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Number 128

Harmonic Analysis

Smooth and Non-smooth

Palle E.T. Jorgensen

AMS American Mathematical Society

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First row, from left to right: Kasso Okoudjou, Palle Jorgensen, Daniel Alpay, and Marius Ionescu. (Credit: Connie S. Steffen, MATH, Iowa State University.)

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Dedicated to the memory of Kiyosi Itô

Contents

Preface

> Mathematics is an experimental science, and definitions do not come first, but later on.
>
> — Oliver Heaviside (1850–1925)

Traditionally the notion of harmonic analysis has centered around analysis of transforms and expansions, and involving *dual variables.* The area of partial differential equations (PDE) has been a source of motivation and a key area of applications; dating back to the days of Fourier. Or rather, the applications might originate in such neighboring areas as signal processing, diffusion equations, and in more general applied inverse problems. The dual variables involved are typical notions of *time* vs *frequency*, or *position* vs *momentum* (in quantum physics). As most students know, Fourier series and Fourier transforms have been a mainstay in analysis courses we teach. In the case of Fourier series, and Fourier transforms, we refer to the variables involved as dual variables. If a function in an x-domain admits a Fourier expansion, the associated transform will be a function in the associated dual variable, often denoted λ, in what is to follow.

Now the x-domain may involve a suitable subset Ω in $\mathbb{R}^d$. The aim of Fourier harmonic analysis is *orthogonal* L^2 *Fourier expansions*, at least initially. Alternatively, the x-domain may refer to a prescribed *measure*, say μ with compact support in $\mathbb{R}^d$. These settings are familiar, at least in the classical case, which we shall here refer to as the "smooth case." Now if the measure μ might be *fractal*, say a Cantor measure, an *iterated function system* (IFS) *measure*, e.g., a Sierpinski construction, then it is not at all clear that the familiar setting of Fourier duality will yield useful orthogonal L^2 decompositions. Take for example the Cantor IFS constructions, arising from scaling by 3, and one gap; or the corresponding IFS measure resulting from scaling by 4, but now with two gaps. In a paper in 1998, Jorgensen and Pedersen showed that the first of these Cantor measures does not admit any orthogonal L^2 Fourier series, while the second does. In the two decades that followed, a rich theory of harmonic analysis for fractal settings has ensued.

If a set Ω, or a measure μ, admits an L^2 spectrum, we shall talk about *spectral pairs* (Ω, Λ), or (μ, Γ), where the second set in the pair will be called a *spectrum.* If the first variable arises as a *fractal in the small*, we will see that associated spectra will arise as *fractals in the large*; in some cases as lacunary Fourier expansions, series with large gaps, or lacunae, between the non-zero coefficients in expansions. We will focus on Fourier series with similar gaps between non-zero coefficients, gaps being a power of a certain scale number. There is a slight ambiguity in modern usage of the term lacunary series. When needed, clarification will be offered in the notes.

In addition to harmonic analyses via *Fourier duality*, there are also *multiresolution wavelet* approaches; work by Dutkay and Jorgensen. In the notes, both will be developed systematically, and it will be demonstrated that the wavelet tools are more flexible but perhaps not as precise for certain fractal applications. A third tool for our fractal harmonic analysis will be L^2 spaces derived from appropriate *Gaussian processes* and their analysis.

In our development of some of these duality approaches, or multiresolution analysis constructions, we shall have occasion to rely on certain *non-commutative* harmonic analyses. They too will be developed from scratch (as needed) in the notes.

The present book is based on a series of 10 lectures delivered in June 2018 at Iowa State University. I am extremely grateful to Feng Tian, who was a big help organizing both the visual material used in the 10 lectures and the text we ended up using for the book.

In addition to my 10 lectures, this CBMS also included the following three featured speakers: Kasso Okoudjou (University of Maryland), Marius Ionescu (United States Naval Academy), and Daniel Alpay (Chapman University). These speakers highlighted connections between the present central themes and a variety of neighboring, current areas of mathematics and its applications. Some relevant references are the following: [**AS17**, **ACQS17**, **AL18**, **AG18**, **IRT12**, **IRS13**, **IR14**, **IOR17**, **WO17**, **BBCO17**].

Palle Jorgensen,
June 2018, University of Iowa

Acknowledgments

I wish to thank the NSF for supporting the present CBMS series of lectures at Iowa State University (ISU). I am also extremely grateful to the three local organizers, Professors Eric Weber, John Herr, and Justin Peters, and to the staff at the ISU mathematics department, especially Ms Connie Steffen. They all did a fantastic job making the conference run smoothly and efficiently and making the participants happy. The 10 CBMS lectures were preceded by a weekend boot camp for students, where colleagues, including graduate students, presented prerequisite material needed in the 10 lectures. This turned out to serve as an extremely effective outreach tool, attracting undergraduate students into mathematics.

Many of the main results presented in the 10 lectures constitute joint research with Jorgensen and co-authors. Two co-authors in some of the initial Jorgensen et al papers are Steen Pedersen and Dorin Dutkay, but there are also many other co-authors involved in subsequent research. They are all cited inside the book. Since the total list is long (see Bibliography), we refer the reader to the many places inside the book where this joint research is cited.

Finally, I wish to thank Feng Tian for his great work with both the visuals for the 10 lectures themselves and the resulting book.

Palle Jorgensen, University of Iowa

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CHAPTER 1

Introduction. Smooth vs the non-smooth categories

> I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.
> — Descartes, René (1596-1650)

There is a recent and increasing interest in understanding the harmonic analysis of non-smooth geometries, typically fractal like. They are unlike the familiar smooth Euclidean geometry. In the non-smooth case, nearby points are not locally connected to each other. Real-world examples where these types of geometry appear include large computer networks, relationships in datasets, and fractal structures such as those found in crystalline substances, light scattering, and other natural phenomena where dynamical systems are present.

The book is based on a series of lectures on smooth and non-smooth harmonic analysis by the author. It aims to demonstrate surprising connections between the two domains of geometry and Fourier spectra, and to bring both experienced and new researchers together to stimulate collaboration on this timely topic. It also aims to advance representation and participation of underrepresented minorities within mathematics, and the development of a globally competitive STEM workforce.

1.1. Preview

Smooth harmonic analysis refers to harmonic analysis over a connected or locally connected domain — typically Euclidean space or locally connected subsets of Euclidean space. The classical example of this is the existence of Fourier series expansions for square integrable functions on the unit interval. *Non-smooth harmonic analysis then refers to harmonic analysis on discrete or disconnected domains* — typical examples of this setting are *Cantor like subsets* of the real line and analogous *fractals in higher dimensions*. In 1998, Jorgensen and Steen Pedersen proved a result: there exists a Cantor like set (of Hausdorff dimension $1/2$) with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the exponentials form an orthonormal basis in the Hilbert space of square integrable functions with respect to that measure. This surprising result, together with results of Robert Strichartz, has lead to a plethora of new research directions in non-smooth harmonic analysis.

Research that has been inspired by this surprising result includes: *fractal Fourier analyses* (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen's research on the *cascade algorithm*, as wavelets already possess a "dual" existence in the *continuous* and *discrete* worlds, and also his research on the *boundary representations* of *reproducing kernels*, as the non-smooth domains appear as boundaries of smooth domains. In work with Dorin Dutkay, Jorgensen showed that the general affine IFS-systems, even if not amenable to Fourier analysis, in fact do admit wavelet bases, and so in particular can be analyzed with the use of multiresolutions. In recent work with Herr and Weber, Jorgensen has shown that fractals that are not spectral (and so do not admit an orthogonal Fourier analysis) still admits a harmonic analysis as boundary values for certain subspaces of the Hardy space of the disc and the corresponding reproducing kernels within them.

The book covers the following overarching themes: the *harmonic analysis* of Cantor spaces (and measures) arising as *fractals* (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. A consequence of the fact that these methods form a bridge between the smooth and non-smooth domain is that the topics to be discussed — while on the surface seem largely unrelated — actually are closely related and together form a tightly focused theme. Hopefully, the breadth of topics will attract a broader audience of established researchers, while the interconnectedness and sharply focused nature of these topics will prove beneficial to beginning researchers in non-smooth harmonic analysis.

Inside the book, we cover a number of theorems due to a diverse list of authors and co-authors. In some cases, the authors and co-authors are simply identified by name; in some cases, if it isn't clear from the context, also one or more research papers are cited. In the latter case, we use the usual citation codes; for example, the paper [**DJP09**] is co-authored by Dorin Ervin Dutkay, Palle E. T. Jorgensen, and Gabriel Picioroaga, and appeared in 2009. And, of course, full details are included in the Reference list. Yet for other theorems, the co-authors' names are listed in parenthesis, in the statement of the theorem itself. For example: Theorem 3.3.9 (Jo-Pedersen); with my name Jorgensen abbreviated "Jo."

The 10 lectures. The material in the present book corresponds to the areas covered in the 10 lectures. But, for pedagogical reasons, we chose to organize the material a bit differently in the book. Readers may wish to compare the book-form *table of contents* with the title of the 10 lectures. The latter list is included below (also see Figure 1.1.1):

Lecture 1.	Harmonic analysis of measures: Analysis on fractals
Lecture 2.	*Spectra of measures*, tilings, and wandering vectors
Lecture 3.	*The universal tiling conjecture* in dimension one and operator fractals
Lecture 4.	*Representations of Cuntz algebras* associated to quasi-stationary Markov measures
Lecture 5.	The Cuntz relations and *kernel decompositions*
Lecture 6.	Harmonic analysis of *wavelet filters*: input-output and state-space models
Lecture 7.	Spectral theory for *Gaussian processes*: reproducing kernels, boundaries, and L^2-wavelet generators with fractional scales

Lecture 8.	Reproducing kernel Hilbert spaces arising from groups
Lecture 9.	Extensions of positive definite functions
Lecture 10.	*Reflection positive* stochastic processes indexed by Lie groups

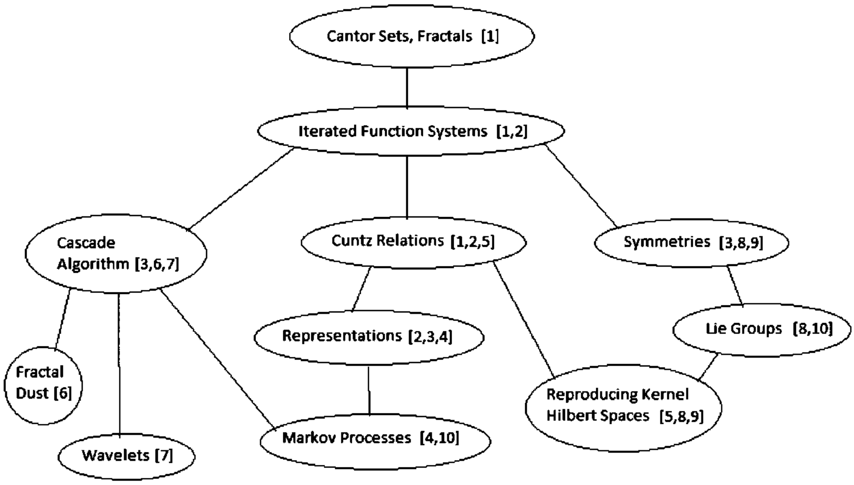


FIGURE 1.1.1. Flow and Connections of Topics: The figure gives a bird's eye view of the main topics in the book, and the lines indicate interconnections. They will be fleshed out in full detail inside the book. The numbers above, inside square brackets, indicate which of the 10 lectures cover the topic in question. The complete title of each of the 10 lectures is listed in the table above.

1.2. Historical context

One of the most fruitful achievements of mathematics in the past two hundred years has been the development of *Fourier series*. Such a series may be thought of as the decomposition of a periodic function into sinusoid waves of varying frequencies. Application of such decompositions are naturally abundant, with waves occurring in all manner of physics, and uses for periodic functions being present in other areas such as *economics* and *signal processing*, just to name a few. The importance of Fourier series is well-known and incontestable.

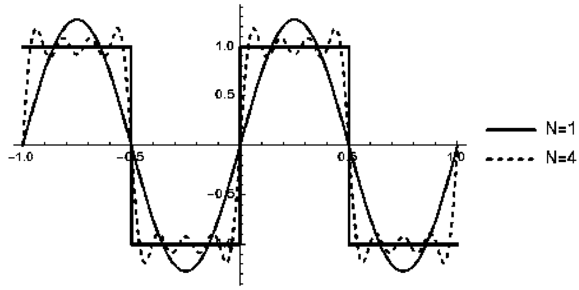
Fourier series. While to many non-mathematicians and undergraduate math majors, a Fourier series is regarded as a breakdown into sine and cosine waves, the experienced analyst will usually think of it (equivalently), as a decomposition into sums of complex exponentials. For instance, in the classical setting of the unit interval $[0,1)$, a Lebesgue integrable function $f:[0,1)\to\mathbb{C}$ will induce a Fourier series

$$f(x)\sim\sum\nolimits_{n\in\mathbb{Z}}\hat{f}(n)\,e^{i2\pi nx} \tag{1.2.1}$$

where

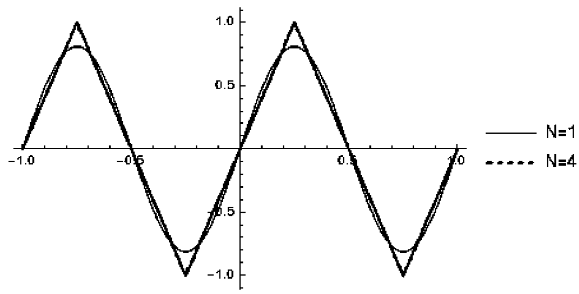
$$\hat{f}(n):=\int_0^1 f(x)\,e^{-i2\pi nx}dx. \tag{1.2.2}$$

See Figures 1.2.1-1.2.2.



$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin(2\pi(2k-1)x)}{2k-1}$$

FIGURE 1.2.1. Fourier series approximation of square wave. The figure illustrates the known difficulty with Fourier series approximation of step functions. In view of this, it seems even more surprising that some fractals admit convergent Fourier series (see Section 2.1.) Also compare the function in Figure 1.2.1 with the mother function for the Haar wavelet, see Section 4.3.



$$f(x) \sim \frac{8}{\pi^2} \sum_{k=1}^{N} \frac{(-1)^{k-1} \sin(2\pi(2k-1)x)}{(2k-1)^2}$$

FIGURE 1.2.2. Fourier series approximation of triangle wave. Of course, here Fourier yields a good fit. But it is also clear that it is not good from the point of view of numerical analysis. For example, most wavelet algorithms will do a lot better; see Section 4.3.

Because the Fourier series is intended to represent the function $f(x)$, it is only natural to ask in what senses, if any, the sum above converges to $f(x)$. One can ask important questions about *pointwise convergence*, but it is more relevant for our purposes to restrict attention to various *normed spaces* of functions or, as we will be most concerned with hereafter, a Hilbert space consisting of square-integrable functions, and then ask about norm convergence. In our present context, if we let $L^2([0,1))$ denote the Hilbert space of (equivalence classes of) functions

$f : [0,1) \to \mathbb{C}$ satisfying

$$\|f\|^2 := \int_0^1 |f(x)|^2 \, dx < \infty \tag{1.2.3}$$

and equipped with the inner product

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx, \tag{1.2.4}$$

then if $f \in L^2([0,1))$, the convergence in (1.2.1) will occur in the norm of $L^2([0,1))$. It is also easy to see that in $L^2([0,1))$,

$$\left\langle e^{i2\pi mx}, e^{i2\pi nx} \right\rangle = \int_0^1 e^{i2\pi mx} e^{-i2\pi mx} dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.5}$$

That is, the set of complex exponentials $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is *orthogonal* in $L^2([0,1))$. Since every function in $L^2([0,1))$ can be written in terms of these exponentials, $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is in fact an *orthonormal basis* of $L^2([0,1))$.

Because there exists a countable set of complex exponential functions that form an orthogonal basis of $L^2([0,1))$, we say that the set $[0,1)$ is *spectral*. The set of frequencies of such an orthogonal basis of exponentials, which is this case is $\mathbb{Z}$, is called a *spectrum*.

Like most areas of analysis, the historical and most common contexts for Fourier series are also the most mundane: The functions they decompose are defined on $\mathbb{R}$, the unit interval $[0,1)$, or sometimes a discrete set. The underlying measure used for integration is *Lebesgue measure*. It is thanks to the work of many individuals, including the author, that modern Fourier analysis has been able to aspire beyond these historical paradigms. Table 1 below provides an overview of *generalized Fourier duality*.

The first paradigm break is to consider a wider variety of domains in a wider variety of dimensions. In general, if C is a compact subset of $\mathbb{R}^n$ of nonzero Lebesgue measure, then we say that C is spectral if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that $\left\{e^{i2\pi\lambda\cdot\vec{x}}\right\}_{n\in\Lambda}$ is an orthogonal basis of $L^2(C)$, where

$$L^2(C) := \left\{ f : C \to \mathbb{C} \mid \int_C |f(\vec{x})|^2 \, d\lambda^n(\vec{x}) < \infty \right\}. \tag{1.2.6}$$

Here λ^n is Lebesgue measure in $\mathbb{R}^n$.

Fuglede's conjecture. The famous Fuglede Conjecture surmised that C would be spectral if and only if it would tessellate by translation to cover $\mathbb{R}^n$. Iosevich, Katz, and Tao proved in 2001 that the conjecture holds for convex planar domains [**IKT03**]. In the same year, they also proved that a smooth, symmetric, convex body with at least one point of nonvanishing Gaussian curvature cannot be spectral [**IKT01**]. However, in 2003 Tao devised counterexamples to the Fuglede Conjecture in $\mathbb{R}^5$ and $\mathbb{R}^{11}$ [**Tao04**]. The conjecture remains open in low dimensions.

The second paradigm break is to substitute a different Borel measure in place of Lebesgue measure. For example, if μ is any Borel measure on $[0,1)$, one can form the Hilbert space

$$L^2(\mu) = \left\{ f : [0,1) \to \mathbb{C} \mid \int_0^1 |f(x)|^2 \, d\mu(x) < \infty \right\} \tag{1.2.7}$$

TABLE 1. Harmonic analysis of measures with the use of *Fourier bases*, *Parseval frames*, or *generalized transforms*: An overview of generalized Fourier duality: Measures vs spectra.

	Measure side	Spectrum side ($e_\lambda(x) = e^{i2\pi\lambda\cdot x}$)
1	$\Omega \subset \mathbb{R}^d$ a Borel set with finite d-dimensional Lebesgue measure λ_d	$\Lambda \subset \mathbb{R}^d$, a subset such that $\{e_\lambda \mid \lambda \in \Lambda\}$ restricts to an orthogonal total system in $L^2(\Omega)$ (w.r.t. λ_d).
2	μ a compactly supported measure in $\mathbb{R}^d$	$\Lambda \subset \mathbb{R}^d$, a subset such that $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthogonal $L^2(\mu)$ basis.
3	μ as above, but $d = 1$, μ assumed singular	$\{g_n \mid n \in \mathbb{N}_0\} \subset L^2(\mu)$ is a Parseval frame, i.e., $\Vert f\Vert^2_{L^2(\mu)} = \sum_0^\infty \vert\langle f, g_n\rangle_{L^2(\mu)}\vert^2$ with $L^2(\mu)$ expansion for $f \in L^2(\mu)$: $f(x) = \sum_0^\infty \langle f, g_n\rangle_{L^2(\mu)} e_n(x)$, i.e., summation over $n \in \mathbb{N}_0$.
4	Symmetric case: μ, ν two Borel measures in $\mathbb{R}^d$ such that $(F_\mu f)(\xi) = \int_{\mathbb{R}^d} f(x)\, e_\xi(x)\, d\mu(x)$, defines an isometric isomorphism onto $L^2(\nu)$, i.e., $$\int_{\mathbb{R}^d} \vert F_\mu(f)(\xi)\vert^2 d\nu(\xi) = \int_{\mathbb{R}^d} \vert f(x)\vert^2 d\mu(x), \ \forall f \in L^2(\mu).$$	

with inner product

$$\langle f, g\rangle_\mu = \int_0^1 f(x)\,\overline{g(x)}\,d\mu(x). \tag{1.2.8}$$

Comparing equations (1.2.7) and (1.2.8) with equations (1.2.3) and (1.2.4), respectively, we see that we can then regard spectrality as a property of measures rather than of sets: The measure μ is spectral if there exists a countable index set Λ such that the set of complex exponentials $\{e^{i2\pi\lambda x}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\mu)$. The index set Λ is then called a spectrum of μ.

Guide to readers. Inside the text in present Introduction, we have been, and will be, using some technical terms which might perhaps not be familiar to all readers. For example, the notion of *iterated function systems* (IFS) are mentioned, and they will be explored in detail in Section 1.3 below, and again later in Chapters 2, 3, and 6. In the present discussion, around the themes of Figure 1.2.4 and the table, we use the concept of *selfadjoint extensions* of densely defined Hermitian symmetric operators in Hilbert space. This is from the theory of unbounded operators in Hilbert space. Again, these tools will be made precise later in the book, for example in Subsection 3.4.1, especially Lemma 3.4.2. In fact these tools will also play an important role in Chapters 3 and 6.

In the Introduction we also refer to *representations of the Cuntz relations* (see eq (1.3.8)), especially in connection with multi-frequency band analysis; see e.g., Remark 1.3.5, and Figure 1.4.1. These notions from representation theory, and their applications, will be taken up in a systematic fashion in Chapter 5 below.

A: *spectrum*	B: *translation tiling*
$\Omega \subset \mathbb{R}^d$, $\lvert\Omega\rvert < \infty$. $\exists \Lambda$ s.t. $\{e_\lambda \,;\, \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$.	$\exists T \subset \mathbb{R}^d$ s.t. $\Omega \dot{+} T = \mathbb{R}^d$
C: *operator theory*	
$\exists$ s.a. commuting extension operators $H_j \supset \frac{1}{i}\frac{\partial}{\partial x_j}\big\rvert_{C_c^\infty(\Omega)}$, $1 \le j \le d$; $H_j = \int_{\mathbb{R}} \lambda \, E_j(d\lambda)$, $E_j(A)\, E_k(B) = E_k(B)\, E_j(A)$, $\forall j, k$, $\forall A, B \in \mathscr{B}(\mathbb{R})$.	

FIGURE 1.2.3. Three related properties for open subsets Ω in $\mathbb{R}^d$: (A) Ω is spectral, (B) Ω admits a translation tiling set, and (C) the minimal symmetric partial derivative operators for Ω admit commuting selfadjoint extensions. Equivalence of (A) and (B) is called the Fuglede conjecture. It is open for $d = 1$, and $d = 2$. But for $d = 3$ and higher, (A) and (B) are known not to be equivalent. In general (A) implies (C); and if Ω is also assumed connected, then (C) implies (A). Also see Table 1.

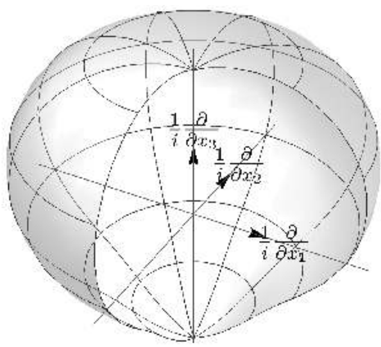


FIGURE 1.2.4. $H_j \supset \frac{1}{i}\frac{\partial}{\partial x_j}\big\rvert_{C_c^\infty(\Omega)}$, $1 \le j \le d$, the partial derivatives.

In our initial discussion and presentation here, we have chosen to start out by first describing a number of applications, and then postpone a more complete treatment of technical issues involved, until later in the book.

1.3. Iterated function systems (IFS)

The title above refers to a class of measures arising naturally in geometric measure theory; they are generated by a prescribed system of functions, and the construction is based on iteration; hence the name Iterated function systems (IFS). The purpose of the brief outline below is to explain a selection of measures. This sample includes measures, and associated maps, the measures with support in an ambient space $\mathbb{R}^d$, and the maps defined in $\mathbb{R}^d$. Examples for all values of d. Here we focus on the case when the initial system of maps are from the class of affine maps in $\mathbb{R}^d$, but the theory of IFSs includes a much wider array of examples and applications; some of which will be taken up later inside the book.

One of the tools we shall employ in our harmonic analysis considerations is as follows: To a particular IFS we shall associate certain systems of operators (S_j); also called systems of *Cuntz isometries*. Even though the initial setting of IFSs is commutative, the consideration of the Cuntz isometries is highly non-commutative. Nonetheless, we wish to demonstrate their use and power in analysis of IFS measures.

There do, of course, exist some measures that are not spectral. Of great interest are measures that arise naturally from affine iterated function systems. An *iterated function system* (IFS) is a finite set of contraction operators $\tau_0, \tau_1, \cdots, \tau_n$ on a complete metric space S. As a consequence of Hutchinson's Theorem [**Hut81**], for an IFS on $\mathbb{R}^n$, there exists a unique compact set $X \subset \mathbb{R}^n$ left invariant by system in the sense that $X = \cup_{j=0}^{n} \tau_j(X)$. There will then exist a unique Borel measure μ on X such that

$$\int_X f(x)\, d\mu(x) = \frac{1}{n+1} \sum\nolimits_{j=0}^{n} \int_X f(\tau_j(x))\, d\mu(x) \tag{1.3.1}$$

for all continuous f.

In many cases of interest, X is a fractal set. In particular, if we take the iterated function system

$$\tau_0(x) = \frac{x}{3}, \quad \tau_1(x) = \frac{x+2}{3}$$

on $\mathbb{R}$, then the attractor is the *ternary Cantor* set C_3. The set C_3 has another construction: One starts with the interval $[0,1]$ and removes the middle third, leaving only the intervals $[0,1/3]$ and $[2/3,1]$, and then successively continues to remove the middle third of each remaining interval. Intersecting the sets remaining at each step yields C_3. The ternary Cantor measure μ_3 is then the measure induced in (1.3.1). Alternatively, μ_3 is the *Hausdorff measure* of dimension $\frac{\ln 2}{\ln 3}$ restricted to C_3.

In [**JP98a**] Jorgensen and Pedersen used the zero set of the Fourier-Stieltjes transform of μ_3 to show that μ_3 is not spectral (see Section 2.2). Equally remarkably, they showed that the quaternary (4-ary) Cantor set, which is the measure induced in (1.3.1) under the IFS

$$\tau_0(x) = \frac{x}{4}, \quad \tau_1(x) = \frac{x+2}{4} \tag{1.3.2}$$

is spectral by using Hadamard matrices and a completeness argument based on the Ruelle transfer operator. The attractor set for this IFS can be described in a manner similar to the ternary Cantor set: The 4-ary set case is as follows,

$$C_4 = \left\{ x \in [0,1] : x = \sum\nolimits_{k=1}^{\infty} \frac{a_k}{4^k},\ a_k \in \{0,2\} \right\},$$

and the invariant measure is denoted by μ_4. Jorgensen and Pedersen prove that

$$\begin{aligned} \Gamma_4 &= \left\{ \sum\nolimits_{n=0}^{N} l_n 4^n : l_n \in \{0,1\},\ N \in \mathbb{N} \right\} \\ &= \{0,1,4,5,16,17,20,21,64,65,\cdots\} \end{aligned} \tag{1.3.3}$$

is a spectrum for μ_4, though there are many spectra [**DHS09, DHL13**]. The proof that this is a spectrum is a two step process: first the orthogonality of the exponentials with frequencies in Γ_4 is verified, and second the completeness of those exponentials is verified.

The orthogonality of the exponentials can be checked in several ways:

(1) checking the zeroes of the Fourier-Stieltjes transform of μ_4;
(2) using the representation of a particular Cuntz algebra on $L^2(\mu_4)$;
(3) generating Γ_4 as the invariant set for a second IFS that is "dual" in a sense to the IFS in (1.3.2) ("fractals in the large").

While these three methods are distinct, they all rely on the fact that a certain matrix associated to the IFSs is a (complex) Hadamard matrix. All three of these methods are, more or less, contained in the original paper [**JP98a**].

As a Borel probability measure, μ_4 is determined uniquely by the following IFS-fixed-point property:

$$\mu_4 = \frac{1}{2}\left(\mu_4 \circ \tau_0^{-1} + \mu_4 \circ \tau_1^{-1}\right), \tag{1.3.4}$$

see (1.3.2) for the affine maps τ_i, $i = 0, 1$; and one checks that the support of μ_4 is *the 4-ary Cantor set* C_4.

The conclusions for the pair (μ_4, Γ_4) from (1.3.3)-(1.3.4) are as follows:

THEOREM 1.3.1 ([**JP98a**]). *Let the pair* (μ_4, Γ_4) *be as described. Then we get a spectral pair; more precisely:*

$$\langle e_\gamma, e_{\gamma'}\rangle_{L^2(\mu_4)} = \widehat{\mu_4}(\gamma - \gamma') = \delta_{\gamma\gamma'} \left(= \begin{cases} 1 & \text{if } \gamma = \gamma' \text{ in } \Gamma_4 \\ 0 & \text{if } \gamma \neq \gamma', \text{ both in } \Gamma_4 \end{cases}\right) \tag{1.3.5}$$

Moreover, if $f \in L^2(\mu_4)$*, and*

$$\widehat{f}(\gamma) = \langle f, e_\gamma\rangle_{L^2(\mu_4)} = \int_{C_4} f(x)\,\overline{e_\gamma(x)}\,d\mu_4(x) \tag{1.3.6}$$

then we have the following $L^2(\mu_4)$ *limit:*

$$\lim_{N\to\infty} \Big\| f - \underbrace{\textstyle\sum_{\Gamma_4(N)} \widehat{f}(\gamma)\, e_\gamma(\cdot)}_{\textit{Fractal Fourier series}} \Big\|_{L^2(\mu_4)} = 0$$

where Γ_4 *is as in (*1.3.3*).*

The proof and the ramifications of Theorem 1.3.1 will be discussed in detail inside the book; especially in the following sections below: Sections 2.4, 3.1, 4.1, and 6.1.

REMARK 1.3.2. It is known that, for classical Fourier series, there are continuous functions whose Fourier series may fail to be pointwise convergent.

Now the Fourier expansion from Theorem 1.3.1 turns out not to have this "defect." The reason is that those gap-fractals which have orthogonal frequency expansions, turn out to also possess a *localization* property (which is not present in the classical setting of Fourier series for functions on an interval.) Indeed, Strichartz [**Str93**] proved that, in the setting of Theorem 1.3.1, *every* continuous function on C_4 has its Γ_4 Fourier expansion be pointwise convergent.

By contrast, when this is modified to (μ_3, C_3), the middle-third Cantor, Jorgensen and Pedersen proved that then there cannot be more than two orthogonal Fourier functions $e_\lambda(x) = e^{i2\pi\lambda x}$, for any choices of points λ in $\mathbb{R}$.

The *completeness* of the exponentials (for the cases when the specified Cantor measure is spectral) can be shown in several ways as well, though the completeness is more subtle. The original argument for completeness given in [**JP98a**] uses a delicate analysis of the spectral theory of a *Ruelle transfer operator*. Jorgensen and Pedersen construct an operator on $C(\mathbb{R})$ using filters associated to the IFS in (1.3.2), which they term a Ruelle transfer operator. The argument then is to check that the eigenvalue 1 for this operator is a simple eigenvalue. An alternative argument for completeness given by Strichartz in [**Str98b**] uses the convergence of the cascade algorithm from *wavelet theory* [**Mal89**, **Dau88**, **Law91**]. Later arguments for completeness were developed in [**DJ09c**, **DJ12b**] again using the representation theory of Cuntz algebras.

The *Cuntz algebra* $\mathscr{O}_N$ for $N \geq 2$ is the universal C^*-algebra generated by a family $\{S_0, \cdots, S_{N-1}\}$ of N isometries satisfying the relation

$$\sum\nolimits_{j=0}^{N-1} S_j S_j^* = I, \quad \text{and} \quad S_i^* S_j = \delta_{ij} I. \tag{1.3.7}$$

When N is fixed, and a system of operators S_j is identified satisfying (1.3.7), we say that $\{S_j\}$ is a system of Cuntz isometries; or that they define a representation of the Cuntz algebra $\mathscr{O}_N$. Equivalently, we say that the operators S_j satisfy the Cuntz relations. In the present book, we shall stress the role of representations of the Cuntz algebras in the study of multi-frequency band signal processing, of wavelet multiresolution generators, as filter-banks, and as generators of an harmonic analysis of iterated function systems (IFSs).

Now the Cuntz algebras $\mathscr{O}_N$ and their representations are of independent interest. And there is a rich literature dealing with them. In fact, it is known that the family of equivalence classes (unitary equivalence) of representations of $\mathscr{O}_N$ (N fixed) *does not admit Borel cross sections*; i.e., it is too big for classification. Nonetheless, we shall show that the class of representations corresponding to solutions to the filter bank systems in Figure 1.4.1 covers an infinite dimensional variety of equivalence classes of representations of $\mathscr{O}_N$.

Solutions to (1.3.7) $\{S_i\}$, realized in Hilbert space $\mathscr{H}$, play an important role in the construction of *multiresolutions*.

1.3.1. $\mathscr{O}_2$ vs $\mathscr{O}_\infty$. We shall discuss sequences $\{F_n\}_{n\in\mathbb{N}}$ of operators in a fixed Hilbert space, say $\mathscr{H}$, so $F_n : \mathscr{H} \to \mathscr{H}$.

Convergence of such sequences will be in the *strong operator topology* (SOT), defined as follows:

If $G : \mathscr{H} \to \mathscr{H}$ is an operator, we say that $F_n \xrightarrow[n\to\infty]{} G$ (SOT) if, for all vectors $h \in \mathscr{H}$, we have:

$$\lim_{n\to\infty} \|F_n h - Gh\|_{\mathscr{H}} = 0.$$

DEFINITION 1.3.3. A system of isometries $\{T_i\}_{i\in\mathbb{N}_0}$ is said to be a solution to the $\mathscr{O}_\infty$-relations iff (Def.)

$$T_i^* T_j = \delta_{ij} I, \quad \text{and} \quad \sum_{i=0}^{\infty} T_i T_i^* = I; \tag{1.3.8}$$

compare with (1.3.7).

LEMMA 1.3.4. *Let $\{S_0, S_1\}$ be a solution to the $\mathscr{O}_2$-relations (1.3.7), and set*

$$T_i = S_0^i S_1, \quad i \in \mathbb{N}_0. \tag{1.3.9}$$

Then $\{T_i\}_{i\in\mathbb{N}_0}$ *satisfies the* $\mathscr{O}_\infty$*-relations (*1.3.8*) if and only if*

$$\lim_{n\to\infty} S_0^{*n} = 0.$$

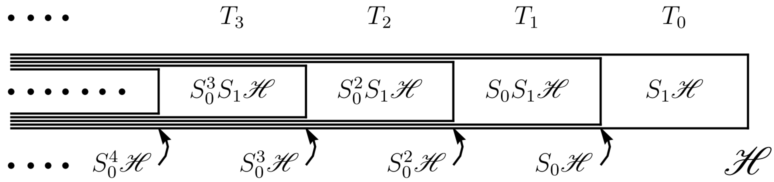
PROOF. Let S_0, S_1, and $T_i = S_0^i S_1$ be as stated. For $k \in \mathbb{N}$, we then have

$$\sum_{i=0}^{k-1} T_i T_i^* = \sum_{i=0}^{k-1} S_0^i \left(I - S_0 S_0^*\right) S_0^{*^i} = I - S_0^k S_0^{*k},$$

and the desired conclusion follows immediately. $\square$

REMARK 1.3.5. To appreciate the role of the lemma in building *multiresolutions*, consider the following diagram, sketching closed subspaces in $\mathscr{H}$.

Assume $\{S_i\}_{i=0}^{1}$ is an $\mathscr{O}_2$-system, then



with the system representing an orthogonal resolution, i.e., a system of orthogonal closed subspaces.

There are many ways to generate such families. For example, consider the isometries S_0, S_1 on $L^2[0,1]$ given by defining their adjoints

$$\left(S_0^* f\right)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \quad \text{and} \quad \left(S_1^* f\right)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+1}{2}\right),$$

$f \in L^2[0,1]$, $x \in [0,1]$. One can check that the range isometries $S_0 S_0^* = \chi_{[0,1/2]}$ and $S_1 S_1^* = \chi_{[1/2,1]}$, so that the Cuntz relations are satisfied.

Developing this example a bit further, we can see a relationship between Cuntz isometries and iterated function systems. Let C be the standard Cantor set in $[0,1]$, consisting of those real numbers whose ternary expansions are of the form $x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}$ where $x_k \in \{0,2\}$ for all k. Let

$$\varphi : C \to [0,1], \quad \varphi\left(\sum\nolimits_{k=1}^{\infty} \frac{x_k}{3^k}\right) = \sum\nolimits_{k=1}^{\infty} \frac{x_k}{2^{k+1}}.$$

Let m be Lebesgue measure on $[0,1]$, and define the Cantor measure μ on C by $\mu\left(\varphi^{-1}(B)\right) = m(B)$ if $B \subset [0,1]$ is Lebesgue measurable. This is well defined since φ is bijective except at countably many points.

Now define isometries R_0, R_1 on $\left(L^2(C), \mu\right)$ by defining their adjoints:

$$R_0^*(f) = S_0^*(f\circ\varphi) \quad \text{and} \quad R_1^*(f) = S_1^*(f\circ\varphi), \quad f \in \left(L^2(C), \mu\right).$$

Then

$$R_0^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{3}\right) \quad \text{and} \quad R_1^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+2}{3}\right),$$

$f \in \left(L^2(C), \mu\right)$, $x \in C$. Thus we see the iterated function system for the Cantor set $\tau_0(x) = x/3$, $\tau_1(x) = (x+2)/3$ arising in the definition of Cuntz isometries on the Cantor set.

TABLE 2. Some popular affine IFSs

	Scaling factor	Number of affine maps τ_i	Ambient dimension	Hausdorff dimension
Middle-third C_3	3	2	1	$\log_3 2 = \frac{\ln 2}{\ln 3}$
The 4-ary C_4	4	2	1	$\frac{1}{2}$
Sierpinski triangle	2	3	2	$\log_2 3 = \frac{\ln 3}{\ln 2}$

Multiresolutions as outlined in Remark 1.3.5 are versatile, they are algorithmic. Here their construction is based on representation theory. We shall discuss their wider use in harmonic analysis, both in the case of traditional wavelet expansions, and in their fractal counterparts. This will be developed in detail in the following three later sections, 2.4, 4.3, and 5.2. For their use in Chapter 4, see especially equations (4.3.6)–(4.3.7), and Figure 4.3.2.

The Cuntz relations can be represented in many different ways. In their paper [**DJ15a**], Dutkay and Jorgensen look at finite *Markov processes*, and the *infinite product* of the state space is a compact set on which different measures can be defined, and these form the setting of representations of the Cuntz relations.

