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# Harmonic Analysis on the Poincaré Group

## II. The Fourier Transform

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**Abstract.** The generalized matrix elements on the Poincaré group are used to write the Fourier transform explicitly. This realizes a mapping between positive type functions on the group and generalized density matrices.

## Introduction

This work is the continuation of our preceding paper [1] hereafter referred to as part I. It is written in three chapters, where we deal successively with

I. The matrix elements.

II. The dual space of the Poincaré group.

III. The Fourier transform in $\mathscr{L}^1$ and $\mathscr{L}^2$ spaces.

One gets a fairly simple view of the structure of the dual space of equivalence classes of unitary irreducible representations of the group.

Our aim is to work out the Fourier transform for distributions on the group and we hope this can be achieved in a following and last article.

## Chapter I. The Matrix Elements

In part I, we computed the generalized matrix elements of the unitary representations of the Poincaré group. For this purpose, we wrote explicitly the matrix elements of the stabilizator groups, $SU(2)$ and $SU(1, 1)$. We obtained them as eigenfunctions of infinitesimal operators; of course they are only defined up to a phase (this corresponds to the relative (and arbitrary) phase of the vectors of the basis). Here, we use this indetermination to redefine the matrix elements, in order to obtain a better analytic behaviour. We give these definitions in the case of the $SU(1, 1)$ subgroup and obtain the case of $SU(2)$ by analytic continuation.

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For the notation one must refer to part I.

$$f_{\varepsilon s\lambda'\lambda}(X) = N(s;\lambda',\lambda)\,\psi_{s\lambda'\lambda}(X)\,, \tag{1}$$

$$\begin{aligned}\psi_{s\lambda'\lambda}(X) = i^{\lambda-\lambda'} \int_0^{2\pi} \frac{d\varphi}{2\pi} (X_{11}e^{i\varphi} + X_{12}e^{-i\varphi})^{s+\lambda'} \\ \cdot (X_{21}e^{i\varphi} + X_{22}e^{-i\varphi})^{s-\lambda'} e^{-2i\lambda\varphi}\,,\end{aligned} \tag{2}$$

$$N(s;\lambda',\lambda) = \sqrt{\frac{(s-\lambda)!\,(-s-1-\lambda')!}{(s-\lambda')!\,(-s-1-\lambda)!}} = \frac{(s-\lambda)!}{(s-\lambda')!}\,Q(s;\lambda',\lambda)\,, \tag{3}$$

$$Q(s;\lambda',\lambda) = \sqrt{\frac{(s-\lambda')!\,(-s-1-\lambda')!}{(s-\lambda)!\,(-s-1-\lambda)!}} = \sqrt{\frac{(s+\lambda)!\,(-s-1+\lambda)!}{(s+\lambda')!\,(-s-1+\lambda')!}}\,. \tag{4}$$

We take the same formulae for all the representations of $SU(1,1)$ and choose the determination of the square root in the following way: Put $z = s + \frac{1}{2}$. Express $Q(s;\lambda',\lambda)$ as product of functions of the form $\sqrt{p-z}$ and $\sqrt{p+z}$ (or inverses of these functions). This is possible as $\lambda - \lambda'$ is an integer. Choose then the determination of the square roots such that

$$|\mathrm{Arg}\sqrt{p \pm z}| < \frac{\pi}{2}\,.$$

In the $z$-plane, one has the following cuts:

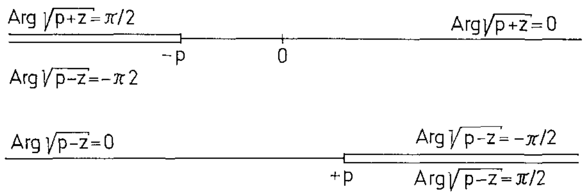


Fig. 1

When $\lambda = \lambda'$, one has $N(s;\lambda',\lambda) = Q(s;\lambda',\lambda) = 1$.
When $\lambda < \lambda'$, one has (with $z = s + \frac{1}{2}$):

$$N(s;\lambda',\lambda) = \prod_{p=-\lambda'+\frac{1}{2}}^{-\lambda-\frac{1}{2}} \frac{\sqrt{p+z}}{\sqrt{p-z}}\,, \tag{5}$$

$$Q(s;\lambda',\lambda) = \prod_{p=-\lambda'+\frac{1}{2}}^{-\lambda-\frac{1}{2}} \frac{1}{\sqrt{p^2-z^2}}\,. \tag{6}$$

When $\lambda > \lambda'$, one has $N(s;\lambda',\lambda) = N(s;\lambda,\lambda')^{-1}$ and

$$Q(s;\lambda',\lambda) = \prod_{p=\lambda'+\frac{1}{2}}^{\lambda-\frac{1}{2}} \sqrt{p^2-z^2}\,.$$

For the $SU(2)$ matrix elements, we take the same formulae and we choose the following determination on the cuts and add a phase

$$N(s;\lambda',\lambda)=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} N(s+i\varepsilon;\lambda',\lambda)\, i^{\lambda'-\lambda}.$$

We get then

$$\begin{aligned} N(s;\lambda',\lambda) &= \prod_{p=\lambda+\frac{1}{2}}^{\lambda'-\frac{1}{2}}\left(-\frac{\sqrt{z-p}}{\sqrt{z+p}}\right) && \text{if } \lambda<\lambda' \\ &=1 && \text{if } \lambda=\lambda' \\ &= N(s;\lambda,\lambda')^{-1} && \text{if } \lambda>\lambda'. \end{aligned} \tag{7}$$

### *1. Analyticity*

The expressions (1) (2) (3) (4) make sense for all values of $s$. In fact, one has to choose a determination of the function in the integral of (2) such that when $X$ approaches the identity $\sigma_0$, one has $f_{\varepsilon s\lambda'\lambda}(X)\to\delta_{\lambda'\lambda}$. This integral is defined for all values of $s$, as $|X_{11}|=|X_{22}|>|X_{12}|=|X_{21}|$, and $\psi_{s\lambda'\lambda}(X)$ is an analytic function of $s$.

From the normalization factors, the matrix element have cuts in the $s$-plane, which are those of the square roots that we find in formula (3).

When $2s$ is an integer (then $2\lambda$ and $2\lambda'$ are of the same parity as $2s$), the matrix elements are zero if

$$-s\leqq\lambda\leqq s \quad\text{and}\quad |\lambda'|>s$$

or

$$-s\leqq\lambda'\leqq s \quad\text{and}\quad |\lambda|>s.$$

Apart from the cuts in the $s$-plane, the matrix elements are analytic functions of $s$ and also are analytic functions of $X$ as one can see by formulae (1), (2) and (3).

For some values of $s$, they defined unitary representations and for $|\mathrm{Re}\,s|<1$, they correspond to continuous representations in Banach space [4]. The composition relations

$$\sum_{\lambda'} f_{\varepsilon s\lambda\lambda'}(X)\, f_{\varepsilon s\lambda'\lambda''}(Y)=f_{\varepsilon s\lambda\lambda''}(XY) \tag{8}$$

which are valid for all $s$ [3] and give rise to the properties of representations (unitary, in Banach space for some $s$), can be used to define "generalized representations" in some sense.

We have in this way an analytic continuation for all $s$. Furthermore, one also has an analytic continuation from $SU(1,1)$ to $SU(2)$, as they are subgroups of $SL(2,C)$. The non unitary finite representations of $SU(1,1)$ (with $2s\in N$ and $-s\leqq\lambda',\lambda\leqq s$) then become the unitary representations of $SU(2)$. For the other representations, formula (2) can be surely continued to the neighbourhood of the origin given by the conditions $|X_{11}|>|X_{12}|$ and $|X_{22}|>|X_{21}|$, but farther there can be cuts which come from the determination of the logarithms to be used in the right-hand side of (2).

