \zeta - \eta \otimes a^\sharp \mid a \in \eta^\perp\} \cong \eta^\perp \wedge \eta$, and $\nabla^*\nabla\zeta = -\xi_{e_i}\xi_{e_i}\zeta$.
3. $\nabla^*\nabla F \in \mathfrak{u}(n) + \eta^\perp \wedge \eta$, i.e. $\nabla^*\nabla F(\varphi X, \varphi Y) = \nabla^*\nabla F(X_{\zeta^\perp}, Y_{\zeta^\perp})$, and $\nabla^*\nabla\eta = -\xi_{e_i}(\xi_{e_i}\eta)$.
4. For all $X, Y \in \mathfrak{X}(M)$, we have:

$$\langle(\nabla^{U(n)}_{e_i}\xi)_{e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle + \langle\xi_{\xi_{e_i}e_i}X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle = 0,$$
$$(\nabla^{U(n)}_{e_i}\xi)_{e_i}\eta + \xi_{\xi_{e_i}e_i}\eta = 0.$$

In particular, if the structure is of type $\mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$, then it is harmonic if and only if $\nabla^*\nabla\zeta = \|\nabla\zeta\|^2\zeta$, that is, ζ is a harmonic unit vector field.

E. Vergara-Díaz and C. M. Wood, Harmonic Almost Contact Structures, *Geom. Dedicata* 123 (2006), 131–151.

Theorem

For an almost contact metric $2n+1$-manifold $(M, \langle\cdot,\cdot\rangle, \varphi, \zeta)$, we have:

(i) *If M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact structure is harmonic if and only if $\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 0$ and $\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 0$, for all $X, Y \in \mathfrak{X}(M)$.*

(ii) *For $n \neq 2$, if M is of type $\mathcal{C}_1 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, then the almost contact metric structure is harmonic if and only if*

$$(n-1)(n-5)\,\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2(n+1)(n-3)\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle,$$
$$\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i}\eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(iii) For $n \neq 2$, if M is of type $\mathcal{C}_1 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then almost contact metric structure is harmonic if and only if

$$(n-1)(n-5)\,\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2(n+1)(n-3)\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle,$$
$$\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i}\eta)(X_{\zeta^\perp}),$$

for all $X, Y \in \mathfrak{X}(M)$.

(iv) For $n \neq 2$, if M is of type $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, then the almost contact metric structure is harmonic if and only if

$$(n-1)\,\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2n\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle,$$
$$\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i}\eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(v) For $n \neq 2$, if M is of type $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is harmonic if and only if, for all $X, Y \in \mathfrak{X}(M)$,

$$\tfrac{n-1}{2n}\,\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle, \quad \mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i}\eta)(X).$$

(vi) If M is normal ($\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$), then the almost contact metric structure is harmonic if and only if

$$\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = -\, 2\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle, \qquad \mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i}\eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(vii) If M is of type $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is harmonic if and only if

$$\mathrm{Ric}^{\mathrm{ac}}_{\mathrm{alt}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = -\, 2\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp}\rangle, \qquad \mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i}\eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(viii) If M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_1 \oplus \mathcal{C}_6 \oplus \mathcal{C}_8$, then the almost contact structure is harmonic if and only if $\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 0$, for all $X \in \mathfrak{X}(M)$.

(ix) For $n \neq 2$, if M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_4 \oplus \mathcal{C}_8$, then the almost contact structure is harmonic if and only if $\mathrm{Ric}^{\mathrm{ac}}(\zeta, X) = 0$, for all $X \in \mathfrak{X}(M)$.

In particular:

(i)* If M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_5$, $\mathcal{C}_1 \oplus \mathcal{C}_8$, $\mathcal{C}_1 \oplus \mathcal{C}_9$, $\mathcal{C}_3 \oplus \mathcal{C}_6$, $\mathcal{C}_3 \oplus \mathcal{C}_7$, $\mathcal{C}_3 \oplus \mathcal{C}_{10}$, $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7$, $\mathcal{C}_5 \oplus \mathcal{C}_8$, $\mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_5 \oplus \mathcal{C}_{10}$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10}$, $\mathcal{C}_8 \oplus \mathcal{C}_9$, $\mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact structure is harmonic.