To construct a Fourier basis for a spectral measure arising from an iterated function system generated by contractions $\{\tau_0, \cdots, \tau_{N-1}\}$, Jorgensen (and others, [**JP98a**, **DPS14**, **DJ15a**, **PW17**]) choose *filters* $m_0, \cdots, m_{N-1}$ and define Cuntz isometries $S_0, \cdots, S_{N-1}$ on $L^2(\mu)$ by

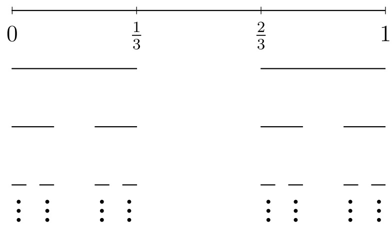
$$S_j f = \sqrt{N} m_j f \circ R,$$

where R is the common left inverse of the τ's. The filters, functions defined on the attractor set of the iterated function system, are typically chosen to be continuous, and are required to satisfy the relation $\sum_{j=0}^{N-1} |m_j|^2 = 1$. The Cuntz relations are satisfied by the S_j's provided the filters satisfy the *orthogonality condition*

$$\mathcal{M}^* \mathcal{M} = I, \quad (M)_{jk} = m_j(\tau_k(\cdot)). \tag{1.3.10}$$

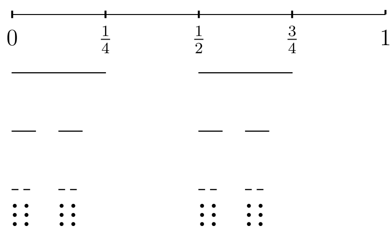
To obtain Fourier bases, the filters m_j are chosen specifically to be exponential functions when possible. This is not possible in general, however, and is not possible in the case of the middle-third Cantor set and its corresponding measure μ_3.

The fact that some measures, such as μ_3, are not spectral leaves us with a conundrum: We still desire Fourier-type expansions of functions in $L^2(\mu)$, that is, a representation as a series of complex exponential functions, but we cannot get such an expansion from an orthogonal basis of exponentials in the case of a non-spectral measure. For this reason, we turn to another type of sequence called a *frame*, which has the same ability to produce series representations that an orthogonal basis does, but has redundancy that orthogonal bases lack and has no orthogonality requirement. Frames for Hilbert spaces were introduced by Dun and Schaeer [**DS52**] in their study of non-harmonic Fourier series. The idea then lay essentially dormant until Daubechies, Grossman, and Meyer reintroduced frames in [**DGM86**]. Frames are now pervasive in mathematics and engineering. For recent applications, we refer the reader to [**Web04**, **ALTW04**, **PW17**].



$$\tau_0(x) = \frac{x}{3}$$
$$\tau_1(x) = \frac{x+2}{3}$$

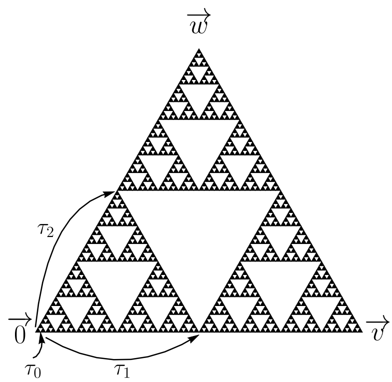
FIGURE 1.3.1. Middle-third Cantor C_3



$$\tau_0(x) = \frac{x}{4}$$
$$\tau_1(x) = \frac{x+2}{4}$$

FIGURE 1.3.2. The 4-ary Cantor C_4

$$\vec{w} = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$
$$\vec{v} = (1, 0)$$



$$\tau_0(\vec{x}) = \frac{1}{2}\vec{x}$$
$$\tau_1(\vec{x}) = \frac{1}{2}(\vec{x} + \vec{v})$$
$$\tau_2(\vec{x}) = \frac{1}{2}(\vec{x} + \vec{w})$$

FIGURE 1.3.3. Sierpinski triangle

1.4. Frequency bands, filters, and representations of the Cuntz-algebras

Our analysis of the Cuntz relations here in the form $\{S_i\}_{i=0}^{N-1}$ turns out to be a modern version of the rule from *signal-processing engineering* (SPEE): When complex frequency response functions are introduced, the (SPEE) version of the Cuntz relations $S_i^* S_j = \delta_{ij} I_{\mathscr{H}}$, $\sum_{i=0}^{N-1} S_i S_i^* = I_{\mathscr{H}}$, where $\mathscr{H}$ is a Hilbert space of time/frequency signals, and where the N isometris S_i are expressed in the following form:

$$(S_i f)(z) = m_i(z) f\left(z^N\right), \ f \in \mathscr{H}, \ z \in \mathbb{C}; \tag{1.4.1}$$

and where $\{m_i\}_{i=0}^{N-1}$ is a system of *bandpass-filters*, m_0 accounting for the low band, and the filters $m_i(z)$, $i > 0$, accounting for the remaining bands in the subdivision into a total of N bands. The diagram form (SPEE) is then as in Figure 1.4.1.

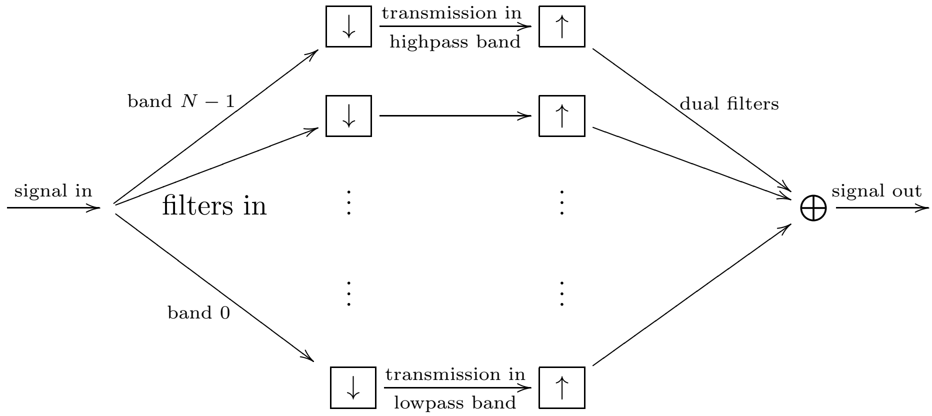


FIGURE 1.4.1. *Down-sampling* $\boxed{\downarrow}$, and *up-sampling* $\boxed{\uparrow}$. The picture is a modern math version of one I (PJ) remember from my early childhood: In our living room, my dad was putting together some of the early versions of low-pass/high-pass frequency band filters for transmitting speech signals over what was then long distance. One of the EE journals had a picture which is much like the one I reproduce here; after hazy memory. Strangely, the same multi-band constructions are still in use for modern wireless transmission, both speech and images. The down/up arrows in the figure stand for down-sampling, up-sampling, respectively. Both operations have easy expressions in the complex frequency domain. For example up-sampling becomes substitution of z^N where N is the fixed total number of bands.

The operators making up the multiband filters in Figure 1.4.1 are expressed in (1.4.1) in the frequency variable z ($\in \mathbb{C}$, or in $\mathbb{T}$). With the usual inner product in the Hilbert space $L^2(\mathbb{T})$, and

$$\int_0^1 \left|\sum\nolimits_{n\in\mathbb{Z}} c_n e_n(x)\right|^2 dx = \sum\nolimits_{n\in\mathbb{Z}} |c_n|^2, \tag{1.4.2}$$

one checks that the adjoint of the above operators (1.4.1) are:

$$\left(S_j^* f\right)(z) = \frac{1}{N} \sum_{w\in\mathbb{T},\, w^N = z} \left(\overline{m}_j f\right)(w), \tag{1.4.3}$$

for $\forall z \in \mathbb{T}$, $0 \le j < N$.

Now, there is a time-frequency duality, and operators in one side of the duality have a counterpart in the other side. As evidenced by (1.4.2), the discrete-time dual version of the frequency function

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e_n(x) \tag{1.4.4}$$

is simply the time-series $(c_n)_{n\in\mathbb{Z}}$:

$$\cdots, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, \cdots \tag{1.4.5}$$

If $\widetilde{m}(x) = \sum_{n\in\mathbb{Z}} h_n e_n(x)$, then the multiplication operator, $f \longmapsto mf$, is simply:

$$(c_n)_{n\in\mathbb{Z}} \longmapsto (\widetilde{m}[c])_n = \sum_{m\in\mathbb{Z}} h_m c_{n-m}. \tag{1.4.6}$$

The up and down-sampling operations $\boxed{\uparrow}$ vs $\boxed{\downarrow}$ acting on the time-series are:

(1.4.7)

$$\begin{aligned}\left(\boxed{\uparrow}[c]\right)_n &= \begin{cases} c_{n/N} & \text{if } N \mid n \\ 0 & \text{if } n \text{ is not divisible by } N \end{cases} \\ &= \Big(\underset{\text{places}}{\cdots}, 0, \underbrace{c_{-1}}_{-N}, 0, \cdots, 0, \underbrace{c_0}_{0}, 0, \cdots, 0, \underbrace{c_1}_{N}, 0, \cdots, 0, \underbrace{c_2}_{2N}, 0, \cdots \Big)\end{aligned}$$

and

$$\left(\boxed{\downarrow}[c]\right)_n = c_{nN}, \quad \forall n \in \mathbb{Z}. \tag{1.4.8}$$

In many applications, the operators from (1.4.1) and (1.4.3) have matrix realizations. The respective matrices are *slanted* (see Figure 1.4.2), and they are used in algorithms for digitized representations of signals and of images.

To appreciate the matrix point of view, we restrict here to the special case where the functions m_i in (1.4.1)-(1.4.3) are polynomials, so M (= one of the functions m_i) has the form

$$M(z) = h_0 + h_1 z + \cdots + h_d z^d. \tag{1.4.9}$$

Assume a signal is given in the form (1.4.4), i.e., with

$$(c) = (c_0, c_1, c_2, \cdots)$$

representing a time series with discrete time $n \in \mathbb{N}_0$. When realized in this form, one checks that the operators

$$(S_M f)(z) = M(z) f\left(z^N\right),$$

and

$$(S_M^* f)(z) = \frac{1}{N} \sum_{w\in\mathbb{T},\, w^N = z} \left(\overline{M} f\right)(w),$$

yield the respective matrix forms:

$$(S_M c)_k = \sum_{n\in\mathbb{Z}} h_{k-Nn} c_n;$$

and

$$(S_M^* c)_k = \sum_{n\in\mathbb{Z}} \overline{h_{Nk-n}} c_n.$$

The slanted matrices themselves are given in Figure 1.4.2 below.

These are wavelet tools, and they will be revisited in a number of applications, later in the book, starting with Section 4.3.

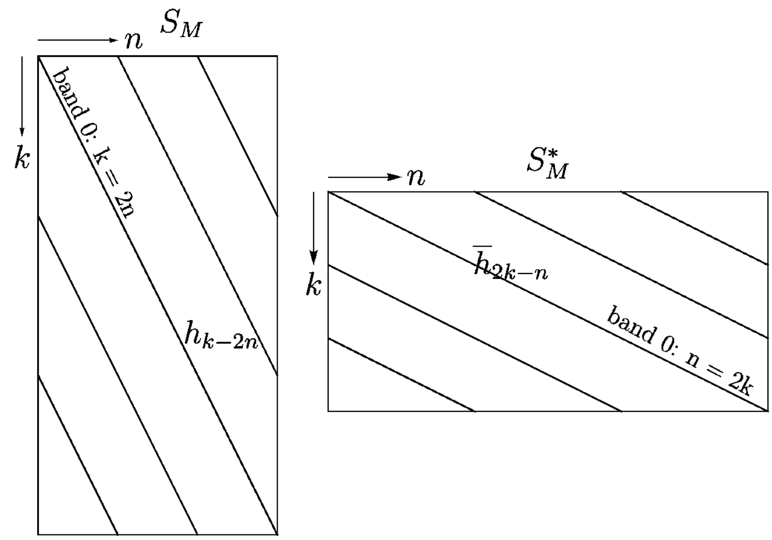


FIGURE 1.4.2. The two slanted matrices in the special case when $N = 2$.

1.5. Frames

Let $\mathscr{H}$ be a separable Hilbert space with *inner product* $\langle \cdot, \cdot \rangle$, and let $\mathbb{J}$ be a *countable index set*. A *frame* for $\mathscr{H}$ is a sequence $\{x_j\}_{j \in \mathbb{J}} \subset \mathscr{H}$ such that there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $v \in \mathscr{H}$,

$$C_1 \|v\|^2 \leq \sum |\langle v, x_j \rangle|^2 \leq C_2 \|v\|^2 .$$

If C_1 and C_2 can be chosen so that $C_1 = C_2 = 1$, we say that $\{x_j\}$ is a *Parseval frame.*

If $\mathbb{X} \subset \mathscr{H}$ is a frame, then any other frame $\tilde{\mathbb{X}} := \{\tilde{x}_j\} \subset \mathscr{H}$ that satisfies

$$\sum \langle v, \tilde{x}_j \rangle x_j = v, \tag{1.5.1}$$

for all $v \in \mathscr{H}$ is called a *dual frame* for $\mathbb{X}$. Every frame possesses a dual frame, and in general, dual frames are not unique. A Parseval frame is self-dual, that is, $v = \sum \langle v, x_j \rangle x_j$.

Returning to our current interest, we say that a measure μ is *frame-spectral* if there exists a countable set $\Lambda \subset \mathbb{R}$ such that $\left\{e^{i2\pi\lambda x}\right\}_{\lambda \in \Lambda}$ is a frame in $L^2(\mu)$. In general, for a compact subset C of $\mathbb{R}^d$ with nonzero measure, Lebesgue measure restricted to that set is not spectral, but it will always be frame spectral. In general, a singular measure will not be frame-spectral [**DHSW11**, **DL14b**], but many singular measures are frame-spectral [**EKW16**, **PW17**]. It is currently unknown whether or not μ_3 is frame-spectral.

The *redundancy* of frames makes them more immune to error in transmission: Multiple frame elements will capture the same dimensions of information, and so if one series coefficient in the frame expansion of a function is transmitted incorrectly, the adverse effect on the reconstructed function will be minimal. However, expansions in terms of a given frame are in general not unique, and this can be a desirable or undesirable quality depending on the application. If we want the best of both worlds — a frame with redundancy but with a unique expansion for each function, then we must turn to the realm of Riesz bases.

A *Riesz basis* in a Hilbert space $\mathscr{H}$ is a sequence $\{x_j\}_{j=1}^{\infty}$ which has dense span in $\mathscr{H}$ and is such that there exist $0 < A \leq B$ such that for any nite sequence of scalars $c_1, c_2, \cdots, c_N$, we have

$$A \sum_{j=1}^{N} |c_j|^2 \leq \| \sum_{j=1}^{N} c_j x_j \|^2 \leq B \sum_{j=1}^{N} |c_j|^2 . \tag{1.5.2}$$

A Riesz basis is a frame that has only one dual frame. Equivalently, $\{x_j\}_{j=1}^{\infty}$ is a Riesz basis if an only if there is a topological isomorphism $T : \mathscr{H} \to \mathscr{H}$ such that $\{Tx_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$.

The unit disk $\mathbb{D}$, for example, as a convex planar body has no orthogonal basis of complex exponential functions, but it does possess a frame of complex exponential functions. However, it is still an open problem whether it possesses a Riesz basis of complex exponential functions.

1.6. Key themes in the book

Beginning with the foundational results in "*Dense analytic subspaces in fractal L^2-spaces*" [**JP98a**], Chapter 2 will cover the construction of spectral measures, the constructions of various spectra, characterizations and invariance of spectra for spectral measures. It will include initial connections to *representation theory* of Cuntz algebras, *spectra* and *tiling* properties in $\mathbb{R}^d$, the Fuglede conjecture, and Reproducing Kernel Hilbert spaces.

The existence of *orthogonal Fourier bases for classes of fractals* came as somewhat of a surprise, referring to the 1998 paper [**JP98a**]. There are several reasons for why existence of orthogonal Fourier bases might have been unexpected: For one, existence of orthogonal Fourier bases, as in the classical case of Fourier, tends to imply a certain amount of "smoothness" which seems inconsistent with fractal geometries, and fractal dimension. Nonetheless, when feasible, such a orthogonal Fourier analysis holds out promise for applications to large chaotic systems, or to analysis of *noisy signals*; areas that had previously resisted analysis by Fourier tools.

When Fourier duality holds, it further yields a *duality of scale*, fractal scales in the small, and for the dual frequency domain, fractals in the large.

While the original framework for the Jorgensen-Pedersen fractals, and associated L^2-spaces, was a rather limited family, this original fractal framework for orthogonal Fourier bases has since been greatly expanded. While the original setting was restricted to that of *affine selfsimilarity*, determined by certain *iterated affine function systems* in one and higher dimension, this has now been broadened to the setting of say conformal selfsimilar IFS systems, and to associated maximal entropy measures. Even when the strict requirements entailed by orthogonal Fourier bases is suitably relaxed, there are computational Fourier expansions (Herr-Jorgensen-Weber) which lend themselves to analysis/synthesis for most *singular measures*.

Inherent in the study of fractal scales is the notion of *multiresolution analyses*, in many ways parallel to the more familiar Daubechies wavelet multiresolutions. Moreover, Strichartz proved that when an orthogonal Fourier expansions exist, they have localization properties which parallel the kind of localization which has made wavelet multiresolutions so useful. The presence of multiresolutions further implies powerful algorithms, and it makes connections to representation theory and to signal/image processing; subjects of the later chapters. Dutkay-Jorgensen proved that all affine IFS fractals have wavelet bases.

Chapter 2 will build on the themes from Chapter 1, detailing the constructions of spectra arising from Cuntz algebras, characterizations of spectra using the spectral theory of *Ruelle operators*, connections between tilings, and *wandering vectors* for unitary groups and unitary systems.

There is an intimate relations between systems of *tiling by translations* on the one hand, and *orthogonal Fourier bases* on the other. Representation theory makes a link between the two, but the tile-spectral question is deep and difficult; so far only partially resolved. One tool of inquiry is that of "wandering vectors" or *wandering subspaces*. The term "wandering" has its origin in the study of systems of isometries in Hilbert space. It has come to refer to certain actions in a Hilbert space which carries representations: When the action generates orthogonal vectors, we refer to them as wandering vectors; similarly for closed subspaces. In the case of representations of groups, this has proved a useful way of generating orthogonal Fourier bases; — when they exist. In the case of representations of the Cuntz algebras, the "wandering" idea has become a tool for generating nested and orthogonal subspaces. The latter includes multiresolution subspaces for wavelet systems and for signal/image processing algorithms.

Chapter 3 will focus on the *tiling properties* arising from the study of spectral measures, specifically in dimension one; advances in the Fuglede conjecture in dimension one, non-commutative fractal analogues in infinite dimensions.

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile $\mathbb{R}^d$ by a set of translates of Ω [**Fug74**]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$, and Iosevich et al. [**IKT01**] proved that no smooth symmetric convex body Ω with *at least one point of nonvanishing Gaussian curvature* can admit an orthogonal basis of exponentials.

Using complex *Hadamard matrices* of orders 6 and 12, Tao [**Tao04**] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$. Tao's results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

Summary of some affirmative recent results: The conjecture has been proved in a great number of special cases (e.g., all convex planar bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling $\mathbb{Z}$ indeed implies one side of Fuglede's conjecture [**CM99**]. Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group [**Ami05**].

Ironically, despite a large number of great advances in the area, Fuglede's original question is still unsolved in the planar case. In the planar case, the question is: *Let Ω be a bounded open and connected subset of $\mathbb{R}^2$. Does it follow that $L^2(\Omega)$ with respect to planar Lebesgue measure has an orthogonal Fourier basis if and only if Ω tiles $\mathbb{R}^2$ with translations by some set of vectors from $\mathbb{R}^2$?* Of course, if Ω is a fundamental domain for some rank-2 lattice, the answer is affirmative on account of early work.

Another direction is to restrict the class of sets Ω in $\mathbb{R}^3$ to be studied. One such recent direction is the following affirmative theorem for the case when Ω is assumed to be a *convex polytope*: Nir Lev et al [**GL17**] proved that a spectral *convex polytope* (i.e., having a Fourier basis) must tile by translations. This implies in particular that Fuglede's conjecture holds true for convex polytopes in $\mathbb{R}^3$.

Chapter 4 is devoted to the RKHSs that appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process, the chapter will set the stage by discussing RKHSs that appear in the context of positive definite functions, and the associated harmonic analysis in such spaces.

Since the measures are spectral, the corresponding *positive definite functions* have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of RKHSs.

Chapter 5 concerns *representations of Cuntz algebras* that arise from the action of stochastic matrices on sequences from $\mathbb{Z}_n$. This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into frequency bands.

Concrete realizations are presented of a class of *explicit representations*. Starting with Hilbert spaces $\mathscr{H}$, the representations produce recursive families of closed subspaces (projections) in $\mathscr{H}$, in such a way that "non-overlapping, or uncorrelated, *frequency bands*" correspond to orthogonal subspaces in $\mathscr{H}$. Since different frequency bands must exhaust the range for signals in the entire system, one looks for orthogonal projections which add to the identity operator in $\mathscr{H}$. Representations of Cuntz algebras (see Figure 1.4.1) achieve precisely this: From representations we obtain classification of families of *multi-band filters*; and representations allow us to deal with non-commutativity as it appears in both time/frequency analysis, and in scale-similarity. The representations further offer canonical selections of special families of commuting orthogonal projections.

The chapter will focus on the connections between harmonic analysis on fractals and the *cascade algorithm* from wavelet theory. Wavelets have a dual existence between the discrete and continuous realms manifested in the discrete and continuous wavelet transforms. Wavelet filters give another bridge between the smooth and non-smooth domains in that the convergence of the cascade algorithm yields wavelets and wavelet transforms in a smooth setting, i.e. $\mathbb{R}^d$, and also the non-smooth setting such as the Cantor dust, depending on the parameters embedded in the choice of wavelet filters.

Chapter 6 concerns *Gaussian processes* for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases). These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related *Reproducing Kernel Hilbert Spaces* (RKHSs).

Chapter 7 will focus on stochastic processes that appear in the representation theory of *Lie groups*. Motivated by reflection symmetries in Lie groups, we will consider representation theoretic aspects of reflection positivity by discussing reflection positive Markov processes indexed by Lie groups, measures on path spaces, and invariant Gaussian measures in spaces of distribution vectors. This provides new constructions of reflection positive unitary representations.

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of *reflection positivity* has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

Below we list suggested papers readers might wish to consult on four central themes:

(1) Fourier analysis on affine fractals [**JP87**, **JP92**, **JP93a**, **JP93b**, **JP94**, **JP95**, **JP96**, **JP98a**, **JP98b**, **JP98c**, **JP98d**, **JP99**, **JP00**, **JPT12**, **JPT14**, **JPT15a**, **JPT15b**]

(2) Multiresolution analyst, fractals, and representations of the Cuntz relations [**DJ05b**, **DJ05a**, **DJ06b**, **DJ06d**, **DJ06c**, **DJ06a**, **DJ07c**, **DJ07a**, **DJ07b**, **DJ07d**, **DJ07e**, **DJ07f**, **DJ08a**, **DJ08b**, **DJ09c**, **DHJS09**, **DJ09b**, **DHJ09**, **DJP09**, **DJ09a**, **DJ11b**, **DJ11a**, **DJS12**, **DJ12a**, **DJ12b**, **DHJP13**, **DJ13b**, **DJ13a**, **DJ14a**, **DJ14b**, **DHJ15**, **DJ15c**, **DJ15b**, **DJ15a**]

(3) Frame analysis of singular measures [**HJW18b**]

(4) Reflection positivity [**JO98**, **JO00**, **JNO16**, **JNO18**, **JT18c**]

The past two decades has seen a rich and diverse flourishing of research in the areas of analysis on fractals, and their applications. While the present lectures have stressed a certain harmonic analysis approaches, and their associated applications, there are others.

And in fact, it will be nearly impossible to cover all directions, even by way of citations, and we apologize for omissions. Nonetheless, we believe that the following supplementary references will help readers broaden their perspective: First, the book [**BP17**] stresses connections to probability and Markov processes. And there is the work by Poltoratski et al with a different perspective on harmonic analysis on fractals; see e.g., [**dRP99**, **dRFP02**, **Pol15**, **Pol13**].

There are many standard textbooks dealing with harmonic analysis and applications. Two of these books might perhaps be more helpful for students; filling in prerequisites. They are [**DM72**, **DM76**] by Dym and McKean. (While they are extremely useful for background material, they do not get into the fractal variants of Fourier series.)

Our present focus as far as the fractals go is harmonic analysis. Many of our tools apply to large and varied classes of fractals, but we have chosen to illustrate most of our results with the fractals called affine iterated function systems (IFS). Fractals are studied in a variety of areas both in mathematics and in diverse applications. And the literature is vast. Readers who want to get started are referred to the book [**Fed88**].

10.1090/cbms/128/02

CHAPTER 2

Spectral pair analysis for IFSs

> In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's the exact opposite.
>
> Dirac, Paul Adrien Maurice (1902–1984)
>
> In H. Eves *Mathematical Circles A*.

In Chapter 1 we outlined the main approaches to an harmonic analysis of iterated function system (IFS) fractals and associated measures. Below we explore these techniques in detail. Our approach in the present chapter is to first concentrate on the special cases of the Cantor fractals. In subsequent chapters, these tools and techniques will then be expanded to cover wider families of fractals. For this we refer readers to Chapters 3, 4, and 5.

Of course, the results are much more explicit for the Cantor fractals which is a good reason for beginning with them.

2.1. The scale-4 Cantor measure, and its harmonic analysis

Let $\nu \in \mathbb{N}$ be fixed. We consider positive measures μ of compact support in $\mathbb{R}^\nu$, and discrete subsets $\Lambda \subset \mathbb{R}^\nu$. We say that (μ, Λ) is a *spectral pair* if $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\mu)$. The following lemma from [**JP98a**] is general, and it will be used often in what follows:

LEMMA 2.1.1. *Let (μ, Λ) be as specified. Then the following two properties are equivalent:*

(1) *(μ, Λ) is a spectral pair; and*
(2) $\sum_{\lambda\in\Lambda} \left|\widehat{\mu}(t-\lambda)\right|^2 \equiv 1$, $\forall t \in \mathbb{R}^\nu$.

A central tool in our work is a certain *double duality:* first the usual duality of Fourier analysis, corresponding to the dual variables on either side of the spectral transform; and secondly a duality which derives from our use of *matrix scaling.* Small scales correspond to *compact* attractors of fractal Hausdorff dimension, while large scales ("fractals in the large") correspond to a *discrete* set of frequencies (in ν dimensions), $\lambda = (\lambda_1, \ldots, \lambda_\nu) \in \mathbb{R}^\nu$ which label our Fourier basis of orthogonal exponentials $e_\lambda(x) := e^{i2\pi\lambda\cdot x}$ where x is restricted to the ("small scale") fractal. In our setup, both scales, small and large, are *finitely generated*, referring to two given finite subsets B and L in $\mathbb{R}^\nu$ (one on each side of the duality) which are paired in a certain unitary matrix $U(B, L)$, defined from the two sets. The unitary matrix $U(B, L)$ is related to one studied by Hadamard. It turns out that not all configurations of sets B, L allow such a unitary pairing, and there is a further constraint from the dimension ν of the ambient Euclidean space.

It is known that there is a unique probability measure μ on $\mathbb{R}$ of compact support such that

$$\int f\,d\mu = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right)d\mu\left(x\right) + \int f\left(\frac{x+2}{4}\right)d\mu\left(x\right)\right) \tag{2.1.1}$$

for all continuous f. In fact, the support K of μ is the Cantor set obtained by dividing $I = [0,1]$ into four equal subintervals, and retaining only the first and third. (See Figure 2.1.1.)

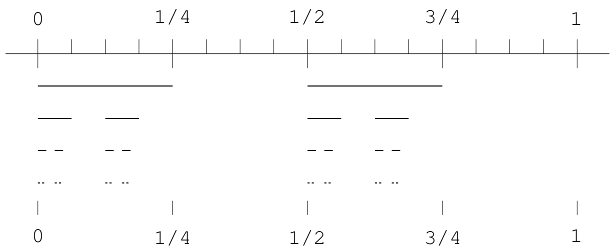


FIGURE 2.1.1. Support of μ. Hausdorff dimension $d_H = \frac{\ln 2}{\ln 4} = \frac{1}{2}$.

Affine IFSs. This is a special case of a more general construction in ν dimensions ($\nu \geq 1$) corresponding to some given real matrix R, and a finite subset $B \subset \mathbb{R}^\nu$. It is assumed that

$$R \text{ has eigenvalues } \xi_i \text{ all satisfying } |\xi_i| > 1. \tag{2.1.2}$$

The subset B is required to satisfy an open-set condition: Introduce

$$\sigma_b x = R^{-1}x + b, \quad x \in \mathbb{R}^\nu. \tag{2.1.3}$$

It is assumed that there is a nonempty, bounded open set V such that

$$\bigcup_{b\in B} \sigma_b V \subset V \tag{2.1.4}$$

with the union disjoint corresponding to distinct points in B. Our present $\{\sigma_b\}$ systems (see below for the axioms) are special cases of *iterated function systems* (IFS) considered in [**Hut81**], see also [**Fal86**]. There are many interesting more general IFSs, and that context also leads to measures μ which satisfy a general version of the invariance property (2.1.5), and there is then a corresponding "open-set assumption". But for our present *affine* systems, the splitting property (2.1.4), for some open subset V in $\mathbb{R}^\nu$, can be shown in fact to be automatic, see [**JP96**]. If $N = \#(B)$, then the corresponding measure μ on $\mathbb{R}^\nu$ (depending on R and B) has compact support, and satisfies

$$\int f\,d\mu = \frac{1}{N}\sum_{b\in B}\int f\left(\sigma_b\left(x\right)\right)\,d\mu\left(x\right) \tag{2.1.5}$$

for all continuous f. (For more details on the "open-set condition" and affinely generated fractal measures, we give the following background references: [**JP94**], [**Str94**], and [**Str98a**]) Define, for $t \in \mathbb{R}^\nu$, the Fourier transform

$$\hat{\mu}\left(t\right) = \int e^{i2\pi t\cdot x}\,d\mu\left(x\right) \tag{2.1.6}$$

with $t \cdot x = \sum_{i=1}^{\nu} t_i x_i$, we then get

$$\hat{\mu}(t) = \chi_B(t)\, \hat{\mu}\left(R^{*\,-1} t\right) \tag{2.1.7}$$

where

$$\chi_B(t) := \frac{1}{N} \sum_{b \in B} e^{i 2\pi b \cdot t} \tag{2.1.8}$$

and R^* is the transposed matrix.

For the example in Figure 2.1.1, this amounts to

$$\hat{\mu}(t) = \frac{1}{2}\left(1 + e^{i\pi t}\right) \hat{\mu}(t/4), \quad t \in \mathbb{R}. \tag{2.1.9}$$

Assume that the matrix R in (2.1.3) has integral entries, and that

$$RB \subset \mathbb{Z}^{\nu}, \quad 0 \in B, \tag{2.1.10}$$

but that none of the differences $b - b'$ is in $\mathbb{Z}^{\nu}$ when $b, b' \in B$ are different. Furthermore, assume that some subset $L \subset \mathbb{Z}^{\nu}$ satisfies $0 \in L$, $\#(L) = N$ $(= \#(B))$, and

$$H_{BL} := N^{-\frac{1}{2}} \left(e^{i 2\pi b \cdot l}\right) \tag{2.1.11}$$

is unitary as an $N \times N$ complex matrix, i.e., $H_{BL}^* H_{BL} = I_N$ (* denotes transposed conjugate.) In fact, it can be checked that the assumed non-integrality of the differences $b - b'$ (when $\neq 0$) follows from assuming that H_{BL} is unitary for some L as described. For our purposes, the assumptions $L \subset \mathbb{Z}^{\nu}$ and $RB \subset \mathbb{Z}^{\nu}$ may actually be weakened as follows:

$$(R^n b) \cdot l \in \mathbb{Z}, \quad \forall n \in \mathbb{N},\ b \in B,\ l \in L. \tag{2.1.12}$$

LEMMA 2.1.2. *With the assumptions, set*

$$P := \{l_0 + R^* l_1 + \cdots : l_i \in L, \text{ finite sums}\}. \tag{2.1.13}$$

Then the functions $\{e_\lambda : \lambda \in P\}$ *are mutually orthogonal in* $L^2(\mu)$ *where*

$$e_\lambda(x) := e^{i 2\pi \lambda \cdot x}. \tag{2.1.14}$$

PROOF. Let $\lambda = \sum R^{*\,i} l_i$, $\lambda' = \sum R^{*\,i} l_i'$ be points in P, and assume $\lambda \neq \lambda'$. Then

$$\begin{aligned}
\langle e_\lambda \mid e_{\lambda'} \rangle_\mu &= \int \overline{e_\lambda} e_{\lambda'}\, d\mu \\
&= \int e^{i 2\pi (\lambda' - \lambda) \cdot x}\, d\mu(x) \\
&= \hat{\mu}(\lambda' - \lambda) \\
&= \hat{\mu}\left(l_0' - l_0 + R^*(l_1' - l_1) + \cdots\right) \\
&= \chi_B(l_0' - l_0)\, \hat{\mu}\left(l_1' - l_1 + R^*(l_2' - l_2) + \cdots\right).
\end{aligned}$$

If $l_0' \neq l_0$ then $\chi_B(l_0' - l_0) = 0$ by (2.1.11). If not, there is a first n such that $l_n' \neq l_n$, and then

$$\begin{aligned}
\hat{\mu}(\lambda' - \lambda) &= \hat{\mu}\left(R^{*\,n}(l_n' - l_n) + R^{*\,n+1}\left(l_{n+1}' - l_{n+1}\right) + \cdots\right) \\
&= \chi_B(l_n' - l_n)\, \hat{\mu}\left(l_{n+1}' - l_{n+1} + \cdots\right) \\
&= 0
\end{aligned}$$

since $\chi_B(l_n' - l_n) = 0$. ∎

COROLLARY 2.1.3. *Let μ be the measure on the line $\mathbb{R}$ given by* (2.1.1) *and with Hausdorff dimension $d_H = \frac{1}{2}$. (We have $R = 4$, $B = \left\{0, \frac{1}{2}\right\}$ and $L = \{0, 1\}$.) Then*

$$P = \left\{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}\,,\ \textit{finite sums}\right\}, \tag{2.1.15}$$

and $\{e_\lambda : \lambda \in P\}$ is an orthonormal *subset of $L^2(\mu)$.*

PROOF. Immediate from the lemma. □

LEMMA 2.1.4. *Let the subsets $B, L \subset \mathbb{R}^\nu$, and the matrix R be as described before* Lemma 2.1.2. *Let*

$$Q_1(t) := \sum_{\lambda \in P} \left|\hat{\mu}(t - \lambda)\right|^2, \quad t \in \mathbb{R}^\nu. \tag{2.1.16}$$

Then $\{e_\lambda : \lambda \in P\}$ is an orthonormal basis for $L^2(\mu)$ if and only if $Q_1 \equiv 1$ on $\mathbb{R}^\nu$.

PROOF. If $\{e_\lambda : \lambda \in P\}$ is an orthogonal basis for $L^2(\mu)$, the Bessel inequality is an identity when applied to e_t; that is,

$$1 = \|e_t\|_\mu^2 = \sum_\lambda \left|\langle e_\lambda, e_t\rangle_\mu\right|^2 = \sum_\lambda \left|\hat{\mu}(t - \lambda)\right|^2.$$

Conversely, if this holds, and if $f \in L^2(\mu) \ominus \{e_\lambda : \lambda \in P\}$, then $\langle e_t, f\rangle_\mu = 0$ for all $t \in \mathbb{R}^\nu$, or equivalently $\int e^{-i2\pi t\cdot x} f(x)\, d\mu(x) = 0$ for all $t \in \mathbb{R}^\nu$. This implies $f = 0$ by Stone–Weierstrass applied to the compact support $\operatorname{supp}(\mu)$. □

We now state the main theorem. A detailed proof can be found in [**JP98a**].

THEOREM 2.1.5. *Let $H_2(P, \mu)$ be the closed span in $L^2(\mu)$ of the functions $\left\{e^{i2\pi n x} : n = 0, 1, 4, 5, 16, 17, 20, 21, \cdots\right\}$ (i.e., $P = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\}$, finite sums$\}$). Then*

$$H_2(P, \mu) = L^2(\mu). \tag{2.1.17}$$

COROLLARY 2.1.6. *There is a canonical isometric embedding Φ of $L^2(\mu)$ into the subspace $H_2\left(z^4\right) + zH_2\left(z^4\right)$ of H_2 where $H_2\left(z^4\right) := \left\{f\left(z^4\right) : f \in H_2\right\}$; and it is given by*

$$\Phi\left(\sum_{\lambda \in P} c_\lambda e_\lambda\right) = \sum_{n \in P} c_{4n} z^{4n} + z \sum_{n \in P} c_{4n+1} z^{4n}. \tag{2.1.18}$$

PROOF. Since $\sum_{\lambda \in P} |c_\lambda|^2 < \infty$, and $P = \bigcup_{l \in \{0,1\}} l + 4P$, with $4P \cap (1 + 4P) = \varnothing$, the representation (2.1.18) is well defined. Note that Φ is everywhere defined on $L^2(\mu)$ by Theorem 2.1.5, and the two functions $f_0(z) = \sum_{n \in P} c_{4n} z^{4n}$ and $f_1(z) = \sum_{n \in P} c_{4n+1} z^{4n}$ are in $H_2\left(z^4\right)$. □

Fractal Hardy spaces. An iteration of the argument from the proof of the corollary yields, for each $n \in \mathbb{N}$, a natural *isometric* embedding Φ_n of $L^2(\mu)$ into the subspace of H_2 characterized as n increases by:

$$\begin{aligned} H_2\left(z^{4^n}\right) + zH_2\left(z^{4^n}\right) + z^4 H_2\left(z^{4^n}\right) + z^5 H_2\left(z^{4^n}\right) \\ + z^{16} H_2\left(z^{4^n}\right) + z^{17} H_2\left(z^{4^n}\right) \cdots + z^{\frac{4^n - 1}{3}} H_2\left(z^{4^n}\right). \end{aligned}$$

Specifically, let $n \in \mathbb{N}$ be fixed, and let $P_n = \left\{ l_0 + 4l_1 + \cdots + 4^{n-1} l_{n-1} : l_i \in \{0,1\} \right\}$. Then the functions in $\Phi_n \left(L^2(\mu) \right)$ $(\subset H_2)$ have the following characteristic module representation:

$$\left\{ \sum\nolimits_{p \in P_n} z^p f_p \left(z^{4^n} \right) : f_p \in H_2 \right\}.$$

For each n, Φ_n maps *into* this space, and not onto.

Spectral pairs. Our interest in the problem of finding dense analytic subspaces in $L^2(\mu)$, for probability measures, grew out of our earlier work on *spectral pairs*. It is known [**Fug74**] that, for $\nu = 2$, the case when Ω is either the triangle, or the disk, does *not* admit any sets Λ such that (Ω, Λ) is a spectral pair. On the other hand, our work in [**JP94**] showed that, when (Ω, Λ) *is* a spectral pair in ν dimensions, then Ω is often "generated" by some amount of self-affine structure, as described by a system of affine transforms σ_b as in (2.1.3).

Since there is a lack of symmetry of the two sets Ω and Λ in a spectral pair, a generalized spectral pair formulation for two measures μ and ρ was introduced in [**JP99**]. The context was locally compact abelian groups:

Definition 2.1.7. Let G be a locally compact abelian group with dual group Γ. Let μ be a Borel measure on G, and ρ one on Γ. For f of compact support and continuous, set

$$F_\mu f(\xi) = \int_G \overline{\langle \xi, x \rangle} f(x)\, d\mu(x)$$

where $\langle \xi, x \rangle$ denotes the pairing between points ξ in Γ and x in G. If $f \mapsto F_\mu f$ extends to an isomorphic isometry (i.e., unitary) of $L^2(\mu)$ *onto* $L^2(\rho)$, then we say that (μ, ρ) is a *spectral pair*.