## 2. Symmetries

When one expands the right-hand side of (2) into series, one gets

$$f_{\varepsilon s\lambda'\lambda}(X)=\frac{Q(s;\lambda',\lambda)}{(\lambda-\lambda')!}X_{11}^{s+\lambda'}X_{21}^{\lambda-\lambda'}X_{22}^{s-\lambda}\cdot F\left(-s-\lambda',\,-s+\lambda;\lambda-\lambda'+1;\frac{X_{12}X_{21}}{X_{11}X_{22}}\right),\tag{9}$$

$$=\frac{Q(s;\lambda,\lambda')}{(\lambda'-\lambda)!}X_{11}^{s+\lambda}(-X_{12})^{\lambda'-\lambda}X_{22}^{s-\lambda'}\cdot F\left(-s-\lambda,\,-s+\lambda';\lambda'-\lambda+1;\frac{X_{12}X_{21}}{X_{11}X_{22}}\right),\tag{10}$$

$$=\frac{Q(s;\lambda',\lambda)}{(\lambda-\lambda')!}X_{21}^{\lambda-\lambda'}X_{22}^{-\lambda-\lambda'}\cdot F(-s-\lambda',s+1-\lambda';\lambda-\lambda'+1;\,-X_{12}X_{21}).\tag{11}$$

One derives formula (11) from formula (9) using a property of the function $F$, which is the hypergeometric function [5].

One may then derive the following symmetries:

$$f_{\varepsilon s\lambda'\lambda}(X)=(-1)^{\lambda-\lambda'}f_{\varepsilon s\lambda\lambda'}(\tilde{X})\tag{12}$$

(from (9) and (10), with $\tilde{X}$ = transposed matrix of $X$).

$$f_{\varepsilon s\lambda'\lambda}(X)=f_{\varepsilon(-s-1)\lambda'\lambda}(X)\tag{13}$$

(from (11)),

$$f_{\varepsilon s\lambda'\lambda}(X)=(-1)^{\lambda-\lambda'}f_{\varepsilon s\lambda'\lambda}(\sigma_3X\sigma_3),\tag{14}$$

$$f_{\varepsilon s\lambda'\lambda}(X)=f^*_{\varepsilon s^*\lambda'\lambda}(X^*)\tag{15}$$

($X^*$ = complex conjugate of $X$).

Formula (14) is evident from (9) $\left(\text{with } \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$. Formula (15) means that $f_{\varepsilon s\lambda'\lambda}(X)$ is a real analytic function of $(s,X)$.

By formula (11), one sees that $s$ and $(-s-1)$ define the same matrix elements, and furthermore, the matrix elements are analytic functions of $-s(s+1)$, which is the eigenvalue of the Casimir operator.

### 3. *Unitarity*

If $X$ belong to $SU(1,1)$, one has

$$X^{-1}=\sigma_3 X^\dagger \sigma_3=\sigma_3 \tilde{X}^* \sigma_3 .$$

Using relations (12), (14) and (15), one gets

$$f_{\varepsilon s \lambda' \lambda}(X^{-1})=f^*_{\varepsilon s^* \lambda \lambda'}(X) .$$

Therefore, the unitarity requirement is fulfilled for real $s$, whenever $s$ is not on the cuts. This is the continuous exceptional series, with $\lambda$ and $\lambda'$ integral and $-1<s<0$. When $2s$ is integral, then $2\lambda$ and $2\lambda'$ are of the same parity as $2s$, many of the matrix elements are zero, and for the remaining, $s$ is not yet on the cuts. This gives rise to the two discrete representations, given by

$$2s \in \mathbb{N} ;$$

$$\lambda \text{ and } \lambda' \geqq s+1, \quad \text{or} \quad -\lambda \text{ and } -\lambda' \geqq s+1 .$$

When $s=-\frac{1}{2}+i\varrho$, one gets the unitarity relations using formula (13). The finite representations of $SU(1,1)$ given by integral $2s$ and $-s \leqq \lambda, \lambda' \leqq +s$, are non unitary; by the non uniformness of the matrix elements, one has:

$$f_{\varepsilon s \lambda' \lambda}(X^{-1})=f^*_{\varepsilon s^* \lambda \lambda'}(X)=(-1)^{\lambda'-\lambda} f^*_{\varepsilon s \lambda \lambda'}(X)$$

(when one takes $s^*$, one chooses the other limits on the cuts of Fig. 1).

The representations of $SU(1,1)$ are very well known [2, 4] and the Fourier transform is well studied [4, 6]. Harmonic analysis on this group differs greatly from the Abelian case.

When $X$ belongs to $SU(2)$ and $2s$ is an integer, we have the unitarity relations for $|\lambda|$ and $|\lambda'| \leqq s$. In fact, we have in that case

$$X^{-1}=X^\dagger$$

$$f_{\varepsilon s \lambda' \lambda}(X^{-1})=(-1)^{\lambda'-\lambda} f^*_{\varepsilon s^* \lambda \lambda'}(X)=f^*_{\varepsilon s \lambda \lambda'}(X) .$$

(As before, one chooses the other side of the cut and therefore introduces a factor $(-1)^{\lambda'-\lambda}$ which cancels.)

### 4. *Composition Relations (Multiplicative Property)*

In our Ref. [3], using series expansions, we proved that the following relation is valid for all $s$:

$$\sum_{\lambda'} \psi_{s \lambda \lambda'}(X) \psi_{s \lambda' \lambda''}(Y)=\psi_{s \lambda' \lambda''}(XY) . \tag{16}$$

The proof is valid for all $X$, $Y$ belonging to $SU(1, 1)$ and the summation is uniformly convergent if $X$ and $Y$ remain bounded. Therefore, one may easily derive (8).

When $2s$ is an integer, the representation breaks down into three parts, as the matrix elements are identically zero whenever one has

$$|\lambda'| \leqq s \quad \text{and} \quad |\lambda| > s$$

or

$$|\lambda'| > s, \ |\lambda| > s \quad \text{and } \lambda, \lambda' \text{ with different signs}.$$

Therefore, $\lambda$ and $\lambda'$ must belong both to one of the intervals $[-\infty, \cdots -s-1], [-s, +s], [s+1, \ldots +\infty]$.

When $s$ is not at the unitary points, relation (16) can still be used to define a representation in an other space than a Hilbert space (see Ref. [4]).

## Chapter II. The Dual Space

Almost all the irreducible continuous unitary representations of the Poincaré group are characterized by the eigenvalues of $P^2$ and $W^2$ and some discrete indices, such as the sign of energy for positive and zero masses, the integral or half-integral character, or the sign of the third component of the spin in the case of zero and negative masses. Excepting those complications, we can give a fairly simple chart of the irreducible representations in Fig. 2, where we plot

$$\omega = -\frac{W^2}{|W^2|}\left(\sqrt{\left|W^2 - \frac{P^2}{4}\right|} - \sqrt{\left|\frac{P^2}{4}\right|}\right) \quad \text{versus} \quad p = \frac{P^2}{|P^2|}\sqrt{|P^2|}\,.$$

For the physical masses $M$, $\omega = Ms$ and $p = M$, each point corresponds to two representations which differ by the sign of the energy.

For the imaginary masses $iM$, $p = -M$; the section by the line $p = -1$ gives all the representations of $SU(1, 1)$ (see Fig. 3).