(ii)* For $n \neq 2$, if M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5$, $\mathcal{C}_4 \oplus \mathcal{C}_6$, $\mathcal{C}_4 \oplus \mathcal{C}_7$, $\mathcal{C}_4 \oplus \mathcal{C}_9$, then the almost contact structure is harmonic.

Corollary

(a) *If an almost contact metric structure is of type* $\mathcal{D}$, *for* $\mathcal{D} = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7$, $\mathcal{C}_5 \oplus \mathcal{C}_8$, $\mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_5 \oplus \mathcal{C}_{10}$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10}$, $\mathcal{C}_8 \oplus \mathcal{C}_9$, $\mathcal{C}_9 \oplus \mathcal{C}_{10}$, *then the characteristic vector field* ζ *is a harmonic unit vector field.*

(b) *For a locally conformally flat* $2n+1$*-manifold* $(M, \langle\cdot,\cdot\rangle)$ *with* $n > 1$*:*

   (i) *If an almost contact structure compatible with* $\langle\cdot,\cdot\rangle$ *is of type* $\mathcal{D}$, *where* $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, $\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, *then it is harmonic.*

   (ii) *For* $n > 2$, *if an almost contact structure compatible with* $\langle\cdot,\cdot\rangle$ *is of type* $\mathcal{D}$, *where* $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_4 \oplus \mathcal{C}_8$, *then it is harmonic.*

$$2d^* \operatorname{Ric} + ds = 0$$

weakly-ac-Einstein, $\operatorname{Ric}^{ac}(X, Y) = \frac{1}{2n} s^{ac}(\langle X, Y\rangle - \eta(X)\eta(Y))$

Lemma

For almost contact manifolds of type $\mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_{10}$*, we have*

$$2d^*(\operatorname{Ric}^{ac})^t(X) + ds^{ac}(X) = 2\langle R_{(e_i,X)}, \xi_{\varphi e_i}\varphi\rangle - 4\operatorname{Ric}^{ac}(X, \xi_{e_i} e_i) + 4\langle \operatorname{Ric}^{ac}, \xi_X\rangle - 2d^* F(\zeta)\operatorname{Ric}^{ac}(\zeta, \varphi X).$$

where $(\operatorname{Ric}^{ac})^t(X, Y) = \operatorname{Ric}^{ac}(Y, X)$ *and* $\xi^\flat_X(Y, Z) = \langle \xi_X Y, Z\rangle$*. In particular, if the manifold is weakly-ac-Einstein, then*

$$(n-1)ds^{ac}(X) + (ds^{ac}(\zeta) + s^{ac} d^*\eta)\eta(X) = 2n\langle R_{(e_i,X)}, \xi_{\varphi e_i}\varphi\rangle - 2s^{ac}\langle \xi_{e_i} e_i, X\rangle.$$

For an almost contact metric $2n+1$-manifold $(M, \langle \cdot, \cdot \rangle, \varphi, \zeta)$, we have:

(i) If M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ or $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is a harmonic map if and only if it is a harmonic almost contact structure and $2d^* \operatorname{Ric}^{\mathrm{ac}} + ds^{\mathrm{ac}} = 0$. In particular:

  (a) If the structure is of type $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ and the manifold is weakly-*ac*-Einstein, then the almost contact structure metric structure is a harmonic map if and only if s^{ac} is such that $n\, ds^{\mathrm{ac}} = s^{\mathrm{ac}}\, d^*\eta\, \eta = -s^{\mathrm{ac}}\, \xi^{\sharp}_{e_i} e_i$.

  (b) If the structure is of type $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ and the manifold is weakly-*ac*-Einstein, then the almost contact metric structure is a harmonic map if and only if s^{ac} is constant.