It is clear how the earlier definition of spectral pairs is a special case, even when G is restricted to the additive group $\mathbb{R}^\nu$. But it is not immediate that there are examples (μ, ρ) of the new spectral pair type which cannot be reduced to the old one.

Theorem 2.1.5 shows that this is indeed the case (i.e., that there are examples): Let $G = \mathbb{R}$, and let μ be the fractal measure as above. Let

$$P = \left\{ l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0,1\},\ \text{finite sums} \right\} = \{0, 1, 4, 5, 16, 17, \cdots\},$$

and let $\rho = \rho_P$ be the counting measure of P. Then the conclusion may be restated to the effect that (μ, ρ_P) is a spectral pair. This is perhaps surprising as earlier work on Fourier analysis of fractal measures, see e.g. [**Str90**], [**Str93**], and [**JP95**], suggested a continuity in the Fourier transform, and also the presence of asymptotic estimates, rather than exact identities.

It can be shown, as a consequence of [**JP99**, Corollary A.5] that if (μ, ρ) satisfies *the spectral-pair property* for any measure ρ, then $\rho = \rho_P$ for some subset $P \subset \mathbb{R}^\nu$, i.e., $L^2(\mu)$ has an orthonormal basis of the form $\{e_\lambda : \lambda \in P\}$. The basis for this argument is the *finiteness* of μ, when generated from (2.1.5).

2.2. The middle third Cantor measure

The significance of the assumptions (2.1.10)–(2.1.11) on the pair R, B lies in the identity (2.2.2) below, and also in orthogonality.

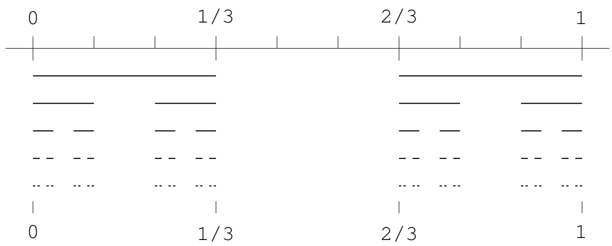


FIGURE 2.2.1. Support of μ_3

LEMMA 2.2.1. *Let the sets $B, L \subset \mathbb{R}^\nu$, and the matrix R, be as in Lemma* 2.1.2. *The function*

$$Q_1(t) := \sum_{\lambda \in P} |\hat{\mu}(t-\lambda)|^2 \tag{2.2.1}$$

(*where $P = \{l_0 + R^* l_1 + \cdots : l_i \in L$, finite sums$\}$*) *satisfies the functional identity* ($t \in \mathbb{R}^\nu$)

$$Q(t) = \sum_{l \in L} |\chi_B(t-l)|^2 \, Q\left(R^{*\,-1}(t-l)\right). \tag{2.2.2}$$

PROOF. Let $t \in \mathbb{R}^\nu$. Then

$$\begin{aligned} Q_1(t) &= \sum_{\lambda \in P} |\hat{\mu}(t-\lambda)|^2 \\ &= \sum_{\lambda \in P} |\chi_B(t-\lambda)|^2 \left|\hat{\mu}\left(R^{*\,-1}(t-\lambda)\right)\right|^2 \\ &= \sum_{l \in L} \sum_{\lambda \in P} |\chi_B(t-l-R^*\lambda)|^2 \left|\hat{\mu}\left(R^{*\,-1}(t-l)-\lambda\right)\right|^2 \\ &= \sum_{l \in L} |\chi_B(t-l)|^2 \sum_{\lambda \in P} \left|\hat{\mu}\left(R^{*\,-1}(t-l)-\lambda\right)\right|^2 \\ &= \sum_{l \in L} |\chi_B(t-l)|^2 \, Q_1\left(R^{*\,-1}(t-l)\right). \end{aligned}$$

□

If, for example, we work with the more traditional triadic Cantor set, then the results in Lemmas 2.1.2 and 2.1.4 no longer are valid. To see this, take $R = 3$ and $B = \left\{0, \frac{2}{3}\right\}$. Let μ_3 denote the corresponding measure on $\mathbb{R}$ with support equal to the triadic Cantor set (see Figure 2.2.1).

It is determined by

$$\int f \, d\mu_3 = \frac{1}{2}\left(\int f\left(\frac{x}{3}\right) d\mu_3(x) + \int f\left(\frac{x+2}{3}\right) d\mu_3(x)\right), \quad \forall f \in C_c(\mathbb{R}),$$

has $d_H = \frac{\ln 2}{\ln 3}$, and satisfies

$$\widehat{\mu_3}(t) = \frac{1}{2}\left(1 + e^{i\frac{4}{3}\pi t}\right)\widehat{\mu_3}\left(\frac{t}{3}\right), \quad t \in \mathbb{R}.$$

Choose $L = \left\{0, \frac{3}{4}\right\}$ so that (2.1.11) is valid; then the subset P_3 (which corresponds to $P = P(L)$ in Lemma 2.1.4), is

$$P_3 = \left\{ \frac{3}{4}\left(l_0 + 3l_1 + 3^2 l_2 + \cdots\right) : l_i \in \{0,1\}, \text{ finite sums} \right\},$$

but the corresponding exponentials $\{e_\lambda : \lambda \in P_3\}$ are now *not* mutually orthogonal in $L^2(\mu_3)$. Take for example the two points $\lambda = \frac{3}{4}$ and $\lambda' = \frac{9}{4}$ both in the set P_3. The corresponding exponentials e_λ and $e_{\lambda'}$ are both orthogonal to e_0, but they are not mutually orthogonal, i.e., $\langle e_\lambda \mid e_{\lambda'} \rangle_\mu \neq 0$. In fact, for the μ_3-inner product:

$$\langle e_\lambda \mid e_{\lambda'} \rangle_\mu = \widehat{\mu_3}\left(\frac{3}{2}\right) = \widehat{\mu_3}\left(\frac{1}{2}\right) = \frac{1}{4}\widehat{u_3}\left(\frac{1}{6}\right) \neq 0.$$

It can further be shown that there is *no subset* $P \subset \mathbb{R}$ such that, if ρ_P denotes the corresponding counting measure on $\mathbb{R}$, then (μ_3, ρ_P) is a spectral pair in the above more general sense (*a fortiori,* fractions don't provide a basis either). Similarly, it can be checked that the identity (2.2.2) in Lemma 2.2.1 fails for this pair μ_3, P_3.

2.3. Infinite Bernoulli convolutions

Infinite Bernoulli convolutions are special cases of affine self-similarity systems, also called iterated function systems (IFSs). Thus IFS measures generalize distributions of Bernoulli convolutions. Bernoulli convolutions in turn generalize Cantor measures.

The term "infinite convolution" is not mysterious at all, since the Fourier transform converts convolutions to products. However, one might ask why these measures have the name "Bernoulli" attached to them. These measures arise in the work of Erdős and others via the study of the random geometric series $\sum \pm\lambda^n$ for $\lambda \in (0,1)$, where the signs are the outcome of a sequence of independent Bernoulli trials. In other words, we could consider the signs to be determined by a string of fair coin tosses. This makes $\sum \pm\lambda^n$ a *random variable*, i.e., a measurable function from a probability space into the real numbers $\mathbb{R}$. In Erdős's language, β is the distribution of the random variable X defined on the probability space Ω of all infinite sequences of ± 1. The measure on Ω is the infinite-product measure resulting from assigning ± 1 equal probability $\frac{1}{2}$. The random variable X takes on a specific real value for each sequence from Ω. This distribution β, then, is the familiar Bernoulli distribution from elementary probability theory. The infinite Bernoulli convolution measure is determined by the distribution D of the random variable $\sum \pm\lambda^n$, which can be constructed from infinite convolution of dilates of β:

$$D := \beta(x) * \beta(\lambda^{-1}x) * \beta(\lambda^{-2}x) * \cdots * \beta(\lambda^{-n}x) * \cdots .$$

These Bernoulli convolution measures have been studied from at least the mid-1930s in various contexts. There seems to have been a flurry of activity in the 1930s and 1940s surrounding these measures. Jessen and Wintner study these measures in their study of the Riemann zeta function in their 1935 paper [**JW35**]; in 1939 and 1940, Erdős published two important papers about these measures [**Erd39**,**Erd40**]. In the 1939 paper, Erdős proved that if α is a Pisot number (that is, α is a real algebraic integer greater than 1 all of whose conjugates $\tilde{\alpha}$ satisfy $|\tilde{\alpha}| < 1$), then the infinite Bernoulli convolution measure associated with $\lambda = \alpha^{-1}$ is singular with respect to Lebesgue measure. However, more recently, Solomyak proved that for almost every $\lambda \in (\frac{1}{2}, 1)$, the measure ν_λ is absolutely continuous with respect to Lebesgue measure [**Sol95**].

Gaps vs overlap

Bernoulli convolutions are special cases of iterated function systems (IFS), and for values of $\lambda < 1/2$, the corresponding IFS is a fractal resulting from iteration of gaps. Its fractal dimension is known, and is < 1. The dimensions are distinct for distinct values of λ.

Of course, if $\lambda = 1/2$, then the IFS measure is simply restriction of Lebesgue measure to the interval. But if $\lambda > 1/2$, then the corresponding IFS has overlaps, i.e., the range of the similarity maps have essential overlaps; see eq (2.6.1). The case of IFSs with overlaps is taken up systematically in Section 2.6 below, covering there the general case.

The Bernoulli convolution measure with scaling factor λ, the measure μ_λ, can be defined in several equivalent ways. Here, we will describe a probabilistic method and and IFS method to obtain the measure μ_λ.

In probability theory, one can define the measure μ_λ with the distribution of a random variable $\mathbf{Y}_{\boldsymbol\lambda}$. For each $k \in \mathbb{N}$, let

$$Y_k : \prod_{k=1}^{\infty} \{-1, 1\} \to \{-1, 1\}$$

be defined by

$$Y_k(\omega_1, \omega_2, \omega_3, \ldots) = \omega_k. \tag{2.3.1}$$

LEMMA 2.3.1. *Define* $\mathbf{Y}_{\boldsymbol\lambda}$ *by*

$$\mathbf{Y}_{\boldsymbol\lambda} = \sum_{k=1}^{\infty} Y_k \lambda^k. \tag{2.3.2}$$

Then

$$\mathbb{E}_\lambda(e^{i\mathbf{Y}_{\boldsymbol\lambda} t}) = \prod_{k=1}^{\infty} \cos(\lambda^k t). \tag{2.3.3}$$

NOTATION. $\mathbf{Y}_{\boldsymbol\lambda}$ is sometimes written $\mathbf{Y}_{\boldsymbol\lambda} = \sum_{k=1}^{\infty} (\pm 1)_k \lambda^k$. Either Y_k or $(\pm 1)_k$ is the outcome of the binary coin-toss where each of the two outcomes, heads $(+1)$ and tails (-1), is equally likely. These coin-tosses are independent of each other and identically distributed.

PROOF. If $\mathbb{E}_\lambda$ denotes the expectation of the random variable $\mathbf{Y}_{\boldsymbol\lambda}$, then for all $t \in \mathbb{R}$,

$$\mathbb{E}_\lambda(e^{i\mathbf{Y}_{\boldsymbol\lambda} t}) = \mathbb{E}_\lambda(e^{\sum_k Y_k \lambda^k}) = \prod_{k=1}^{\infty} \mathbb{E}_\lambda(e^{iY_k \lambda^k t}), \tag{2.3.4}$$

where independence of the random variables Y_k is used to obtain the second equality in (2.3.4). Because the two outcomes -1 and $+1$ are equally likely, we obtain

$$\mathbb{E}_\lambda(e^{i\mathbf{Y}_{\boldsymbol\lambda}(\cdot) t}) = \prod_{k=1}^{\infty} \Big(\frac{1}{2} e^{i\lambda^k t} + \frac{1}{2} e^{-i\lambda^k t}\Big) = \prod_{k=1}^{\infty} \cos(\lambda^k t). \tag{2.3.5}$$

□

For more details about random Fourier series and this approach to the measure μ_λ, see [**Kah85**] and [**Jor06**, Chapter 5].

Another way to generate the measure μ_λ is from an iterated function system (IFS) with two affine maps

$$\tau_+(x) = \lambda(x+1) \quad \text{and} \quad \tau_-(x) = \lambda(x-1). \tag{2.3.6}$$

By Banach's fixed point theorem, there exists a compact subset of the line, denoted X_λ and called the attractor of the IFS, which satisfies the invariance property

$$X_\lambda = \tau_+(X_\lambda) \cup \tau_-(X_\lambda). \tag{2.3.7}$$

Hutchinson proved that there exists a unique measure μ_λ corresponding to the IFS (2.3.6), which is supported on X_λ and is invariant in the sense that

$$\mu_\lambda = \frac{1}{2}\Big(\mu_\lambda \circ \tau_+^{-1}\Big) + \frac{1}{2}\Big(\mu_\lambda \circ \tau_-^{-1}\Big) \tag{2.3.8}$$

[**Hut81**, Theorems 3.3(3) and 4.4(1)]. The property in (2.3.8) defines the measure μ_λ and can be used to compute its Fourier transform. The Fourier transform of μ_λ is precisely the same function we saw in (2.3.5):

$$\widehat{\mu}_\lambda(t) = \prod_{k=1}^{\infty} \cos(\lambda^k t). \tag{2.3.9}$$

See Figure 2.3.1.

Bernoulli convolution measures have been studied in various settings, long before IFS theory was developed. Some of the earliest papers on Bernoulli convolution measures date to the 1930s and work with an infinite convolution definition for μ_λ [**JW35**, **KW35**, **Win35**, **Erd39**]. The history of Bernoulli convolutions up to 1998 is detailed in [**PSS00**].

We might ask under which conditions the measure μ_λ has a Fourier basis (i.e. an orthonormal basis of complex exponential functions) for the Hilbert space $L^2(\mu_\lambda)$. When such an orthonormal Fourier basis exists, we say that μ_λ is a *spectral measure*.

Some lacunary spectra. This line of questioning has its origins in [**JP98a**], in which the duality can be highly non-intuitive. For example, when the scaling factor is $\frac{1}{3}$ — that is, $\mu_{\frac{1}{3}}$ is the Cantor-Bernoulli measure for the omitted third Cantor set construction — there is no Fourier basis. In other words, there is no Fourier series representation in $L^2(\mu_{\frac{1}{3}})$. In fact, there can be at most two orthogonal Fourier frequencies in $L^2(\mu_{\frac{1}{3}})$. But if we modify the Cantor-Bernoulli construction, using scale $\frac{1}{4}$, as opposed to $\frac{1}{3}$, then a Fourier basis does exist in $L^2(\mu_{\frac{1}{4}})$. In fact, *each* of the Cantor-Bernoulli measures $\mu_{\frac{1}{2n}}$ with $n \in \mathbb{N}$ has a Fourier basis. For each of these measures, there is a canonical choice for a Fourier dual set Γ.

Other scale numbers. Jorgensen and Pedersen demonstrated Fourier bases for $L^2(\mu_\lambda)$ when $\lambda = \frac{1}{2n}$ for each $n \in \mathbb{N}$. They also showed that when $\lambda = \frac{1}{2n+1}$, there is no orthonormal basis (ONB) consisting of exponential functions, and in fact, every orthogonal collection of exponentials is finite when the denominator of λ is odd. The Fourier basis is indexed by a discrete set $\Gamma \subset \mathbb{R}$ given by

$$\Gamma = \Gamma\Big(\frac{1}{2n}\Big) = \left\{ \sum_{i=0}^{m} a_i (2n)^i \ : \ a_i \in \Big\{0, \frac{n}{2}\Big\}, m \text{ finite} \right\}. \tag{2.3.10}$$

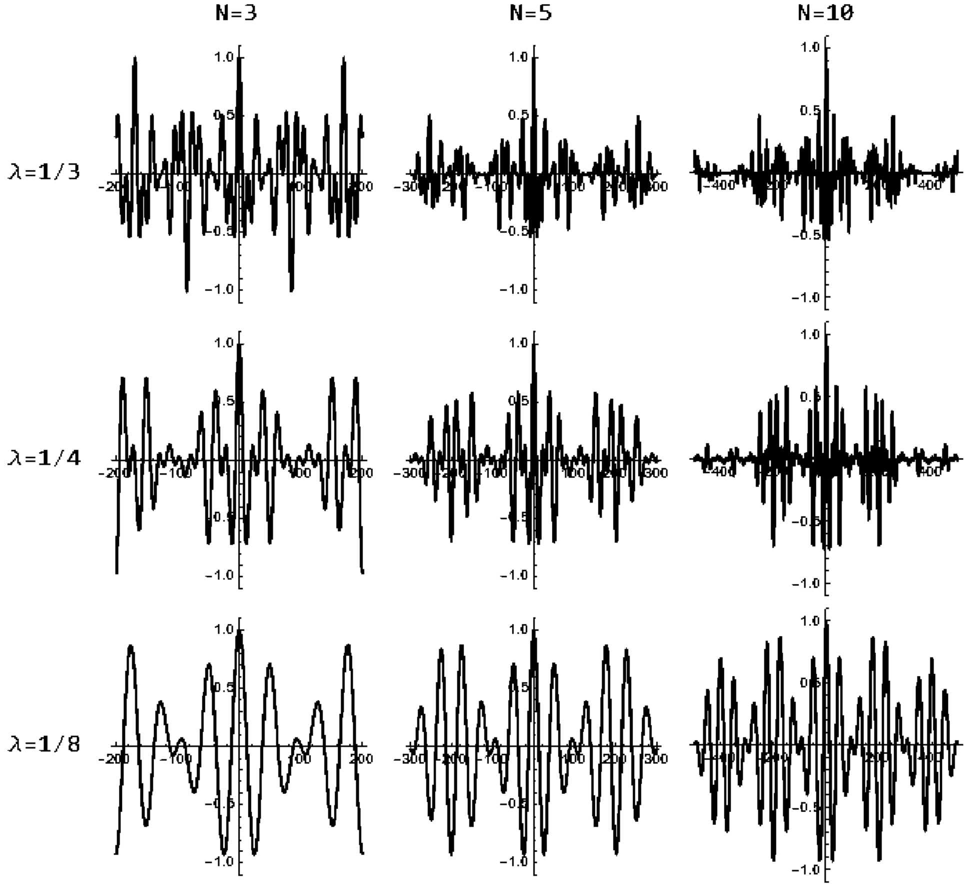


FIGURE 2.3.1. It is often difficult to compute fractal Fourier series, and Fourier expansions, directly. Starting with an IFS measure μ, it is typically easier to do an infinite product expression like (2.3.9) for $\widehat{\mu}(t)$. Of course, for computations, one must select a suitable finite number of factors. The figure illustrates some choices. As noted, the question of whether μ is part of a spectral pair depends on the configuration of the real zeros of $\widehat{\mu}(t)$. Computation of the corresponding μ Fourier expansions will involve derivatives of $\widehat{\mu}(t)$ evaluated at the real zeros.

Note that the elements of Γ are integers when n is even and are all in $\frac{1}{2}\mathbb{Z}$ when n is odd.

We observe that Γ has a self-similarity by scaling in the large. For instance, Γ is invariant under scaling by the value $2n$, the reciprocal of λ. But there is even a stronger scaling invariance:

$$\Gamma = 2n\Gamma \sqcup \left(2n\Gamma + \frac{n}{2}\right).$$

It is known that if $\lambda > \frac{1}{2}$, there is no Fourier basis [**DHJ09**]. It has also been shown that if $\lambda = \frac{q}{2n}$ where q is odd, the set of exponentials to be an ONB for

$\mu_{\frac{1}{2n}}$ are also orthogonal for $\mu_{\frac{a}{2n}}$ [**JKS08**], [**HL08**]. There are still a variety of open questions regarding the existence and classification of Fourier bases for Bernoulli convolution measures.

We note that, since we wish to discuss ONBs for *Bernoulli measures*, we will be restricting here to the case where the scale factor $\lambda = \frac{1}{2n}$ for $n \in \mathbb{N}$. To keep the notation simple, we will write μ for $\mu_{\frac{1}{2n}}$ and X for $X_{\frac{1}{2n}}$.

Given the invariance equation (2.3.8) , there is a standard convenient expression for the integral of an exponential function e_t. We denote the resulting function in t by $\widehat{\mu}$ since this produces a Fourier transform of μ:

$$\begin{aligned}
\widehat{\mu}(t) &= \int_{X_\lambda} e^{2\pi i x t} d\mu(x) \\
&= \frac{1}{2}\int_{X_\lambda} \left(e^{2\pi i\lambda(x+1)t} + e^{2\pi i\lambda(x-1)t}\right) d\mu(x) \\
&= \cos(2\pi\lambda t)\widehat{\mu}(\lambda t) \\
&= \cos(2\pi\lambda t)\cos(2\pi\lambda^2 t)\widehat{\mu}(\lambda^2 t) \\
&= \quad \vdots \quad .
\end{aligned}$$

Continuing the iteration, we find an infinite product formula for $\widehat{\mu}$:

$$\widehat{\mu}(t) = \prod_{k=1}^{\infty} \cos(2\pi\lambda^k t). \tag{2.3.11}$$

Given exponential functions e_γ and $e_{\gamma'}$, we note that

$$\langle e_{\gamma'}, e_\gamma\rangle_{L^2(\mu)} = \int_X e^{2\pi i(\gamma-\gamma')x} d\mu(x) = \widehat{\mu}(\gamma - \gamma'). \tag{2.3.12}$$

We are considering orthogonal collections of exponential functions, the zeroes of the function $\widehat{\mu}$. By (2.3.11), $\widehat{\mu}$ is zero if and only if one of the factors in the infinite product is zero. The cosine function is zero at the odd multiples of $\frac{\pi}{2}$, which yields the set of zeroes for $\widehat{\mu}$, denoted $\mathcal{Z}$:

$$\mathcal{Z}(\widehat{\mu}_{\frac{1}{2n}}) = \left\{ \frac{(2n)^k(2m+1)}{4} \mid m \in \mathbb{Z},\ k \geq 1 \right\}. \tag{2.3.13}$$

Given a discrete set Γ, then, the collection of exponential functions $E(\Gamma)$ is an orthogonal collection if $\langle e_{\gamma'}, e_\gamma\rangle_{L^2(\mu)} = \delta_{\gamma,\gamma'}$, which occurs if and only if

$$\gamma - \gamma' \in \mathcal{Z} \quad \text{ for all } \gamma, \gamma' \in \Gamma \text{ with } \gamma \neq \gamma'.$$

Note that the set from (2.4.3) does indeed satisfy this condition.

Rational values of λ

It was shown in [**JKS08**], that *given $\lambda = \frac{a}{b}$, if b is even, then there exist infinite families of orthogonal exponentials in the corresponding Hilbert space. If b is odd, then every collection of mutually orthogonal exponentials must be finite.*

2.4. The scale-4 Cantor measure, and scaling by 5 in the spectrum

Fractal sets which are invariant under a collection of contractive maps (an iterated function system) exhibit *scaling self-similarity*; one sees the same shape when zooming in to look more closely at one part of the set. We might call this scaling *in the small* to find self-similarity. It is also common to find that the discrete sets which index Fourier bases for fractal L^2 spaces have an expansive self-similarity themselves. We say that these sets (called *spectra*) have a self-similarity *in the large*. When a measure μ has such a discrete index set Γ for a Fourier basis, we call (μ, Γ) a *spectral pair*.

For decades, it has been known that a subclass of IFS measures μ have associated Fourier bases for $L^2(\mu)$ [**JP98a**]. If $L^2(\mu)$ does have a Fourier ONB with Fourier frequencies $\Gamma \subset \mathbb{R}$, we then say that (μ, Γ) is a spectral pair. In the case that a set of Fourier frequencies exist for $L^2(\mu)$, we say Γ is a Fourier dual set for μ or that Γ is a spectrum for μ; we say μ is a spectral measure. The goal of the present section is to examine the operator U which scales one spectrum into another spectrum. We observe how the intrinsic scaling (by 4) which arises in our set Γ interacts with the spectral scaling (to 5Γ) that defines U. We call U an *operator-fractal* due to its self-similarity, which is described in detail in [**JKS12**].

For illustration, consider first the simplest case — the Bernoulli convolution formed by recursive scaling by $\frac{1}{4}$ with two affine maps on the real line. The resulting measure $\mu_{\frac{1}{4}}$ is an infinite Bernoulli convolution, also called a Cantor measure, or a Hutchinson measure. The Hilbert space $L^2(\mu_{\frac{1}{4}})$ has a Fourier basis which has a self-similarity under scaling in the large by 4. It is somewhat surprising to find that that $\mu_{\frac{1}{4}}$ also gives rise to a symmetry based on scaling by 5. It turns out that scaling by 5 transforms the Fourier spectrum Γ into another ONB 5Γ. As a result, we have a natural unitary operator U acting in $L^2(\mu_{\frac{1}{4}})$ which maps one ONB to the other. Its spectral properties turn out to reveal a surprising level of symmetry and self-similarity which lead us to the nomenclature *operator-fractal* [**JKS12**].

Not all measures are spectral measures. For example, the middle-third Cantor measure $\mu_{\frac{1}{3}}$ does not have a spectrum — in fact, there cannot be more than two orthogonal complex exponentials in $L^2(\mu)$. On the other hand, many Cantor measures are spectral [**JP98a**]. If μ is determined by scaling by $\frac{1}{4}$ at each Cantor iteration step, then the corresponding space $L^2(\mu_{\frac{1}{4}})$ does have a Fourier basis.

Typically, a spectrum for μ is a relatively "*thin*" subset of $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$ which has its own scaling properties related to the scaling invariance of μ. Sometimes a spectrum displays invariance with respect to two different scales, and in these cases, many questions arise. A particularly interesting example is the Jorgensen-Pedersen spectrum

$$\Gamma(\frac{1}{4}) = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \cdots\},$$

which has self-similarity when scaled by 4; in addition, the set $5\Gamma(\frac{1}{4})$ is also a spectrum for $\mu_{\frac{1}{4}}$. The intertwining properties of the two scaling operations were first discovered in [**DJ12a**] and later considered in [**JKS12**] and [**JKS14b**].

We consider here a particular additional symmetry relation for the subclass of Cantor-Bernoulli measures that form spectral pairs. Starting with a spectral pair (μ, Γ), we consider an action which scales the set Γ. In the special case of $\mu_{\frac{1}{4}}$, we scale Γ by 5. Scaling by 5 induces a natural unitary operator U in $L^2(\mu_{\frac{1}{4}})$, and we study the spectral-theoretic properties of U.

The measure $\mu_{\frac{1}{4}}$ and its support $X_{\frac{1}{4}}$ admit the similarity scaling laws shown in Equations (2.3.7) and (2.3.8) — scaling by $\frac{1}{4}$ — which we call affine scaling in the small. The canonical construction of the dual set Γ of Fourier frequencies in

[**JP98a**] uses scaling by powers of 4 in the large. Elements of $L^2(\mu_{\frac{1}{4}})$ are lacunary Fourier series, with the lacunary Fourier bases involving powers of 4.

The 5-scaling operator U was studied in [**JKS14a**]. Its spectral theory is surprisingly subtle. While U is induced by scaling a spectrum for $\mu_{\frac{1}{4}}$ by 5, U is not the lifting of a $\mu_{\frac{1}{4}}$-measure preserving endomorphism. But the operator U has a "fractal" nature of its own — it is the countable infinite direct sum of the operator MU plus a rank-one projection. Here, M is multiplication by z in a Fourier representation of $L^2(\mu_{\frac{1}{4}})$.

SETTING. In details, we consider $L^2(\mu_{\frac{1}{4}})$ and operators in the following context:

$$\text{(2.4.1)} \qquad \text{Hilbert space, operators:} \begin{cases} \mathscr{H} \text{ a Hilbert space} \\ \{S_i\}_{i=0}^{1} \in Rep(\mathscr{O}_2, \mathscr{H}) \\ U \text{ a normal operator on } \mathscr{H} \\ M \text{ a unitary operator on } \mathscr{H}. \end{cases}$$

and

$$\text{(2.4.2)} \qquad \text{Operator relations:} \begin{cases} S_1^* U S_1 = MU \\ S_0 \text{ commutes with } U. \end{cases}$$

The two operators $\{S_0, S_1\}$ which form a representation of $\mathscr{O}_2$ on $\mathscr{H}$; see (1.3.7).

The Hilbert space $L^2(\mu_{\frac{1}{4}})$ has an associated spectrum

$$\text{(2.4.3)} \qquad \Gamma\Big(\frac{1}{4}\Big) = \left\{ \sum_{i=0}^{m} a_i 4^i \;:\; a_i \in \Big\{0, 1\Big\} \right\} = \{0, 1, 4, 5, 16, 17, 20, \ldots\}$$

which in turn gives rise to an orthonormal basis (ONB) for $L^2(\mu_{\frac{1}{4}})$:

$$\text{(2.4.4)} \qquad E\Big(\Gamma\Big(\frac{1}{4}\Big)\Big) = \Big\{ e_\gamma(t) = e^{2\pi i \gamma t} \mid \gamma \in \Gamma\Big(\frac{1}{4}\Big) \Big\}$$

By [**DJ12a**, Proposition 5.1], the scaled set $5\Gamma(\frac{1}{4})$ *is also a spectrum* for $\mu_{\frac{1}{4}}$, which leads us to define the unitary operator U by

$$\text{(2.4.5)} \qquad U e_\gamma = e_{5\gamma}.$$

The operator $M = M_{e_1}$ is multiplication by the exponential e_1:

$$\text{(2.4.6)} \qquad M_{e_1} e_\gamma = e_{\gamma+1}.$$

In [**JKS12**], the authors showed that the operator U has a fractal-like nature which arises from a representation of $\mathscr{O}_2$ on $L^2(\mu_{\frac{1}{4}})$ given by the operators

$$\text{(2.4.7)} \qquad S_0 e_\gamma = e_{4\gamma} \quad \text{and} \quad S_1 e_\gamma = e_{4\gamma+1}.$$

THEOREM 2.4.1 (Jo-Kornelson-Shuman). *The Cuntz operators S_0 and S_1 give rise to an ordering of the basis $E(\Gamma(\frac{1}{4}))$ and a resulting* orthogonal decomposition *of $L^2(\mu_{\frac{1}{4}})$ given by*

$$\text{(2.4.8)} \qquad L^2(\mu_{\frac{1}{4}}) = span\{e_0\} \oplus \bigoplus_{k=0}^{\infty} S_0^k S_1 L^2(\mu_{\frac{1}{4}}).$$

The subspaces $S_0^k S_1 L^2(\mu_{\frac{1}{4}})$ have the property that the matrix of U restricted to each subspace is the same.

The second set of axioms (2.4.2) is satifsied by U, M_{e_1}, S_0 and S_1. We have

$$S_0 U = U S_0 \quad \text{and} \quad S_1^* U S_1 = M_{e_1} U; \tag{2.4.9}$$

see e.g., [**JKS11**].

THEOREM 2.4.2 (Jo-Kornelson-Shuman). *The operator U can be written as an orthogonal sum of "identical" copies*

$$U = P_{e_0} \oplus \bigoplus_{k=0}^{\infty} M_{e_1} U, \tag{2.4.10}$$

where P_{e_0} is the orthogonal projection onto span$\{e_0\}$; P_{e_0} is the projection onto the unitary part of the Wold decomposition of S_0.

THEOREM 2.4.3 ([**JKS14a**]). *The relations* (2.4.1) *and* (2.4.2) *have an irreducible representation on $L^2(\mu_{\frac{1}{4}})$.*

PROOF. Consider the $*$-algebra $\mathfrak{A}$ generated by U, M_{e_1}, and the representation of $\mathscr{O}_2$ in $L^2(\mu_{\frac{1}{4}})$. Note that the representation of $\mathscr{O}_2$ carries its own relations, but as of now, with an abuse of notation, we have specified no relations among U, M_{e_1}, and the representation of $\mathscr{O}_2$.

Let $\mathfrak{I}$ be the two-sided ideal generated by the relations which have already been established in (2.4.9):

$$S_0 U - U S_0 = 0 \quad \text{and} \quad S_1^* U S_1 - M_{e_1} U = 0. \tag{2.4.11}$$

We want to establish that S_0 and S_1 are not in $\mathfrak{I}$. But neither can be in $\mathfrak{I}$ because if for $i = 0$ or $i = 1$

$$S_i = c_1 X_1 (S_0 U - U S_0) Y_1 + c_2 X_2 (S_1^* U S_1 - M_{e_1} U) Y_2$$

for $X_1, X_2, Y_1, Y_2 \in \mathfrak{A}$, then we could multiply both sides by S_i^*, which tells us that $I \in \mathfrak{I}$, or that $S_i = 0$ in the $*$-algebra $\mathfrak{A}/\mathfrak{I}$, which is not true.

We explicitly establish that the representation of $\mathscr{O}_2$ is irreducible in so the representation of $\mathfrak{A}/\mathfrak{I}$ is also irreducible. □

In [**JKS12**, Theorem 4.10], it is proved that U is an orthogonal sum of a one-dimensional projection and an infinite number of copies of the operator MU. In other words, U is an "operator fractal"; i.e., it is a geometric representation of an infinite number of scaled versions of itself. By "orthogonal sum" we mean that $L^2(\mu_{\frac{1}{4}})$ is an orthogonal sum of closed invariant subspaces for U.

The matrix of U with respect to the decomposition we have just described is given by

	span$\{e_0\}$	S_1	S_0S_1	$S_0^2S_1$	$S_0^3S_1$	$\cdots$
span$\{e_0\}$	1	0	0	0	0	$\cdots$
S_1	0	$M_{e_1}U$	0	0	0	$\cdots$
S_0S_1	0	0	$M_{e_1}U$	0	0	$\cdots$
$S_0^2S_1$	0	0	0	$M_{e_1}U$	0	$\cdots$
$S_0^3S_1$	0	0	0	0	$M_{e_1}U$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2.5. IFS measures and admissible harmonic analyses

Fuglede's conjecture [**Fug74**] asserts that a Lebesgue measurable subset Ω of $\mathbb{R}^d$ is spectral if and only if it tiles $\mathbb{R}^d$ by translations. Tao [**Tao04**] found a union of cubes, in dimension 5 or higher, which is spectral but does not tile. Later, Tao's counterexample was improved by Matolcsi and his collaborators [**KM06**, **FMM06**], to disprove Fuglede's conjecture in both directions, down to dimension 3. In dimension 1 and 2, the conjecture is still open in both directions.

From previous discussion, we have seen that Lebesgue measure is not the only measure that provides examples of spectral sets. For example, Hausdorff measure on a fractal Cantor set with scale 4 is also spectral [**JP98a**]. There are many more spectra for the same measure as shown in [**DHS09**]. Many more examples of fractal spectral measures have been constructed since [**Str00**, **LaW02**, **DJ07d**].

Question. Which sets appear as spectra of some measure?

This section presents a characterization of spectra of measures in terms of the existence of a strongly continuous representation of the ambient group which has a *wandering vector* for the given set. The material below is based primarily on ideas in [**DJ15b**].

Definition 2.5.1. Let $\mathcal{U}$ be a family of unitary operators acting on a Hilbert space $\mathscr{H}$. We say that a vector $v_0 \neq 0$ in $\mathscr{H}$ is a *wandering vector* if $\{Uv_0 : U \in \mathcal{U}\}$ is an orthogonal family of vectors.

Theorem 2.5.2 ([**DJ15b**]). *Let $S \subset \Gamma$ be an arbitrary subset. Then the subset S is a spectrum/frame spectrum with bounds A, B for a Borel probability measure μ_0 on G if and only if there exists a triple $(\mathscr{H}, v_0, U)$ where $\mathscr{H}$ is a complex Hilbert space, $v_0 \in \mathscr{H}$, $\|v_0\| = 1$ and $U(\cdot)$ is a strongly continuous representation of Γ on $\mathscr{H}$ such that $\{U(\gamma)v_0 : \gamma \in S\}$ is an orthonormal basis/frame with bounds A, B for $\mathscr{H}$.*

Moreover, in this case μ_0 can be chosen such that

$$\langle v_0, U(\xi)v_0\rangle_{\mathscr{H}} = \int_G e_\xi\left(g\right) d\mu_0\left(g\right), \quad \xi \in \Gamma, \tag{2.5.1}$$

and there is an isometric isomorphism $W : L^2(G, \mu_0) \to \mathscr{H}$ such that W

$$We_\gamma = U(\gamma)v_0 \quad \textit{for all } \gamma \in \Gamma. \tag{2.5.2}$$

Proof. Suppose S is a spectrum for μ_0. Set $\mathscr{H} = L^2(G, \mu_0)$, $v_0 =$ the constant function 1 in $L^2(G, \mu_0)$ and take, for $\xi \in \Gamma$, $U(\xi)$ on $L^2(G, \mu_0)$ to be the multiplication operator , i.e.,

$$(U(\xi)f)(g) = e_\xi(g)f(g) \quad f \in L^2(G, \mu_0),\ \xi \in \Gamma, g \in G \tag{2.5.3}$$

A simple check shows that all the requirements are satisfied and the isomorphism W is just the identity.

Conversely, suppose $(\mathscr{H}, v_0, U)$ is a triple such that $\{U(\gamma)v_0 : \gamma \in S\}$ is orthonormal in $\mathscr{H}$. Then by the Stone-Naimark-Ambrose-Godement theorem (the SNAG theorem [**Mac92**, **Mac04**]), there is an orthogonal projection valued measure P_U defined on the Borel subsets of G, such that

$$U(\xi) = \int_G e_\xi\left(g\right) dP_U\left(g\right), \quad \xi \in \Gamma. \tag{2.5.4}$$

Now set

$$d\mu_0(g) := \|dP_U(g)v_0\|^2_{\mathscr{H}} \tag{2.5.5}$$

and note that μ_0 will then be a Borel probability measure on G.

We prove that (2.5.1) holds. Let $\xi \in \Gamma$. We have

$$\begin{aligned} \int_G e_\xi(g)\, d\mu_0(g) &= \int_G e_\xi\,(g)\, \|dP_U\,(g)\, v_0\|^2_{\mathscr{H}} \\ &= \int_G e_\xi\,(g)\, \langle v_0, dP_U\,(g)\, v_0\rangle \\ &= \left\langle v_0, \int_G e_\xi\,(g)\, dP_U\,(g)\, v_0 \right\rangle \\ &= \langle v_0, U(\xi)v_0\rangle\,. \end{aligned} \tag{2.5.6}$$

We now show that there is an isometric isomorphism $W : L^2(G, \mu_0) \to \mathscr{H}$ that satisfies (2.5.2). The fact that $\{e_\gamma : \gamma \in S\}$ is an orthonormal basis will follow from this. Define $We_\gamma = U(\gamma)v_0$ for $\gamma \in \Gamma$. We prove that the inner products are preserved by W and this shows that W can be extended to a well defined isometry from $L^2(G, \mu_0)$ onto $\mathscr{H}$; it is onto because $U(\gamma)v_0$ with $\gamma \in S$ is an orthonormal basis for $\mathscr{H}$, and it will be defined everywhere because the functions e_γ, $\gamma \in \Gamma$ are uniformly dense on any compact subset of G so they are dense in $L^2(G, \mu_0)$. But according to (2.5.6), we have for $\gamma, \gamma' \in \Gamma$:

$$\langle U(\gamma)v_0, U(\gamma')v_0\rangle = \int_G \overline{e_\gamma\,(g)} e_{\gamma'}\,(g)\, d\mu_0\,(g)\,.$$

□

2.6. Harmonic analysis of IFS systems with overlap

Previous work on IFSs without overlap was extended in [**JKS07**]. These methods involve systems of operators generalizing the more familiar Cuntz relations from operator algebra theory, and from subband filter operators in signal processing. Before turning to the details, below we recall briefly the operator-theoretic approach to IFSs.