The structure of the representations is more involved when $P^2 = 0$. The half-line $p = 0$, $\omega > 0$ contains all the infinite spin representations of zero masses, each point corresponds to four representations: positive or negative energy, integral or half-integral spin.

The chart of unitary irreducible representations of the Poincaré group.

1) The oblique hachures indicate the continuous integral spin and negative square mass representations ($\varepsilon = 0$).

2) The horizontal hachures indicate the continuous half integral and negative square mass representations ($\varepsilon = 1$).

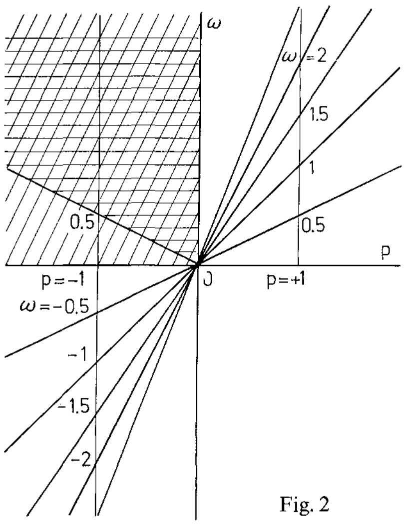


Fig. 2

3) All the lines $\omega = kp$ with positive integral or half integral $k$ and a half line with $k = -\frac{1}{2}$ and $p < 0$ correspond to discrete spin representations ($\varepsilon = \pm$).

4) Zero mass and infinite spin are localized on the half line $\omega > 0$ and $p = 0$.

5) At $p = \omega = 0$, one has zero mass and finite spin and zero 4-momentum representations.

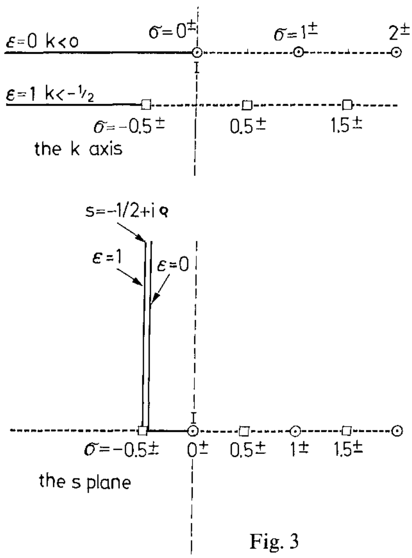


Fig. 3

The multiplicity at the line $p=-1$ of Fig. 2. $k=\omega/p$; $s=k$ if $k \geqq -\frac{1}{2}$, $s=-\frac{1}{2}+i(k+\frac{1}{2})$ if $k<-\frac{1}{2}$; $\sigma=(s,\varepsilon)$ is the spin index:

$\varepsilon=0$: integral continuous spin representations ($k<0$, $s=-\frac{1}{2}+i\varrho$ or $-\frac{1}{2}<s<0$),

$\varepsilon=1$: half integral continuous spin representation ($k<-\frac{1}{2}$, $s=-\frac{1}{2}+i\varrho$),

$\varepsilon=\pm$: 1) $s$ half integral $\geqq -\frac{1}{2}$: half integral discrete spin with positive or negative helicity representations,

2) $s$ integral $\geqq 0$: integral discrete spin with positive or negative helicity representations.

Some values of $\sigma$ are given on the figures. $I$ is the zero spin representation (trivial representation).

The point $P=0$, $\omega=0$ is rather tangled: it contains all the finite spin, zero mass representations (four-fold: signs of the energy and of the helicity) and also all the zero-momentum representations.

Let us call $\hat{\mathscr{P}}$ the set of equivalence classes of irreducible unitary representations of $\mathscr{P}$. $\hat{\mathscr{P}}$ is called the dual space of the group $\mathscr{P}$, and is usually endowed with the Jacobson topology [7, 8]. One may feature it as follows: Let $\mathscr{S}$ be a subset of $\hat{\mathscr{P}}$, a representation $\mathscr{U} \in \hat{\mathscr{P}}$ belongs to $\bar{\mathscr{S}}$ if $\mathscr{U}$ is weakly contained in $\mathscr{S}$, that is to say if one nonzero continuous positive functional associated with $\mathscr{U}$ is a weak limit of continuous positive functionals associated with $\mathscr{S}$; one may suppose these functionals bounded in norm. The weak topology is the topology of simple convergence over $\mathscr{L}^1(\mathscr{P})$. But the continuous positive functionals on $\mathscr{L}^1(\mathscr{P})$ can be identified with the continuous positive definite functions on the group (see Ref. [8], Theorem 13.4.5), and as one may restrict oneself to uniformly bounded families of such functions, the weak convergence is equivalent to the vague convergence (i.e. pointwise on $\mathscr{L}(\mathscr{P})$, which is defined to be the set of continuous functions on $\mathscr{P}$ with compact support) or to the convergence on any dense subset of $\mathscr{L}^1(\mathscr{P})$.

Now, this topology can be replaced by the equivalent one which is obtained by the vague convergence of the generalized matrix elements (which are measures):

First, any positive type continuous function $\psi$ associated with an irreducible unitary continuous representation (see Ref. [8], No. 13. 4.6) is in the vague closure of its generalized matrix elements. This function can always be expressed as

$$\psi(\alpha)=\langle\xi,\mathscr{U}^{\nu\sigma}(\alpha)\xi\rangle=\int_{\mathscr{T}_{\nu\sigma}\times\mathscr{T}_{\sigma\nu}} \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\,\eta^*(\tau)\,\eta(\tau')\,d\mu(\tau)\,d\mu(\tau')$$

where $\xi$ is some vector given by a square integrable function $\eta$ (see formulae (17) and (18) in Part I). As a measure, $\psi$ takes its value on a function $\varphi\in\mathscr{L}(\mathscr{P})$ (the symbol $\mathscr{L}$ stands for "the space of continuous functions with compact support on...”):

$$I(\varphi) = \int_{\mathscr{P}} \psi(\alpha)\, \varphi(\alpha)\, d^{10}\alpha$$
$$= \int_{\mathscr{T}_{\nu\sigma} \times \mathscr{T}_{\nu\sigma}} \eta^*(\tau)\, \eta(\tau')\, \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\varphi)\, d\mu(\tau')\, d\mu(\tau)$$

$\left(\text{with } \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\varphi) = \int_{\mathscr{P}} \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\, \varphi(\alpha) d^{10}\alpha\right)$.

At this point, one sees that one may restrict oneself to the $\eta$'s belonging to $\mathscr{L}(\mathscr{T}_{\nu\sigma})$ and as $\mathscr{U}^{\nu\sigma}_{\tau\tau'}(\varphi)$ is a continuous function of $\tau$ and $\tau'$, there exists a sequence of Riemann sums which converges to $I(\varphi)$ for any $\varphi \in \mathscr{L}(\mathscr{P})$. As a Riemann sum is the value on $\varphi$ of a finite linear combination of the measures $\mathscr{U}^{\nu\sigma}_{\tau\tau'}$, the assertion is proved, as consequence of the separability of $\mathscr{L}(\mathscr{P})$.