  (c) If the manifold is nearly-K-cosymplectic ($\mathcal{C}_1$), then almost contact metric structure is a harmonic map. Furthermore, if the nearly-K-cosymplectic structure is flat, then it is cosymplectic, i.e., $\xi = 0$.

(ii) If M is of type $\mathcal{D}$, where $\mathcal{D} = \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ or $\mathcal{D} = \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is a harmonic map if and only if it is a harmonic structure and

$$\begin{aligned} 2d^*(\mathrm{Ric}^{\mathrm{ac}})^t(X) + ds^{\mathrm{ac}}(X) + 4\,\mathrm{Ric}^{\mathrm{ac}}(X, \xi_{e_j} e_j) & \\ -4\langle \mathrm{Ric}^{\mathrm{ac}}, \xi_X^\flat\rangle + 2d^*F(\zeta)\,\mathrm{Ric}^{\mathrm{ac}}(\zeta, \varphi X) &= 0, \end{aligned}$$

for all $X \in \mathfrak{X}(M)$. In particular,

(a)* If $\mathrm{Ric}^{\mathrm{ac}}$ is symmetric, then the almost contact structure is a harmonic map if and only if $\xi_{\xi_{e_i} e_i} = 0$ and $2d^*\,\mathrm{Ric}^{\mathrm{ac}} + ds^{\mathrm{ac}} + 4\xi_{e_i} e_i \lrcorner\, \mathrm{Ric}^{\mathrm{ac}} = 0$. Furthermore, if the manifold is weakly-*ac*-Einstein, then the almost contact structure is a harmonic map if and only if $\xi_{\xi_{e_i} e_i} = 0$ and $(n-1)ds^{\mathrm{ac}} + (ds^{\mathrm{ac}}(\zeta) - s^{\mathrm{ac}} d^*\eta)\eta + 2s^{\mathrm{ac}} \xi^\flat_{e_i} e_i = 0$.

(b)* If the almost contact metric structure is of type $\mathcal{C}_3 \oplus \mathcal{C}_i$, $i = 6, 7, 10$, then it is a harmonic map if and only if

$$2d^*\,\mathrm{Ric}^{\mathrm{ac}} + ds^{\mathrm{ac}} = 0.$$

For an oriented and compact Riemannian n-manifold $(M, \langle\cdot,\cdot\rangle)$ equipped with a G-structure, $G \subseteq SO(n)$ and a differential p-form ϕ preserved by the action of G, the following Bochner type formula is satisfied

$$\int_M \left(\tfrac{1}{p+1}\|d\phi\|^2 + p\|d^*\phi\|^2 - \|\nabla\phi\|^2 \right) = \int_M \langle \widetilde{\mathcal{R}}\phi, \phi \rangle$$

G. Bor and L. Hernández Lamoneda, Bochner formulae for orthogonal G-structures on compact manifolds, *Diff. Geom. Appl.* **15** (2001), 265–286.

$$\operatorname{alt} \alpha(x_1, \ldots, x_p) = \sum_\tau \operatorname{sign}(\tau)\alpha(x_{\tau(1)}, \ldots, x_{\tau(p)})$$

$$(R^c\phi)(x_1, \ldots, x_p) = (R(x_1, e_i)\phi)(e_i, x_2, \ldots, x_p)$$

$$\widetilde{\mathcal{R}}(\phi) = \operatorname{alt}(R^c\phi)$$

$$\int_M \left(\tfrac{1}{3}\|dF\|^2 + 2\|d^*F\|^2 - \|\nabla F\|^2 \right) = 2\int_M (s - s^{\mathrm{ac}} - \mathrm{Ric}(\zeta,\zeta)).$$

$$\int_M \left(\tfrac{1}{2}\|d\eta\|^2 + \|d^*\eta\|^2 - \|\nabla\eta\|^2 \right) = \int_M \mathrm{Ric}(\zeta,\zeta).$$