For each N, there is a simple Cuntz C^*-algebra on generators and relations, and its representations offer a useful harmonic analysis of general IFSs, but there is a crucial difference between IFSs without overlap and those with essential overlap.

Definition 2.6.1. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $(\tau_i)_{i=1}^N$ be a finite system of measurable endomorphisms, $\tau_i \colon X \to X$, $i = 1, \ldots, N$; and suppose μ is some normalized equilibrium measure. We then say that the system has *essential overlap* if

$$\sum_{i \neq j}\sum \mu\,(\tau_i\,(X) \cap \tau_j\,(X)) > 0. \tag{2.6.1}$$

The operators generating the appropriate Cuntz relations are composition operators, e.g., $F_i \colon f \to f \circ \tau_i$ where (τ_i) is the given IFS. If the particular IFS is essentially non-overlapping, it is relatively easy to compute the adjoint operators $S_i = F_i^*$, and the S_i operators will be isometries in $L^2(\mu)$ with orthogonal ranges. In a way, for the more difficult case of essential overlap, we can use the extra terms entering into the computation of the adjoint operators F_i^* as a "measure" of the essential overlap for the particular IFS we study. Here the adjoint operators F_i^* refer to the Hilbert space $L^2(\mu)$ where μ is carefully chosen. When the IFS is given, there are special adapted measures μ. We will be using the equilibrium measure μ for the given IFS (τ_i), which contains much essential information about the IFS, even in the classical cases of IFSs coming from number theory.

While there is already a rich literature on IFSs without overlap, the more difficult case of overlap has received relatively less attention; see, however, [**FLP94**, **Sol95**, **Sol98**]. The point of view here is generalized number expansions in the form of random variables: Real numbers are expanded in a basis which is a fraction, although the "digits" are bits; with infinite strings of bits identified in a Bernoulli probability space. It turns out the distribution of the resulting random variables is governed by the measures which arise as a special cases of Hutchinson's analysis of IFSs with overlap. However, concrete results about these measures have been elusive. For example, it is proved in [**Sol95**] that the measures for expansions in a basis corresponding to IFSs with overlap and given by a scaling parameter are known to be absolutely continuous for a.e. value of the parameter.

We shall consider a rather general class of IFSs with overlap, and illustrate that they can be understood in terms of the spectral theory of Cuntz-like column isometries. Moreover, such column isometries yield exact representations of the Cuntz relations precisely when the IFS has overlap of measure zero, where the measure is an equilibrium measure μ of the Hutchinson type.

Multivariable operator theory

In quantum communication (the study of (quantum) error-correction codes), certain algebras of operators and *completely positive mappings* form the starting point. They take the form of a finite number of channels of Hilbert space operators F_i which are assumed to satisfy certain compatibility conditions. The essential one is that the operators from a partition of unity, or rather a partition of the identity operator I in the chosen Hilbert space. Here we call such a system (F_i) a column isometry. An extreme case of this is when a certain Cuntz relation is satisfied by (F_i). Referring back to our IFS application, the extreme case of the operator relations turn out to correspond to the limiting case of non-overlap, i.e., to the case when our IFSs have no essential overlap.

Now, we outline some uses of ideas from *multivariable operator theory* (see, e.g., [**Arv04**]) in iterated function systems (IFS) with emphasis on IFSs with overlap. The tools we use are column isometries and systems of composition operators.

DEFINITION 2.6.2. Let $\mathscr{H}$ be a complex Hilbert space, and let $N \in \mathbb{N}$, $N \geq 2$. A system $(F_1, \dots, F_N)$ of bounded operators in $\mathcal{H}$ is said to be a *column isometry* if the mapping

$$\mathbb{F}\colon \mathscr{H} \longrightarrow \begin{pmatrix} \mathscr{H} \\ \oplus \\ \vdots \\ \oplus \\ \mathscr{H} \end{pmatrix} : \xi \longmapsto \begin{pmatrix} F_1\xi \\ \vdots \\ F_N\xi \end{pmatrix} \tag{2.6.2}$$

is isometric. Here we write the N-fold orthogonal sum of $\mathscr{H}$ in column form, but we will also use the shorter notation $\mathscr{H}_N$. As a Hilbert space, it is the same as $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$, but for clarity it is convenient to identify the *adjoint operator* $\mathbb{F}^*$ as a row

$$\mathbb{F}^*\colon \underbrace{\mathscr{H} \oplus \cdots \oplus \mathscr{H}}_{N \text{ times}} \longrightarrow \mathcal{H} : (\xi_1, \ldots, \xi_N) \longrightarrow \sum_{i=1}^{N} F_i^* \xi_i. \tag{2.6.3}$$

The inner product in $\mathscr{H}_N$ is $\sum_{i=1}^{N} \langle \xi_1, \eta_i \rangle$, and relative to the respective inner products on $\mathscr{H}_N$ and on $\mathscr{H}$, we have

$$\left\langle \underbrace{\mathbb{F}}_{\substack{\text{column}\\ \text{operator}}} \xi, \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \right\rangle_{\mathscr{H}_N} = \left\langle \xi, \underbrace{\mathbb{F}^*}_{\substack{\text{row}\\ \text{operator}}} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \right\rangle_{\mathscr{H}} \tag{2.6.4}$$

It follows that the matrix operator $\mathbb{F}\mathbb{F}^*\colon \mathscr{H}_N \to \mathscr{H}_N$ is a proper *projection*. By this we mean that the block matrix

$$\mathbb{F}\mathbb{F}^* = \left(F_i F_j^*\right)_{i,j=1}^{N} \tag{2.6.5}$$

satisfies the following system of identities:

$$\sum_{k=1}^{N} \left(F_i F_k^*\right)\left(F_k F_j^*\right) = F_i F_j^*, \quad 1 \le i,\ j \le N. \tag{2.6.6}$$

Note that $\mathbb{F}\mathbb{F}^* = I_{\mathscr{H}_N}$ if and only if

$$F_i F_j^* = \delta_{i,j} I, \quad 1 \le i,\ j \le N. \tag{2.6.7}$$

Definition 2.6.3. A column isometry $\mathbb{F}$ satisfies $\mathbb{F}\mathbb{F}^* = I_{\mathscr{H}_N}$ if and only if it defines a representation of the Cuntz algebra $\mathscr{O}_N$. In that case, the operators $S_i := F_i^*$ are isometries in $\mathscr{H}$ with orthogonal ranges, and

$$\sum_{i=1}^{N} S_i S_i^* = I_{\mathscr{H}}. \tag{2.6.8}$$

Remark. The distinction between the operator relations in Definitions 2.6.2 and 2.6.3 is much more than a technicality: Def. 2.6.3 is the more restrictive. Because of *the orthogonality axiom*, it is easy to see that if a Hilbert space $\mathscr{H}$ carries a nonzero representation of the Cuntz relations, then $\mathscr{H}$ must be infinite-dimensional, reflecting the infinitely iterated and orthogonal subdivision of projections, a hallmark of fractals.

In contrast, the condition of Def. 2.6.2, or equivalently $\sum_{i=1}^{N} F_i^* F_i = I_{\mathscr{H}}$, may easily be realized when the dimension of the Hilbert space $\mathscr{H}$ is finite. In fact such representations are used in quantum computation; see, e.g., [**LS05**, Theorem 2] and [**Kri05**].

Operator theory of essential overlap

Proposition 2.6.4. *Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $\tau_1, \ldots, \tau_N$ be measurable endomorphisms. Then some μ is an equilibrium measure if and only if the associated linear operator*

$$\mathbb{F}_\tau\colon L^2(\mu) \longrightarrow \begin{pmatrix} L^2(\mu) \\ \oplus \\ \vdots \\ \oplus \\ L^2(\mu) \end{pmatrix} : f \longmapsto \frac{1}{\sqrt{N}} \begin{pmatrix} f \circ \tau_1 \\ \vdots \\ f \circ \tau_N \end{pmatrix} \tag{2.6.9}$$

is isometric, i.e., if and only if the individual operators

$$F_i\colon f \longmapsto \frac{1}{\sqrt{N}} f \circ \tau_i \quad \text{in } L^2(\mu) \tag{2.6.10}$$

define a column isometry.

PROOF. Using polarization for the inner product in $L^2(\mu)$, we first note that μ is an equilibrium measure if and only if

$$\frac{1}{N}\sum_{i=1}^{N}\int_X |f|^2 \circ \tau_i \, d\mu = \int_X |f|^2 \, d\mu \tag{2.6.11}$$

holds for all $f \in L^2(\mu)$. □

The terms on the left-hand side in (2.6.11) are $\frac{1}{N}\int_X |f|^2 \circ \tau_i \, d\mu = \|F_i f\|^2_{L^2(\mu)}$, so (2.6.11) is equivalent to

$$\langle f, \sum\nolimits_{i=1}^{N} F_i^* F_i f\rangle_\mu = \sum\nolimits_{i=1}^{N} \langle F_i f, F_i f\rangle = \sum\nolimits_{i=1}^{N} \|F_i f\|^2_{L^2(\mu)} = \|f\|^2_{L^2(\mu)},$$

which in turn is the desired operator identity

$$\sum_{i=1}^{N} F_i^* F_i = I_{L^2(\mu)} \tag{2.6.12}$$

that defines $\mathbb{F}$ as a column isometry.

DEFINITION 2.6.5. A measurable endomorphism $\tau\colon X \to X$ is said to be of *finite type* if there are a *finite partition* $E_1, \ldots, E_k$ of $\tau(X)$ and measurable mappings $\sigma_i\colon E_i \to X$, $i = 1, \ldots, k$ such that

$$\sigma_i \circ \tau|_{E_i} = id_{E_i}, \quad 1 \le i \le k. \tag{2.6.13}$$

THEOREM 2.6.6 ([**JKS07**]). *Let $(X, \mathcal{B}, \mu)$ and $(\tau_i)_{i=1}^N$ be as in Definition* 2.6.1*; in particular we assume that μ is some (τ_i)-equilibrium measure. We assume further that each τ_i is of finite type. Let $\mathscr{H} = L^2(\mu)$,*

$$F_i\colon f \longmapsto \frac{1}{\sqrt{N}} f \circ \tau_i,$$

and let $\mathbb{F} = (F_i)$ be the corresponding column isometry. Then $\mathbb{F}$ maps onto $\bigoplus_1^N L^2(\mu)$ if and only if (τ_i) has zero μ-essential overlap.

Note that the conclusion of the theorem states that the operators $S_i := F_i^*$ define a representation of the Cuntz C^*-algebra $\mathscr{O}_N$ if and only if the system has non-essential overlap, i.e., if and only if $\mu(\tau_i(X) \cap \tau_j(X)) = 0$ for all $i \ne j$.

The intricate geometric features of IFSs can be understood nicely by specializing the particular affine transformations making up the IFS to have a single scale number (which we call λ). Examples in 1D & 2D can be found in [**JKS07**] for illustrating the operator theory behind Theorem 2.6.6. These special 1D examples also go under the name infinite Bernoulli convolutions.

In passing from 1D to 2D, the possible geometries of the IFS-recursions increase; for example, new fractions and new gaps may appear simultaneously at each iteration step. Specifically, (i) fractal (i.e., repeated gaps) and (ii) "essential overlap" co-exist in the 2D examples.

Shifts. In fact, every IFS with essential overlap has a canonical and *minimal dilation* to one with non-overlap. We shall need the following:

DEFINITION 2.6.7. Suppose $(\tau_i)_{i=1}^N$ is a contractive IFS with attractor X. Set $\Omega = \mathbb{Z}_N^{\mathbb{N}}$. If $\omega \in \Omega$ is given, let $\pi(\omega)$ be the (unique) point in the intersection $\bigcap_{n=1}^{\infty} \tau_{\omega|n}(X)$. The mapping $\pi\colon \Omega \to X$ is called the *encoding*.

Points in Ω are denoted $\omega = (\omega_1\omega_2\dots)$, $\omega_i \in \mathbb{Z}_N$, $i = 1, 2, \dots$. For $n \in \mathbb{N}$, set $\omega|n = (\omega_1\omega_2\dots\omega_n)$. Further, we shall need the one-sided shifts

$$\sigma_j(\omega_1\omega_2\omega_3\dots) = (j\omega_1\omega_2\omega_3\dots), \quad j \in \mathbb{Z}_N,\ \omega \in \Omega, \tag{2.6.14}$$

$$\sigma(\omega_1\omega_2\omega_3\dots) = (\omega_2\omega_3\dots), \qquad \omega \in \Omega. \tag{2.6.15}$$

LEMMA 2.6.8. *Let $(\tau_i)_{i=1}^N$ and X be as above, i.e., assumed contractive. Then the coding mapping $\pi\colon \Omega \to X$ is continuous, and we have*

$$\pi \circ \sigma_j = \tau_j \circ \pi, \quad j = 1, \dots, N \quad \text{or} \quad j \in \mathbb{Z}_N. \tag{2.6.16}$$

PROOF. The continuity is clear from the definitions. We verify (2.6.16): Let $\omega = (\omega_1\omega_2\dots) \in \Omega$. Then

$$\begin{aligned}(\pi \circ \sigma_j)(\omega) = \pi(j\omega_1\omega_2\dots) &= \bigcap_n \tau_j\tau_{\omega_1}\cdots\tau_{\omega_n}(X)\\ &= \tau_j\left(\bigcap_n \tau_{\omega_1}\tau_{\omega_2}\cdots\tau_{\omega_n}(X)\right) = \tau_j(\pi(\omega)) = (\tau_j \circ \pi)(\omega),\end{aligned}$$

where we used contractivity of the mappings τ_j. ∎

THEOREM 2.6.9 ([**JKS07**]). *Let $N \in \mathbb{N}$, $N \geq 2$, be given, and let $(\tau_i)_{i\in\mathbb{Z}_N}$ be a contractive IFS in a complete metric space. let (X, μ) be the Hutchinson data. Let P $(= P_{1/N})$ be the Bernoulli measure on $\Omega = \mathbb{Z}_N^{\mathbb{N}}$. Let $\pi\colon \Omega \to X$ be the encoding mapping of Lemma* 2.6.8. *Set*

$$F_i f := \frac{1}{\sqrt{N}} f \circ \tau_i \quad f \in L^2(X, \mu), \text{ and} \tag{2.6.17}$$

$$S_i^*\psi := \frac{1}{\sqrt{N}} \psi \circ \sigma_i \quad \psi \in L^2(\Omega, P), \tag{2.6.18}$$

where σ_i denotes the shift map of (2.6.14).

(1) *Then the operator $V\colon L^2(X, \mu) \to L^2(\Omega, P)$ given by*

$$Vf = f \circ \pi \tag{2.6.19}$$

is isometric.

(2) *The following intertwining relations hold:*

$$VF_i = S_i^* V, \quad i \in \mathbb{Z}_N. \tag{2.6.20}$$

(3) *The isometric extension $L^2(X, \mu) \hookrightarrow L^2(\Omega, P)$ of the (F_i)-relations is minimal in the sense that $L^2(\Omega, P)$ is the closure of*

$$\bigcup_n \bigcup_{i_1 i_2 \dots i_n} S_{i_1} S_{i_2} \cdots S_{i_n} V L^2(X, \mu). \tag{2.6.21}$$

PROOF. (1)–(2): Let $f \in L^2(X,\mu)$, and let $\|\cdot\|_\mu$ and $\|\cdot\|_P$ denote the respective L^2-norms in $L^2(\mu)$ and $L^2(P)$. Then

$$\|Vf\|_P^2 \quad = \quad \int_\Omega |f\circ\pi|^2 \, dP = \int_X |f|^2 \, d\left(P\circ\pi^{-1}\right) = \int_X |f|^2 \, d\mu = \|f\|_\mu^2 .$$

Moreover,

$$VF_i f = (F_i f)\circ\pi = \frac{1}{\sqrt{N}} f\circ\tau_i\circ\pi = \frac{1}{\sqrt{N}} f\circ\pi\circ\sigma_i = S_i^* V f,$$

which is assertion (2).

(3): Let $\psi \in L^2(\Omega,P)$, and let $\langle\cdot,\cdot\rangle_\mu$ and $\langle\cdot,\cdot\rangle_P$ denote the respective Hilbert inner products of $L^2(\mu)$ and $L^2(P)$. To show that the space in (2.6.21) is dense in $L^2(P)$, suppose

$$0 = \langle S_{i_1}\cdots S_{i_n} V f, \psi\rangle_P \tag{2.6.22}$$

for all n, all multi-indices $(i_1\dots i_n)$, and all $f \in L^2(\mu)$. We will prove that then $\psi = 0$.

When $(i_1\dots i_n)$ is fixed, we denote the cylinder set in Ω by

$$C(i_1,\dots,i_n) = \{\omega\in\Omega \mid \omega_j = i_j,\ 1\le j\le n\} . \tag{2.6.23}$$

We then get

$$S_{i_n}^*\cdots S_{i_1}^*\psi = N^{-n/2}\psi\circ\sigma_{i_1}\circ\cdots\circ\sigma_{i_n}.$$

Substitution into (2.6.22) yields

$$\int_\Omega \chi_{C(i_1,\dots,i_n)}\psi \, dP = 0.$$

We used the fact that (2.6.22) holds for all $f \in L^2(\mu)$. But the indicator functions $\chi_{C(i_1,\dots,i_n)}$ span a dense subspace in $L^2(\Omega,P)$ when n varies, and all finite words of length n are used. We conclude that $\psi = 0$, and therefore that the space in (2.6.21) is dense in $L^2(\Omega,P)$. $\square$

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CHAPTER 3

Harmonic analyses on fractals, with an emphasis on iterated function systems (IFS) measures

> Mathematicians have long since regarded it as demeaning to work on problems related to elementary geometry in two or three dimensions, in spite of the fact that it it precisely this sort of mathematics which is of practical value.
>
> Grünbaum, Branko (1926–),
>
> *Handbook of Applicable Mathematics.*

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile $\mathbb{R}^d$ by a set of translates of Ω [**Fug74**]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$ and Iosevich et al. [**IKT01**] proved that no smooth symmetric convex body Ω with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

Using *complex Hadamard matrices* of orders 6 and 12, Tao [**Tao04**] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$. Tao's results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

3.1. Harmonic analysis in the smooth vs the non-smooth categories

Definition 3.1.1. Let μ be a positive measure support in $\mathbb{R}^N$, and let Λ be a (discrete) subset of $\mathbb{R}^N$; we say that (μ, Λ) is a *spectral pair* iff (Def.) $\{e_\lambda \,;\, \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\mu)$. The case when $d\mu(x) = \chi_\Omega(x)(dx)^N$ is of special interest; we shall say that (Ω, Λ) is a spectral pair.

Definition 3.1.2. A subset $\Omega \subset \mathbb{R}^2$ is said to be *admissible* iff (Def.) there is an $\Lambda \subset \mathbb{R}^2$ such that (Ω, Λ) is a spectral pair with respect to Lebesgue measure.

Example 3.1.3. The case of $N = 2$. See Figure 3.1.1.

Example 3.1.4. Figure 3.1.2 shows two examples of non-admissible sets Ω:

- Ω_1: $\exists$ at most a finite number of orthogonal e_λ functions in $L^2(\Omega_1)$.
- Ω_2: many choices of infinite sets Λ s.t. $\{e_\lambda \mid \lambda \in \Lambda\}$ is orthogonal in $L^2(\Omega_2)$ but none is total.

Example 3.1.5. $N = 3$. Figure 3.1.3 shows a union of 12 cubes. The idea is to get the union Ω of the 12 cubes to be connected and admissible. We will thus get a connected set $\Omega \subset \mathbb{R}^3$, which has a spectrum that is not a (rank-3) lattice. None of the sets which serve as spectrum for this set Ω can be chosen to be a lattice.

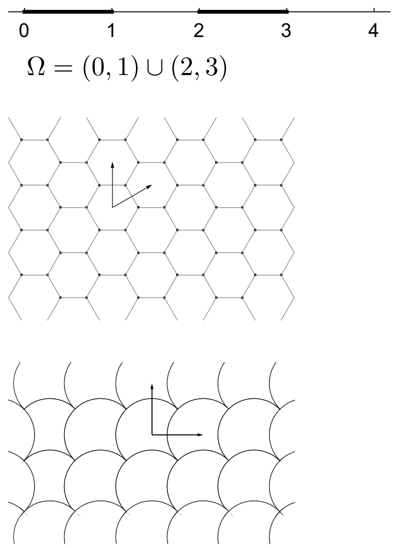


FIGURE 3.1.1. Examples of planar admissible domains. Both tile, but by distinct lattices in $\mathbb{R}^2$.

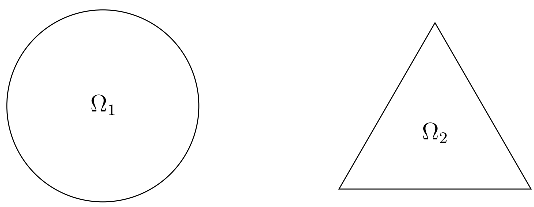


FIGURE 3.1.2. Examples of non-admissible domains in $\mathbb{R}^2$

There is no example known in the plane: open, connected, and having a spectrum which is not a rank-2 lattice.

Table 1 summarizes connections between the spectral properties and PDEs.

3.2. Spectral pairs and the Fuglede conjecture

The setting of *spectral pairs* in d real dimensions involves two subsets Ω and Λ in $\mathbb{R}^d$ such that Ω has finite and positive d-dimensional Lebesgue measure, and Λ is an index set for an orthogonal $L^2(\Omega)$-basis e_λ of exponentials. Some of the interest in spectral pairs derives from their connection to *tilings.*

It follows that the spectral pair property for a pair (Ω, Λ) is equivalent to the nonzero elements of the set

$$\Lambda - \Lambda = \{\lambda - \lambda' : \lambda, \lambda' \in \Lambda\}$$

being contained in the *zero-set* of the complex valued function

$$z \longmapsto \int_\Omega e^{i2\pi z \cdot x}\, dx =: F_\Omega(z) \tag{3.2.1}$$

and the corresponding e_λ-set $\{e_\lambda : \lambda \in \Lambda\}$ being *total* in $L^2(\Omega)$.

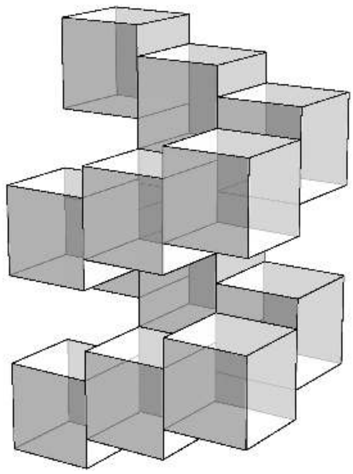
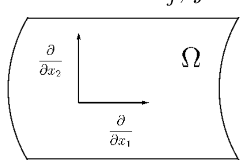


FIGURE 3.1.3. $\Omega = \bigcup_{i=1}^{12} Q_i$, connected and open. First 3 cubes in front, moving from left to right; then the next 3 move back by one unit, moving now from right to left; and the next 3 then return to the front, moving from left to right; and finally the last 3 move back again, then moving back to the left. Front vs back is indicated with a y-coordinate. In each move from cube 1 through cube 12, the vertical z-coordinate increases by 1/3. This ensures that the union of the 12 is a connected open set.

TABLE 1. Connection to PDEs.

SPECTRAL	TILING
$\{e_\lambda \mid \lambda \in \Lambda\}\big\|_\Omega$ is assumed to be an orthogonal basis in $L^2(\Omega)$	There is a sec $\Gamma \subset \mathbb{R}^N$ s.t. $\Gamma \dotplus \Omega = \mathbb{R}^N$

PDE: Commuting selfadjoint extensions H_j, $j = 1, 2$, of $\frac{1}{i}\frac{\partial}{\partial x_j}\Big|_{C_c^\infty(\Omega)}$ in $L^2(\Omega)$.



DEFINITION 3.2.1. A subset $\Omega \subset \mathbb{R}^d$ with nonzero measure is said to be a *tile* if there is a set $L \subset \mathbb{R}^d$ such that the translates $\{\Omega + l : l \in L\}$ cover $\mathbb{R}^d$ up to measure zero, and if the intersections

$$(\Omega + l) \cap (\Omega + l') \quad \text{for } l \neq l' \text{ in } L \tag{3.2.2}$$

have measure zero.

We will call (Ω, L) a *tiling pair* and we will say that L is a *tiling set*.

The *Spectral-Set conjecture* due to Fuglede states [**Fug74**]:

CONJECTURE 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ have positive and finite Lebesgue measure. Then Ω is a spectral set if and only if Ω is a tile, i.e., there exists a set L so that (Ω, L) is a spectral pair if and only if there exists a set L' so that (Ω, L') is a tiling pair.*

We formulate a "dual" conjecture.

CONJECTURE 3.2.3. *Let $L \subset \mathbb{R}^d$. Then L is a spectrum if and only if L is a tiling set, i.e., there exists a set Ω so that (Ω, L) is a spectral pair if and only if there exists a set Ω' so that (Ω', L) is a tiling pair.*

CONJECTURE 3.2.4. *Let $L \subset \mathbb{R}^d$. Then $\left(I^d, L\right)$ is a spectral pair if and only if $\left(I^d, L\right)$ is a tiling pair.*

The significance of the special case $\Omega = I^d$ (= the d-dimensional unit cube) lies in part in the results below where we show, for $d = 1, 2, 3$, that $\left(I^d, \Lambda\right)$, $\Lambda \subset \mathbb{R}^d$, is a *spectral pair* if and only if I^d *tiles* $\mathbb{R}^d$ *by* Λ*-translates*. Our proofs also construct all possible spectra for the unit cube when $d = 1, 2, 3$.

Tiling questions for $I \subset \mathbb{R}$ are trivial, but not so for $I^d \subset \mathbb{R}^d$ when $d \geq 2$. The connection between *tiles* and *spectrum* is more direct for $\Omega = I^d$ than for other examples of sets Ω. This is explained by the following (easy) lemma relating the problems to the function F_Ω from (3.2.1) above.

LEMMA 3.2.5. *If $\Omega = I^d$, then the zero-set for the function F_Ω in* (2.1.3) *is*

$$\mathbf{Z}_{I^d} = \left\{ z \in \mathbb{C}^d \setminus \{0\} : \exists j \in \{1, \ldots, d\} \text{ s.t. } z_j \in \mathbb{Z} \setminus \{0\} \right\}. \tag{3.2.3}$$

PROOF. The function $F_{I^d}(\cdot)$ factors as follows.

$$F_{I^d}(z) = \prod_{j=1}^{d} \frac{e^{i2\pi z_j} - 1}{i2\pi z_j} \tag{3.2.4}$$

for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, with the interpretation that the function $z \mapsto \frac{e^{i2\pi z} - 1}{i2\pi z}$ is 1 when $z = 0$ in $\mathbb{C}$. □

In particular, if (I^d, Λ) is a spectral pair, then $\Lambda - \Lambda \subset \mathbf{Z}_{I^d} \cup \{0\}$. The corresponding result for tilings is non-trivial, it was proved by Keller [**Kel30**, **Kel37**], a detailed proof appears in [**Per40**]. The precise statement of Keller's theorem is:

THEOREM 3.2.6. *If (I^d, Λ) is a tiling pair, then $\Lambda - \Lambda \subset \mathbf{Z}_{I^d} \cup \{0\}$, where $\mathbf{Z}_{I^d}$ is given by (*3.2.3*).*

DEFINITION 3.2.7. Let μ, ν be two Borel measures on $\mathbb{R}^d$. We will say that (μ, ν) is a *tiling pair* if the convolution, $\mu * \nu$, of μ and ν is Lebesgue measure on $\mathbb{R}^d$.

This coincides with the previous definition of a tiling pair in the sense that if (Ω, L) is a pair of subsets of $\mathbb{R}^d$ so that Ω has finite positive Lebesgue measure, L is discrete, ω denotes Lebesgue measure restricted to Ω, and ℓ denotes counting measure on L, then (Ω, L) is tiling pair if and only if (ω, ℓ) is a tiling pair. Since convolution is commutative, (μ, ν) is a tiling pair if and only if (ν, μ) is a tiling pair. In the appendix we introduce (and investigate properties of) a notion of a spectral pair of measures (μ, ν). In particular, we show that (μ, ν) is a spectral pair if and only if (ν, μ) is a spectral pair.

3.2.1. Spectral pairs in Cartesian coordinates. In this section, we are concerned with the structure of the discrete sets Λ which at the same time serve as spectra for I^d (i.e., the basis property), and also are sets of vectors λ which make the translates $\lambda + I^d$ tile $\mathbb{R}^d$. The material below is based primarily on ideas in the paper [**JP99**].

There is a *recursive procedure* for constructing spectral pairs in higher dimensions from *"factors"* in lower dimension. It is a *cross-product* construction, and it applies to any two spectral pairs, (Ω_i, Λ_i), $i = 1, 2$, in arbitrary dimensions d_1 and d_2. It is known that the "spectral-pair category" is *closed under tensor product* (see [**JP92**, **JP94**]), and we also have the following:

THEOREM 3.2.8. *Let (Ω_1, Λ_1) be a spectral pair in dimension d_1, let Ω_2 be a set of finite positive measure in dimension d_2. Suppose that for each $\lambda_1 \in \Lambda_1$, $\Lambda(\lambda_1)$ is a discrete subset of $\mathbb{R}^{d_2}$ such that $(\Omega_2, \Lambda(\lambda_1))$ is a spectral pair. If $\Lambda = \{(\lambda_1, \lambda_2) : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda(\lambda_1)\}$ then $(\Omega_1 \times \Omega_2, \Lambda)$ is a spectral pair in $d_1 + d_2$ dimensions.*

PROOF. We first show that the exponentials $\{e_\lambda : \lambda \in \Lambda\}$ are mutually orthogonal in $\mathcal{L}^2(\Omega_1 \times \Omega_2)$. The inner product in $\mathcal{L}^2(\Omega_1 \times \Omega_2)$ of e_λ and $e_{\lambda'}$ factors as follows:

$$\int_{\Omega_1} e_{\lambda_1 - \lambda_1'}(x) \left(\int_{\Omega_2} e_{\lambda_2 - \lambda_2'}(y)\, dy \right) dx.$$

If $\lambda_1 \neq \lambda_1'$ in Λ_1, then it vanishes since (Ω_1, Λ_1) is a spectral pair; and, if $\lambda_1 = \lambda_1'$ but $\lambda_2 \neq \lambda_2'$, it vanishes since $(\Omega_2, \Lambda(\lambda_1))$ is one. This proves orthogonality of Λ. To see that it is total, let $f \in \mathcal{L}^2(\Omega_1 \times \Omega_2)$ and suppose f is orthogonal to Λ. The inner products (vanishing) are:

$$\langle e_\lambda, f \rangle = \int_{\Omega_2} e_{\lambda_2}(y)\, e_{\lambda_1}(y) \left(\int_{\Omega_1} e_{\lambda_1}(x)\, \overline{f(x,y)} dx \right) dy.$$

If λ_1 is fixed, and the double integral vanishes for all $\lambda_2 \in \Lambda(\lambda_1)$, then the integral $\int_{\Omega_1} e_{\lambda_1}(x)\, \overline{f(x,y)} dx = 0$ for almost all y, by the totality of $\Lambda(\lambda_1)$ on Ω_2. But λ_1 is arbitrary so the totality of Λ_1 on Ω_1 implies $f = 0$. We conclude, that Λ is total on $\Omega_1 \times \Omega_2$ as claimed. $\square$

A more concrete version of Theorem 3.2.8 is:

THEOREM 3.2.9. *Let (Ω_i, Λ_i), $i = 1, 2$, be spectral pairs in the respective dimensions d_1 and d_2, and let $\beta \colon \Lambda_1 \to \mathbb{R}^{d_2}$ be an arbitrary function. Let*

$$\Lambda_\beta := \left\{ \begin{pmatrix} \lambda_1 \\ \beta(\lambda_1) + \lambda_2 \end{pmatrix} ; \lambda_1 \in \Lambda_1 \text{ and } \lambda_2 \in \Lambda_2 \right\}. \tag{3.2.5}$$

Then $(\Omega_1 \times \Omega_2, \Lambda_\beta)$ is a spectral pair in $d_1 + d_2$ dimensions.

PROOF. If (Ω_2, Λ_2) is a spectral pair, then so is $(\Omega_2, \Lambda_2 + \beta)$ for any vector β. An application of Theorem 3.2.8 completes the proof. $\square$

By repeatedly applying Theorem 3.2.9 if follows that if Λ is the set of points given by:

$$\begin{pmatrix} \alpha + k_1 \\ \beta_1(k_1) + k_2 \\ \beta_2(k_1, k_2) + k_3 \\ \vdots \\ \beta_{d-1}(k_1, k_2, \ldots, k_{d-1}) + k_d \end{pmatrix} \tag{3.2.6}$$

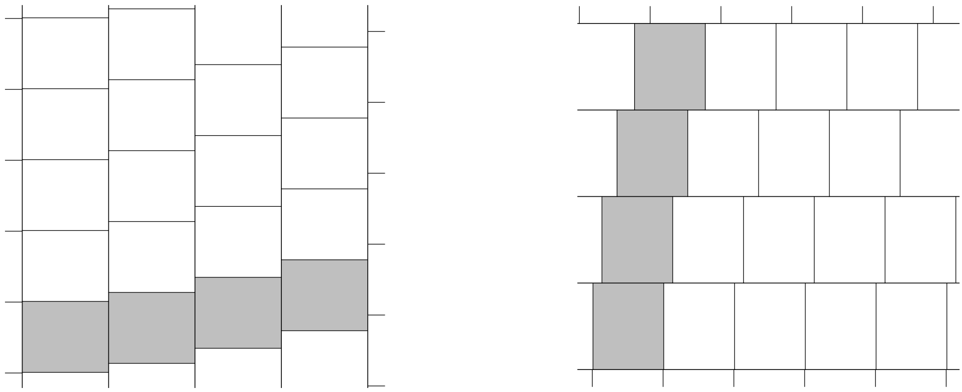


FIGURE 3.2.1. Illustrating tiling with (3.2.8) (left) and (3.2.9) (right)

with $k_1, k_2, \ldots, k_d \in \mathbb{Z}$, where $\beta_i \colon \mathbb{Z}^i \longrightarrow [0,1\rangle$ are fixed functions, then (I^d, Λ) is a spectral pair. Of course, there are the obvious modifications resulting from permutation of the d coordinates; but, when $d \geq 10$, these configurations do *not* suffice for cataloguing all the possible spectra Λ which turn $\left(I^d, \Lambda\right)$ into a spectral pair on $\mathbb{R}^d$.

Dimensions two and three

Now we prove Conjecture 3.2.4 for $d = 1, 2, 3$. Furthermore we give a complete classification of the possible spectra for the unit cube in those dimensions.

We begin with the following simple observation in one dimension for $\Omega = I = [0,1\rangle$.

PROPOSITION 3.2.10. *The only subsets $\Lambda \subset \mathbb{R}$ such that (I, Λ) is a spectral pair are the translates*

$$\Lambda_\alpha := \alpha + \mathbb{Z} = \{\alpha + n : n \in \mathbb{Z}\} \tag{3.2.7}$$

where α is some fixed real number.

In two dimensions, the corresponding result is more subtle, but the possibilities may still be enumerated as follows:

THEOREM 3.2.11. *The only subsets $\Lambda \subset \mathbb{R}^2$ such that $\left(I^2, \Lambda\right)$ is a spectral pair must belong to either one or the other of the two classes, indexed by a number α, and a sequence $\{\beta_m \in [0,1\rangle : m \in \mathbb{Z}\}$, where*

$$\Lambda = \left\{ \begin{pmatrix} \alpha + m \\ \beta_m + n \end{pmatrix} : m, n \in \mathbb{Z} \right\} \tag{3.2.8}$$

or

$$\Lambda = \left\{ \begin{pmatrix} \beta_n + m \\ \alpha + n \end{pmatrix} : m, n \in \mathbb{Z} \right\}. \tag{3.2.9}$$

Each of the two types occurs as the spectrum of a pair for the cube I^2, and each of the sets Λ as specified is a tiling set for the cube I^2.

Replacing the appeal to Lemma 3.2.5 in this proof with an appeal to Theorem 3.2.6 it follows that any tiling set Λ for the cube I^2 must be given by (3.2.8)–(3.2.9), we leave the details for the reader. The fact that this simple tiling pattern for the cube I^d in d dimensions is broken for $d = 10$ follows from examples of Lagarias and Shor [**LS92**]. It is shown there that for each $d \geq 10$ there exists a tiling of $\mathbb{R}^d$ by translates of I^d such that no two tiles have a complete facet in common. These examples also demonstrate that if $d \geq 10$, then the corresponding combinations (2.1.17) do not supply all possible spectra for I^d.

The following result shows that spectra for I^3 and tilings of $\mathbb{R}^3$ by I^3 are the same by fully determining each. No complete description of such tilings or spectra is known for $d > 3$.

THEOREM 3.2.12. *$\left(I^3, \Lambda\right)$ is a tiling pair, or a spectral pair, if and only if, after a possible translation by a single vector and a possible permutation of the coordinates (x_1, x_2, x_3), Λ can be brought into the following form: there is a partition of $\mathbb{Z}$ into disjoint subsets A, B (one possibly empty) with associated functions*

$$\begin{aligned} \alpha_0 \colon A &\longrightarrow [0,1\rangle\,, \\ \alpha_1 \colon A \times \mathbb{Z} &\longrightarrow [0,1\rangle\,, \\ \beta_0 \colon B &\longrightarrow [0,1\rangle\,, \\ \beta_1 \colon B \times \mathbb{Z} &\longrightarrow [0,1\rangle \end{aligned}$$

such that Λ is the (disjoint) union of

$$\begin{pmatrix} a \\ \alpha_0\,(a) + k \\ \alpha_1\,(a,k) + l \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ \beta_1\,(b,n) + m \\ \beta_0\,(b) + n \end{pmatrix} \tag{3.2.10}$$

as $a \in A$, $b \in B$, and $k, l, m, n \in \mathbb{Z}$.

For the proofs of Theorems 3.2.11 and 3.2.12, we refer to the paper [**JP99**]. There are a number of technical steps involved, and it would take us too far afield if we were to include them here. However, the underlying main ideas can be gleaned from the discussion above.

COROLLARY 3.2.13. *The commuting selfadjoint extensions $\{H_j : j = 1, \ldots, d\}$ in (2.1.6) are completely classified and determined, for $d = 1, 2, 3$ and $\Omega = I^d$, by Proposition 3.2.10 for $d = 1$, Theorem 3.2.11 for $d = 2$, and Theorem 3.2.12 for $d = 3$.*

PROOF. The stated conclusion follows from combining the results in the present section with Theorem 3.3.5; for I^d the spectral condition is equivalent to the operator extension property. □

3.3. Spectral pairs

Fuglede's conjecture was stated for arbitrary finite dimension. It asserts that the tiling and the spectral properties are equivalent. Tao [**Tao04**] disproved one direction in the Fuglede conjecture in dimensions 5 or higher: there exists a union of cubes which is spectral but does not tile. Later, Tao's counterexample was improved to disprove both directions in Fuglede's conjecture for dimensions 3 or higher [**KM06**, **FMM06**]. In the cases of dimensions 1 and 2, both directions are still open.