The converse is also true: a generalized matrix element is a vague limit of finite sums of positive type functions. With two sequences of functions of $\mathscr{L}(\mathscr{T}_{\nu\sigma})$, $\eta_n$ converging to $\delta(\tau - \tau_1)$ and $\zeta_n$ to $\delta(\tau' - \tau'_1)$, one has

$$\mathscr{U}^{\nu\sigma'}_{\tau_1\tau_1}(\varphi) = \lim_{n\to\infty} \int \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\varphi)\, \eta^*_n(\tau)\, \zeta_n(\tau')\, d\mu(\tau)\, d\mu(\tau')$$

and the function

$$\int \mathscr{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\, \eta^*_n(\tau)\, \zeta_n(\tau')\, d\mu(\tau)\, d\mu(\tau')$$

can be expressed as a linear combination of four positive type functions

$$\langle \eta^*_n + \varepsilon\zeta_n, \mathscr{U}^{\nu\sigma}(\alpha)\, (\eta_n + \varepsilon\zeta_n)\rangle \quad (\text{with } \varepsilon = \pm 1, \pm i)\,.$$

This convergence is the consequence of the continuity of $\mathscr{U}^{\nu\sigma}_{\tau\tau'}(\varphi)$ as a function of $(\tau, \tau')$. This can be easily proved using the Lebesgue theorem in formula (23). In fact, when $\varphi$ is of compact support, one may have even the equicontinuity in $(\tau, \tau')$ for all $\nu$ and all $\sigma$ with the real part of $s$ bounded; this is a direct consequence of the uniform bound of $f_{\varepsilon s \lambda' \lambda}$ one may get by formulae (1), (2) and (3). We have a more complete

**Theorem 1.** *The generalized matrix elements are vaguely continuous functions of the eigenvalues of the differential operators, when* $P^2 \neq 0$ *or* $W^2 \neq 0$.

This means that

1. for unphysical masses, the matrix elements are vaguely continuous functions of $s$, and not only of $\tau = (k, \lambda)$ and $\tau' = (k', \lambda')$ (this can easily be seen from formulae (23); one may even get the analyticity in $s$);

2. the matrix elements have the same limits as the masses approach zero from $P^2 > 0$ and $P^2 < 0$ (see Fig. 2). Some similar results are known [10].

This can be seen in the following.

First, one notices that for fixed $\lambda$ and $\lambda'$, and variable $k$ and $k'$, all the generalized matrix elements of an irreducible unitary representation are obtained by translating one of them, therefore one may show that one appropriate generalized matrix element for the zero mass representation is a vague limit of a sequence of generalized matrix elements of irreducible unitary representations from each side of the line $P=0$ of Fig. 2. Therefore we deal with

### *a) Physical Masses and Infinite Spin*

First, when $k$ and $k'$ approach $\overset{03}{1}$ ($\overset{03}{1}$ is defined in Part I), the measure $\omega\delta^3(Xk, k')$ has the limit $\nu\delta^3(X\overset{03}{1}, \overset{03}{1})$. The surface defined by the equation $Xk=k'$ approaches uniformly over any compact the surface defined by $X\overset{03}{1}=\overset{03}{1}$, and as the invariant measures on these surfaces have well-correlated normalizations, one can conclude.

In a second step, as the matrix elements are the product of this measure by the continuous functions $e^{ik\cdot x} f_{s\lambda'\lambda}(U)$, one shows that the multiplicative factor converge also uniformly over any compact to the limit $e^{i\overset{03}{1}\cdot x} f_{r\varepsilon\lambda'\lambda}(S)$, when one lets $M^2 s(s+1)$ converge to $r^2$ as $M^2\to 0$; $S$ is related to $U$ by

$$S=\lim_{k\to\overset{03}{1}} H_k^{-1}\,U H_k\,.$$

The calculation is straightforward with Stirling formula (in fact the index $\varepsilon$ is a function of $\lambda$ and can be dropped out). Therefore, we conclude that we have

$$\underset{\substack{k\to\overset{03}{1}\\ k^2 s(s+1)\to r^2}}{v\text{-lim}} (k^{-2})\,\psi(skk\lambda'\lambda;\alpha)=\psi(r\varepsilon\overset{03}{1}\;\overset{03}{1}\;\lambda'\lambda;\alpha)\,. \tag{17}$$

The symbol $v$-lim stands for vague-limit. For other matrix elements, one can translate this equality.

### *b) Unphysical Masses and Infinite Spin*

The same techniques can be used and one gets the following equality

$$\underset{\substack{k\to\overset{03}{1}\\ k^2 s(s+1)\to r^2}}{v\text{-lim}} (-k^{-2})\,\psi(\varepsilon skk\lambda'\lambda;\alpha)=\tfrac{1}{2}\psi(r\varepsilon\overset{03}{1}\;\overset{03}{1}\;\lambda'\lambda;\alpha)\,. \tag{18}$$

Except the factor $\frac{1}{2}$, the two limits are identical. In fact, this is obtained because we chose an additional phase in formula (7); the natural continuation on the Poincaré group is not the continuation given in $SL(2, C)$. The factor $\frac{1}{2}$ can be understood easily if we remember that the same unphysical mass representations give rise to two representations with zero mass, but with the opposite signs of energy. At the same time, we see that those representations are not separated in the Jacobson topology.

### *c) Physical Masses and Finite Spin – Unphysical Masses and Finite Spin*

The proof is the same as before, and this time, the multiplicative factor has trivially, in the two cases, the same limit:

$$\begin{aligned} \lim f_{\varepsilon s \lambda' \lambda}(O) &= e^{i\lambda\varphi}\delta_{\lambda\lambda'} \\ \lim f_{s \lambda' \lambda}(U) &= e^{i\lambda\varphi}\delta_{\lambda\lambda'} \end{aligned} \quad \text{with} \quad \varphi = \operatorname{Arg} X_{11}\,. \tag{19}$$

Therefore, we get;

$$\begin{aligned} v\text{-}\lim_{\substack{k \to \overset{03}{1} \\ k^2 s(s+1) \to 0}} (k^{-2})\, \psi(skk\lambda'\lambda;\alpha) &= \psi(0\,\varepsilon\overset{03}{1}\ \overset{03}{1}\ \lambda\lambda;\alpha)\delta_{\lambda'\lambda} \\ v\text{-}\lim_{\substack{k \to \overset{03}{1} \\ -k^2 s(s+1) \to 0}} (-k^{-2})\, \psi(\varepsilon skk\lambda'\lambda;\alpha) &= \tfrac{1}{2}\psi(0\,\varepsilon\overset{03}{1}\ \overset{03}{1}\ \lambda\lambda;\alpha)\delta_{\lambda'\lambda} \end{aligned} \tag{20}$$

### *d) The Zero-momentum Point*

Let the momentum converge to zero, for example by setting $k = cq$ and $k' = cq'$ and letting $c$ go to zero. Then the matrix elements converge to those of the representations of $SL(2, C)$ induced by the respective representations of $SU(2)$, $ST(2)$ and $SU(1, 1)$. These induced representations are highly reducible, for instance a representation of $SU(2)$ of index $s$, induces a representation of $SL(2, C)$ which is a continuous sum of all the representations $\sigma_{m,\varrho}$ (with $\varrho \in \mathbb{R}$, $m = -s/2,\ -s/2+1, \ldots s/2$) of the principal series in the Naimark notation [9]. Therefore all the representations $\sigma_{m\varrho}$ are not separated in the dual space of the Poincaré group.

We can now exhibit the structure of the dual space $\hat{\mathscr{P}}$:

First, by a theorem of Bernat and Dixmier [11], the topology of $\hat{\mathscr{P}}$ is stronger than the one given by the plane $\{W^2 \times P^2\}$, i.e. by Fig. 2. According to our considerations, both topologies turn out to be identical. Non separated points are given by the three half axes:

a) $P^2 = 0$; $W^2 \leqq 0$: zero mass and zero momentum;

b) $P^2 < 0$: $W^2 = -\frac{1}{4}P^2$: imaginary mass and discrete half-integral spin, $s = -1/2$;

c) $P^2 < 0$; $W^2 = 0$: imaginary mass and discrete integral spin, $s = 0$.