$$\int_M \Big(8\|\xi_{(1)}\|^2 - 4\|\xi_{(2)}\|^2 + \|\xi_{(5)}\|^2 + (2n-1)\|\xi_{(6)}\|^2 - \|\xi_{(7)}\|^2 + \|\xi_{(8)}\|^2 - \|\xi_{(9)}\|^2 - \|\xi_{(10)}\|^2 - 4\|\xi_{(11)}\|^2 - \|\xi_{(12)}\|^2 + \|2(\nabla\eta)_{(10)}\circ\varphi - (\nabla_\zeta F)_{(11)}\|^2 + 2\| - e_i \lrcorner (\nabla_{e_i} F)_{(4)} + (\nabla_\zeta \eta)_{(12)} \circ \varphi\|^2 \Big) = 2\int_M (s - s^{\mathrm{ac}} - \mathrm{Ric}(\zeta,\zeta))$$

$$\int_M \Big((2n-1)\|\xi_{(5)}\|^2 + \|\xi_{(6)}\|^2 + \|\xi_{(7)}\|^2 - \|\xi_{(8)}\|^2 - \|\xi_{(9)}\|^2 + \|\xi_{(10)}\|^2 \Big) = 2\int_M \mathrm{Ric}(\zeta,\zeta)$$

Theorem

Let $(M, \langle \cdot, \cdot \rangle)$ be a $(2n+1)$-dimensional conformally flat compact Riemannian manifold, where $n > 1$. If s is the scalar curvature and $C_{\langle \cdot,\cdot \rangle} = \int_M s$, that is, a constant depending on the metric $\langle \cdot, \cdot \rangle$, then every almost contact structure σ compatible with $\langle \cdot, \cdot \rangle$ satisfies

$$\begin{aligned} \tfrac{2(n-1)}{2n-1} C_{\langle \cdot,\cdot \rangle} &= \int_M \Big(4\|\xi_{(1)}(\sigma)\|^2 - 2\|\xi_{(2)}(\sigma)\|^2 + (n-1)\|\xi_{(5)}(\sigma)\|^2 \\ &+ (n - \tfrac{1}{2n-1})\|\xi_{(6)}(\sigma)\|^2 - \tfrac{1}{2n-1}\|\xi_{(7)}(\sigma)\|^2 + \tfrac{1}{2n-1}\|\xi_{(8)}(\sigma)\|^2 \\ &- (1 - \tfrac{1}{2n-1})\|\xi_{(9)}(\sigma)\|^2 - \tfrac{1}{2n-1}\|\xi_{(10)}(\sigma)\|^2 - 2\|\xi_{(11)}(\sigma)\|^2 \\ &- \|\xi_{(12)}(\sigma)\|^2 + \tfrac{1}{2}\|2(\nabla \eta_\sigma)_{(10)} \circ \varphi_\sigma - (\nabla_{\zeta_\sigma} F_\sigma)_{(11)}\|^2 \\ &+ \| - e_i \lrcorner (\nabla_{e_i} F_\sigma)_{(4)} + (\nabla_{\zeta_\sigma} \eta_\sigma)_{(12)} \circ \varphi_\sigma \|^2 \Big). \end{aligned}$$

Theorem

Moreover:

(i) *If* σ_0 *is an almost contact structure compatible with* $\langle\cdot,\cdot\rangle$ *of type* $\mathcal{C}_1 \oplus \mathcal{C}_4$ *and* $n = 3$, *then* σ_0 *is an energy minimiser such that its total bending is* $B(\sigma_0) = \frac{1}{10} C_{\langle\cdot,\cdot\rangle}$. *Furthermore, in this situation any other energy minimiser is of type* $\mathcal{C}_1 \oplus \mathcal{C}_4$.

(ii) *If* $n = 2$ *or* $n \geq 4$, *and* σ_0 *is an almost contact structure compatible with* $\langle\cdot,\cdot\rangle$ *of type* $\mathcal{C}_4$, *then* σ_0 *is an energy minimiser such that its total bending is* $B(\sigma_0) = \frac{1}{2(2n-1)} C_{\langle\cdot,\cdot\rangle}$. *Furthermore, in this situation any other energy minimiser is of type* $\mathcal{C}_4$.