3.3.1. From spectrum to tile: the uniform tiling conjecture. We focus on the *spectral-tile implication* in the Fuglede conjecture and present some equivalent statements. One of the main ingredients that we will use is the fact that any spectrum is *periodic* (see [**BM11**,**IK13**]).

There are good reasons for our focus on the cases of the conjectures in one dimension. One reason is periodicity (see the next definition): it is known, in 1D, that the possible sets Λ serving as candidates for spectra, in the sense of Fuglede's conjecture, must be periodic. A second reason lies in the difference, from 1D to 2D, in the possibilities for geometric configurations of translation sets.

The Universal Tiling Conjecture (Conjecture 3.3.2) suggests a reduction of the implication from spectrum to tile in Fuglede's conjecture, to a consideration of finite subsets of $\mathbb{Z}$. Hence computations for the problems in 1D are arithmetic in nature, as opposed to geometric; and connections to classical Fourier series may therefore be more direct in 1D.

If Ω is spectral then any spectrum Λ is *periodic* with some period $p \neq 0$, i.e., $\Lambda + p = \Lambda$, and p is an integer multiple of $\frac{1}{|\Omega|}$. We call p a period for Λ. If $p = \frac{k(p)}{|\Omega|}$ with $k(p) \in \mathbb{N}$, then Λ has the form

$$\Lambda = \{\lambda_0, \dots, \lambda_{k(p)-1}\} + p\mathbb{Z}, \tag{3.3.1}$$

with $\lambda_0, \dots, \lambda_{k(p)-1} \in [0, p)$. The reason that there are $k(p)$ elements of Λ in the interval $[0, p)$ can be seen also from the fact that the Beurling density of a spectrum Λ has to be $|\Omega|$ [**Lan67**].

These assertions follow from [**IK13**]. According to [**IK13**], if Ω has Lebesgue measure 1 and is spectral with spectrum Λ, with $0 \in \Lambda$, then Λ is periodic, the period p is an integer and Λ has the form

$$\Lambda = \{\lambda_0 = 0, \lambda_1, \dots, \lambda_{p-1}\} + p\mathbb{Z}, \tag{3.3.2}$$

where λ_i in $[0, p)$ are some distinct real numbers.

DEFINITION 3.3.1. Let A be a finite subset of $\mathbb{R}$. We say that A is *spectral* if there exists a finite set Λ in $\mathbb{R}$ such that $\{e_\lambda : \lambda \in \Lambda\}$ is a Hilbert space for $L^2(\delta_A)$ where δ_A is the atomic measure $\delta_A := \sum_{a \in A} \delta_a$ and δ_a is the Dirac measure at a. We call Λ a *spectrum* for A.

We formulate the following "*Universal Tiling Conjecture*":

CONJECTURE 3.3.2 (UTC(p) [**DJ13b**]). *Let $p \in \mathbb{N}$. Let*

$$\Gamma := \{\lambda_0 = 0, \lambda_1, \dots, \lambda_{p-1}\}$$

be a subset of $\mathbb{R}$ with p elements. Assume Γ has a spectrum of the form $\frac{1}{p}A$ with $A \subset \mathbb{Z}$. Then for every finite family $A_1, A_2, \dots, A_n$ of subsets of $\mathbb{Z}$ such that $\frac{1}{p}A_i$ is a spectrum for Γ for all i, there exists a common tiling subset $\mathcal{T}$ of $\mathbb{Z}$ such that the set A_i tiles $\mathbb{Z}$ by $\mathcal{T}$ for all $i \in \{1, \dots, n\}$.

THEOREM 3.3.3 ([**DJ13b**]). *The following affirmations are equivalent.*

1. *The Universal Tiling Conjecture is true for all $p \in \mathbb{N}$.*
2. *Every bounded Lebesgue measurable spectral set tiles by translations.*

Moreover, if these statements are true and if Ω, $|\Omega| = 1$, is a bounded Lebesgue measurable set which has a spectrum with period p, then Ω tiles by a subset $\mathcal{T}$ of $\frac{1}{p}\mathbb{Z}$.

The term "universal tiling" appears also in [**PW01**] for a special class of tiles of $\mathbb{R}$, namely those that tile $\mathbb{R}^+$. On the dual side, the *Universal Spectrum Conjecture* was introduced in [**LW97**] where it is proved that some sets Ω which tile by some special tiling set $\mathcal{T}$ have a spectrum Λ which depends only on $\mathcal{T}$. In [**FMM06**] it is proved that the Universal Spectrum Conjecture is equivalent to the tile-spectral implication in Fuglede's conjecture, in the case of finite abelian groups. The notion of universal tiling complements is also introduced in [**FMM06**], and it is remarked that the spectral-tile implication in Fuglede's conjecture is equivalent to a universal tiling conjecture, again for finite abelian groups. In dimension 1 this means that for cyclic groups the spectral-tile implication is equivalent to all spectral sets possessing a universal tiling complement. This result in full generality is proved in [**DJ13b**] for any bounded Lebesgue measurable sets Ω as in Fuglede's original setting [**Fug74**]. It is also shown in that the spectral-tile implication in the Fuglede conjecture is equivalent to some formulations of this implication for some special classes of sets Ω: unions of intervals with rational or integer endpoints.

THEOREM 3.3.4 ([**DJ13b**]). *The following affirmations are equivalent:*

(1) *For every finite union of intervals with rational endpoints $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$ with $|\Omega| = 1$, if Ω has a spectrum Λ with period p, then Ω tiles $\mathbb{R}$ by a subset $\mathcal{T}$ of $\frac{1}{p}\mathbb{Z}$.*
(2) *For every finite union of intervals with integer endpoints $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$, $|\Omega| = N$, if Ω has a spectrum Λ with minimal period $\frac{r}{N}$, $r \in \mathbb{Z}$, then $\frac{N}{r}$ is an integer and Ω tiles $\mathbb{R}$ with a subset $\mathcal{T}$ of $\frac{N}{r}\mathbb{Z}$.*
(3) *Every bounded Lebesgue measurable spectral set tiles by translations.*

Commuting selfadjoint extensions of symmetric operators

If $\Omega \subset \mathbb{R}^d$ is open, then we consider the partial derivatives $\frac{\partial}{\partial x_j}$, $j = 1, \ldots, d$, defined on $C_c^\infty(\Omega)$ as unbounded skew-symmetric operators in $L^2(\Omega)$. The corresponding versions $\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$ are symmetric of course. We say that Ω has the *extension property* if there are commuting selfadjoint extension operators H_j, i.e.,

$$\frac{1}{i}\frac{\partial}{\partial x_j} \subset H_j, \quad j = 1, \ldots, d. \tag{3.3.3}$$

THEOREM 3.3.5 (Fuglede, Jorgensen, Pedersen). *Let $\Omega \subset \mathbb{R}^d$ be open and connected with finite and positive Lebesgue measure. Then Ω has the extension property if and only if it is a spectral set. If Ω is only assumed open, then the spectral-set property implies the extension property, but not conversely.*

Commuting selfadjoint extensions. The problem of understanding commuting symmetric, but non-selfadjoint, unbounded operators also has an origin in mathematical physics. The terminology from physics is "Hermitian", or *"formally* selfadjoint", for symmetry, i.e., for the identity $\langle Sf, h\rangle = \langle f, Sh\rangle$ for all vectors f, h in the domain of the operator S. The simplest case of this is the problem of assigning quantum mechanical boundary conditions for free particles confined in a box. More specifically, the problem here corresponds to the quantum-mechanical trajectories of a particle confined to a region of tube type, e.g., a unit cube. It is "free" except for the boundary conditions, and variations of the boundary conditions (as considered here) correspond to different physics. For single operators, von Neumann solved (or made precise) the problem by use of the Cayley transform, and considering instead the extension problem for partial isometries. But this approach does not work well in the case of several operators. Powers (in [**Pow71**,**Pow74**]) introduced an algebraic approach for understanding several operators, but the present problem is very concrete and does not lend itself easily to the algebraic techniques introduced by Powers.

We recall that Fuglede showed [**Fug74**] that the disk and the triangle in two dimensions are *not spectral sets.* By the disk and the triangle we mean the usual versions, respectively,

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\}$$

and

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1,\ 0 < x_2,\ x_1 + x_2 < 1\right\}.$$

Note that, for the present discussion, it is inessential whether or not the sets Ω are taken to be open, but it is essential for the following theorem which we will need.

If $\Omega \subset \mathbb{R}^d$ is open, then we consider the partial derivatives $\frac{\partial}{\partial x_j}$, $j = 1, \ldots, d$, defined on $C_c^\infty(\Omega)$ as unbounded skew-symmetric operators in $\mathcal{L}^2(\Omega)$. The corresponding versions $\frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial x_j}$ are symmetric of course. We say that Ω has the *extension property* if there are commuting selfadjoint extension operators H_j, i.e.,

$$\frac{1}{2\pi i} \frac{\partial}{\partial x_j} \subset H_j, \quad j = 1, \ldots, d. \tag{3.3.4}$$

We say that the containment $A \subset B$ holds for two operators A and B if the graph of A is contained in that of B. (For details, see [**RS75**] and [**DS88**].) *Commutativity* for the extension operators H_j is in the strong sense of spectral resolutions. Since the H_j's are assumed selfadjoint, each one has a projection-valued *spectral resolution* E_j, i.e., an $\mathcal{L}^2(\Omega)$-projection-valued Borel measure on $\mathbb{R}$, such that $E_j(\mathbb{R}) = I_{\mathcal{L}^2(\Omega)}$, and

$$H_j = \int_{-\infty}^{\infty} \lambda E_j(d\lambda), \tag{3.3.5}$$

for $j = 1, \ldots, d$. The strong commutativity is taken to mean

$$E_j(\Delta) E_{j'}(\Delta') = E_{j'}(\Delta') E_j(\Delta) \tag{3.3.6}$$

for all $j, j' = 1, \ldots, d$, and all Borel subsets $\Delta, \Delta' \subset \mathbb{R}$. Extensions commuting in a weaker sense were considered in [**Fri87**].

Our analysis is based on von Neumann's deficiency-space characterization of the selfadjoint extensions of a given symmetric operator [**vN30**]. Let Ω be an open set with finite Lebesgue measure. For each j, the deficiency spaces corresponding to $\frac{1}{i}\frac{\partial}{\partial x_j}$ are infinite-dimensional. It follows that each $\frac{1}{i}\frac{\partial}{\partial x_j}$ has "many" selfadjoint extensions. The main problem (not addressed by von Neumann's theory) is the selection of a commuting set $H_1, H_2, \ldots, H_d$ of selfadjoint extensions. In fact, for some Ω (e.g., when $d = 2$, the disk and the triangle) it is impossible to select a commuting set $H_1, H_2, \ldots, H_d$ of selfadjoint extensions. Now, for each j fixed, indeed, there are selfadjoint extensions. But, in the case of disk and the triangle (see Example 3.1.4), any choice of two selfadjoint extensions must consist of two non-commuting operators.

There are a number of technical steps involved, and it would take us too far afield if we were to include them here. The reader is referred to the original paper [**JP00**].

3.3.2. Spectral pairs of measures and a Heisenberg uncertainty theorem for pairs of quantum observables in duality. In this section, we consider, generalized spectral transforms for a certain Fourier duality in $\mathbb{R}^d$. Our results are motivated by considerations of the transform

$$\xi \longmapsto \int_{\Omega} e^{-i\xi\cdot x} f(x)\, dx, \quad \xi \in \mathbb{R}^d,\ f \in L^2(\Omega),$$

for a given measurable subset $\Omega \subset \mathbb{R}^d$ of finite Lebesgue measure. Instead we consider pairs of measures (μ, ν) on $\mathbb{R}^d$ such that the following generalized transform,

$$F_\mu f \colon \lambda \longmapsto \int e^{-i2\pi\lambda\cdot x} f(x)\, d\mu(x), \tag{3.3.7}$$

induces an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$, specifically making precise the following unitarity:

$$\int |f(x)|^2\, d\mu(x) = \int |(F_\mu f)(\lambda)|^2\, d\nu(\lambda).$$

When applied to the case when μ is a measure of compact support with fractal Hausdorff dimension, we identify some candidates for pairs (μ, ν), in concrete examples, when duality does hold.

The material below is based primarily on ideas in the paper [**JP98d**] by Jorgensen et al.

New pairs from old pairs

Definition 3.3.6. Let μ and ν be Borel measures on $\mathbb{R}^d$. We say that (μ, ν) is a spectral pair if the map F_μ from (3.3.7) above, defined for $f \in L^1 \cap L^2(\mu)$, extends by continuity to an isometric isomorphism mapping $L^2(\mu)$ onto $L^2(\nu)$.

It was shown in [**Ped87**] that if μ is the restriction of Lebesgue measure to a connected set of infinite measure, then the result on extensions of the directional derivatives, described above, remains valid.

It turns out that the class of measures that can be part of a spectral pair is limited in the following sense: If two measures μ and ν yield a spectral pair (μ, ν); see Definition 3.3.6, then any pair of measurements with respect to this duality must satisfy the following strong version of Heisenberg-uncertainty:

Theorem 3.3.7 (*Jo-Pedersen,* an uncertainty relation extending the Heisenberg uncertainty principle). *Suppose (μ, ν) is a spectral pair, $f \in L^2(\mu)$, $f \neq 0$, and $A, B \subset \mathbb{R}^d$. If $\|f - \chi_A f\|_\mu \leq \varepsilon$ and $\|Ff - \chi_B Ff\|_\mu \leq \delta$, then $(1 - \varepsilon - \delta)^2 \leq \mu(A)\, \nu(B)$.*

Theorem 3.3.8 (*Jo-Pedersen*). *Suppose (μ, ν) is a spectral pair, and $t \in \mathbb{R}^d$. If $\mathcal{O}$ and $\mathcal{O} + t$ are subsets of the support of μ, then $\mu(\mathcal{O}) = \mu(\mathcal{O} + t)$.*

Our work on generalized spectral pairs is motivated by M.N. Kolountzakis and J.C. Lagarias who in [**KL96**] discuss related tilings of the real line $\mathbb{R}^1$ by a function.

The following result establishes a direct connection to the spectral pairs mentioned above.

Theorem 3.3.9 (Jo-Pedersen). *Suppose (μ, ν) is a spectral pair. If $\mu\left(\mathbb{R}^d\right) < \infty$, then ν must be a counting measure with uniformly discrete support.*

Dual iteration systems

Recall the triplet (R, B, L) where R is an expansive $d \times d$ matrix with real entries, and B and L are subsets of $\mathbb{R}^d$ such that $N := \#B = \#L$,

$$R^n b \cdot l \in \mathbb{Z}, \text{ for any } n \in \mathbb{N},\ b \in B,\ l \in L, \tag{3.3.8}$$

$$H_{B,L} := N^{-1/2} \left(e^{i2\pi b \cdot l}\right)_{b \in B,\, l \in L} \tag{3.3.9}$$

is a unitary $N \times N$ matrix.

We introduce two dynamical systems, $\sigma_b(x) := R^{-1}x + b$ and $\tau_l(x) := R^* x + l$, and the corresponding "attractors", $X_\sigma := \left\{\sum_{k=0}^{\infty} R^{-k} b_k : b_k \in B\right\}$ and

$$\mathcal{L} = X_\tau := \left\{\sum_{k=0}^{n} R^{*k} l_k : n \in \mathbb{N},\ l_k \in L\right\}. \tag{3.3.10}$$

The set X_ρ is then the support of the unique probability measure which solves the equation

$$\mu = N^{-1} \sum_{b \in B} \mu \circ \sigma_b^{-1}. \tag{3.3.11}$$

If we set

$$\chi_B(t) := N^{-1} \sum_{b \in B} e_b(t), \tag{3.3.12}$$

then the expansiveness property of R and (3.3.11) imply an explicit product formula for the Fourier transform of μ,

$$\widehat{\mu}(t) := \int \overline{e_t\left(x\right)} d\mu\left(x\right) = \prod_{k=0}^{\infty} \chi_B(R^{*\,-k} t), \tag{3.3.13}$$

the convergence being uniform on bounded subsets of $\mathbb{R}^d$. We introduce the function

$$Q(t) := \sum_{\lambda \in \mathcal{L}} \left|\widehat{\mu}(t - \lambda)\right|^2, \quad t \in \mathbb{R}^d, \tag{3.3.14}$$

and the Ruelle operator C given by

$$\left(Cq\right)(t) := \sum_{l \in L} \left|\chi_B(t - l)\right|^2 q(\rho_l(t)), \tag{3.3.15}$$

where $\rho_l(x) := R^{*\,-1}(x - l)$. Both Q and the constant function $\mathbb{1}$ are eigenfunctions for the Ruelle operator C with eigenvalue 1, and the issue becomes one of multiplicity. The attractor

$$X_\rho := \left\{\sum_{k=0}^{\infty} -R^{*\,-k} l_k : l_k \in L\right\}, \tag{3.3.16}$$

corresponding to the system $\{\rho_l\}$, will also be used below.

Fractal Hardy spaces

One way to construct systems (R, B, L) satisfying (3.3.8)–(3.3.9) is to pick R, B and L so that

$$R \in M_d(\mathbb{Z}), \quad RB \subset \mathbb{Z}^d, \quad L \subset \mathbb{Z}^d. \tag{3.3.17}$$

In fact (3.3.17) implies (3.3.8) since $R^n b \cdot l = Rb \cdot R^{*(n-1)} l$ for $n = 1, 2, 3, \ldots$. The only condition that is hard to satisfy is (3.3.9). Equation (3.3.17) is closely related to a condition use in the study of certain multi-dimensional wavelets. Some results for systems (R, B, L) satisfying (3.3.17) and (3.3.9) were established in [**JP96**].

If R has non-negative integer entries, we will often end up with $\{e_\lambda : \lambda \in \mathcal{L}\}$ being an orthonormal basis for $L^2(\mu)$, and each element in $\mathcal{L}$ only having non-negative coordinates. This is an interesting situation because the basis property leads to the expansion

$$f = \sum_{\lambda \in \mathcal{L}} \langle f, e_\lambda \rangle_\mu \, e_\lambda,$$

so setting $z_j := e^{i2\pi x_j}$ we see that

$$f(x) = \sum_{\lambda \in \mathcal{L}} \langle f, e_\lambda \rangle_\mu \, z^\lambda$$

for $f \in L^2(\mu)$, where $z^\lambda := \prod_{k=1}^d z_k^{\lambda_k}$. It follows that $f(x)$, $x \in X_\sigma$, is the a.e. boundary value of a function analytic in the polydisc $\{z \in \mathbb{C}^d : |z_j| < 1\}$. Hence our construction shows that many fractal L^2-spaces are Hardy spaces. This is in sharp contrast to the Lebesgue spaces, for example, if $m_{[0,1]}$ is Lebesgue measure restricted to the unit interval $[0, 1]$.

The Hardy spaces have served as a classical tool in harmonic analysis over decades, but not really in harmonic analysis of fractals. So we will be revisiting the Hardy spaces, and their closed subspaces later, especially in Section 6.2 in connection with a new harmonic analysis of singular measures.

From the above discussion, one can prove the following:

THEOREM 3.3.10. *Suppose $d = 1$, $N = 2$, $B = \{0, a\}$, with $a \in \mathbb{R} \setminus \{0\}$, R is an integer with $|R| \geq 2$, and μ is given by (3.3.11). If R is odd, then $L^2(\mu)$ does not have a basis of exponentials for any $a \in \mathbb{R} \setminus \{0\}$. If R is even and $|R| \geq 4$, then $L^2(\mu)$ has a basis of exponentials for all $a \in \mathbb{R} \setminus \{0\}$.*

EXAMPLE 3.3.11. Let μ_0 be the probability measure solving (3.3.11) when $R = 4$ and $B = \{0, 1/2\}$. Let $L = \{0, 1\}$ and let $\mathcal{L}$ be given by (3.3.10). Set $\Omega := [0, 1] + \mathcal{L}$, and define two measures μ and ν: $\mu(\Delta) := m(\Delta \cap \Omega)$, $\nu(\Delta) := \sum_{k=0}^{\infty} \mu_0(\Delta + k)$. Then (μ, ν) is a spectral pair, and Ω is a tile with tiling set $-2\mathcal{L}$. (This is an example of a spectral set of infinite measure whose spectrum is not periodic.)

Three dimensions

EXAMPLE 3.3.12. The conditions are satisfied in the following example ([**JP98a**, Example 7.4], the Eiffel Tower, see Figure 3.3.1):

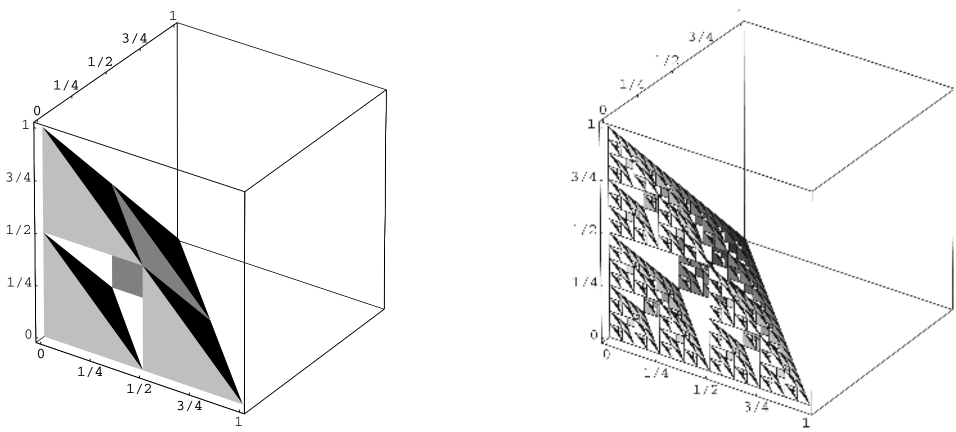
$$\begin{aligned}
&\nu = 3,\\
&R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},\\
&B = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \right\},\\
&L = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.
\end{aligned} \tag{3.3.18}$$

The natural candidate for a subset $P \subset \mathbb{R}^3$ such that $\{e_\lambda : \lambda \in P\}$ is an orthonormal basis in $L^2(\mu)$ is

$$P = \left\{ l_0 + 2l_1 + 2^2 l_2 + \cdots : l_i \in L, \text{ finite sums} \right\}. \tag{3.3.19}$$

If $\lambda \in P$, the three coordinates $\lambda = \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ are all in $\mathbb{N}_0$.

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertrieben künnen. — D. Hilbert [**Hil26**, p. 170]



(A) First iteration

(B) Fourth iteration (shading increases with depth)

Figure 3.3.1. The Eiffel Tower (Example 3.3.12)

Conclusion: Let μ denote the IFS measure corresponding to the Eiffel tower-iteration from Figure 3.3.1, and let P be the discrete subset in $\mathbb{R}^3$ specified in (3.3.19). Then (μ, P) is a spectral pair.

Proof sketch. One checks that the Hadamard matrix (see (3.3.9)) corresponding to the vector system (R, B, L) in (3.3.18) is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

□

3.4. Spectral theory of multiple intervals

In this section, we present a model for spectral theory of *families of selfadjoint operators*, and their corresponding *unitary one-parameter groups* (acting in Hilbert space.) The models allow for a scale of complexity, indexed by the natural numbers $\mathbb{N}$. For each $n \in \mathbb{N}$, we get families of selfadjoint operators indexed by:

(i) the unitary matrix group $U(n)$, and by
(ii) a prescribed set of n non-overlapping intervals.

Take Ω to be the complement in $\mathbb{R}$ of n fixed closed finite and disjoint intervals, and let $L^2(\Omega)$ be the corresponding Hilbert space. Moreover, given $B \in U(n)$, then both the lengths of the respective intervals, and the gaps between them, show up as spectral parameters in our corresponding *spectral resolutions* within $L^2(\Omega)$.

Our models have two advantages:

One, they encompass realistic features from quantum theory, from acoustic wave equations and their obstacle scattering; as well as from harmonic analysis.

Secondly, each choice of the parameters in our models, $n \in \mathbb{N}$, $B \in U(n)$, and interval configuration, allows for explicit computations, and even for closed-form formulas: Computation of spectral resolutions, of generalized eigenfunctions in $L^2(\Omega)$ for the continuous part of spectrum, and for scattering coefficients.

Our models further allow us to identify embedded point-spectrum (in the continuum), corresponding, for example, to *bound-states* in scattering, to trapped states, and to *barriers in quantum scattering*. The possibilities for the discrete atomic part of spectrum includes both periodic and non-periodic distributions.

The results here are closely related to Fuglede's conjecture in one dimension. We say that a bounded Borel subset Ω of $\mathbb{R}^d$ is *spectral* if the restriction of the Lebesgue measure to Ω is a spectral measure. We say that a finite subset A of $\mathbb{R}^d$ is *spectral* if the counting measure on A is a spectral measure.

There is recent work dealing with dimension 1. The relevance of the present section, to the Fuglede problem in dimension 1, is the case when Ω is a union of non-overlapping open intervals. Indeed this case is generally considered to offer a key to the resolution of the Fuglede problem in dimension 1. About dimension 1, see for example [**DL14a**, **DH15**, **DJ13b**].

The study of unitary one-parameter groups ([**vN49**]) is used in such areas as quantum mechanics, in PDE, and more generally in dynamical systems, and in harmonic analysis. A *unitary one-parameter group* $U(t)$ is a representation of the additive group of the real line $\mathbb{R}$, $t \in \mathbb{R}$, with each unitary operator $U(t)$ acting on a complex Hilbert space $\mathscr{H}$. By a theorem of Stone (see [**Sto90**, **LP68**, **DS88**] for details), we know that there is a bijective correspondence between:

(i) strongly continuous unitary one-parameter groups $U(t)$ acting on $\mathscr{H}$; and
(ii) selfadjoint operators P with dense domain in $\mathscr{H}$.

In these applications, the first question for $(\mathscr{H}, U(t))$ relates to spectrum. We take the spectrum for $U(t)$ to be the spectrum of its selfadjoint generator. Hence one is led to study $(\mathscr{H}, U(t))$ up to unitary equivalence. The gist of Lax-Phillips theory [**LP68**] is that $(\mathscr{H}, U(t))$, up to multiplicity, will be unitarily equivalent to the translation representation, i.e., to the group of translation operators acting in $L^2(\mathbb{R}, \mathscr{M})$, the square-integrable functions from $\mathbb{R}$ into a complex Hilbert space $\mathscr{M}$. The dimension of $\mathscr{M}$ is called multiplicity. For interesting questions one may take $\mathscr{M}$ to be of finite small dimension; see details below, and [**JPT13**, **JPT12**].

To make concrete *the geometric possibilities*, we study here $L^2(\Omega)$ when Ω is a fixed open subset of $\mathbb{R}$ with two unbounded connected components. For many questions, we may restrict to the case when there is only a finite number of bounded connected components in Ω.

In other words, Ω is the complement of a finite number of closed, bounded and disjoint intervals. We begin with Dirichlet boundary conditions for the derivative operator d/dx, i.e., defined on absolutely continuous L^2 functions with $f' \in L^2(\Omega)$

and vanishing on the boundary of Ω, $f = 0$ on $\partial\Omega$. Using deficiency index theory [**vN49**, **DS88**], we then arrive at all the skew-selfadjoint extensions, and the corresponding unitary one-parameter groups $U(t)$ acting on $L^2(\Omega)$.

3.4.1. Unbounded operators. We recall the following fundamental result of von Neumann on extensions of Hermitian operators.

In order to make precise our boundary conditions, we need a:

LEMMA 3.4.1. *Let $\Omega \subset \mathbb{R}$ be as above. Suppose f and $f' = \frac{d}{dx}f$ (distribution derivative) are both in $L^2(\Omega)$; then there is a continuous function $\tilde{f}$ on $\overline{\Omega}$ (closure) such that $f = \tilde{f}$ a.e. on Ω, and $\lim_{|x|\to\infty} \tilde{f}(x) = 0$.*

PROOF. Let $p \in \mathbb{R}$ be a boundary point. Then for all $x \in \Omega$, we have:

$$f(x) - f(p) = \int_p^x f'(y)dy. \tag{3.4.1}$$

Indeed, $f' \in L^1_{loc}$ on account of the following Schwarz estimate

$$|f(x) - f(p)| \leq \sqrt{|x-p|}\, \|f'\|_{L^2(\Omega)}\,.$$

Since the RHS in (3.4.1) is well-defined, this serves to make the LHS also meaningful. Now set

$$\tilde{f}(x) := f(p) + \int_p^x f'(y)dy,$$

and it can readily be checked that $\tilde{f}$ satisfies the conclusions in the Lemma. ☐

LEMMA 3.4.2 (see e.g. [**DS88**]). *Let L be a closed Hermitian operator with dense domain $\mathscr{D}_0$ in a Hilbert space. Set*

$$\mathscr{D}_\pm = \{\psi_\pm \in dom(L^*) \mid L^*\psi_\pm = \pm i\psi_\pm\}$$
$$\mathscr{C}(L) = \left\{U : \mathscr{D}_+ \to \mathscr{D}_- \mid U^*U = P_{\mathscr{D}_+}, UU^* = P_{\mathscr{D}_-}\right\} \tag{3.4.2}$$

where $P_{\mathscr{D}_\pm}$ denote the respective projections. Set

$$\mathscr{E}(L) = \{S \mid L \subseteq S,\ S^* = S\}\,.$$

Then there is a bijective correspondence between $\mathscr{C}(L)$ and $\mathscr{E}(L)$, given as follows:

If $U \in \mathscr{C}(L)$, and let L_U be the restriction of L^* to

$$\{\varphi_0 + f_+ + Uf_+ \mid \varphi_0 \in \mathscr{D}_0, f_+ \in \mathscr{D}_+\}\,. \tag{3.4.3}$$

Then $L_U \in \mathscr{E}(L)$, and conversely every $S \in \mathscr{E}(L)$ has the form L_U for some $U \in \mathscr{C}(L)$. With $S \in \mathscr{E}(L)$, take

$$U := (S - iI)(S + iI)^{-1}\mid_{\mathscr{D}_+} \tag{3.4.4}$$

and note that

1. *$U \in \mathscr{C}(L)$, and*
2. *$S = L_U$.*

Vectors $f \in dom(L^*)$ admit a unique decomposition $f = \varphi_0 + f_+ + f_-$ where $\varphi_0 \in dom(L)$, and $f_\pm \in \mathscr{D}_\pm$. For the boundary-form $\mathbf{B}(\cdot,\cdot)$, we have

$$\begin{aligned} i\mathbf{B}(f,f) &= \langle L^*f, f\rangle - \langle f, L^*f\rangle \\ &= \|f_+\|^2 - \|f_-\|^2\,. \end{aligned} \tag{3.4.5}$$

3.4.2. Momentum operators. In this section we outline our model, and we list the parameters of the family of boundary value problems to be studied. We will need a technical lemma on reproducing kernels.

By momentum operator P we mean the generator for the group of translations in $L^2(-\infty, \infty)$, see (3.4.10) below. There are several reasons for taking a closer look at restrictions of the operator P. In our analysis, we study spectral theory determined by the complement of n bounded disjoint intervals, i.e., the union of n bounded component and two unbounded components (details below.) Our motivation derives from quantum theory, and from the study of spectral pairs in geometric analysis; see e.g., [**DJ07d**], [**Fug74**], [**JP99**], [**La01**], and [**PW01**]. In our model, we examine how the spectral theory depends on both variations in the choice of the n intervals, as well as on variations in the von Neumann parameters.

Granted that in many applications, one is faced with vastly more complicated data and operators; nonetheless, it is often the case that the more subtle situations will be unitarily equivalent to a suitable model involving P. This is reflected for example in the conclusion of *the Stone-von Neumann uniqueness theorem*: The Weyl relations for quantum systems with a finite number of degree of freedom are unitarily equivalent to the standard model with momentum and position operators P and Q.

The boundary form, spectrum, and the group $U(n)$. Fix $n > 2$, let $-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_n < \alpha_n < \infty$, and let

$$\Omega := \mathbb{R} \backslash \left(\bigcup_{k=1}^{n} [\beta_k, \alpha_k] \right) = \bigcup_{k=0}^{n} J_k \tag{3.4.6}$$

be the exterior domain, where

$$J_0 := (-\infty, \beta_1)\,, J_1 := (\alpha_1, \beta_2)\,, \ldots, J_{n-1} := (\alpha_{n-1}, \beta_n), J_n := (\alpha_n, \infty)\,. \tag{3.4.7}$$

Moreover, we set

$$J_- := J_0,\ J_+ := J_n \tag{3.4.8}$$

for the two unbounded components; see Figure 3.4.1 below.

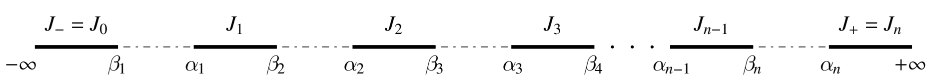


FIGURE 3.4.1. $\Omega = \bigcup_{k=0}^{n} J_k = \left(\bigcup_{k=1}^{n-1} J_k \right) \cup (J_- \cup J_+)$, i.e., $\Omega =$ the complement in $\mathbb{R}$ of n finite and disjoint intervals.

We shall write $\boldsymbol{\alpha} = (\alpha_i)$ for all the left-hand side endpoints, and $\boldsymbol{\beta} = (\beta_i)$ for the right-hand side endpoints in $\partial\Omega$.

Let $L^2(\Omega)$ be the Hilbert space with respect to the inner product

$$\langle f, g \rangle := \sum_{k=0}^{n} \int_{J_k} \overline{f(x)} g(x) dx. \tag{3.4.9}$$

The *maximal momentum operator* is

$$P := \frac{1}{i2\pi} \frac{d}{dx} \tag{3.4.10}$$

with domain $\mathscr{D}(P)$ equal to the set of absolutely continuous functions on Ω where both f and Pf are square-integrable.

The *boundary form* associated with P is defined as the form

$$i2\pi\,\mathbf{B}\,(g,f) := \langle g, Pf\rangle - \langle Pg, f\rangle \tag{3.4.11}$$

on $\mathscr{D}\,(P)$. This is consistent with (3.4.5): If $L = P_{min}$, then L^* in (3.4.5) is P. Recall, $\mathscr{D}(P_{min}) = \{f \in \mathscr{D}(P)\,;\ f = 0 \ \text{ on } \partial\Omega\}$.

LEMMA 3.4.3. *Let $\boldsymbol{\alpha} = (\alpha_i)$, $\boldsymbol{\beta} = (\beta_i)$ be the system of interval endpoints in (*3.4.7*), and set*

$$\rho_1(f) := f(\boldsymbol{\beta}) = \begin{pmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_n) \end{pmatrix},\ \rho_2(f) := f(\boldsymbol{\alpha}) = \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_n) \end{pmatrix}$$

for all $f \in \mathscr{D}(P)$; then

$$i2\pi\,\mathbf{B}(g,f) = \langle g(\boldsymbol{\alpha}), f(\boldsymbol{\alpha})\rangle_{\mathbb{C}^n} - \langle g(\boldsymbol{\beta}), f(\boldsymbol{\beta})\rangle_{\mathbb{C}^n} \tag{3.4.12}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ is the usual Hilbert-inner product in $\mathbb{C}^n$.

PROOF. First note that for the domain of the operator L^* in $L^2(\Omega)$, we have

$$dom(L^*) = \{f \in L^2(\Omega)\,;\ f' \in L^2(\Omega)\}.$$

This means that every $f \in dom(L^*)$ has a realization in $C(\overline{\Omega})$, so continuous up to the boundary. As a result the following boundary analysis is justified by von Neumann's formula (3.4.5) in Lemma 3.4.2; and valid for for all $f, g \in dom(L^*)$:

$$\begin{aligned}
-i2\pi\boldsymbol{B}(g,f) &= \langle L^*g, f\rangle_\Omega - \langle g, L^*f\rangle_\Omega \\
&= \int_\Omega \frac{d}{dx}\left(\overline{g(x)}f(x)\right)dx \\
&= \left(\int_{-\infty}^{\beta_1} + \sum_{j=1}^{n-1}\int_{\alpha_j}^{\beta_{j+1}} + \int_{\alpha_n}^{\infty}\right)\frac{d}{dx}\left(\overline{g(x)}f(x)\right)dx \\
&= \overline{g(\beta_1)}f(\beta_1) + \sum_{j=1}^{n-1}\left(\overline{g(\beta_{j+1})}f(\beta_{j+1}) - \overline{g(\alpha_j)}f(\alpha_j)\right) - \overline{g(\alpha_n)f(\alpha_n)} \\
&= \langle g(\boldsymbol{\beta}), f(\boldsymbol{\beta})\rangle_{\mathbb{C}^n} - \langle g(\boldsymbol{\alpha}), f(\boldsymbol{\alpha})\rangle_{\mathbb{C}^n}\,.
\end{aligned}$$

□

COROLLARY 3.4.4. *It follows that the system $(\mathbb{C}^n, \rho_1, \rho_2)$, $\rho_1(f) = f(\boldsymbol{\beta})$ and $\rho_2(f) = f(\boldsymbol{\alpha})$, represents a boundary triple, and we get all the selfadjoint extension operators for P_{min} indexed by $B \in U(n)$; we shall write P_B. Explicitly, see e.g.,* [**dO09**],

$$\mathscr{D}\,(P_B) := \{f \in \mathscr{D}\,(P) \mid B\rho_1(f) = \rho_2(f)\}\,. \tag{3.4.13}$$

3.4.3. Spectral theory. We identify a number of sub-classes within the family of all selfadjoint extensions P_B of the minimal operator in $L^2(\Omega)$.

If the open set Ω is chosen (as the complement of a fixed system consisting of n bounded, closed and disjoint intervals), then the set of all selfadjoint extensions is indexed by elements B in the matrix group $U(n)$. The possibilities for the spectral resolution of a particular P_B are twofold:

(i) pure Lebesgue spectrum with uniform multiplicity one; or
(ii) still Lebesgue spectrum but with *embedded point spectrum* (within the continuum).

While all the operators within class (i) are unitarily equivalent, it is still the case that, within each of the two sides in the rough subdivision, there is a rich variety of possibilities: Via a set of scattering poles, we show that the fine-structure of the spectral theory for each of the selfadjoint operators of P_B, and the corresponding unitary one-parameter groups $U_B(t)$, depends on all the geometric data: The number n, the choice of intervals, their respective lengths, and the location of the gaps; see Figure 3.4.1. More precisely, these spectral/scattering differences reflect themselves in detailed properties of an associated system of scattering coefficients. To identifying particulars for a given unitary one-parameter group $U_B(t)$ we study the location of a set of scattering poles.

The resolution of these questions is closely related with a more coarse distinction: This has to do with decomposition properties for the unitary one-parameter groups $U_B(t)$ in $L^2(\Omega)$.

Definition 3.4.5. Fix $n > 2$, and let

$$B = \begin{pmatrix} \boldsymbol{u} & B' \\ c & \boldsymbol{w}^* \end{pmatrix} \in U(n) \tag{3.4.14}$$

where $\boldsymbol{u}, \boldsymbol{w} \in \mathbb{C}^{n-1}$, and $c \in \mathbb{C}$.

An element $B \in U(n)$ is said to be *indecomposable* iff (Def.) it does not have a presentation

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}, \tag{3.4.15}$$

$1 \leq k < n$, $B_1 \in U(k)$, $B_2 \in U(n-k)$; i.e., iff B as a transformation in $\mathbb{C}^n$ does not have a *non-trivial* splitting $B_1 \oplus B_2$ as a sum of two unitaries.