We may explain the Jacobson topology in a simple way: take the space $\mathcal{M}$ of measures over $\mathcal{P}$, which has a separated vague topology. To each irreducible unitary representation of $\mathcal{P}$, associate a subset of $\mathcal{M}$ given by the vague closure of the matrix elements of the representation. The Jacobson topology of $\hat{\mathcal{P}}$ is the natural topology of these subsets of $\mathcal{M}$, therefore it can be very pathological.

## Chapter III. The Fourier Transform in $\mathcal{L}^1$ and $\mathcal{L}^2$ Spaces

### *1. $\mathcal{L}^1$-space Transform*

The generalized matrix elements are measures on the Poincaré group and their values on fixed function of $\mathcal{L}(\mathcal{P})$ are continuous functions of the parameters. But we can also generalize their values to integrable functions, i.e. define the Fourier transform in $\mathcal{L}^1(\mathcal{P})$. For definiteness we take (a measure defined by) a matrix element of physical mass and we define (with $M=\sqrt{k^2}$; $\varepsilon = k_0/|k_0|$, and $\varphi \in \mathcal{L}^1$)

$$\hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda) = \int_{\mathcal{P}} \psi(sk'k\lambda'\lambda;(x,X))\,\varphi(x,X)\,d^4x\,d^6X\,. \tag{21}$$

The matrix element is carried by the submanifold of equation $Xk=k'$, and for all $k'$ and almost all $k$ the function $\varphi$ is integrable, that is to say, its restriction to the submanifold is integrable:

$$\hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda) = \int e^{ik'\cdot x} f_{s\lambda'\lambda}(U)\,\varphi(x, H_{k'}^{-1}UH_k)\,d^4x\,d^3U\,. \tag{21a}$$

$\varphi(x,X)$ is integrable, and $\varphi(x,H_{k'}^{-1}X)$ too belongs to $\mathcal{L}^1(\mathcal{P})$. By the Fubini theorem, if one parametrizes by $X=UH_k$, for almost all values of $k$, $\varphi(x, H_{k'}^{-1}UH_k)$ is integrable for the measure $d^4x\,d^3U$ of the submanifold, and furthermore, the same is true for the matrix element as it has a finite continuous density $(=e^{ik'\cdot x} f_{s\lambda'\lambda}(U))$ when related to $d^4x\,d^3U$. The proof is the same for all the other cases, and we get

**Theorem 2.** *For any $\varphi \in \mathcal{L}^1(\mathcal{P})$, all $\nu\sigma$, all $\tau'$ and almost all $\tau$, the formula*

$$\hat{\varphi}(\nu\sigma;\tau'\tau) = \int_{\mathcal{P}} \mathcal{U}^{\nu\sigma}_{\tau'\tau}(\alpha)\,\varphi(\alpha)\,d^{10}\alpha \tag{22}$$

*makes sense, and defines a finite function $\hat{\varphi}$ of the parameters $\nu\sigma\tau'\tau$. ($\hat{\varphi}$ is called the Fourier transform of $\varphi$.)*

Let us call $\tilde{\mathcal{P}}$ the space of all the indices $\nu\sigma\tau'\tau$. First, $\hat{\varphi}$ is defined only almost everywhere, and is not a continuous function, it is not bounded and does not decrease at infinity. (However, $\hat{\varphi}$ is integrable for the measure $d^3\mathbf{k}/|k^2k_0|$, and when integrated, gives a continuous function of the remaining parameters, bounded by $\|\varphi\|_1$, and decreasing at infinity.) We see that the Fourier transform is more singular on the Poincaré group than in the Abelian case; this reflects the fact that the matrix elements are measures and not bounded continuous functions.

The explicit formulae are

$$\hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda) = \int e^{ik'\cdot x} f_{s\lambda'\lambda}(U)\, \varphi(x, H_{k'}^{-1} U H_k) d^4x\, d^3U\,, \quad (23\text{a})$$

$$\hat{\varphi}(0\varepsilon\varepsilon' r; k'\lambda' k\lambda) = \int e^{ik'\cdot x} f_{r\varepsilon'\lambda'\lambda}(S)\, \varphi(x, T_{k'}^{-1} S T_k) d^4x\, d^3S\,, \quad (23\text{b})$$

$$\hat{\varphi}(iMs\varepsilon'; k'\lambda' k\lambda) = \int e^{ik'\cdot x} f_{\varepsilon' s\lambda'\lambda}(O)\, \varphi(x, F_{k'}^{-1} O F_k) d^4x\, d^3O\,, \quad (23\text{c})$$

where $\varepsilon$ is the sign of energy ($\varepsilon = k_0/|k_0|$) and $\varepsilon'$ is a spin index ($\varepsilon' = 0, 1, +, -$), $M = \sqrt{|k^2|}$.

When one defines

$$\overline{\varphi}(k', X) = \int e^{ik'\cdot x} \varphi(x, X) d^4x$$

formulae (23) can be shortened by using $\overline{\varphi}(k', X)$, the only integration left is that over the stabilizator subgroup ($d^3U$ or $d^3S$ or $d^3O$).

## 2. $\mathscr{L}^2$-space Transform

We come now to the Fourier transform in $\mathscr{L}^2$ with the standard procedure.

For any $\varphi \in \mathscr{L}^1 \cap \mathscr{L}^2$, we can define $\hat{\varphi}$. Let us consider

$$N^2(X) = \int |\varphi(x, X)|^2 d^4x\,.$$

As $\varphi \in \mathscr{L}^2(\mathscr{P})$, for almost all $X$ and as a function of $x$, $|\varphi|^2$ is integrable and by classical Fourier theorems we have

$$N^2(X) = \frac{1}{(2\pi)^4} \int |\overline{\varphi}(k', X)|^2 d^4k'\,.$$

According to the values of $k'$, we use the parametrizations:

$$X = H_{k'}^{-1} U H_k; \qquad d^6X = \frac{d^3\boldsymbol{k}}{|k^2 k_0|} d^3U$$

(when $k'^2 > 0$; $k^2 = k'^2$ and $k_0 k'_0 > 0$),

$$X = F_{k'}^{-1} O F_k; \qquad d^6X = \frac{d^3\boldsymbol{k}}{|k^2 k_0|} d^3O$$

(when $k'^2 < 0$; $k'^2 = k^2$), and we set

$$\overline{\overline{\varphi}}(k', k; U) = \overline{\varphi}(k', H_{k'}^{-1} U H_k)\,,$$

$$\overline{\overline{\varphi}}(k', k; O) = \overline{\varphi}(k', F_{k'}^{-1} O F_k)\,.$$

$N^2(X)$ is integrable for the measure $d^6 X$ and we have

$$\|\varphi\|_2^2 = \int N^2(X) d^6 X = \int_{k'^2>0} |\bar{\bar{\varphi}}(k', k; U)|^2 d^4 k' \frac{d^3 \boldsymbol{k}}{|k^2 k_0|} d^3 U + \int_{k'^2<0} |\bar{\bar{\varphi}}(k', k; O)|^2 d^4 k' \frac{d^3 \boldsymbol{k}}{|k^2 k_0|} d^3 O . \tag{24}$$

In this formula $d^3\boldsymbol{k}/|k^2 k_0|$ is the invariant measure on the mass shell defined by $k'$, and the two integrals of the right-hand side are to be performed respectively on positive and negative square masse domains.