(iii) *If* σ_0 *is an almost contact structure compatible with* $\langle\cdot,\cdot\rangle$ *of type* $\mathcal{C}_2$, *then* σ_0 *is an energy minimiser such that its total bending is* $B(\sigma_0) = -\frac{n-1}{2(2n-1)} C_{\langle\cdot,\cdot\rangle}$. *Furthermore, in this situation any other energy minimiser is of type* $\mathcal{C}_2$.

Theorem

Let $(M, \langle\cdot,\cdot\rangle)$ be a $(2n+1)$-dimensional compact Einstein manifold. If s is the scalar curvature, then every almost contact structure σ compatible with $\langle\cdot,\cdot\rangle$ satisfies

$$\int_M \Big((2n-1)\|\xi_{(5)}(\sigma)\|^2 + \|\xi_{(6)}(\sigma)\|^2 + \|\xi_{(7)}(\sigma)\|^2 - \|\xi_{(8)}(\sigma)\|^2 - \|\xi_{(9)}(\sigma)\|^2 + \|\xi_{(10)}(\sigma)\|^2 \Big) = \frac{2s}{2n+1}\mathrm{Vol}(M).$$

Moreover:

(i) For $n = 1$, if σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_5 \oplus \mathcal{C}_6$ (trans-Sasakian), then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{s}{3}\mathrm{Vol}(M)$. Furthermore, in this situation any other energy minimiser is trans-Sasakian.

(ii) For $n > 1$, if σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_5$ (α-Kenmotsu, where $2n\alpha = -d^*\eta_{\sigma_0}$), then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{s}{4n^2-1}\mathrm{Vol}(M)$. Furthermore, in this situation any other energy minimiser is of type α-Kenmotsu with $4n\int_M \alpha^2 = \frac{s}{4n^2-1}\mathrm{Vol}(M)$.

(iii) If σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_8 \oplus \mathcal{C}_9$, then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = -\frac{s}{2n+1}\mathrm{Vol}(M)$. Furthermore, in this situation any other energy minimiser is of type $\mathcal{C}_8 \oplus \mathcal{C}_9$.

Example

1. $\mathbb{C}^{n+1}$, $(\langle\cdot,\cdot\rangle, J)$ $S^{2n+1}(r)$, U is a unit normal vector field, on $S^{2n+1}(r)$, a-Sasakian structure, $\langle\cdot,\cdot\rangle$, $\varphi = \tan \circ J$, $\zeta = JU$

$$a^2 = \frac{1}{r^2}, \quad \mathrm{Ric} = \frac{2n}{r^2}\langle\cdot,\cdot\rangle, \quad \mathrm{Ric}^{\mathrm{ac}} = \frac{1}{r^2}\left(\langle\cdot,\cdot\rangle - \eta\otimes\eta\right).$$

   $S^{2n+1}(r)$ is Einstein, ac-Einstein and conformally flat

   $\pm\frac{1}{r}$-Sasakian structure is a harmonic map

2. Different generalisations of the Heisenberg group admit almost contact structures of type $\mathcal{C}_6 \oplus \mathcal{C}_7$ and $\mathcal{C}_8 + \mathcal{C}_9$. Therefore, they are harmonic almost contact structures. Most of them satisfy $2d^* \mathrm{Ric}^{\mathrm{ac}} + ds^{\mathrm{ac}} = 0$. As a consequence, these almost contact structures are harmonic maps.

   The characteristic vector fields are harmonic unit vector fields.

3. $S^6 \times S^1$ admits a structure of strict type $\mathcal{C}_1$. Therefore, it is an energy minimiser.

4. $S^{2n+1} \times S^1 \times S^1$ admits a structure of strict type $\mathcal{C}_4$. Therefore, it is an energy minimiser.