Definition 3.4.6. Let $B \in U(n)$ as in (3.4.14). We say B is *degenerate* if $1 \in sp(B')$, i.e., there exists $\boldsymbol{\zeta} \in \mathbb{C}^{n-1} \backslash \{0\}$ such that $B'\boldsymbol{\zeta} = \boldsymbol{\zeta}$.

In this generality we are able to establish (Theorem 3.4.8) the complete and detailed spectral resolution for P_B, and therefore for the one-parameter group $U_B(t)$ as it acts on the Hilbert space $L^2(\Omega)$. In addition, if the first coefficient in the formula for the generalized eigenfunction system is chosen to be 1, then the measure σ_B in the spectral resolution for $U_B(t)$ becomes Lebesgue measure. Moreover, the multiplicity is uniformly one.

Theorem 3.4.7 (Jo-Pedersen-Tian). *If B is non-degenerate, then the continuous spectrum is the real line with uniform multiplicity one and the spectral measure is absolutely continuous with respect to Lebesgue measure.*

More specifically, we have:

THEOREM 3.4.8 (Jo-Pedersen-Tian). *Let* $\boldsymbol{\alpha} = (\alpha_i)$ *and* $\boldsymbol{\beta} = (\beta_i)$ *be a system of interval endpoints:*

$$-\infty < \beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_n < \alpha_n < \infty, \tag{3.4.16}$$

with $J_0 = J_- = (-\infty, \beta_1)$, $J_n = J_+ = (\alpha_n, \infty)$, *and* $J_i = (\alpha_i, \beta_{i+1})$, $i = 1, \dots, n-1$. *Let* $B \in U(n)$ *be chosen non-degenerate (fixed), and let*

$$\psi_\lambda(x) := \psi_\lambda^{(B)}(x) = \left(\sum_{i=0}^{n} \chi_i(x) A_i^{(B)}(\lambda)\right) e_\lambda(x) \tag{3.4.17}$$

where $\Omega = \bigcup_{i=0}^n J_i$, $\chi_i := \chi_{J_i}$, $0 \le i \le n$, *and where the functions* $\left(A_i^{(B)}(\cdot)\right)_{i=0}^n$ *are chosen with* $A_0^{(B)} \equiv 1$.

For $f \in L^2(\Omega)$, *setting*

$$(V_B f)(\lambda) = \langle \psi_\lambda, f \rangle_\Omega = \int \overline{\psi_\lambda(y)} f(y)\, dy, \tag{3.4.18}$$

we then get the following orthogonal expansions:

$$f = \int_{\mathbb{R}} (V_B f)(\lambda)\, \psi_\lambda(\cdot)\, d\lambda \tag{3.4.19}$$

*where the convergence in (*3.4.19*) is to be taken in the* L^2*-sense via*

$$\|f\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} |(V_B f)(\lambda)|^2\, d\lambda,\ f \in L^2(\Omega). \tag{3.4.20}$$

Moreover, we have

$$V_B U_B(t) = M_t V_B,\ t \in \mathbb{R} \tag{3.4.21}$$

where

$$(M_t g)(\lambda) = e_\lambda(-t)\, g(\lambda)$$

for all $t, \lambda \in \mathbb{R}$, *and all* $g \in L^2(\mathbb{R})$.

$$\begin{tikzcd} L^2(\Omega) \arrow[r, "U_B(t)"] \arrow[d, "V_B"'] & L^2(\Omega) \arrow[d, "V_B"] \\ L^2(\mathbb{R}) \arrow[r, "M_t"'] & L^2(\mathbb{R}) \end{tikzcd}$$

FIGURE 3.4.2. Intertwining

The reason for the word "generalized" referring to the family (3.4.17) of *generalized eigenfunctions* is that, for a fixed value of the spectral parameter λ, the function ψ_λ is not in $L^2(\Omega)$, so strictly speaking it is not an eigenfunction for the unbounded selfadjoint operator P_B in $L^2(\Omega)$. But there is a fairly standard way around the difficulty, involving distributions [**Mau68**, **JPT12**].

EXAMPLE 3.4.9. Set $n = 2$, $B = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. With the normalization $A_0^{(B)} \equiv 1$, we get the following representation of the two function $\mathbb{R} \ni \lambda \mapsto A_i^{(B)}(\lambda)$, $i = 1, 2$: Fix $-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \infty$; set $L := \beta_2 - \alpha_1$, and $G := \alpha_2 - \beta_1$; then

$$\begin{cases} A_1^{(B)}(\lambda) = \dfrac{a\, e_\lambda(\beta_1 - \alpha_1)}{1 - b\, e_\lambda(L)}, \text{ and} \\ A_2^{(B)}(\lambda) = \dfrac{e_\lambda(L - G) - \overline{b}\, \overline{e_\lambda(G)}}{1 - b\, e_\lambda(L)}. \end{cases} \tag{3.4.22}$$

Note the poles in the presentation of the two functions in (3.4.23). In the meromorphic extensions of the two functions, we have, for $z \in \mathbb{C}$,

$$\begin{cases} A_1^{(B)}(z) = \dfrac{a\, e(z(\beta_1 - \alpha_1))}{1 - b\, e(zL)} \\ A_2^{(B)}(z) = \dfrac{e(z(L - G)) - \overline{b}\, e(-zG)}{1 - b\, e(zL)}. \end{cases} \tag{3.4.23}$$

In some cases the analysis is very explicit for discrete spectrum:

THEOREM 3.4.10. *If* $B = \begin{pmatrix} \boldsymbol{u} & B' \\ c & \boldsymbol{w}^* \end{pmatrix}$, *where* $c = e(\theta_1)$, $B' = \operatorname{diag}(e(\theta_2), \ldots, e(\theta_n))$ *then the continuous spectrum of* P_B *is the real line and the discrete spectrum of* P_B *is* $\bigcup_{k=1}^{n-1} \left(\frac{\theta_{k+1}}{\ell_k} + \frac{1}{\ell_k}\mathbb{Z} \right)$, *the multiplicity of each eigenvalue* λ *is* $\#\{2 \le k \le n \mid \ell_k \lambda - \theta_k \in \mathbb{Z}\}$, *and counting multiplicities the discrete spectrum has density* $\sum_{k=1}^{n-1} \ell_k$.

THEOREM 3.4.11 (Jo-Pedersen-Tian). *If* $B \in U(n)$ *is non-degenerate (see Definition* 3.4.6*), then there is a system of bounded generalized eigenfunctions* $\{\psi_\lambda^{(B)}; \lambda \in \mathbb{R}\}$, *and a positive Borel function* $F_B(\cdot)$ *on* $\mathbb{R}$ *such that the unitary one-parameter group* $U_B(t)$ *in* $L^2(\Omega)$ *generated by* P_B *has the form*

$$(U_B(t) f)(x) = \int_{\mathbb{R}} e_\lambda(-t) \left\langle \psi_\lambda^{(B)}, f \right\rangle_\Omega \psi_\lambda^{(B)}(x) F_B(\lambda) d\lambda \tag{3.4.24}$$

for all $f \in L^2(\Omega)$, $x \in \Omega$, *and* $t \in \mathbb{R}$; *where*

$$\left\langle \psi_\lambda^{(B)}, f \right\rangle_\Omega := \int_\Omega \overline{\psi_\lambda^{(B)}(y)} f(y) dy.$$

Our study of duality pairs x and λ in systems of *generalized eigenfunctions* ψ_λ is related to, but different from another part of spectral theory, dual variables for bispectral problems [**Grfrm[o]–1, GR10, DG09**].

Let Q be a measurable subset of $\mathbb{R}^d$ and let p be a regular positive Borel measure on $\mathbb{R}^d$. We will say that (Q, p) is a spectral pair, if (1) for each f in $L^1(Q) \cap L^2(Q)$ the continuous function $Ff(\lambda) := (f, e_\lambda)$ is in $L^2(p)$ and (2) the map $f \mapsto Ff$ of $L^1(Q) \cap L^2(Q) \subset L^2(Q)$ into $L^2(p)$ is isometric and has dense range. We say Q is a spectral set, when there is a p such that (Q, p) is a spectral pair.

COROLLARY 3.4.12. *The exterior domains, i.e., the sets forming the exterior to a finite union of intervals, are not a spectral sets.*

PROOF. When Ω has infinite measure and is a spectral set then every point in the spectrum is an accumulation point of the spectrum [**Ped87**]. In fact, if λ

is an isolated point in the spectrum, then it is an eigenvalue with corresponding eigenvector e_λ, but e_λ is not in $L^2(\Omega)$, contradiction. □

There are a number of technical points involved, and it would take us too far afield if we were to include them here. However, the underlying main ideas can be gleaned from the discussion above. The reader is referred to the original paper [**JPT15a**].

Some of the issues addressed may be summarized briefly as follows.

(1) An element $B \in U(n)$ is decomposable as a unitary matrix, i.e., it has at least two non-trivial unitary summands B_1 and B_2. Note however, that this definition presupposes a choice of an ordered orthonormal basis (ONB) in $\mathbb{C}^n$.
(2) As a selfadjoint operator in $L^2(\Omega)$, P_B is a corresponding orthogonal sum of the two operators P_i, $i = 1, 2$.
(3) The unitary one-parameter group $U_B(t)$ generated by P_B decomposes as an orthogonal sum of two one-parameter groups with generators P_i, each unitary in a proper subspace in $L^2(\Omega)$.

The two infinite intervals

If a particular B in $U(n)$ is decomposable, then the corresponding summands in $L^2(\Omega)$ arise from lumping together the L^2 spaces of the intervals J_j, j from 0 to n, each corresponding to a closed subspace in $L^2(\Omega)$. But when lumping together these closed subspaces, there is the following restriction: one of the two infinite half-lines cannot occur alone: the two infinite half-lines must merge together. The reason is that L^2 for an infinite half-line, by itself yields deficiency indices $(1, 0)$ or $(0, 1)$.

The finite intervals

If a subspace $L^2(J_j)$ for j from 1 to $n-1$ occurs as a summand, there must be embedded point-spectrum (called bound-states in physics), embedded in the continuum.

Caution about "matrix decomposition." The notion of decomposition for B in $U(n)$ is basis-dependent in a strong sense: it depending on prescribing an ONB in $\mathbb{C}^n$, as an ordered set, so depends on permutations of a chosen basis. Hence an analysis of an action of the permutation group S_n enters. So a particular property may hold before a permutation is applied, but not after.

This means that some B in $U(n)$ might be decomposable in some ordered ONB (in $\mathbb{C}^n$) , but such a decomposition may *not* lead to an associated $(P_B,\ L^2(\Omega))$-decomposition.

For our matrix analysis we work with two separate notions, "non-degenerate" and "indecomposable", but a direct comparison is not practical. The reason is that they naturally refer to different orderings of the canonical ONB in $\mathbb{C}^n$.

3.4.4. Scratching the surface of infinity. In this section we consider some cases when the give open set Ω has an infinite number of connected components. As in the discussion above, we still assume that two of the components are the infinite half-lines. Our motivation for studying the infinite case is four-fold:

One is the study of geometric analysis of Cantor sets; so the infinite case includes a host of examples when Ω is the complement in $\mathbb{R}$ of one of the Cantor sets studied in earlier recent papers [**DJ07d, DJ11a, JP98a, PW01**]. The other is our interest in boundary value problems when the boundary is different from the more traditional choices. And finally, the case when the von Neumann-deficiency indices are (∞, ∞) offers new challenges; involving now reproducing kernels, and more refined spectral theory.

Finally we point out how the spectral theoretic conclusions for the *infinite case* differ from those that hold in the *finite case* (see details above for the finite case.) For example, for finitely many intervals we computed that the Beurling density of embedded point spectrum equals the total length of the finite intervals. By contrast, we show below that when Ω has an infinite number of connected components, there is the possibility of dense point spectrum; see Example 3.4.17.

Let $I_k = (r_k, s_k)$ be a sequence of pairwise disjoint open subintervals of the open interval $(0, 1)$. Let

$$\Omega = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{k=0}^{\infty} I_k.$$

The functions satisfying the eigenfunction equation $\frac{1}{i2\pi}\frac{d}{dx}\psi_\lambda = \lambda\psi_\lambda$ are the functions

$$\psi_\lambda(x) = \left(A_{-\infty}(\lambda)\chi_{(-\infty,0)}(x) + A_\infty(\lambda)\chi_{(1,\infty)}(x) + \sum_{k=0}^{\infty} A_k(\lambda)\chi_{I_k}(x) \right) e_\lambda(x),$$

where $A_{-\infty}, A_\infty$, and A_k are constants depending on λ. Let $r_0 = 1$ and $s_0 = 0$.

Example 3.4.13. An example of this is the complement of the middle thirds Cantor set C. We can write the complement of the Cantor set C as

$$(-\infty, 0) \cup (1, \infty) \cup \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{2^j} \left(a_{j,k}, a_{j,k} + 3^{-(j+1)} \right)$$

where in base 3

$$a_{0,1} = .1,$$
$$a_{1,1} = .01, \quad a_{1,2} = .21$$
$$a_{2,1} = .001, \quad a_{2,1} = .021, \quad a_{2,3} = .201, \quad a_{2,4} = .221$$

and so on. So $a_{j,k}$, $k = 1, \ldots, 2^j$ are the numbers with finite base three expansions of the form

$$0.x_1x_2\cdots x_j1, \quad x_\ell \in \{0, 2\}.$$

In this case the generalized eigenfunctions are

$$\psi_\lambda(x) = \Big(A_{-\infty}(\lambda)\chi_{(-\infty,0)}(x) + A_\infty(\lambda)\chi_{(1,\infty)}(x) + \sum_{j=0}^{\infty}\sum_{k=1}^{2^j} A_{j,k}(\lambda)\chi_{\left(a_{j,k}a_{j,k}+3^{-(j+1)}\right)}(x) \Big) e_\lambda(x).$$

Consider a selfadjoint restriction P_B of the maximal momentum operator on Ω such that $A_k \in \ell^2$ and

$$BD_r(\lambda) \begin{bmatrix} A_\infty \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} = D_s(\lambda) \begin{bmatrix} A_{-\infty} \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix},$$

where

$$D_r(\lambda) = \operatorname{diag}(e(\lambda r_0), e(\lambda r_1), e(\lambda r_2), \cdots) = \operatorname{diag}(e(\lambda), e(\lambda r_1), e(\lambda r_2), \cdots)$$
$$D_s(\lambda) = \operatorname{diag}(e(\lambda s_0), e(\lambda s_1), e(\lambda s_2), \cdots) = \operatorname{diag}(1, e(\lambda s_1), e(\lambda s_2), \cdots)$$

and B is some unitary on ℓ^2.

THEOREM 3.4.14. *If $B = \operatorname{diag}(1, 1, \ldots)$, then the spectrum of P_B is the real line and the embedded point spectrum is $\Lambda_p = \bigcup_{k=1}^{\infty} \frac{1}{\ell_k}\mathbb{Z}$, where $\ell_k = s_k - r_k$ is the length of I_k. The multiplicity of $\lambda \in \Lambda_p$ equals the cardinality of the set $\{k \mid \lambda \ell_k \in \mathbb{Z}\}$.*

EXAMPLE 3.4.15. Some examples illustrating this are:

(1) If $\ell_k = 2^{-k}$, then $\Lambda_p = 2\mathbb{Z}$. Let $\mathbb{Z}_{\text{odd}}$ be the odd integers. The eigenvalues in $2^k\mathbb{Z}_{\text{odd}}$ have multiplicity k and 0 has infinite multiplicity.
(2) For the complement of the middle thirds Cantor set $\Lambda_p = 3\mathbb{Z}$. The eigenvalues that are multiples of 3^k but not of 3^{k+1} have multiplicity $2^k - 1$ and 0 has infinite multiplicity.
(3) If ℓ_k/ℓ_j is irrational for all $j \neq k$, then 0 has infinite multiplicity and all other eigenvalues have multiplicity one.

COROLLARY 3.4.16. *If $B = \operatorname{diag}(e(\theta_0), e(\theta_1), \ldots)$, then the spectrum of P_B is the real line and the embedded point spectrum is $\Lambda_p = \bigcup_{k=1}^{\infty} \left(\frac{\theta_k}{\ell_k} + \frac{1}{\ell_k}\mathbb{Z}\right)$, where $\ell_k = s_k - r_k$ is the length of I_k. The multiplicity of $\lambda \in \Lambda_p$ equals the cardinality of the set $\{k \mid \lambda\ell_k - \theta_k \in \mathbb{Z}\}$.*

When we have a finite number of intervals *the point spectrum* has uniform density equal to the sum of the lengths of the intervals. The following example shows that this need not be the case for infinitely many intervals.

EXAMPLE 3.4.17. Suppose $B = \operatorname{diag}(e(\theta_0), e(\theta_1), \ldots)$ and $\ell_k = 2^{-k}$. Then $2^k(\theta_k + m) = 2^j(\theta_j + n)$ if and only if $2^{j+k}(\theta_k - \theta_j) = 2^{k+j}(n - m)$. Hence, if $\theta_k - \theta_j$ is not an integer when $k \neq j$, then each eigenvalue has multiplicity one. Note $2^k\theta_k$ is an eigenvalue for each k. Hence, if $2^k\theta_k \to \lambda_0$ then λ_0 is a limit point of Λ_p. Similarly, by a suitable choice of the sequence θ_k, we can arrange that P_B has dense point spectrum.

THEOREM 3.4.18. [**JPT15a**]*If we write $\ell^2 = \mathbb{C} \oplus \ell^2$, then B takes the form*

$$B = \begin{pmatrix} c & \mathbf{w}^* \\ \mathbf{u} & B' \end{pmatrix}.$$

If the spectrum of B' does not intersect the unit circle, then the spectrum P_B is the real line and each point in the spectrum has multiplicity one, in particular, the point spectrum is empty.

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CHAPTER 4

Four kinds of harmonic analysis

> A large part of mathematics which becomes useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so. By and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and the moment when it is useful; and that this lapse of time can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful.
> — John von Neumann (1903–1957)

Representations of Cuntz algebras that arise from the action of stochastic matrices on sequences from $\mathbb{Z}_n$ are considered. This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into *frequency bands*.

The constructions of *spectral measures* often utilize “Cuntz isometries”, namely isometries that satisfy the Cuntz relations. The present chapter will discuss how understanding specific representations of the Cuntz algebras yields information concerning other spectra for a spectral measure. Conversely, beginning with a representation of a Cuntz algebra, a *Markov measure* can be associated to the representation which gives spectral information about the representation.

4.1. Orthogonal Fourier expansions

While it is not true in general, for given measures μ, that the Hilbert space $L^2(\mu)$ is amenable to Fourier analysis, at least not in a direct way, the notion of spectrum was introduced for the analysis of certain singular measures μ. Here our aim is to fix sets Γ that serve as spectrum, and ask for the variety of measures μ that have Γ as their spectrum.

The present section is based primarily on ideas in the paper [**DJ13a**] by Jorgensen et al.

Our study of spectral pairs (μ, Γ) extends the more familiar theory of Pontryagin duality for locally compact abelian groups. The simplest instance of interesting spectral pairs include the compact d-torus $\mathbb{T}^d$ and its Fourier dual the rank-d lattice $\mathbb{Z}^d$, the setting of multivariable Fourier series. In this context, the required and more standard Fourier tools for $d = 1$ do carry over to $d > 1$. It will be convenient to model $\mathbb{T}^d$ as the d-cube in $\mathbb{R}^d$, i.e., as $Q_d := I^d$, where $I := [0, 1)$.

There are many differences between classical multivariable Fourier analysis on the one hand, and spectral pairs (μ, Γ) on the other; for example this: the absence of groups in the context of general spectral pairs. Indeed, typically for general spectral pairs, neither of the two sets in the pair, the support of the measure nor its spectrum, is a group.

Nonetheless, there are important spectral theoretic questions for those particular spectral pairs where the measure μ is d-dimensional Lebesgue measure restricted to Q_d. There are several questions here; first: What sets Γ in $\mathbb{R}^d$ make (Q_d, Γ) a spectral pair? As shown in Chapter 2, a discrete subset Γ in $\mathbb{R}^d$ is a spectrum for Q_d if and only if it tiles $\mathbb{R}^d$ by translations of Q_d.

This spectral/tile duality in fact is a part of a wider duality theory. In particular, higher dimensions are of interest because of the existence of exotic "cube-tilings" in $\mathbb{R}^d$ for $d = 10$ and higher, found by Lagarias and Shor [**LS92**]. Our purpose here is to turn the question around: Rather than fixing one part in a particular spectral pair, in this case Q_d, instead we pick the simplest spectrum $\Gamma = \mathbb{Z}^d$, and we then ask what are the possibilities for measures μ in spectral pairs $(\mu, \mathbb{Z}^d)$. The theorem below answers this question.

THEOREM 4.1.1 ([**DJ13a**]). *Let μ be a Borel probability measure on $\mathbb{R}^d$. The following statements are equivalent:*

(1) *The set $\{e_n : n \in \mathbb{Z}^d\}$ forms an orthonormal set in $L^2(\mu)$.*
(2) *There exists a bounded measurable function $\varphi \geq 0$ that satisfies*

$$\sum_{k \in \mathbb{Z}^d} \varphi(x + k) = 1, \text{ for Lebesgue a.e. } x \in \mathbb{R}^d, \tag{4.1.1}$$

such that $d\mu = \varphi \, dx$.

This work is motivated in part by recent problems in spectral theory and geometric measure theory. The applications include Schroedinger operators from physics, especially their scattering theory [**And09**, **Abd08**, **FLZ08**]. In these problems, it is helpful to have at hand concrete model-examples involving measures amenable to direct computations. In stochastic processes and stochastic integration, key tools depend on underlying spectral densities. For problems involving fluctuations and chaotic dynamics, the measures are often singular, and model-measures are helpful. In determining the nature of orbits in ergodic theory, the first question is often "what is the spectral type?" The measures in these applications are typically not compactly supported. Nonetheless, there is a procedure from geometric measure theory which produces compactly supported measures, and much of the earlier literature has focused on measures of compact support.

Here we explore the general theory, and we find very general and varied families: a rich family of iso-spectral fractals.

To help the reader understand the ideas, consider Borel probability measures that are iso-spectral; i.e., every measure in the family has the same spectrum, say Γ. Specifically, we fix a spectrum. What are natural families of Borel probability measures with this spectrum?

Indeed, consider the most general spectral pair (μ, Γ) in $\mathbb{R}^d$; and then develop algorithms yielding indexed families of measures (μ_a) with the index a in a specific index set A. A natural choice for A is a suitable family of partitions of a base-point measure μ in the family. We find extensive families, but there are probably other bigger and intriguing families of iso-spectral measures.

We elaborate on the set A below in Theorem 4.1.3. The two steps in the algorithm consist in choosing measurable partitions of the d-cube and translations by $\mathbb{Z}^d$ defining a translation congruence. The set A of all translation congruences labels the iso-spectral measures.

In the simplest case, take $a =$ the trivial partition; and this yields back μ itself, up to a translation. There are at least four reasons this is of interest:

(1) There is a big classical literature going back to Mark Kac's question: "Can one hear the shape of a drum?" In our context, we will be considering the Fourier transform of μ as a function on $\mathbb{R}^d$, and the possible drums will be iso-spectral data, typically iso-spectral fractals, or fractal measures. The idea of making connections between geometric features and spectral theoretic data, of course dates back to Fourier, but it was made popular by Mark Kac in [**Kac66**]. While Kac had in mind a Laplacian on a planar domain, the question has generated a host of formulations involving various forms of spectral data, and various geometries; see for example [**Lap08**] and the references cited there. Here we are concerned with Fourier frequencies on one side of the divide, and geometric measure theory on the other.

(2) The Fourier transform of μ_a, $F_a(t) = \widehat{\mu}_a(t)$, $t \in \mathbb{R}^d$ is interesting as we vary $a \in A$. We can get $F_a(t)$ non-differentiable; and anything in between continuous and entire analytic. In a more fundamental setting, the problem of recovering a function of a measure from a Fourier transform lies at the root of obstacle scattering, but in a classical context.

(3) And going back to Paley-Wiener there is much literature on the interplay between the possibility of analytic continuations of geometric Fourier transforms and the geometry itself. Here, by Paley-Wiener, we mean questions dealing with asymptotic estimates on a complex Fourier transform. However, by contrast, there are relatively few parallel results dealing with classes of fractal measures.

(4) There is an analogy of these families to wavelet sets that play an important role in the spectral theory of wavelets in higher dimensions. Here we refer to tiling properties for wavelet sets. This is only a parallel as wavelet sets involve two operations, translation and scaling. Our focus here is on translations by integer lattices.

DEFINITION 4.1.2. We say that a Borel subset E of $\mathbb{R}^d$ is *translation congruent* to $Q = [0,1)^d$ if there exists a measurable partition $\{E_k : k \in \mathbb{Z}^d\}$ of Q such that

$$E = \bigcup_{k\in\mathbb{Z}^d} (E_k + k).$$

Fix a set Γ arising as a spectrum. We now ask for the variety of measures μ that have Γ as their spectrum. The result below answers the question for the special case when $\Gamma = \mathbb{Z}^d$. The question in [**JP99**], inspired in part by [**LS92**], deals with the possibility of spectral pairs when one term in the pair is the d-cube Q in $\mathbb{R}^d$. The classification of the spectra was found for small d. The authors of [**JP99**] further suggested that spectra have the additional property that they are translation sets for additive tilings. Theorem 4.1.3 about translation congruence offers a possible answer to this, and we refer the reader to the original paper for the proof.

THEOREM 4.1.3 ([**DJ13a**]). *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then μ has spectrum $\mathbb{Z}^d$ if and only if μ is the Lebesgue measure restricted to a set E which is translation congruent to Q.*

EXAMPLE 4.1.4 ($d = 1$). Let

$$\varphi = \frac{2}{3}\chi_{[0,1)} + \frac{1}{3}\chi_{[1,2)}.$$

Let $d\mu = \varphi\, dx$. By Theorem 4.1.1, the set $\{e_n : n \in \mathbb{Z}\}$ is orthogonal in $L^2(\mu)$. We have

$$\widehat{\mu}(t) = \frac{1}{6\pi i t}\left(e^{2\pi i 2t} + e^{2\pi i t} - 2\right), \quad t \in \mathbb{R}.$$

This shows that $\widehat{\mu}(t) = 0$ if and only if $t \in \mathbb{Z}$. From this we see that there is no $t \in \mathbb{R}\setminus\mathbb{Z}$ such that e_t is orthogonal to e_n for $n \in \mathbb{Z}$, because $\langle e_t, e_n\rangle_{L^2(\mu)} = \widehat{\mu}(t-n)$.

Therefore $\mathbb{Z}$ yields a *maximal* set of orthogonal exponentials, which is incomplete by Theorem 4.1.3.

Next, we characterize measures that have spectrum $\{0, \dots, N-1\}$ for some finite integer N. The simplest example is of course $\frac{1}{N}\sum_{k=0}^{N-1}\delta_{1/k}$, where δ_x is the Dirac measure at x.

THEOREM 4.1.5. *Let $N \geq 2$ be an integer. Let A be a set in $\mathbb{R}$ such that the atomic measure $\delta_A = \frac{1}{N}\sum_{a\in A}\delta_a$ has spectrum $\{0, 1, \dots, N-1\}$. The A is of the form $A = \frac{1}{N}A'$ where A' is a compete set of representatives for $\mathbb{Z}/N\mathbb{Z}$.*

PROOF. It is easy to see, by writing the orthogonality of the exponential functions, that A has spectrum $\{0, \dots, N-1\}$ if and only if the matrix

$$\frac{1}{\sqrt{N}}\left(e^{2\pi i a k}\right)_{a\in A,\, k\in\{0,\dots,N-1\}}$$

is unitary.

If A has the form given in the statement of the theorem, then this matrix is unitary; it is the matrix of the Fourier transform on the group $\mathbb{Z}/N\mathbb{Z}$.

For the converse, assume the matrix is unitary. Then for any pair of distinct points a, a' in A, we must have

$$\sum_{k=0}^{N-1} e^{2\pi i(a-a')k} = 0$$

so $e^{2\pi i(a-a')}$ is a root of the polynomial $\sum_{k=0}^{N-1} z^k$. Then $a - a' = \frac{l}{N}$ for some $l \in \mathbb{Z}$, not a multiple of N. Since there are N elements in A, the pigeon hole principle implies that NA is a complete set of representatives for $\mathbb{Z}/N\mathbb{Z}$. $\square$

REMARK. Theorems 4.1.3 and 4.1.5 might lead one think that if two measures μ and μ' have a common spectrum Γ contained in $\mathbb{Z}$, then they must be translation equivalent. However, this is not true, as the following example shows.

EXAMPLE 4.1.6. Consider the atomic measures δ_A and $\delta_{A'}$, where

$$A = \left\{0, \frac{1}{8}, \frac{4}{8}, \frac{5}{8}\right\}, \quad \text{and} \quad A' = \left\{0, \frac{3}{8}, \frac{4}{8}, \frac{7}{8}\right\}.$$

They have the common spectrum $\Gamma = \{0, 1, 4, 5\}$. This can be seen by computing the matrices, $\frac{1}{\sqrt{4}}(e^{2\pi i a\lambda})_{a\in A,\lambda\in\Gamma}$ and similarly for A', with $\rho = e^{2\pi i/8}$:

$$\frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \rho & -1 & -\rho \\ 1 & -1 & 1 & -1 \\ 1 & -\rho & -1 & \rho \end{pmatrix}, \quad \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \rho^3 & -1 & -\rho^3 \\ 1 & -1 & 1 & -1 \\ 1 & -\rho^3 & -1 & \rho^3 \end{pmatrix}$$

which are unitary. However, the measures δ_A and $\delta_{A'}$ are not translation equivalent.

Theorem 4.1.3 is a characterization of probability measures μ supported in $\mathbb{R}^d$ which allow $L^2(\mu)$-orthogonal Fourier series indexed by $\mathbb{Z}^d$. We argue how our result fits into a wider context of making links between geometric shapes, on one side, and spectral data on the other. Here we are concerned with Fourier frequencies on one side of the divide, and geometric measure theory on the other.

4.2. Frame and related non-orthogonal Fourier expansions

The material below is based primarily on ideas in the paper [**Jor08**]. Also see, e.g., [**Dut04**, **DJ07a**, **Cas00**].

Frames are redundant bases which turn out in certain applications to be more flexible than the better known orthonormal bases (ONBs) in Hilbert space. The frames allow for more symmetries than ONBs do, especially in the context of signal analysis, and of wavelet constructions; see, e.g., [**CD93**, **Lan67**, **Dut06**]. Since frame bases (although containing redundancies) still allow for efficient algorithms, they have found many applications, even in finite dimensions.

As is well known, when a vector f in a Hilbert space $\mathscr{H}$ is expanded in an orthonormal basis B, there is then automatically an associated Parseval identity. In physical terms, this identity typically reflects a *stability* feature of a decomposition based on the chosen ONB B. Specifically, Parseval's identity reflects a conserved quantity for a problem at hand, for example, energy conservation in quantum mechanics.

The theory of frames begins with the observation that there are useful vector systems which are in fact not ONBs but for which a Parseval formula still holds. In fact, in applications it is important to go beyond ONBs. While this viewpoint originated in signal processing (in connection with frequency bands, aliasing, and filters), the subject of frames appears now to be of independent interest in mathematics.

On occasion, we may have a system of vectors S in $\mathscr{H}$ for which Parseval's identity is still satisfied, but such that a generalized Parseval's identity might only hold up to a fixed constant c of scale. (For example, in sampling theory, a scale might be introduced as a result of "oversampling".) In this case, we say that the constant c scales the expansion. Suppose a system of vectors S in a given Hilbert space $\mathscr{H}$ allows for an expansion, or decomposition of every f in $\mathscr{H}$, but the analogue of Parseval's identity holds only up to a fixed constant c of scale. In that case, we say that S is a *tight frame* with frame constant c. So the special case $c = 1$ is the case of a Parseval frame.

Aside from applications, at least three of the other motivations for frame theory come from: (1) wavelets, e.g., [**CD93**] and [**BJMP05**]; (2) from non-harmonic Fourier expansions [**DS52**]; and (3) from computations with polynomials in several variables, and their generalized orthogonality relations [**DX01**].

While frames already have impressive uses in signal processing, they have recently been shown to be central in our understanding of a fundamental question in operator algebras, the Kadison–Singer conjecture. We refer the reader to [**Cas00**, **CFTW06**] for up-to-date research, and to [**Chr99**, **KR83**, **Nel58a**, **Nel59**] for background.

Recursive Algorithms. In all these cases, the authors work with recursive algorithms, and the issue of *stability* plays a crucial role. Stability, however, may obtain in situations that are much more general than the context of traditional ONBs, or even tight frames. In fact, stability may apply even when we have only *a priori* estimates, as opposed to identities: for example, when the scaled version of Parseval's identity is replaced with a pair of estimates, a fixed lower bound and an upper bound; see (4.2.21) below. If such bounds exist, they are called lower and upper frame bounds.

If a system S of vectors in a Hilbert space $\mathscr{H}$ satisfies such a pair of *a priori* estimates, we say that S is simply a *frame*. And if such an estimate holds only with an *a priori* upper bound, we say that S is a *Bessel sequence*. It is known that for a fixed Hilbert space $\mathscr{H}$, the various classes of frames S in $\mathscr{H}$ may be obtained from some ambient Hilbert space $\mathscr{K}$ and an orthonormal basis B in $\mathscr{K}$, i.e., when the pair $(S, \mathscr{K})$ is given, there are choices of $\mathscr{K}$ such that the frame S may be obtained from applying a certain bounded operator T to a suitable ONB B in $\mathscr{K}$. Passing from the given structure in $\mathscr{H}$ to the ambient Hilbert space is called *dilation* in operator theory. The properties of the operator T which does the job depend on the particular frame in question. For example, if S is a Parseval frame, then T will be a projection of the ambient Hilbert space $\mathscr{K}$ onto $\mathscr{H}$. But this operator-theoretic approach to frame theory has been hampered by the fact that the ambient Hilbert space is often an elusive abstraction. Starting with a frame S in a fixed Hilbert space $\mathscr{H}$, then by dilation, or extension, we pass to an ambient Hilbert space $\mathscr{K}$. In this paper we make concrete the selection of the "magic" operator $T\colon \mathscr{K} \to \mathscr{H}$ which maps an ONB in $\mathscr{K}$ onto S. While existence is already known, the building of a dilation system $(\mathscr{K}, T, \text{ONB})$ is often rather non-constructive, and the various methods for getting $\mathscr{K}$ are fraught with choices that are not unique.

Nonetheless, it was shown in [**Dut04**, **Dut06**] that when the dilation approach is applied to Parseval frames of wavelets in $\mathscr{H} = L^2(\mathbb{R})$, i.e., to wavelet bases which are not ONBs, then the ambient Hilbert space $\mathscr{K}$ can be made completely explicit, and the constructions are algorithmic. Moreover, the "inflated" ONB in $\mathscr{K}$ then takes the form of a traditional ONB-wavelet basis, a so-called "super-wavelet".

It is the purpose of the present section to show that the techniques which work well in this restricted context, "super-wavelets" and redundant wavelet frames, apply to a more general and geometric context, one which is motivated in turn by extension principles in probability theory.

A key idea in our present approach is the use of reproducing Hilbert spaces, and their reproducing kernels in the sense of [**Aro50**]. See also [**Nel58a**] for an attractive formulation. Indeed, for every Hilbert space $\mathscr{H}$, and every frame S in $\mathscr{H}$ (even if S is merely a Bessel sequence), there is a way of constructing the ambient Hilbert space $\mathscr{K}$ in such a way that the operator T has a concrete reproducing kernel.

Settings. Let S be a countable set, finite or infinite, and let $\mathscr{H}$ be a complex or real Hilbert space. We shall be interested in a class of spanning families of vectors $(\mathbf{v}(s))$ in $\mathscr{H}$ indexed by points $s \in S$. Their properties will be defined precisely below, and the families are termed *frames*. The simplest instance of this is when $\mathscr{H} = \ell^2(S) =$ the Hilbert space of all square-summable sequences, i.e., all $f\colon S \to \mathbb{C}$ such that $\sum_{s\in S} |f(s)|^2 < \infty$. In that case, set

$$\langle f_1, f_2 \rangle := \sum_{s\in S} \overline{f_1(s)} f_2(s) \tag{4.2.1}$$

for all $f_1, f_2 \in \ell^2(S)$.

It is then immediate that the delta functions $\{\delta_s \mid s \in S\}$ given by

$$\delta_s(t) = \begin{cases} 1, & t = s, \\ 0, & t \in S \setminus \{s\}, \end{cases} \tag{4.2.2}$$

form an *orthonormal basis* (ONB) for $\mathscr{H}$, i.e., that

$$\langle \delta_{s_1}, \delta_{s_2} \rangle = \begin{cases} 1 & \text{if } s_1 = s_2 \text{ in } S, \\ 0 & \text{if } s_1 \neq s_2, \end{cases} \tag{4.2.3}$$

and that this is a maximal orthonormal family in $\mathscr{H}$. Moreover,

$$f = \sum_{s\in S} f(s)\, \delta_s \quad \text{for all } f \in \ell^2(S). \tag{4.2.4}$$

It also is immediate from (4.2.1) that Parseval's formula

$$\|f\|^2 = \sum_{s\in S} |\langle \delta_s, f \rangle|^2 \tag{4.2.5}$$

holds for all $f \in \ell^2(S)$.

We shall consider pairs $(S, \mathscr{H})$ and indexed families

$$\{\mathbf{v}(s) \mid s \in S\} \subset \mathscr{H} \tag{4.2.6}$$

such that for some $c \in \mathbb{R}_+$, the identity

$$\|f\|^2 = c \sum_{s\in S} |\langle \mathbf{v}(s), f \rangle|^2 \tag{4.2.7}$$

holds for all $f \in \mathscr{H}$.

When a Hilbert space $\mathscr{H}$ is given, our main result states that solutions to (4.2.7) exist if and only if $\mathscr{H}$ is isometrically embedded in $\ell^2(S)$. But we further characterize these embeddings, and we use this in understanding the geometry of tight frames.

DEFINITION 4.2.1. Let $\left(S, \mathscr{H}, c, (\mathbf{v}(s))_{s\in S}\right)$ be as above. We shall say that this system constitutes a *tight frame* with frame constant c if (4.2.7) holds.

Tight frames with frame constant equal to one are called *Parseval frames*.

(Note that if $(\mathbf{v}(s))_{s\in S}$ satisfies (4.2.7), then the scaled system $(\sqrt{c}\,\mathbf{v}(s))_{s\in S}$ has the property with frame constant one.)

EXAMPLE 4.2.2. (S finite.) Let $\mathscr{H}$ be the two-dimensional real Hilbert space, and let $n \geq 3$. Set $S := \{1, 2, \dots, n\} =: S_n$, and

$$\mathbf{v}(s) := \begin{pmatrix} \cos\left(\dfrac{2\pi s}{n}\right) \\ \sin\left(\dfrac{2\pi s}{n}\right) \end{pmatrix}, \quad s \in S. \tag{4.2.8}$$

Then it is easy to see that this constitutes a tight frame with frame constant $c = \frac{2}{n}$. Examples are presented in Figures 4.2.1 and 4.2.2.