Now, again by the Fubini theorem, we can perform first the integration over the subgroups, and use the classical theorems of Peter Weyl [12] on $SU(2)$ or of Bargmann [2] on $SU(1,1)$ to replace these integrations by an integration over the dual spaces of these subgroups.

As we have

$$\hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda) = \int f_{s\lambda'\lambda}(U)\, \bar{\bar{\varphi}}(k', k; U) d^3 U \qquad (\text{when } k'^2 > 0),$$

$$\hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda) = \int f_{\varepsilon s\lambda'\lambda}(O)\, \bar{\bar{\varphi}}(k', k; O) d^3 O \qquad (\text{when } k'^2 < 0),$$

we use the Bessel-Parseval formulae:

$$\int_{SU(2)} |\bar{\bar{\varphi}}(k', k; U)|^2 d^3 U = \sum_{2s\in\mathbb{N}} \sum_{\lambda', \lambda\in I_{\varepsilon s}} (2s+1)\, |\hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda)|^2 , \tag{25}$$

$$\int_{SU(1,1)} |\bar{\bar{\varphi}}(k', k; O)|^2 d^3 O = \sum_{2s\in\mathbb{N}, \varepsilon=\pm} \sum_{\lambda', \lambda\in I_{s\varepsilon}} 2(2s+1)\, |\hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda)|^2 + \sum_{\varepsilon=0,1} \sum_{\lambda', \lambda\in I_{s\varepsilon}} \int_{\substack{0 \\ s=-\frac{1}{2}+i\varrho}}^{\infty} C_\varepsilon(\varrho)\, |\hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda)|^2 d\varrho . \tag{26}$$

We define the summation domains of $\lambda$ and $\lambda'$:

$$\begin{aligned} I_{\varepsilon s} &= \{-s, -s+1, \ldots +s\} \quad (\text{for formula (25)}), \\ I_{s\varepsilon} &= \{\varepsilon(s+1), \varepsilon(s+2), \ldots\} \quad \text{when} \quad \varepsilon = \pm , \\ I_{s\varepsilon} &= Z + \varepsilon/2 \quad \text{when} \quad \varepsilon = 0, 1 \quad (\text{for formula (26)}). \end{aligned} \tag{27}$$

One sums on $s$ with $2s\in\mathbb{N}$, and in the last integral of formula (26), $s = -\frac{1}{2} + i\varrho$, $C_\varepsilon(\varrho) = 4\varrho(\operatorname{th}\pi\varrho)^{1-2\varepsilon}$.

When the right-hand side expressions of formulae (25) and (26) are replaced in Eq. (24), we get a generalized Bessel-Parseval formula. It shows that the Fourier transform is an isometric mapping of $\mathscr{L}^1(\mathscr{P}) \cap \mathscr{L}^2(\mathscr{P})$ into the space of square integrable functions over the index space $\tilde{\mathscr{P}}$, with the natural measure as it appears in the formulae. By density considerations we can therefore define a Fourier-Plancherel transform from $\mathscr{L}^2(\mathscr{P})$ into $\mathscr{L}^2(\tilde{\mathscr{P}})$, and these formulae give an explicit construction of this transformation.

The inverse Fourier transform is easily obtained by classical theorems. For almost all $X$, we have

$$\varphi(x, X) = \frac{1}{(2\pi)^4} \int \bar{\varphi}(k', X) e^{-ik'x} d^4k' . \tag{28}$$

According to the sign of $k'^2$, one has

$$\begin{aligned} \bar{\varphi}(k', X) &= \bar{\bar{\varphi}}(k', k; U) \qquad (k'^2 > 0; X = H_{k'}^{-1} U H_k), \\ &= \bar{\bar{\varphi}}(k', k; O) \qquad (k'^2 < 0; X = F_{k'}^{-1} O F_k), \end{aligned}$$

and respectively

$$\begin{aligned} \bar{\bar{\varphi}}(k', k; U) &= \sum_{2s \in \mathbb{N}} \sum_{\lambda', \lambda \in I_{\varepsilon s}} (2s+1)\, \hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda) f^*_{s\lambda'\lambda}(U), \\ \bar{\bar{\varphi}}(k', k; O) &= \sum_{\substack{\varepsilon = \pm \\ 2s \in \mathbb{N}}} \sum_{\lambda', \lambda \in I_{s\varepsilon}} 2(2s+1)\, \hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda) f^*_{\varepsilon s\lambda'\lambda}(O), \\ &\quad + \sum_{\varepsilon = 0,1} \sum_{\lambda', \lambda \in I_{s\varepsilon}} \int_{\substack{0 \\ s = -\frac{1}{2} + i\varrho}}^{\infty} C_\varepsilon(\varrho)\, \hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda) f^*_{\varepsilon s\lambda'\lambda}(O) d\varrho \end{aligned} \tag{29}$$

(for the summation domains, see formulae (25), (26) and (27)).

All these formulae are to be taken as limits in $\mathscr{L}^2(\mathscr{P})$.

Formulae (28) to (29) define the inverse transform, which can be written with the matrix elements:

$$\begin{aligned} (2\pi)^4 \varphi(x, X) = & \sum_{2s \in \mathbb{N}} \sum_{\lambda, \lambda' \in I_{s\varepsilon}} (2s+1) \int_{k'^2 > 0} \psi^*(sk'k\lambda'\lambda; (x, X)) \\ & \cdot \hat{\varphi}(M\varepsilon s; k'\lambda' k\lambda)\, d^4k' \frac{d^3\boldsymbol{k}}{|k^2 k_0|} \\ & + \sum_{\varepsilon = \pm, 2s \in \mathbb{N}} \sum_{\lambda', \lambda \in I_{\varepsilon s}} 2(2s+1) \int_{k'^2 < 0} \psi^*(\varepsilon sk'k\lambda'\lambda; (x, X)) \\ & \cdot \hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda)\, d^4k' \frac{d^3\boldsymbol{k}}{|k^2 k_0|} \\ & + \sum_{\varepsilon = 0,1} \sum_{\lambda', \lambda \in I_{\varepsilon s}} \int_{\substack{0 \\ s = -\frac{1}{2} + i\varrho}}^{\infty} C_\varepsilon(\varrho)\, d\varrho \\ & \cdot \int_{k'^2 < 0} \psi^*(\varepsilon sk'k\lambda'\lambda; (x, X))\, \hat{\varphi}(iMs\varepsilon; k'\lambda' k\lambda)\, d^4k' \frac{d^3\boldsymbol{k}}{|k^2 k_0|} . \end{aligned} \tag{30}$$

The summation domains of $\lambda$ and $\lambda'$ are given in (27) and $d^3\boldsymbol{k}/|k^2 k_0|$ is the invariant measure on the mass shell defined by $k'$.