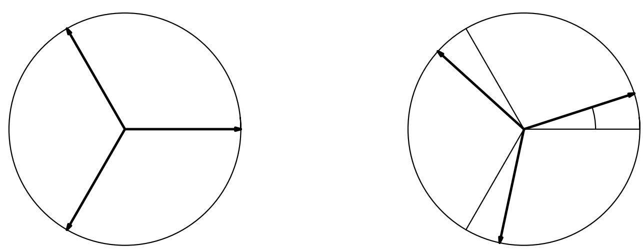


FIGURE 4.2.1. Two illustrations for $n = 3$

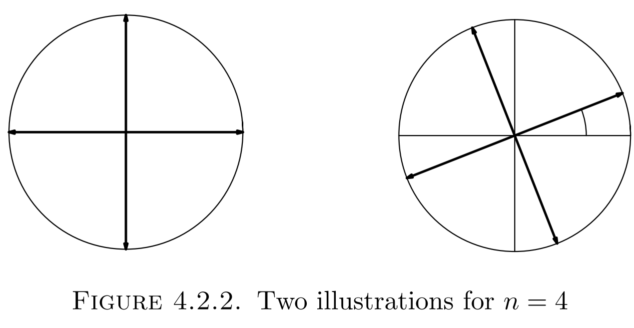


FIGURE 4.2.2. Two illustrations for $n = 4$

DEFINITION 4.2.3. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces over $\mathbb{C}$ or $\mathbb{R}$, and let $V\colon \mathscr{H} \to \mathscr{K}$ be a linear mapping. We say that V is an *isometry*, and that $\mathscr{H}$ is *isometrically embedded* in $\mathscr{K}$ (via V) if

$$\|Vf\|_{\mathscr{K}} = \|f\|_{\mathscr{H}}, \quad f \in \mathscr{H}. \tag{4.2.9}$$

Given a linear operator $V\colon \mathscr{H} \to \mathscr{K}$, we then denote the adjoint operator $V^*\colon \mathscr{K} \to \mathscr{H}$. It is easy to see that V is *isometric* if and only if $V^*V = I_{\mathscr{H}}$, where $I_{\mathscr{H}}$ denotes the identity operator in $\mathscr{H}$. Moreover, if V is isometric then $P = P_V = VV^*\colon \mathscr{K} \to \mathscr{K}$ is a projection, i.e.,

$$P = P^* = P^2 \tag{4.2.10}$$

holds, and the subspace

$$P\mathscr{K} \subset \mathscr{K} \tag{4.2.11}$$

may be identified with $\mathscr{H}$ via the isometric embedding.

We state our next result only in the case of frame constant $c = 1$, but as noted it easily generalizes.

Theorem 4.2.4. *Let S be a countable set, and let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$ (or $\mathbb{R}$). Then the following two conditions are equivalent:*

(1) *There is a tight frame $\{\mathbf{v}(s) \mid s \in S\} \subset \mathscr{H}$ with frame constant $c = 1$;*
(2) *$\mathscr{H}$ is isometrically embedded (as a closed subspace) in $\ell^2(S)$.*

In the Hilbert space $L^2(\mathbb{R})$ we will consider the usual Fourier transform

$$\hat{f}(t) := \int_{\mathbb{R}} e^{-i2\pi tx} f(x)\, dx, \tag{4.2.12}$$

We shall omit the proof here, and consider the following example.

Example 4.2.5. The familiar interpolation formula of Shannon [**Ash90**] applies to band-limited functions, i.e., to functions f on $\mathbb{R}$ such that the Fourier transform $\hat{f}$ is of compact support. Now, pick the following normalization

$$supp(\hat{f}) \subset \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{4.2.13}$$

and let $\mathscr{H}$ denote the subspace in $L^2(\mathbb{R})$ defined by this support condition. In particular, $\mathscr{H}$ is the range of the projection operator P in $L^2(\mathbb{R})$ defined by

$$(Pf)(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi xt} \hat{f}(t)\, dt. \tag{4.2.14}$$

Shannon's interpolation formula applies to $f \in \mathscr{H}$, and it reads:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}. \tag{4.2.15}$$

Let $S \subset \mathbb{R}$, and set

$$\mathbf{v}(s)(x) := \mathbf{v}(s,x) = \frac{\sin \pi (x-s)}{\pi (x-s)}, \quad s \in S. \tag{4.2.16}$$

Hence if we take as index set $S := \mathbb{Z}$, then we may observe that the functions on the right-hand side in Shannon's formula (4.2.15) are $\mathbf{v}(n)$ frame vectors, $n \in \mathbb{Z}$. We shall be interested in other index sets S, so-called sets of *sampling points.*

The following is well known but is included as an application of Theorem 4.2.4. It is also an example of a pair $(S, \mathscr{H})$ where S is *infinite.*

PROPOSITION 4.2.6. *Let $S \subset \mathbb{R}$ be a fixed discrete subgroup, and assume that $\mathbb{Z} \subset S$. Then $\{ \mathbf{v}(s) \mid s \in S \}$ is a tight frame in $\mathscr{H}$ if and only if the group index $(S:\mathbb{Z})$ is finite, and in that case the frame constant c is $c = (S:\mathbb{Z})^{-1}$. For the Gram matrix, we have:*

$$K(s_1, s_2) = \begin{cases} \dfrac{\sin \pi (s_1 - s_2)}{\pi (s_1 - s_2)} & \text{for } s_1, s_2 \in S,\ s_1 \neq s_2, \\ 1 & \text{if } s_1 = s_2. \end{cases} \tag{4.2.17}$$

PROOF. Follows immediately from Theorem 4.2.4. □

The significance of using a larger subgroup S, i.e., $\mathbb{Z} \subset S$, in a modified version of Shannon's interpolation formula (4.2.15) is that a larger (discrete) group represents "*oversampling*". However, note that the oversampling changes the frame constant.

As a contrast showing stability, we now recast a result on oversampling from [**BJMP05**] in the present context. It is for tight frames of *wavelet bases* in $L^2(\mathbb{R})$, and it represents an instance of stability: a case when oversampling leaves invariant the frame constant.

PROPOSITION 4.2.7. *Let $\psi \in L^2(\mathbb{R})$, and suppose that the family*

$$\psi_{j,k}(x) := 2^{j/2}\psi\left(2^j x - k\right), \qquad j, k \in \mathbb{Z}, \tag{4.2.18}$$

is a Parseval frame in $L^2(\mathbb{R})$. Let $p \in \mathbb{N}$ be odd, $p > 1$, and set

$$\tilde{\psi}_p(x) := \frac{1}{p}\,\psi\left(\frac{x}{p}\right), \tag{4.2.19}$$

and

$$\tilde{\psi}_{p,j,k}(x) := 2^{j/2}\,\tilde{\psi}_p\left(2^j x - k\right), \qquad j, k \in \mathbb{Z}. \tag{4.2.20}$$

Then the "oversampled" family (4.2.20) *is again a Parseval frame in the Hilbert space $L^2(\mathbb{R})$.*

PROOF. We refer the reader to the argument in [**BJMP05**, section 2]. □

More general frames

As before, we will consider a pair $(S, \mathscr{H})$, where S is a fixed countable set, and where $\mathscr{H}$ is a Hilbert space. Recall that a system of vectors $(\mathbf{v}(s))_{s\in S}$ in $\mathscr{H}$ is called a *frame* for $\mathscr{H}$ if there are constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1 \|f\|^2 \leq \sum_{s\in S} |\langle \mathbf{v}(s), f\rangle|^2 \leq A_2 \|f\|^2 \quad \text{for all } f \in \mathscr{H}. \tag{4.2.21}$$

DEFINITION 4.2.8. A system of vectors $(\mathbf{v}(s))_{s\in S}$ in $\mathscr{H}$ is called a *Bessel sequence* if only the estimate on the right-hand side in (4.2.21) is assumed, i.e., if for some finite constant A,

$$\sum_{s\in S} |\langle \mathbf{v}(s), f\rangle|^2 \leq A \|f\|^2 \quad \text{for all } f \in \mathscr{H}. \tag{4.2.22}$$

If (4.2.22) is assumed, then the analysis operator $V = V_{(\mathbf{v}(s))}$ given by

$$\mathscr{H} \ni f \xmapsto{V} (\langle \mathbf{v}(s), f\rangle)_{s\in S} \in \ell^2(S) \tag{4.2.23}$$

is well defined and bounded. Hence, the adjoint operator $V^*\colon \ell^2(S) \to \mathscr{H}$ is bounded as well, and

$$V^*(\xi_s) = \sum_{s\in S} \xi_s \,\mathbf{v}(s) \quad \text{for all } (\xi_s) \in \ell^2(S), \tag{4.2.24}$$

where the sum on the right-hand side in (4.2.24) is convergent in $\mathscr{H}$ for all $(\xi_s) \in \ell^2(S)$.

THEOREM 4.2.9. *Let $(S, \mathscr{H})$ be as above, and let $(\mathbf{v}(s))_{s\in S}$ be a Bessel sequence with Bessel constant A.*

(1) *Then the closed span $\mathscr{H}_{in}$ of $(\mathbf{v}(s))_{s\in S}$ contains a derived Parseval frame.*
(2) *The derived Parseval frame is a Parseval frame for $\mathscr{H}$ if and only if $(\mathbf{v}(s))_{s\in S}$ is a frame for $\mathscr{H}$, i.e., if and only if $\mathscr{H}_{in} = \mathscr{H}$.*
(3) *In the general case when $(\mathbf{v}(s))_{s\in S}$ is a Bessel sequence, the operator $W := V(V^*V)^{-1/2}$ is well defined and isometric on $\mathscr{H}_{in}$, and*

$$\mathbf{w}(s) := (V^*V)^{-1/2}\,\mathbf{v}(s), \qquad s \in S, \tag{4.2.25}$$

is a Parseval frame in $\mathscr{H}_{in}$.

PROOF. We refer the reader to [**Jor08**]; and also see e.g., [**PW17**, **DJ07a**]. □

4.3. Wavelet expansions

Following Figure 1.4.1 and Lemma 1.3.4, we begin with a *multiresolution* in $L^2(\mathbb{R})$ for the space on which the scale number $N = 2$. We choose the filters

$$m_0(z) = \sum_k h_k z^k \quad \text{(low-pass)} \tag{4.3.1}$$

$$m_1(z) = \sum_k g_k z^k \quad \text{(high-pass)} \tag{4.3.2}$$

such that

$$\begin{cases} S_0 f(z) = m_0(z)\, f\left(z^2\right) \\ S_1 f(z) = m_1(z)\, f\left(z^2\right) \end{cases} \tag{4.3.3}$$

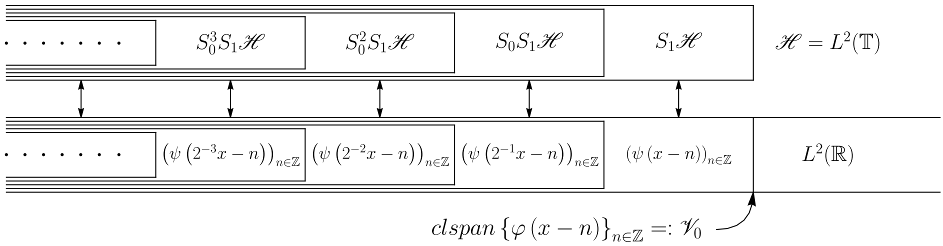
defines an $\mathscr{O}_2$-system (see Remark 1.3.5) in $L^2(\mathbb{T}) \simeq l^2$. For example

$$g_k = (-1)^k\, \overline{h_{1-k}}, \quad k \in \mathbb{Z}. \tag{4.3.4}$$

With suitable restrictions on (4.3.1)-(4.3.2), one shows that there is an associated father function φ, and mother function ψ such that

$$\begin{cases} \varphi(x) = \sqrt{2} \displaystyle\sum_{k\in\mathbb{Z}} h_k \varphi(2x-k) \\ \psi(x) = \sqrt{2} \displaystyle\sum_{k\in\mathbb{Z}} g_k \varphi(2x-k) \end{cases} \tag{4.3.5}$$

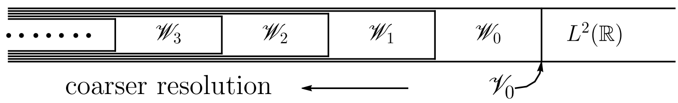
and identify operators with the Figure in Remark 1.3.5, as follows:



In detail,

$$\mathscr{W}_k = clspan\left\{\{\psi\left(2^{-k}x - n\right)\}_{n\in\mathbb{Z}}\right\}, \quad k \in \mathbb{N}_0, \tag{4.3.6}$$

and the orthogonal multiresolution subspaces in $L^2(\mathbb{R})$



The unitary operator $(Uf)(x) = 2^{-1/2} f(x/2)$ yields

$$U^k(\mathscr{W}_0) = \mathscr{W}_k, \quad \forall k \in \mathbb{Z}. \tag{4.3.7}$$

The reader is referred to Section 1.3 for the use of representation theory (for the Cuntz relations) in constructing wavelet multiresolutions.

Wavelets on fractals

The wavelet algorithm in matrix form is sketched below: Let $\{h_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be a system of wavelet coefficients, and consider for simplicity that they are finite and real valued, i.e.,

$$h_0, h_1, h_2, \cdots \quad \text{and} \quad g_0, g_1, g_2, \cdots,$$

see (4.3.1)-(4.3.2).

The reason for the (A, D) notation is but illustrated as the case of the Haar wavelet. In that example the combined (A, D) matrix operation from Figure 4.3.1 simplifies as follows:

For images, the signal in is of the form $\left(x_k^{(1)}, x_k^{(2)}\right)$, $k \in \mathbb{Z}_A$ (gray scale numbers); and for color images, such a vector-time series for each of the three basic colors.

Hence, in the image case (gray scale numbers), the operator takes a tensor form as follows

$$\begin{pmatrix} A\otimes A & A\otimes D \\ D\otimes A & D\otimes D \end{pmatrix};$$

see also Figure 4.3.3.

We have described the traditional wavelet multiresolution constructions. Below we show that the general machinery also applies to a variety of IFS-constructions. To be specific, pick a rational function $r(z) = \frac{p(z)}{q(z)}$ of one complex variable, p, q fixed polynomials, such that $r(z)$ has a compact Julia set $M(r)$ with maximal entropy measure μ [**Mil06**]. Specifically, let $\{\tau_j\}$ be branches of the inverses of

$$
\underset{\substack{\uparrow\\ \text{signal out}}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_N \\ y_{N+1} \\ y_{N+2} \\ \vdots \\ \vdots \\ \vdots \\ y_{2N} \end{bmatrix}}
=
\left[\begin{array}{cccccccccc}
h_0 & h_1 & h_2 & h_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & h_0 & h_1 & h_2 & h_3 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\\
\cdots & \cdots & \multicolumn{7}{c}{\text{generalized average operators: } A} & \cdots \\
\hline
g_0 & g_1 & g_2 & g_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & g_0 & g_1 & g_2 & g_3 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\\
\cdots & \cdots & \multicolumn{7}{c}{\text{generalized difference operators: } D} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right]
\underset{\substack{\uparrow\\ \text{signal in}}}{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \\ x_{N+1} \\ x_{N+2} \\ \vdots \\ \vdots \\ x_{2N} \end{bmatrix}}
$$

FIGURE 4.3.1. (A, D) processing. A concrete example of this, applied to digital image of the author, is included in Figure 4.3.2 below.

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_N \\ y_{N+1} \\ y_{N+2} \\ \vdots \\ \vdots \\ \vdots \\ y_{2N} \end{bmatrix}
= \frac{1}{\sqrt{2}}
\left[\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \\
\hline
1 & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -1
\end{array}\right]
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \\ x_{N+1} \\ \vdots \\ \vdots \\ \vdots \\ x_{2N} \\ x_{2N+1} \end{bmatrix}
$$

FIGURE 4.3.2. (A, D) processing, Haar wavelet.

$r(z)$ on $M(r)$; for example if $r(z) = z^2 + c$, c fixed, then $\tau_{\pm}(z) := \pm\sqrt{z-c}$, see Figure 5.2.1.

In general, μ is determined by

$$\mu = \frac{1}{N} \sum_{j=1}^{N} \mu \circ \tau_j^{-1}. \tag{4.3.8}$$

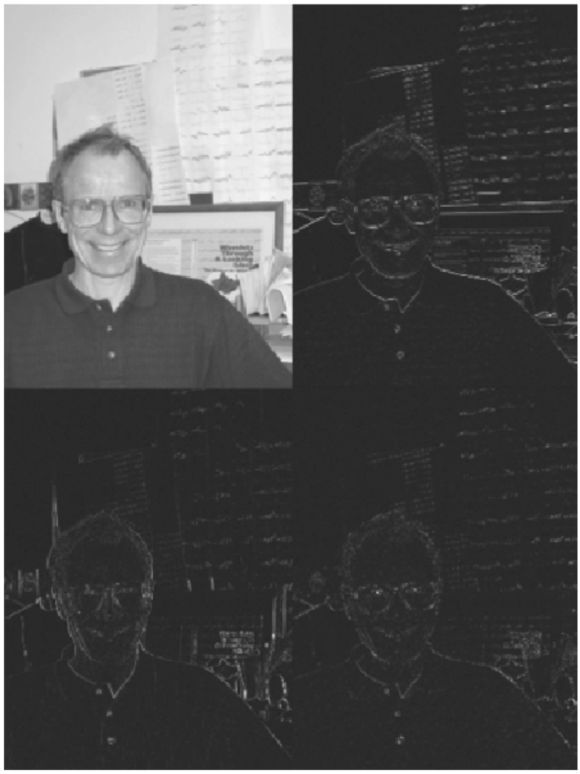


FIGURE 4.3.3. An example of wavelet decomposition

DEFINITION 4.3.1. The solenoid $Sol\,(r)$ is defined by

$$Sol\,(r) = \left\{ (z_n)_{n\in\mathbb{N}_0} \in M\,(r)^{\mathbb{N}_0} \mid r\,(z_{n+1}) = z_n \right\}. \tag{4.3.9}$$

The following result follows from the ideas presented in Chapter 2.

THEOREM 4.3.2. *Let r, $M\,(r)$, and $Sol\,(r)$ be as above. Set*

$$X_n\left((z_k)_{k\in\mathbb{N}_0} \right) = z_n, \quad X_n : Sol\,(r) \longrightarrow M\,(r)\,; \tag{4.3.10}$$

then there is a unique measure $\mathbb{P}$ on the cylinder σ-algebra of $Sol\,(r)$ such that

$$\mathbb{P} \circ X_0^{-1} = \mu, \tag{4.3.11}$$

and

$$U \mathcal{M}\,(f \circ X_0)\,U^{-1} = \mathcal{M}\,(f\,(r\,(X_0)))$$

for all $f \in L^\infty\,(M\,(r))$, and U is a unitary dilation of the isometry $Sf = f \circ \sigma$ defined on $L^2\,(M\,(r)\,,\mu)$. Here, $\mathcal{M}$ denotes the multiplication operator, corresponding to the function in question. (Recall

$$\|Sf\|_{L^2(M(r),\mu)} = \|f\|_{L^2(M(r),\mu)} \tag{4.3.12}$$

holds for functions f on $M\,(r)$.)

4.3.1. Wavelet algorithms and multiresolutions: the smooth vs the non-smooth categories. The material below is adapted primarily from [**DJ06a**] by Jorgensen et al.

The present section has three interrelated themes:

(1) construction of wavelet bases in separable Hilbert spaces built on affine fractals and Hausdorff measure;
(2) approximation of the corresponding wavelet scaling functions, using the cascading approximation algorithm; and
(3) an associated spectral theoretic analysis of a transfer operator, often called the Ruelle operator.

There are surprises when our results are compared to what is known for the traditional multiresolution approach for $L^2(\mathbb{R}^d)$, and even when compared to known results for special classes of affine fractals.

Some comments on (1)–(3): Due to earlier work by Jorgensen, Pedersen [**JP98a**] and Strichartz et al [**Str00**], it is known that a subclass of the affine fractals admits Fourier duality. Affine fractals arise from the specification of an expansive matrix, and a finite set of translations. The fractal X itself then arises from this data and an iteration "in the small" of the corresponding affine maps. Let $L = L(X)$ be the associated iteration "in the large". We say that (X, L) is a Fourier duality, if an orthonormal basis on X may be built from the frequencies in L. While it is known that, if X is the middle third Cantor set, then there is no L which makes a duality pair, we show that nonetheless, every affine fractal admits an orthonormal wavelet basis. In our discussion of wavelets, we start with the middle third Cantor set; and we then pass on to the general affine fractals.

As for the approximation issues in (2), we know that for $L^2(\mathbb{R}^d)$, there is a rich family of wavelet filters which yield cascade approximation. This family of filters is much more restricted for the fractals. Our results for the affine fractals even offer a certain dichotomy: If the cascades do not converge in the Hilbert space, then the terms in the cascading approximation sequence are typically orthogonal, and thus very far from being convergent.

Our analysis of (1)–(2) is based on spectral theory of the associated transfer operator, and we show shall show how this spectral theory differs in the three cases, the standard $L^2(\mathbb{R}^d)$–wavelets, and the special duality fractals versus the general class of affine fractals.

We develop the theory of multiresolutions in the context of Hausdorff measure of fractional dimension between 0 and 1. While our fractal wavelet theory has points of similarity that it shares with the standard case of Lebesgue measure on the line, there are also sharp contrasts.

It is well known that the Hilbert spaces $L^2(\mathbb{R})$ has a rich family of orthonormal bases of the following form:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z},$$

where ψ is a single function in $L^2(\mathbb{R})$, with

$$\|\psi\|_2 = \left(\int_{\mathbb{R}} |\psi(x)|^2 \, dx\right)^{1/2} = 1,$$

and the integration refers to the usual Lebesgue measure on $\mathbb{R}$. Take for example

$$\psi(x) = \chi_I(2x) - \chi_I(2x-1) \tag{4.3.13}$$

where $I = [0, 1]$ is the unit interval.

Clearly I satisfies

$$2I = I \cup (I + 1).$$

The Cantor subset $\mathbf{C} \subset I$ satisfies

$$3\mathbf{C} = \mathbf{C} \cup (\mathbf{C} + 2) \tag{4.3.14}$$

and its indicator function $\varphi_{\mathbf{C}} := \chi_{\mathbf{C}}$ satisfies

$$\varphi_{\mathbf{C}}(\frac{x}{3}) = \varphi_{\mathbf{C}}(x) + \varphi_{\mathbf{C}}(x - 2). \tag{4.3.15}$$

Since both constructions, the first one for the Lebesgue measure, and the second one for the Hausdorff version $(dx)^s$, arise from scaling and subdivision, it seems reasonable to expect multiresolution wavelets also in Hilbert spaces constructed on the scaled Hausdorff measures $\mathcal{H}^s$. The latter are basic for the kind of iterated function systems which give Cantor constructions built on scaling and translations by lattices. We show this to be the case, but there are still striking differences between the two settings, and we spell out some of them in this section.

The practical applications are to fractals arising in physics and in symbolic dynamical systems from theoretical computer science. There is already a considerable body of work on harmonic analysis on fractals, where much of it is based on subdivision techniques, and algorithms which use cascade constructions. Our emphasis here is direct wavelet algorithms and wavelet analysis for fractals.

IFS and gap-filling wavelets

The middle-third Cantor set $\mathbf{C}$ is a special case of an Iterated Function System (IFS). It falls in the subclass of the IFSs which are called *affine.*

Specifically, let $d \in \mathbb{Z}_+$, and let A be a $d \times d$ matrix of $\mathbb{Z}$. Suppose that the eigenvalues λ_i of A satisfy $|\lambda_i| > 1$. Set $N := |\det A|$. These matrices are called *expansive.* Then note that the quotient group $\mathbb{Z}^d/A(\mathbb{Z}^d)$ is of order N. A subset $\mathcal{D} \subset \mathbb{Z}^d$ is said to represent the A-residues if the natural quotient mapping

$$\gamma{:}\mathbb{Z}^d \longrightarrow \mathbb{Z}^d/A(\mathbb{Z}^d) \tag{4.3.16}$$

restricts to a bijection $\gamma_{\mathcal{D}}$ of $\mathcal{D}$ onto $\mathbb{Z}^d/A(\mathbb{Z}^d)$. For example, if $d = 1$, and $A = 3$, then we may take either one of the two sets $\{0, 1, 2\}$ or $\{0, 1, -1\}$ as $\mathcal{D}$.

The IFSs will be constructed from finite subsets $\mathcal{S} \subset \mathbb{Z}^d$ which represent the A-residues for some given expansive matrix A. If $(A, \mathcal{S})$ is a pair with these properties, define the maps

$$\sigma_s(x) := A^{-1}(x + s), \quad s \in \mathcal{S},\ x \in \mathbb{R}^d. \tag{4.3.17}$$

Using a theorem of Hutchinson [**Hut81**], we conclude that there is a unique measure $\mu = \mu_{(A,\mathcal{S})}$ with compact support $\mathbf{C} = \mathbf{C}_{(A,\mathcal{S})}$ on $\mathbb{R}^d$ such that

$$\mu = \frac{1}{\#(\mathcal{S})} \sum_{s \in \mathcal{S}} \mu \circ \sigma_s^{-1}, \tag{4.3.18}$$

or equivalently

$$\int f(x)\, d\mu(x) = \frac{1}{\#(\mathcal{S})} \sum_{s \in \mathcal{S}} \int f(\sigma_s(x))\, d\mu(x). \tag{4.3.19}$$

The quotient mapping

$$\gamma : \mathbb{R}^d \longrightarrow \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \tag{4.3.20}$$

restricts to map $\mathbf{C}$ bijectively onto a compact subset of $\mathbb{T}^d$. The Hausdorff dimension h of μ and of the support $\mathbf{C}$ is

$$h = \frac{\log \#(\mathcal{S})}{\log N}.$$

The system $(\mathbf{C}, \mu)$ is called a Hutchinson pair.

If $d = 1$, we will look at two examples:

(i) $(A, \mathcal{S}) = (3, \{0, 2\})$ which is the middle-third Cantor set $\mathbf{C}$, and

(ii) $(A, \mathcal{S}) = (4, \{0, 2\})$ which is the corresponding construction, but starting with a subdivision of the unit interval I into 4 parts, and in each step of the iteration omitting the second and the fourth quarter interval. As noted, then

$$h_{(i)} = \log_3(2) = \frac{\log 2}{\log 3}, \quad \text{and} \quad h_{(ii)} = \frac{1}{2}. \tag{4.3.21}$$

For more details, see [**JP98a**].

We will only sketch the general statements of results for the affine IFSs, those based on pairs $(A, \mathcal{S})$ in $\mathbb{R}^d$ where the matrix A and the subset $\mathcal{S} \subset \mathbb{Z}^d$ satisfy the stated conditions. The number h will be $h = \frac{\log(\#(\mathcal{S}))}{\log|\det A|}$; i.e., the Hausdorff dimension of the measure μ, and its support $\mathbf{C}$ which are determined from the given pair $(A, \mathcal{S})$. We will then be working with the corresponding Hausdorff measure $\mathcal{H}^h$, but now as a measure defined on subsets of $\mathbb{R}^d$. For example, for the middle-third Cantor set, we have

$$A\mathbf{C} = \bigcup_{s \in \mathcal{S}} (\mathbf{C} + s) \tag{4.3.22}$$

where $A\mathbf{C} := \{Ax \mid x \in \mathbf{C}\}$, and $\mathbf{C} + s := \{x + s \mid x \in \mathbf{C}\}$, or equivalently

$$\mathbf{C} = \bigcup_{s \in \mathcal{S}} \sigma_s(\mathbf{C}) \tag{4.3.23}$$

where $\sigma_s(\mathbf{C}) := \{\sigma_s(x) \mid x \in \mathbf{C}\}$. The conditions on the pair $(A, \mathcal{S})$ guarantees that the sets in the union on the right-hand side in (4.3.22) or in (4.3.23), are mutually non-overlapping. This amounts to the so-called open-set-condition of Hutchinson. Moreover, we get

$$\mathcal{R} = \bigcup_{n \geq 0} \bigcup_{k \in \mathbb{Z}^d} A^{-n}(\mathbf{C} + k) = \mathcal{R} = \bigcup_{n \in \mathbb{Z}} \bigcup_{k \in \mathbb{Z}^d} A^{-n}(\mathbf{C} + k) \tag{4.3.24}$$

where $\mathbf{C}$ is the (unique) compact set determined by (4.3.23), of Hutchinson's theorem [**Hut81**]. One shows that for every $k \in \mathbb{Z}^d$ and every $n \in \mathbb{Z}$,

$$\mathcal{R} + A^{-n}k = \mathcal{R}, \quad \text{and} \quad A^n\mathcal{R} = \mathcal{R}.$$

Set $H := L^2(\mathcal{R}, \mathcal{H}^h)$, and we have unitary operators T and U:

$$(T_k f)(x) := f(x - k), \quad f \in H,\, x \in \mathcal{R},\, k \in \mathbb{Z}^d, \tag{4.3.25}$$

and

$$(Uf)(x) = \frac{1}{\sqrt{\#(\mathcal{S})}} f(A^{-1}x), \quad f \in H,\, x \in \mathcal{R}, \tag{4.3.26}$$

satisfying

$$UT_kU^{-1} = T_{Ak}, \quad k \in \mathbb{Z}^d. \tag{4.3.27}$$

We now need the familiar duality between the two groups $\mathbb{Z}^d$, and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, which identifies points $n \in \mathbb{Z}^d$ with monomials on $\mathbb{T}^d$ as follows:

$$z^n = z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d} = e^{i2\pi n_1\theta_1} e^{i2\pi n_2\theta_2} \cdots e^{i2\pi n_d\theta_d}. \tag{4.3.28}$$

Note that (4.3.28) identifies the torus $\mathbb{T}^d$ with the d-cube

$$\{(\theta_1, \ldots, \theta_d) \mid 0 \le \theta_i < 1,\ i = 1, \ldots, d\}.$$

Since $\mathbf{C}$ is naturally identified with a subset of $\mathbb{T}^d$, we may view the monomials $\{z^n \mid n \in \mathbb{Z}^d\}$ as functions on $\mathbf{C}$ by restriction. We say that the system $(A, \mathcal{S})$ is of *orthogonal type* if there is a subset $\mathcal{T}$ of $\mathbb{Z}^d$ such that the set of functions $\{z^n \mid n \in \mathcal{T}\}$ is an orthonormal basis (ONB) in the Hilbert space $L^2(\mathbf{C}, \mu_{(A,\mathcal{S})})$. If there is no subset $\mathcal{T}$ with this ONB-property we say that $(A, \mathcal{S})$ is of *non-orthogonal type*. The authors of [**JP98a**] showed that $(4, \{0, 2\})$ is of orthogonal type, while $(3, \{0, 2\})$ is not. Hence, for the Cantor set $\mathbf{C}_4$ there is an ONB $\{z^n \mid n \in \mathcal{T}\}$ for a subset $\mathcal{T}$ of $\mathbb{Z}$; in fact we may take

$$\mathcal{T} = \{0, 1, 4, 5, 16, 17, 20, 21, 24, 25, \cdots\} = \left\{ \sum_{0}^{\text{finite}} n_i 4^i \mid n_i \in \{0, 1\} \right\}. \tag{4.3.29}$$

For the middle-third Cantor set $\mathbf{C}_3$ it can be checked that $\{z^n \mid n \in \mathbb{Z}\}$ contains no more than two elements which are orthogonal in $L^2(\mathbf{C}_3, \mu_3)$.

THEOREM 4.3.3. *Let $(A, \mathcal{S})$ be an affine IFS in $\mathbb{R}^d$, and suppose $\mathcal{S}$ has an extension to a set of A-residues in $\mathbb{Z}^d$. Let*

$$h = \frac{\log \#(\mathcal{S})}{\log |\det A|},$$

and let $(\mathbf{C}, \mu)$ be as above; i.e., depending on $(A, \mathcal{S})$, and let $\mathcal{R}$ be defined from $\mathbf{C}$ in the usual way as in (4.3.24). *Assume further that*

$$\mathbf{C} \cap (\mathbf{C} + k) = \emptyset, \quad k \in \mathbb{Z}^d \setminus \{0\}. \tag{4.3.30}$$

Then the system $(A, \mathcal{S})$ is of orthogonal type if and only if there is a subset $\mathcal{T}$ in $\mathbb{Z}^d$ such that

$$\Big\{ (\#(\mathcal{S}))^{n/2}\, e^{i2\pi A^n k\cdot x} \chi_{\mathbf{C}}(A^n x - \ell) \mid k \in \mathcal{T},\ (n = 0 \text{ and } \ell \in \mathbb{Z}^d) \text{ or}$$

$$(n \ge 1 \text{ and } \ell \not\equiv s \mod A \text{ for all } s \in \mathcal{S})\Big\} \tag{4.3.31}$$

is an orthonormal basis in the Hilbert space $L^2(\mathcal{R}, \mathcal{H}^h)$.

PROOF. A simple check shows that

$$\mathcal{R} = \bigcup \big\{ A^{-n}(\mathbf{C} + l) \mid (n = 0 \text{ and } \ell \in \mathbb{Z}^d)$$
$$\text{or } (n \ge 1 \text{ and } \ell \not\equiv s \mod A \text{ for all } s \in \mathcal{S}) \big\},$$

and the union is disjoint. Suppose $(A, \mathcal{S})$ is of orthogonal type. Then the restriction of the Hausdorff measure $\mathcal{H}^h$ to $\mathbf{C}$ agrees with the Hutchinson measure $\mu = \mu_{(A,\mathcal{S})}$ on $\mathbf{C} = \mathbf{C}_{(A,\mathcal{S})}$. Hence density of $\{z^n \mid n \in \mathcal{T}\}$ in $L^2(\mathbf{C}, \mu)$ implies density of $\{e^{i2\pi k\cdot x} \chi_{\mathbf{C}}(x) \mid k \in \mathcal{T}\}$ in the subspace $L^2(\mathbf{C}, \mathcal{H}^h)$ of $L^2(\mathcal{R}, \mathcal{H}^h)$. Now the formula for $\mathcal{R}$ implies that the functions in (4.3.31) are dense in $L^2(\mathcal{R}, \mathcal{H}^h)$.

Suppose conversely that the family (4.3.31) is dense in $L^2(\mathcal{R}, \mathcal{H}^h)$. Then $\{z^n \mid n \in \mathcal{T}\}$ must be dense in $L^2(\mathbf{C}, \mu)$ since $\mathbf{C}$ is the support of Hutchinson's measure μ, and since μ restricts $\mathcal{H}^h$. $\square$

COROLLARY 4.3.4. *Let* $(\mathbf{C}_4, \mu_4)$ *be the Cantor construction in the unit interval* $I \cong \mathbb{T}^1$ *defined by the IFS* $\sigma_0(x) = \frac{x}{4}$, $\sigma_2(x) = \frac{x+2}{4}$; *i.e., by* $(A, \mathcal{S}) = (4, \{0, 2\})$, *and let* $\mathcal{R}$ *be the subset of* $\mathbb{R}$ *defined in* (4.3.24). *Then the family of functions*

$$\Big\{2^{n/2} e^{i2\pi 4^n kx} \chi_{\mathbf{C}}(4^n x - \ell) \mid k \in \{0, 1, 4, 5, 16, 17, \cdots\},$$

$$\ell \in \left\{ \begin{array}{ll} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z} \setminus (4\mathbb{Z} + \{0, 2\}) & \text{if} \quad n \geq 1. \end{array} \right\} \tag{4.3.32}$$

forms an orthonormal basis in the Hilbert space $L^2(\mathcal{R}, \mathcal{H}^{\frac{1}{2}})$.

THEOREM 4.3.5. *This is a direct application of the theorem as the subset*

$$\mathcal{T} = \{0, 1, 4, 5, 16, 17, \cdots\}$$

from (4.3.29) *and* (4.3.32) *satisfies the basis property for* $\mathbf{C}_4, \mu_4$ *by Theorem 3.4 in* [**JP98a**].

The next result makes clear the notion of gap-filling wavelets in the context of iterated function systems (IFS). While it is stated just for a particular example, the idea carries over to general IFSs. Note that in the system (4.3.33) below of wavelet functions, the two ψ_2 and ψ_3 are gap-filling.

COROLLARY 4.3.6. *Let* $\mathbf{C} = \mathbf{C}_4$ *be the Cantor set determined from the IFS,* $\sigma_0(x) = \frac{x}{4}$, $\sigma_2(x) = \frac{x+2}{4}$, *from the previous corollary. Then the three functions*

$$\begin{aligned} \psi_1(x) &:= \chi_{\mathbf{C}}(4x) - \chi_{\mathbf{C}}(4x - 2) \\ \psi_2(x) &:= \sqrt{2}\chi_{\mathbf{C}}(4x - 1) \\ \psi_3(x) &:= \sqrt{2}\chi_{\mathbf{C}}(4x - 3) \end{aligned} \tag{4.3.33}$$

generate an orthonormal wavelet basis in the Hilbert space $L^2(\mathcal{R}, \mathcal{H}^{\frac{1}{2}})$. *Specifically, the family*

$$\Big\{2^{\frac{k}{2}} \psi_i(4^k x - \ell) \mid i = 1, 2, 3,\ k \in \mathbb{Z},\ \ell \in \mathbb{Z}\Big\} \tag{4.3.34}$$

is an orthonormal basis in $L^2(\mathcal{R}, \mathcal{H}^{\frac{1}{2}})$.

PROOF. The result amounts to checking the general orthogonality relations for the functions m_0, m_1, m_2, m_3 on $\mathbb{T}$ which define wavelet filters for the system in (4.3.34). Note that from (4.3.34) the subband filters $\{m_i\}_{i=0}^3$ are as follows, $z \in \mathbb{T}$:

$$\begin{aligned} m_0(z) &= \frac{1}{\sqrt{2}}(1 + z^2) \\ m_1(z) &= \frac{1}{\sqrt{2}}(1 - z^2) \\ m_2(z) &= z \\ m_3(z) &= z^3. \end{aligned}$$

Since the 4×4 matrix in the system

$$\begin{pmatrix} m_0(z) \\ m_1(z) \\ m_2(z) \\ m_3(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \end{pmatrix}$$

is clearly unitary, the result follows from a direct computation. To verify that the Ruelle operator $R = R_{m_0}$ given by

$$\begin{aligned}(Rf)(z) &= \frac{1}{4}\sum_{w^4=z} |m_0(w)|^2 f(w) \\ &= \frac{1}{4}\sum_{w^4=z}\left(1+\frac{w^2+w^{-2}}{2}\right) f(w)\end{aligned}$$

satisfies the two conditions

(a) $\dim\{f \in C(\mathbb{T}) \mid Rf = f\} = 1$, and

(b) for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $\lambda \neq 1$, $\dim\{f \in C(\mathbb{T}) \mid Rf = \lambda f\} = 0$, we may apply the theorem from [**Nus98**]. □

For the more general affine IFSs the results above extend as follows.

Consider the affine IFS $(\sigma_i)_{i=1}^p$ with

$$\sigma_i(x) = \frac{1}{N}(x + a_i), \quad x \in \mathbb{R},$$

where $N \geq 2$ is an integer and $(a_i)_{i=1}^p$ are distinct integers in $\{0, \cdots, N-1\}$. Then there is a unique compact subset K of $\mathbb{R}$ which is the attractor of the IFS, i.e.,

$$\mathbf{C} = \cup_{i=1}^p \sigma_i(\mathbf{C}).$$

Actually, one can give a more explicit description of this attractor, namely

$$\mathbf{C} = \left\{\sum\nolimits_{j\geq 1} d_j N^{-j} \mid d_j \in \{a_1, ..., a_p\},\ j \geq 1\right\}.$$

Since the digits a_i are distinct and less then N, K is contained in $[0, 1]$, and the sets $\sigma_i(K)$ are almost disjoint (they have at most one point in common, those of the form k/N for some $k \in \{1, ..., N-1\}$.