We may give another form for the inverse formula by making the integration over $d^3\mathbf{k}/|k^2 k_0|$:

$$
\begin{aligned}
(2\pi)^4\,\varphi(x,X) = & \sum_{2s\in\mathbb{N}} \sum_{\lambda,\lambda'\in I_{s\varepsilon}} (2s+1) \int_{k^2>0} e^{-ik\cdot x} f^*_{s\lambda'\lambda}(H_k X H^{-1}_{X^{-1}k}) \\
& \cdot \hat{\varphi}(M\varepsilon s; k\lambda' X^{-1}k\lambda)\,d^4k \\
& + \sum_{\varepsilon=\pm,2s\in\mathbb{N}} \sum_{\lambda',\lambda\in I_{\varepsilon s}} 2(2s+1) \int_{k^2<0} e^{-ik\cdot x} f^*_{\varepsilon s\lambda'\lambda}(F_k X F^{-1}_{X^{-1}k}) \\
& \cdot \hat{\varphi}(iMs\varepsilon; k\lambda' X^{-1}k\lambda)\,d^4k \\
& + \sum_{\varepsilon=0,1} \sum_{\lambda',\lambda\in I_{\varepsilon s}} \int_{\substack{0\\ s=-\frac{1}{2}+i\varrho}}^{\infty} C_\varepsilon(\varrho)\,d\varrho \\
& \cdot \int_{k^2<0} e^{-ik\cdot x} f^*_{\varepsilon s\lambda'\lambda}(F_k X F^{-1}_{X^{-1}k})\,\hat{\varphi}(iMs\varepsilon; k\lambda' X^{-1}k\lambda)\,d^4k\,.
\end{aligned}
\tag{30a}
$$

We can state:

**Theorem 3.** *There exists a Fourier-Plancherel transform, which is an isometric mapping between $\mathscr{L}^2(\mathscr{P})$ and the space $\mathscr{L}^2(\tilde{\mathscr{P}})$ of square integrable functions over the index space $\tilde{\mathscr{P}}$, with a natural measure as it appears through formulae* (24), (25) *and* (26). *This mapping is given by formulae* (23) *and* (30).

We may notice that the measure on $\tilde{\mathscr{P}}$ is related to the measures on the space $\mathscr{T}_{v\sigma}$ of $(k,\lambda)$, which are defined in Part I.

The Fourier-Plancherel transform on the Poincaré group has been studied by Rideau [13]. By classical theorems, one knowns this transformation to exist: The Poincaré group is postliminar [8, 13, 14] and such a group admits a Fourier-Plancherel transform (in $\mathscr{L}^2$) [16].

The unitary representations of the Poincaré group which contribute to this transform are given by the indices $(v,\sigma)$ which can be classified in three subsets of $\hat{\mathscr{P}}$:

$$
\begin{aligned}
&\text{a) } v^2>0; \quad \sigma=(\varepsilon,s) \quad \text{with} \quad \varepsilon=\pm\,;\ 2s\in\mathbb{N}\,,\\
&\text{b) } v^2<0; \quad \sigma=(s,\varepsilon) \quad \text{with} \quad \varepsilon=\pm\,;\ 2s\in\mathbb{N}\,,\\
&\text{c) } v^2<0; \quad \sigma=(s,\varepsilon) \quad \text{with} \quad \varepsilon=0,1\,;\ s=-\tfrac{1}{2}+i\varrho \quad \text{with} \quad \varrho>0\,.
\end{aligned}
\tag{31}
$$

These subsets are open in $\hat{\mathscr{P}}$, and can be taken as the carrier (which is called the Plancherel set) of the Plancherel measure we introduce now:

$$
\begin{aligned}
d\mu(v,\sigma) &= \frac{v^2\,dv^2}{2(2\pi)^4} \sum_{2s\in\mathbb{N}} (2s+1)\delta_s \quad \text{(on the set 31a)}\,,\\
&= \frac{v^2\,dv^2}{(2\pi)^4} \sum_{2s\in\mathbb{N}} (2s+1)\delta_s \quad \text{(on the set 31b)}\,,\\
&= \frac{v^2\,dv^2}{2(2\pi)^4}\, C_\varepsilon(\varrho)\,d\varrho \qquad \text{(on the set 31c)}\,.
\end{aligned}
\tag{32}
$$

We recall that $v^2$ is the eigenvalue of the square mass operator $P^2$, $\delta_s$ is the unit mass at the point $s$ of the discrete space $\mathbb{N}$, and $d\varrho$ is the Lebesgue measure on $\mathbb{R}$.

In Part I, we introduced the space $\mathscr{T}_{v\sigma}$ of the indices $\tau = (k, \lambda)$ with a measure

$$d\mu(\tau) = \frac{d^3\boldsymbol{k}}{|k^2 k_0|} \sum_{\lambda \in I_\sigma} \delta_\lambda \quad \text{(see formulae (27))}$$

and the generalized matrix elements, denoted by $\mathscr{U}^{v\sigma}_{\tau'\tau}(\alpha)$, act in the space $\mathscr{L}^2(\mathscr{T}_{v\sigma}, d\mu(\tau))$. Now for each integrable function $\varphi$, we have defined its Fourier transform which can be denoted by

$$\mathscr{U}^{v\sigma}_{\tau'\tau}(\varphi) = \int_{\mathscr{P}} \mathscr{U}^{v\sigma}_{\tau'\tau}(\alpha)\, \varphi(\alpha)\, d^{10}\alpha\,. \tag{33}$$

For all $\tau'$ and almost $\tau$, this function is defined (and finite). The space $\tilde{\mathscr{P}}$ of indices is the natural space for the indices $v\sigma\tau'\tau$, and the space $\mathscr{L}^2(\tilde{\mathscr{P}})$ has a very simple meaning. Let us denote by $\mathscr{H}_{v\sigma}$ the Hilbert space of Hilbert-Schmidt operators on $\mathscr{L}^2(\mathscr{T}_{v\sigma}, d\mu(\tau))$, (with the norm $\langle \mathscr{U}^{v\sigma}, \mathscr{V}^{v\sigma}\rangle = \mathrm{Trace}\,\mathscr{U}^{v\sigma\dagger}\cdot \mathscr{V}^{v\sigma}$). The space $\mathscr{L}^2(\tilde{\mathscr{P}})$ is then the Hilbert integral (see for example Ref. [17]) of the spaces $\mathscr{H}_{v\sigma}$ with the measure $d\mu(v, \sigma)$:

$$\mathscr{L}^2(\tilde{\mathscr{P}}) = \int_{\hat{\mathscr{P}}}^{\oplus} \mathscr{H}_{v\sigma}\, d\mu(v, \sigma)\,. \tag{34}$$

For any square integrable function $\varphi$, formula (33) gives the generalized matrix elements of an operator $\mathscr{U}^{v\sigma}(\varphi)$ acting in $\mathscr{L}^2(\mathscr{T}_{v\sigma}, d\mu(\tau))$, which is defined for almost all $v\sigma$, and a decomposable operator

$$\mathscr{U}(\varphi) = \int_{\hat{\mathscr{P}}}^{\oplus} \mathscr{U}^{v\sigma}(\varphi)\, d\mu(v, \sigma)$$

acting on the space

$$\int_{\hat{\mathscr{P}}}^{\oplus} \mathscr{L}^2(\mathscr{T}_{v\sigma}, d\mu(\tau))\, d\mu(v, \sigma)\,.$$

Formula (24) then becomes

$$\begin{aligned} \int_{\mathscr{P}} |\varphi(\alpha)|^2\, d^{10}\alpha &= \int_{\hat{\mathscr{P}}} \langle \mathscr{U}^{v\sigma}(\varphi), \mathscr{U}^{v\sigma}(\varphi)\rangle\, d\mu(v, \sigma) \\ &= \int_{\hat{\mathscr{P}}} \mathrm{Trace}\,\mathscr{U}^{v\sigma}(\varphi)^\dagger\, \mathscr{U}^{v\sigma}(\varphi)\, d\mu(v, \sigma) \end{aligned} \tag{35}$$

and the inversion formula (30) also:

$$\varphi(\alpha) = \int_{\hat{\mathscr{P}}} \mathrm{Trace}\,\mathscr{U}^{v\sigma}(\varphi)\, \mathscr{U}^{v\sigma}(\alpha)^\dagger\, d\mu(v, \sigma)\,, \tag{36}$$

while the Fourier transform formula can be rewritten:

$$\mathcal{U}^{\nu\sigma}(\varphi) = \int_{\mathcal{P}} \mathcal{U}^{\nu\sigma}(\alpha)\, \varphi(\alpha)\, d^{10}\alpha\,. \tag{37}$$

Formulae (35), (36), (37) give the results in their usual form in harmonic analysis [16].

In this very condensed notation, we list some properties of the Fourier transform:

$$\begin{aligned}
\mathcal{U}(I\varphi) &= \mathcal{U}(\varphi^*)^\dagger \\
\mathcal{U}(\varphi * \psi) &= \mathcal{U}(\varphi)\,\mathcal{U}(\psi) \qquad (\varphi \text{ and } \psi \in \mathcal{L}^1(\mathcal{P})) \\
\mathcal{U}(U(\alpha)\varphi) &= \mathcal{U}(\alpha)\,\mathcal{U}(\varphi) \\
\mathcal{U}(V(\alpha)\varphi) &= \mathcal{U}(\varphi)\,\mathcal{U}^\dagger(\alpha) \\
\int_{\mathcal{P}} \varphi(\alpha)\,\overline{\psi}(\alpha)\, d^{10}\alpha &= \int_{\hat{\mathcal{P}}} \mathrm{Trace}\,\mathcal{U}^{\nu\sigma}(\varphi)\,\mathcal{U}^{\nu\sigma}(\psi)^\dagger\, d\mu(\nu,\sigma)
\end{aligned} \tag{38}$$

($\varphi$ and $\psi \in \mathcal{L}^2(\mathcal{P})$) and we recall the definitions of:

$$\begin{aligned}
I:&\quad (I\varphi)(\alpha) = \varphi(\alpha^{-1}) \\
U(\alpha):&\quad (U(\alpha)\varphi)(\beta) = \varphi(\alpha^{-1}\beta) \\
V(\alpha):&\quad (V(\alpha)\varphi)(\beta) = \varphi(\beta\alpha)\,.
\end{aligned} \tag{39}$$

## *3. The Bochner Theorem on Positive Type Functions*

A function $\varphi$ on a group is said of positive type if for any set of $n$ points $\alpha_1 \dots \alpha_n$ of the group, the $n \times n$ matrix whose elements are $\varphi(\alpha_i \alpha_j^{-1})$, is Hermitian positive. Such a function $\varphi$ fulfills the following relations

$$\begin{aligned}
&|\varphi(\alpha)| \leqq \varphi(\varepsilon) \quad (\varepsilon = \text{unit element of the group}), \\
&I\varphi = \varphi^* \qquad \text{(see formula (39))}, \\
&|\varphi(\alpha) - \varphi(\beta)|^2 \leqq 2\varphi(\varepsilon)\,\mathrm{Re}(\varphi(\varepsilon) - \varphi(\alpha\beta^{-1}))\,.
\end{aligned}$$

One has the following generalization of the

**Bochner Theorem** [18]. *Let $G$ be a separable, locally compact, type I group; a function is of positive type on $G$ if and only if one has*

$$\varphi(\alpha) = \int_{\hat{G}} \mathrm{Trace}\,\mathcal{U}_\zeta^\dagger(\alpha)\, d\varrho(\zeta)\,,$$

*where $\hat{G}$ is the dual space of $G$, whose points, denoted by $\zeta$, are equivalence classes of irreducible unitary representations $\mathcal{U}_\zeta$, and $d\varrho(\zeta)$ is a Hermitian positive operator valued measure of finite mass (i.e. $\int_{\hat{G}} \mathrm{Trace}\, d\varrho(\zeta) < \infty$).*

This theorem is valid for the Poincaré group [15] and gives a Fourier representation for any positive type function $\varphi$:

$$\varphi(\alpha) = \int_{\hat{\mathscr{P}}} \mathrm{Trace}\, \mathscr{U}^{\nu\sigma}(\alpha)^{\dagger}\, d\varrho(\nu\sigma). \tag{40}$$

But in the above theorem the unicity of the measure $d\varrho(\nu\sigma)$ is not established. Let us recall the proof (in the notation of Ref. [8]).

To any function $\varphi$ of positive type, one associates a representation $\pi_\varphi$ and a cyclic vector $\xi_\varphi$ such that

$$\varphi(\alpha) = \langle \pi_\varphi(\alpha)\, \xi_\varphi, \xi_\varphi \rangle$$

and the couple $(\pi_\varphi, \xi_\varphi)$ is unique within an isomorphism. As the group is of type I, one may expand $\pi_\varphi$ into multiples of irreducible representations in a unique way: there exist positive measures $\mu_1, \mu_2, \ldots \mu_\infty$ on $\hat{\mathscr{P}}$, foreign one to another, such that [1]

$$\begin{aligned} \pi_\varphi &\simeq \int^{\oplus} \zeta\, d\mu_1(\zeta) \oplus 2 \int^{\oplus} \zeta\, d\mu_2(\zeta) \oplus \cdots \oplus \aleph_0 \int^{\oplus} \zeta\, d\mu_\infty(\zeta) \\ \xi_\varphi &= \int^{\oplus} \xi_1(\zeta)\, d\mu_1(\zeta) \oplus \int^{\oplus} \xi_2(\zeta)\, d\mu_2(\zeta) \oplus \cdots \oplus \int^{\oplus} \xi_\infty(\zeta)\, d\mu_\infty(\zeta) \end{aligned} \tag{41}$$

where $\xi_n(\zeta) \in \mathscr{H}(\zeta) \oplus \cdots \oplus \mathscr{H}(\zeta)$ ($n$ times), and can be written as direct sum of $n$ vectors $\xi_{ni}(\zeta) \in \mathscr{H}(\zeta)$:

$$\xi_n(\zeta) = \bigoplus_{i=1}^{n} \xi_{ni}(\zeta). \tag{42}$$

The measure $d\varrho(\zeta)$ is defined by

$$d\varrho(\zeta) = \sum_n \sum_{i=1}^{n} |\xi_{ni}(\zeta)\rangle \langle \xi_{ni}(\zeta)|\, d\mu_n(\zeta). \tag{43}$$

The unicity of the measure $d\varrho(\zeta)$ is obtained if one sees that the measure $d\varrho(\zeta)$ gives the decomposition (41) within an equivalence. The measure $d\mu_n(\zeta)$ is equivalent to the trace of $d\varrho_n(\zeta)$ ($d\varrho_n(\zeta)$ is the restriction of $d\varrho(\zeta)$ to the subset of $\hat{G}$ where $d\varrho(\zeta)$ is of rank $n$); furthermore, formula (43) determines the vectors $\xi_{ni}(\zeta)$ up to a constant factor and a substitution, that is to say, determines $\pi_\varphi$ and $\xi_\varphi$. The unicity of the representation $(\pi_\varphi, \xi_\varphi)$ implies the unicity of the measure $d\varrho(\zeta)$. So we may add a

**Proposition.** The measure $d\varrho(\zeta)$ of the Bochner Theorem is unique.

We therefore get a Fourier transform for positive type functions, but this does not show the complicated structure of the dual space. Furthermore, one needs the Fourier transform for distributions if one intends to look at the Fourier transform for infinitesimal operators for example, and these operators have all a physical significance! That is the reason why we hope to study this generalization in a following and last paper.

[1] For the notations and the theorems which are used here, see Ref. [8], § 8.6.

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