The Hausdorff dimension of K is $\log_N p$.

Now consider the set

$$\mathcal{R} = \Big\{\sum\nolimits_{j\geq -m} d_j N^{-j} \mid m \in \mathbb{Z},\ d_j \in \{a_1, ..., a_p\} \text{ for all but finitely many indices } j\Big\}.$$

$\mathcal{R}$ is invariant under integer translations

$$\mathcal{R} + k = \mathcal{R}, \quad k \in \mathbb{Z},$$

and it is invariant under dilation by N,

$$N\mathcal{R} = \mathcal{R}.$$

Endow $\mathcal{R}$ with the Hausdorff measure $\mathcal{H}^s$ for $s = \log_N p$, and on $L^2(\mathcal{R}, \mathcal{H}^s)$, define the translation operator

$$Tf(x) = f(x-1), \quad x \in \mathcal{R},\ f \in L^2(\mathcal{R}, \mathcal{H}^s),$$

and the dilation operator

$$Uf(x) = \sqrt{\frac{1}{p}} f\left(\frac{x}{N}\right), \quad x \in \mathcal{R},\ f \in L^2(\mathcal{R}, \mathcal{H}^s).$$

These are unitary operators satisfying the commutation relation

$$UTU^{-1} = T^N.$$

Let $\varphi := \chi_{\mathbf{C}}$. The function φ is an orthogonal scaling function for $L^2(\mathcal{R}, \mathcal{H}^s)$, with filter

$$m_0(z) = \sqrt{\frac{1}{p}} \sum_{i=1}^{p} z^{a_i}, \tag{4.3.35}$$

so it satisfies the following conditions:

(1) [**Orthogonality**]

$$\left\langle T^k \varphi, \varphi \right\rangle = \delta_k, \quad k \in \mathbb{Z}.$$

(2) [**Scaling equation**]

$$U\varphi = \sum_{i=1}^{p} \sqrt{\frac{1}{p}} T^{a_i} \varphi = m_0(T).$$

(3) [**Cyclicity**]

$$\overline{span} \left\{ U^{-n} T^k \varphi \mid n, k \in \mathbb{Z} \right\} = L^2(\mathcal{R}, \mathcal{H}^s).$$

Next, we define the wavelets. For this, we need the "high-pass" filters $m_1, \cdots, m_{N-1}$ such that the matrix

$$\frac{1}{\sqrt{N}} \left(m_i(\rho^j z) \right)_{i,j=0}^{N-1}$$

is unitary for almost every z. ($\rho = e^{2\pi i/N}$).

First, we define the filters for the gap-filling wavelets $\psi_1, \cdots, \psi_{N-p}$. The set $G = \{0, \cdots, N-1\} \setminus \{a_1, \cdots, a_p\}$ has $N - p$ elements. We label the functions $z \mapsto z^d$ for $d \in G$, by $m_1, \cdots, m_{N-p}$.

The remaining $p - 1$ filters are for the detail-filling wavelets. Let $\eta = e^{2\pi i/p}$. Define

$$m_{N-p+k}(z) = \sqrt{\frac{1}{p}} \sum_{i=1}^{p} \eta^{k(i-1)} z^{a_i}, \quad k \in \{1, ..., p-1\}.$$

We have to check that

$$\frac{1}{N} \sum_{w^N = z} m_i(w) \overline{m_j(w)} = \delta_{ij}, \quad z \in \mathbb{T},\ i, j \in \{0, ..., N-1\}. \tag{4.3.36}$$

For this we use the following identity:

$$\sum_{w^N = z} w^k = 0, \quad z \in \mathbb{T},\ k \not\equiv 0 \bmod N.$$

Therefore, if $f_1(z) = \sum_{i=0}^{N-1} \alpha_i z^i$, $f_2 = \sum_{i=0}^{N-1} \beta_i z^i$, then

$$\frac{1}{N} \sum_{w^N = z} f_1(w) \overline{f_2(w)} = \frac{1}{N} \sum_{i,j=0}^{N-1} \alpha_i \overline{\beta}_j \sum_{w^N = z} w^{i-j} = \sum_{i=0}^{N-1} \alpha_i \overline{\beta}_j.$$

Applying these to the filters $m_i, i \in \{0, ..., N-1\}$, we obtain (4.3.36).

With these filters, we construct the wavelets in the usual way:

$$\psi_i = U^{-1} m_i(T) \varphi, \quad i \in \{1, ..., N-1\},$$

and

$$\{U^m T^n \psi_i \mid m, n \in \mathbb{Z}, i \in \{1, ..., N-1\}\}$$

is an orthonormal basis for $L^2(\mathcal{R}, \mathcal{H}^s)$.

Let $N \in \mathbb{Z}_+$ be as above, and consider $\mathcal{S} = \{a_1, \cdots, a_p\} \subset \{0, 1, 2, \cdots, N-1\}$. A second subset $\mathcal{B} = \{b_1, \cdots, b_p\} \subset \mathbb{Z}$ is an N-dual if the $p \times p$ matrix

$$M_N(\mathcal{S}, \mathcal{B}) = \frac{1}{\sqrt{p}} \left(\exp\left(i \frac{2\pi a_j b_k}{N} \right) \right)_{1 \le j,k \le p} \tag{4.3.37}$$

is unitary. When N and $\mathcal{S}$ are given as specified, it is not always true that there is a subset $\mathcal{B} \subset \mathbb{Z}$ for which $M_N(\mathcal{S}, \mathcal{B})$ is unitary. If for example $N = 3$ and $\mathcal{S} = \{0, 2\}$, then no $\mathcal{B}$ exists, while for $N = 4$ and $\mathcal{S} = \{0, 2\}$, we may take $\mathcal{B} = \{0, 1\}$, and

$$M_4(\mathcal{S}, \mathcal{B}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is unitary.

Lemma 4.3.7 ([**JP98a**]). *Let N and $\mathcal{S}$ be as specified above, and suppose*

$$\mathcal{B} = \{b_1, \cdots, b_p\} \subset \mathbb{Z}$$

is an N-dual subset. Suppose $0 \in \mathcal{B}$, and set

$$\Lambda = \Lambda_N(\mathcal{B}) := \left\{ \sum_{i=0}^{finite} n_i N^i \mid n_i \in \mathcal{B} \right\}. \tag{4.3.38}$$

Let $(\mathbf{C}, \mu) = (\mathbf{C}_{(N,\mathcal{S})}, \mu_{(N,\mathcal{S})})$ be the Hutchinson pair. Then the set of functions $\{z^n \mid n \in \Lambda\}$ is orthogonal in $L^2(\mathbf{C}, \mu)$; i.e.,

$$\int_{\mathbf{C}} z^{n-n'} d\mu(z) = \delta_{n,n'}, \quad n, n' \in \Lambda \tag{4.3.39}$$

where we identify $\mathbf{C}$ as a subset of $\mathbb{T}^1$ via

$$\mathbf{C} \ni \theta \longrightarrow e^{i2\pi\theta} \in \mathbb{T}^1.$$

Proof. Set $e(\theta) = e^{i2\pi\theta}$, and for $k \in \mathbb{R}$

$$B(k) := \int_{\mathbf{C}} e(k\theta) d\mu(\theta). \tag{4.3.40}$$

Using (4.3.19), we get

$$B(k) = \frac{1}{\sqrt{p}} m_0 \left(\frac{k}{N} \right) B \left(\frac{k}{N} \right), \tag{4.3.41}$$

where m_0 is defined in (4.3.35).

If $n, n' \in \Lambda$, and $n \ne n'$, we get the representation

$$n' - n = b' - b + mN^\ell, \quad b, b' \in \mathcal{B}, \ m, \ell \in \mathbb{Z}, \ \ell \ge 1.$$

As a result, the inner product in $L^2(\mathbf{C}, \mu)$ is

$$\left\langle z^n, z^{n'} \right\rangle_\mu = B(n' - n) = \frac{1}{\sqrt{p}} m_0 \left(\frac{b' - b}{N} \right) B \left(\frac{n' - n}{N} \right). \tag{4.3.42}$$

Since the matrix $M_N(\mathcal{S}, \mathcal{B})$ is unitary,

$$m_0 \left(\frac{b' - b}{N} \right) = 0$$

when $b' \ne b$ in $\mathcal{B}$, and the result follows. ∎

Even if the matrix $M_N(\mathcal{S},\mathcal{B})$ is unitary, the orthogonal functions $\{z^n \mid n \in \Lambda\}$ might not form a basis for $L^2(\mathbf{C},\mu)$. From [**JP98a**], we know that it is an orthonormal basis (ONB) if and only if

$$\sum_{n\in\Lambda} |B(\xi-n)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{4.3.43}$$

Introducing the function

$$\Omega(\xi) := \frac{1}{p}\sum_{b\in\mathcal{B}} |B(\xi-n)|^2\,, \tag{4.3.44}$$

and the dual Ruelle operator

$$(R_{\mathcal{B}}f)(\xi) := \frac{1}{p}\sum_{b\in\mathcal{B}} \left|m_0\left(\frac{\xi-b}{N}\right)\right|^2 f\left(\frac{\xi-b}{N}\right), \tag{4.3.45}$$

we easily verify that Ω and the constant function $\mathbb{1}$ both solve the eigenvalue problem $R_{\mathcal{B}}(f)=f$, both functions Ω and $\mathbb{1}$ are continuous on $\mathbb{R}$, even analytic.

THEOREM 4.3.8. *If the space*

$$\{f \in Lip(\mathbb{R}) \mid f \geq 0,\ f(0)=1, \quad R_{\mathcal{B}}(f)=f\} \tag{4.3.46}$$

is one-dimensional, then $\Lambda\,(=\Lambda_N(\mathcal{B}))$ *induces an ONB; i.e.,* $\{z^n \mid n \in \Lambda\}$ *is an ONB in* $L^2(\mathbf{C},\mu)$.

PROOF. The result follows from the discussion and the added observation that $\Omega(0)=1$. This normalization holds since $0\in\mathcal{B}$ was assumed, and so $\langle e_0, e_n\rangle_\mu = 0$ for all $n \in \Lambda\backslash\{0\}$. □

DEFINITION 4.3.9. A $\mathcal{B}$-cycle is a finite set $\{z_1, z_2, \ldots, z_{k+1}\} \subset \mathbb{T}$, with a pairing of points in $\mathcal{B}$, say $b_1, b_2, \ldots, b_{k+1} \in \mathcal{B}$, such that

$$z_i = \sigma_{-b_i}(z_{i+1}), \qquad z_{k+1} = z_1, \tag{4.3.47}$$

and $|m_0(z_i)|^2 = p$. Equivalently, a $\mathcal{B}$-cycle may be given by $\{\xi_1,\ldots,\xi_{k+1}\}\subset\mathbb{R}$ satisfying

$$\begin{gathered}\xi_{i+1} \equiv b_i + N\xi_i \mod N\mathbb{Z}\\ \left(N^k-1\right)\xi_1 \equiv b_k + Nb_{k-1} + \cdots + N^{k-1}b_1 \mod N^k\mathbb{Z}.\end{gathered}$$

THEOREM 4.3.10. *Let* $N\in\mathbb{Z}_+$, $N\geq 2$ *be given. Let* $\mathcal{S}\subset\{0,1,\cdots,N-1\}$, *and suppose there is a* $\mathcal{B}\subset\mathbb{Z}$ *such that* $0\in\mathcal{B}$, $\#(\mathcal{S})=\#(\mathcal{B})=p$, *and the matrix*

$$M_N(\mathcal{S},\mathcal{B}) = \frac{1}{\sqrt{p}}\left(\exp\left(i\frac{2\pi ab}{N}\right)\right)$$

is unitary. Then $\{z^n \mid n\in\Lambda_N(\mathcal{B})\}$ *is an ONB for* $L^2(\mathbf{C},\mu)$ *where* $\Lambda_N(\mathcal{B})$ *is defined in* (4.3.38) *if the only* $\mathcal{B}$*-cycles are the singleton* $\{1\}\subset\mathbb{T}$.

PROOF. By Theorem 4.3.8, we need only verify that the absence of $\mathcal{B}$-cycles of order ≥ 2 implies that the Perron-Frobenius eigenspace (4.3.45) is one-dimensional. But this follows from [**BJ02**, Theorem 5.5.4]. In fact, the argument from Chapter 5 in [**BJ02**] shows that the absence of $\mathcal{B}$-cycles of order ≥ 2 implies that the $\mathcal{B}$-Ruelle operator $R_{\mathcal{B}}$ with $\sigma_{-b}(\xi) := \frac{\xi-b}{N}$,

$$(R_{\mathcal{B}}f)(\xi) = \frac{1}{p}\sum_{b\in\mathcal{B}}\left(|m_0(\sigma_{-b}(\xi))|^2 f(\sigma_{-b}(\xi))\right)$$

satisfies the two Perron-Frobenius properties:

(i) the only bounded continuous solutions f to $R_{\mathcal{B}}(f) = f$ are the multiples of $\mathbb{1}$, and

(ii) for all $\lambda \in \mathbb{T} \setminus \{1\}$, the eigenvalue problem $R_{\mathcal{B}}(f) = \lambda f$ has no non-zero bounded continuous solutions.

□

EXAMPLE 4.3.11 (An application). Let $N = 4$, $\mathcal{S} = \{0, 2\}$, and $\mathcal{B} = \{0, 1\}$. Then

$$M_4(\mathcal{S}, \mathcal{B}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\Lambda_4(\mathcal{B}) = \{0, 1, 4, 5, 16, 17, 20, 21, \cdots\}, \text{ and}$$

$$(R_{\mathcal{B}} f)(\xi) = \cos^2(2\pi\xi)\, f\left(\frac{\xi}{4}\right) + \sin^2(2\pi\xi)\, f\left(\frac{\xi - 1}{4}\right)$$

and there is only on $\mathcal{B}$-cycle, the singleton $\{1\} \subset \mathbb{T}$. Recall from [**Hut81**] that the Hutchinson construction of $(\mathbf{C}, \mu)$ identifies $\mathbf{C}$ as the Cantor set arising by the subdividing algorithm starting with the unit interval I dividing into four equal subintervals and dropping the second and the fourth at each step in the algorithm. The measure μ is the restriction of $\mathcal{H}^{\frac{1}{2}}$ to $\mathbf{C}$, and it follows from the last theorem that $\{z^n \mid n \in \Lambda_4(\mathcal{B})\}$ is an ONB for $L^2(\mathbf{C}, \mu)$. The dual system $\{\sigma_{-b} \mid b \in \mathcal{B}\}$; i.e., $\sigma_0(\xi) = \frac{\xi}{4}$, $\sigma_{-1}(\xi) = \frac{\xi - 1}{4}$, generates a Cantor subset $\mathbf{C}_{\mathcal{B}} \subset [-1, 0]$ also of Hausdorff dimension $\frac{1}{2}$. Note that the fractional version of the Ruelle operator $R_{\mathcal{B}}$ does not map 1-periodic functions into themselves; in general

$$(R_{\mathcal{B}} f)(\xi) \neq (R_{\mathcal{B}} f)(\xi + 1);$$

in fact

$$(R_{\mathcal{B}} f)(\xi + 1) = \cos^2(2\pi\xi) f\left(\frac{\xi + 1}{4}\right) + \sin^2(2\pi\xi) f\left(\frac{\xi}{4}\right);$$

so

$$R_{\mathcal{B}} f(\xi) = R_{\mathcal{B}} f(\xi + 1)$$

holds only if

$$\cos^4(2\pi\xi) f\left(\frac{\xi + 1}{4}\right) = \sin^4(2\pi\xi) f\left(\frac{\xi - 1}{4}\right).$$

4.3.2. Wavelet systems via solenoid Hilbert spaces. The material below is adapted primarily from [**AJL18**] and [**JT16a**].

Why the solenoids? A number of reasons. Given an endomorphism σ in a measure space, the associated solenoid Sol_σ is then a useful tool for the study of scales of multiresolutions. The latter includes those resolutions arising naturally from discrete wavelet algorithms, as well as from the study of non-reversible dynamics in ergodic theory in and physics. In fact it is not so much Sol_σ itself that is central in this program, but rather probability spaces $(Sol_\sigma, \mathscr{F}, \mathbb{P})$ where the solenoid is the sample space. It is the pair $(\mathscr{F}, \mathbb{P})$ which carries the information about the relevant scales of multiresolutions for the problem at hand, and the nature and the details of $(\mathscr{F}, \mathbb{P})$ change from one algorithm to the next; much like traditional wavelet analysis depend on scaling functions, father function, mother functions etc in $L^2(\mathbb{R}^d)$. But the latter is too restrictive a framework; see e.g. [**Bag00, BJ02**].

By "discrete wavelet algorithms" we mean recursive algorithms with selfsimilarity given by a scaling matrix. In one dimension, this may be just the N-adic scaling, but in general we allow for discrete time to be modeled by higher rank lattices, by more general discrete abelian groups, or even by infinite discrete sets with some given structure. For a given time-series, even in this general form, we may always introduce an associated generating function. This will be a function in "dual frequency variables" in one or more complex variables, and called the frequency response function (see e.g., [**BJ02**]). In many classical wavelet settings the given discrete wavelet algorithms may be realized in $L^2(\mathbb{R}^d)$ for some d, but such a realization places very strong restrictions and limitations on the given multi-band filters making up the discrete wavelet algorithm at hand. We show that with the Hilbert space $L^2(Sol_\sigma, \mathscr{F}, \mathbb{P})$, we can get around this difficulty, and still retain the useful features of multi-scale resolutions and selfsimilarity which makes the wavelet realizations so useful.

In our discussion of solenoids and multiresolutions, we have here restricted the discussion to the commutative case, as our motivation is from stochastic processes. But in the recent literature, there is also an exciting, and somewhat parallel non-commutative theory of solenoids and their multiresolutions. It too is motivated (at least in part) by developments in the analysis of wavelet-multiresolutions, and the corresponding scaling operators. However, the relevant questions in the non-commutative theory are quite different from those addressed here. The relevant questions are simply different in the non-commutative theory. The differences between the two in fact reflect the dichotomy for two different notions of probability theory, the difference between (classical) commutative, versus non-commutative probability theory. Among the recent results on the non-commutative theory, we mention [**BFMP09**, **BFMP10**, **BMPR12**, **LP13**, **LP15**], and the literature cited there.

We shall use the notion of "transfer operator" in a wide sense so that our framework will encompass diverse settings from mathematics and its applications, including statistical mechanics where the relevant operators are often referred to as Ruelle-operators. See, e.g,. [**Sto13**, **Rug16**, **MU15**, **JR05**, **Rue04**]. But we shall also consider families of transfer operators arising in harmonic analysis, including spectral analysis of wavelets, in ergodic theory of endomorphisms in measure spaces, in Markov random walk models, in the study of transition processes in general; and more.

In the setting of endomorphisms and solenoids, we obtain new multiresolution orthogonality relations in the Hilbert space of square integrable random variables. We shall further draw parallels between our present infinite-dimensional theory and the classical finite-dimensional Perron-Frobenius theorems (see, e.g., [**JR05**, **Rue04**, **GH16**, **MU15**, **Pap15**, **FT15**]); the latter referring to the case of finite positive matrices.

To make this parallel, it is helpful to restrict the comparison of the infinite-dimensional theory to the case of the Perron-Frobenius (P-F) for finite matrices in the special case when the spectral radius is 1.

The general setting is as follows:

Definition 4.3.12. Let X be a non-empty set.

(1) $(X, \mathscr{B})$ is a fixed *measure space*, i.e., $\mathscr{B}$ is a fixed sigma-algebra of subsets of X. Usually, we assume, in addition, that $(X, \mathscr{B})$ is a Borel space.

(2) Notation: $\sigma : X \to X$ is a measurable *endomorphism*, i.e., $\sigma^{-1}(\mathscr{B}) \subset \mathscr{B}$, $\sigma^{-1}(A) \in \mathscr{B}$ for all $A \in \mathscr{B}$; and we assume further that $\sigma(X) = X$, i.e., σ is onto.
(3) $\mathscr{F}(X, \mathscr{B})$ = the algebra of all measurable functions on $(X, \mathscr{B})$.
(4) By a *transfer operator* R, we mean that $R : \mathscr{F}(X, \mathscr{B}) \longrightarrow \mathscr{F}(X, \mathscr{B})$ is a linear operator s.t. (4.3.48) & (4.3.49) hold, where:

$$f \geq 0 \Longrightarrow R(f) \geq 0; \text{ and} \tag{4.3.48}$$

$$R((f \circ \sigma) g) = f R(g), \ \forall f, g \in \mathscr{F}(X, \mathscr{B}). \tag{4.3.49}$$

(See, e.g., [**Sto13**, **Rug16**, **MU15**, **JR05**, **Rue04**].)
(5) We assume that

$$R\mathbb{1} = \mathbb{1} \tag{4.3.50}$$

where $\mathbb{1}$ denotes the constant function "one" on X, and we restrict consideration to the case of *real* valued functions. Subsequently, condition (4.3.50) will be relaxed.
(6) If λ is a measure on $(X, \mathscr{B})$, we set λR to be the measure specified by

$$\int_X f \, d(\lambda R) := \int_X R(f) \, d\lambda, \ \forall f \in \mathscr{F}(X, \mathscr{B}). \tag{4.3.51}$$

(7) We shall assume separability, for example we assume that $(X, \mathscr{B}, \lambda)$, as per (1)–(6), has the property that $L^2(X, \mathscr{B}, \lambda)$ is a separable Hilbert space.

The role of the endomorphism $X \xrightarrow{\sigma} X$ is fourfold:

(a) σ is a point-transformation, generally not invertible, but assumed *onto*.
(b) We also consider σ as an endomorphism in the fixed measure space $(X, \mathscr{B})$ and so $\sigma^{-1} : \mathscr{B} \to \mathscr{B}$ where

$$\sigma^{-1}(\mathscr{B}) = \left\{\sigma^{-1}(A) \mid A \in \mathscr{B}\right\}, \text{ and}$$
$$\sigma^{-1}(A) := \{x \in X \mid \sigma(x) \in A\},$$

so $\sigma^{-1}(\mathscr{B}) \subset \mathscr{B}$.
(c) We shall assume further that σ is ergodic [**Yos80**, **KP16**], i.e., that

$$\bigcap_{n=1}^{\infty} \sigma^{-n}(\mathscr{B}) = \{\emptyset, X\}$$

modulo sets of λ-measure zero.
(d) σ defines an endomorphism in the space $\mathscr{F}(X, \mathscr{B})$ of all measurable functions via $f \mapsto f \circ \sigma$.

Sets of measures for $(X, \mathscr{B}, \sigma, R)$

We shall undertake our analysis of particular transfer operators/endomorphisms in a fixed measure space $(X, \mathscr{B})$ with the use of certain sets of measures on $(X, \mathscr{B})$. These sets play a role in our theorems, and they are introduced below. We present examples of transfer operators associated to iterated function systems (IFSs) in a stochastic framework.

For positive measures λ and μ on $(X, \mathscr{B})$, we shall work with absolute continuity, written $\lambda \ll \mu$.

DEFINITION 4.3.13. $\lambda \ll \mu$ iff (Def.) $[A \in \mathscr{B},\ \mu(A) = 0 \Longrightarrow \lambda(A) = 0]$. Moreover, when $\lambda \ll \mu$, we denote the Radon-Nikodym derivative $\frac{d\lambda}{d\mu}$. In detail,

$$\int_B \left(\frac{d\lambda}{d\mu}\right) d\mu = \lambda(B),\ B \in \mathscr{B}.$$

Note that $\frac{d\lambda}{d\mu} \in L^1(\mu)$.

DEFINITION 4.3.14. Let σ be an endomorphism in the measure space $(X, \mathscr{B})$, assuming σ is onto. Introduce the corresponding *solenoid*

$$Sol_\sigma(X) := \left\{ (x_n)_0^\infty \in \prod_0^\infty X \mid \sigma \circ \pi_{n+1} = \pi_n \right\}; \tag{4.3.52}$$

where $\pi_n((x_k)) := x_n$, and we set

$$\tilde{\sigma}(x_0, x_1, x_2 \cdots) := (\sigma(x_0), x_0, x_1, x_2, \cdots),\ \forall x = (x_i)_0^\infty \in Sol_\sigma(X). \tag{4.3.53}$$

EXAMPLE 4.3.15. The following considerations cover an important class of transfer operators which arise naturally in the study of controlled Markov-processes, and in analysis of iterated function system (IFS), see, e.g., [**GS79**, **LW15**, **DLN13**] and [**DF99**].

Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be two measure spaces. We equip $Z := X \times Y$ with the product sigma-algebra induced from $\mathscr{B}_X \times \mathscr{B}_Y$, and we consider a fixed measurable function $G : Z \to X$. For $\nu \in M(Y, \mathscr{B}_Y)$ (= positive measures on Y), we set

$$(Rf)(x) = \int_Y f(G(x, y))\, d\nu(y), \tag{4.3.54}$$

defined for all $f \in \mathscr{F}(X, \mathscr{B}_X)$. This operator R from (4.3.54) is a transfer operator; it naturally depends on G and ν.

If $\nu \in M_1(Y, \mathscr{B}_Y)$ (= the probability measures), then $R\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ denotes the constant function "one" on X.

For every $x \in X$, $G(x, \cdot)$ is a measurable function from Y to X, which we shall denote G_x. It follows from (4.3.54) that the marginal measures $\mu(\cdot \mid x)$ from the representation

$$(Rf)(x) = \int_X f(t)\, \mu(dt \mid x) \tag{4.3.55}$$

may be expressed as

$$\mu(\cdot \mid x) = \nu \circ G_x^{-1}, \tag{4.3.56}$$

pull-back from ν via G_x.

Set $M_1(X, \mathscr{B}) :=$ all probability measures on $(X, \mathscr{B})$, and

$$\mathscr{L}_1(R) := \{\lambda \in M_1(X, \mathscr{B}) \mid \lambda R = \lambda\}$$

where $\int_X f\, d(\lambda R) := \int_X R(f)\, d\lambda$, for all $f \in \mathscr{F}(X, \mathscr{B}_X)$.

The following lemma is now immediate.

LEMMA 4.3.16. *Let G, ν, and R be as above, with R given by (4.3.54), or equivalently by (4.3.55); then a fixed measure λ on $(X, \mathscr{B}_X)$ is in $\mathscr{L}_1(R)$ if and only if*

$$\lambda(B) = \int_X \nu(\{y : G(x,y) \in B\})\, d\lambda(x) \tag{4.3.57}$$

for all $B \in \mathscr{B}_X$.

PROOF. Immediate from the definitions. □

The purpose of the next theorem is to make precise the direct connections between the following three notions, a given positive transfer operator, an induced probability space, and an associated Markov chain [**PU16**, **HHSW16**].

THEOREM 4.3.17. *Fix $h \geq 0$ on $(X, \mathscr{B}_X)$ s.t. $Rh = h$, and $\int_X h\, d\lambda = 1$.*

(1) *Then $\Omega_X := \prod_0^\infty X$ supports a probability space $(\Omega_X, \mathscr{F}, \mathbb{P})$ (Definition 4.3.19), such that $\mathbb{P}$ is determined by the following:*

$$\begin{aligned}&\int_{\Omega_X} (f_0 \circ \pi_0)(f_1 \circ \pi_1) \cdots (f_n \circ \pi_n)\, d\mathbb{P} \\ &= \int_X f_0(x)\, R(f_1 R(f_2 \cdots R(f_n h)) \cdots)(x)\, d\lambda(x),\end{aligned} \tag{4.3.58}$$

where π_n is the coordinate mapping, $\pi_n((x_i)) = x_n$.

More generally,

$$\begin{aligned}&Prob(\pi_0 = x, \pi_1 \in B_1, \pi \in B_2, \cdots, \pi_n \in B_n) \\ &= \int_{B_1}\int_{B_2} \cdots \int_{B_n} \mu(dy_1 \mid x)\, \mu(dy_2 \mid y_1) \cdots \mu(dy_n \mid y_{n-1})\, h(y_n) \\ &= R(\chi_{B_1} R(\chi_{B_2} \cdots R(\chi_{B_n} h)) \cdots)(x),\ \forall B_j \in \mathscr{B}_X.\end{aligned} \tag{4.3.59}$$

(2) *If $d(\lambda R) = W d\lambda$, then*

$$\mathbb{P} \circ \pi_1^{-1} = ((W \circ \pi_0)\, d\mathbb{P}) \circ \pi_0^{-1}. \tag{4.3.60}$$

(3) *Moreover,*

$$\begin{aligned}&suppt(\mathbb{P}) = Sol_\sigma(X) \\ &\Updownarrow \\ &R[(f \circ \sigma) g] = f R(g),\ \forall f, g \in \mathscr{F}(X, \mathscr{B}).\end{aligned} \tag{4.3.61}$$

PROOF. Follows from Kolmogorov's inductive limit construction. For details, see [**JT15b**, **DJ14b**] and also [**Hid80**, **Moh14**, **SSBR71**]. □

Multiresolutions in $L^2(\Omega, \mathscr{C}, \mathbb{P})$

Here we aim to realize multiresolutions in probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$; and we now proceed to outline the details.

We first need some preliminary facts and lemmas.

LEMMA 4.3.18. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $A : \Omega \to X$ be a random variable with values in a fixed measure space $(X, \mathscr{B}_X)$, then $V_A f := f \circ A$ defines an isometry $L^2(X, \mu_A) \to L^2(\Omega, \mathbb{P})$ where μ_A is the law (distribution) of A, i.e., $\mu_A(\Delta) := \mathbb{P}(A^{-1}(\Delta))$, for all $\Delta \in \mathscr{B}_X$; and $V_A^*(\psi)(x) = \mathbb{E}_{\{A=x\}}(\psi \mid \mathscr{F}_A)$, for all $\psi \in L^2(\Omega, \mathbb{P})$, and all $x \in X$.*

We shall apply Lemma 4.3.18 to the case when $(\Omega, \mathscr{F}, \mathbb{P})$ is realized on an infinite product space as follows:

DEFINITION 4.3.19. Let $(\Omega_X, \mathscr{F}, \mathbb{P})$ be a probability space, where $\Omega_X = \prod_{n=0}^{\infty} X$. Let $\pi_n : \Omega_X \to X$ be the random variables given by

$$\pi_n (x_0, x_1, x_2, \cdots) = x_n, \ \forall n \in \mathbb{N}_0. \tag{4.3.62}$$

The sigma-algebra generated by π_n will be denoted $\mathscr{F}_n$, and the isometry corresponding to π_n will be denoted V_n.

REMARK 4.3.20. Suppose the measure space $(X, \mathscr{B}_X)$ in Lemma 4.3.18 is specialized to $(\mathbb{R}, \mathscr{B})$; it is then natural to consider Gaussian probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$ where Ω is a suitable choice of sample space, and $A : \Omega \to X$ is replaced with Brownian motion $B_t : \Omega \to \mathbb{R}$, see e.g., [**Hid80**]. We instead consider samples

$$0 < t_1 < t_2 < \cdots < t_n,$$

and functions F on $\mathbb{R}^n$ with now $f \to f \circ A$ replaced with a suitable Malliavin derivative

$$DF_n (B_{\varphi_1}, \cdots, B_{\varphi_n}) = \sum_{i=1}^{n} \frac{\partial F_n}{\partial x_i} (B_{\varphi_1}, \cdots, B_{\varphi_n}) \varphi_i, \tag{4.3.63}$$

where $B_\varphi = \int \varphi (t) \, dB_t$.

We computed the adjoint of (4.3.63) in [**JT16c**] and identified it as a multiple Ito-integral. For more details, we refer the reader to the [**BNBS14**, **HRZ14**, **AH84**, **HPP00**, **CH13**], and also see [**Bog98**, **HKPS13**].

DEFINITION 4.3.21. Let R be a positive transfer operator, i.e., $f \geq 0 \Rightarrow Rf \geq 0$, $R\mathbb{1} = \mathbb{1}$, let λ be a probability measure on a fixed measure space $(X, \mathscr{B}_X)$. We further assume that

$$R ((f \circ \sigma) g) = f R (g), \ \forall f, g \in \mathscr{F} (X, \mathscr{B}_X). \tag{4.3.64}$$

Denote $\mu (\cdot \mid x)$, $x \in X$, the conditional measures determined by

$$Rf (x) = \int_X f (y) \, \mu (dy \mid x), \tag{4.3.65}$$

for all $f \in C (X)$, representing R as an integral operator. Set

$$\begin{aligned} \mu (B \mid x) :=& R (\chi_B) (x), \ \forall B \in \mathscr{B}_X \\ =& \mathbb{P} (\pi_1 \in B \mid \pi_0 = x). \end{aligned} \tag{4.3.66}$$

Note the RHS of (4.3.65) extends to all measurable functions on X, and we shall write R also for this extension.

LEMMA 4.3.22. *Let $\{\mu (\cdot \mid x)\}_{x \in X}$ be as in (*4.3.65*), and $W := \frac{d\lambda R}{d\lambda}$ = Radon-Nikodym derivative. If $B \in \mathscr{B}_X$ then*

$$\int_X \mu (B \mid x) \, d\lambda (x) = \int_B W (x) \, d\lambda (x).$$

PROOF. Let $B \in \mathscr{B}_X$, then

$$\begin{aligned} \text{LHS} &= \int_X R (\chi_B) (x) \, d\lambda (x) \\ &= \int_X \chi_B d (\lambda R) = \int_B W (x) \, d\lambda (x) = \text{RHS}. \end{aligned}$$

□

LEMMA 4.3.23. *Suppose R has a representation*

$$R(\chi_B)(x) = \mu(B \mid x), \ B \in \mathscr{B}_X, \ x \in X.$$

Then the following are equivalent:

(1) $R[(f \circ \sigma) g](x) = f(x) R(g)(x)$, $\forall x \in X$, $\forall f, g \in \mathscr{F}(X, \mathscr{B})$;
(2) $\mu\left(\sigma^{-1}(A) \cap B \mid x\right) = \chi_A(x) \mu(B \mid x)$, $\forall A, B \in \mathscr{B}$, $\forall x \in X$.

PROOF. Recall that, by assumption, $(Rf)(x) = \int_X f(x) \mu(dy \mid x)$. The conclusion follows by setting $f = \chi_A$, and $g = \chi_B$. □

PROPOSITION 4.3.24. *Let $\{\mu(\cdot \mid x)\}_{x \in X}$ be the Markov process indexed by $x \in X$ (see (4.3.65)), where $(X, \mathscr{B}_X)$ is a fixed measure space, and let $\mathbb{P}$ be the corresponding path space measure (see, e.g., [**CFS82**, **HKPS13**]) determined by (4.3.58)-(4.3.59). Let $\sigma \in End(X, \mathscr{B}_X)$ as in Def. 5.2.9. Then*

$$\begin{gathered} suppt(\mathbb{P}) \subset Sol_\sigma(X) \\ \Updownarrow \\ \mathbb{P}\left(\pi_{k+1} \in B \cap \sigma^{-1}(A) \mid \pi_k = x\right) = \chi_A(x) \mathbb{P}(\pi_{k+1} \in B \mid \pi_k = x). \end{gathered} \tag{4.3.67}$$

The next result will serve as a tool in our subsequent study of multiresolutions, orthogonality relations, and scale-similarity, each induced by a given endomorphism.

THEOREM 4.3.25. *Let $(X, \sigma, R, h, \lambda, W)$ be as above, $W = \frac{d\lambda R}{d\lambda}$; then*

(1) *there exists a unique path space measure $\mathbb{P}$ on $Sol_\sigma(X)$, such that*

$$L^2(X, \mu_n) \xrightarrow{V_n} L^2(Sol_\sigma, \mathbb{P}), \ V_n f = f \circ \pi_n \tag{4.3.68}$$

is isometric, where $\mu_n := dist(\pi_n)$, and $\int_X f \, d\mu_n = \int_X R^n(fh) \, d\lambda$;

(2) *$\mathbb{P}$ has the property:*

$$\frac{d\mathbb{P} \circ \tilde{\sigma}}{d\mathbb{P}} = W \circ \pi_0, \tag{4.3.69}$$

where $\tilde{\sigma}$ is as in (4.3.53).

PROOF. See [**JT15b**, **DJ14b**]. □

4.4. Harmonic analysis via reproducing kernel Hilbert spaces (RKHS)

Below we introduce a powerful tool, going by the name *reproducing kernel Hilbert spaces*, and abbreviated (RKHS). Our initial purpose below is to outline their use in the class of harmonic analysis questions considered in earlier sections of this book. But the RKHSs have many other applications, some of which will be discussed in later chapters of the present book. (Readers are also referred to the cited papers and books in the References.) The single author whose name is usually cited in connection with the general setting of reproducing kernel Hilbert spaces (RKHS) is Aronszajn [**Aro43**, **Aro50**], but there are other pioneers as well (details later.)

The idea of RKHSs seems to have been rediscovered, over decades, in diverse areas of mathematics; each area typically associated with an application. Such applications include the theory of analytic functions in one or several complex variables, complex geometry, PDE theory, probability theory, calculus of Gaussian processes and stochastic PDE; and Martin boundary theory for Markov processes; to list just a few.

The underlying idea of a RKHS is surprisingly simple, and yet very powerful. A particular reproducing kernel Hilbert spaces (RKHS) $\mathscr{H}$ refers to functions on some prescribed set, say S. In summary, a RKHS is then a Hilbert space of functions f on S having the following property: When $s \in S$ is fixed, we consider $f(x)$ as a linear functional on $\mathscr{H}$, and it is assumed that this functional is continuous in the norm of $\mathscr{H}$. From this we then arrive at an associated *positive definite kernel* K. Conversely, every positive definite kernel is the kernel naturally associated with a RKHS, usually written $\mathscr{H}(K)$.

The theory of positive definite functions has a large number of applications in a host of areas; for example, in harmonic analysis, in representation theory (of both algebras and groups), in physics, and in the study of probability models, such as stochastic processes.

One reason for this is the theorem of Bochner which links continuous positive definite functions on locally compact abelian groups G to measures on the corresponding dual group. Analogous uses of positive definite functions exist for classes for non-abelian groups. Even the seemingly modest case of $G = \mathbb{R}$ is of importance in the study of spectral theory for *Schrüdinger equations*. And further, counting the study of *Gaussian stochastic processes*, there are even generalizations (Gelfand-Minlos) to the case of continuous positive definite functions on Früchet spaces of test functions which make up part of a Gelfand triple. These cases will be explored below, but with the following important change in the starting point of the analysis; – we focus on the case when the given positive definite function is only partially defined, i.e., is only known on a proper subset of the ambient group, or space. How much of duality theory carries over when only partial information is available?

The purpose of the present paper is to explore what can be said when the given continuous positive definite function is only given on a subset of the ambient group. For this problem of partial information, even the case of positive definite functions defined only on bounded subsets of $G = \mathbb{R}$ (say an interval), or on bounded subsets of $G = \mathbb{R}^n$, is of substantial interest.

The material below is adapted primarily from [**JPT15b**, **JPT16**] by Jorgensen et al.

4.4.1. Two extension problems.

We study two classes of extension problems, and their interconnections:

(i) Extension of positive definite (p.d.) continuous functions defined on subsets in locally compact groups G;

(ii) In case of Lie groups, *representations of the associated Lie algebras* $La(G)$ by unbounded skew-Hermitian operators acting in a reproducing kernel Hilbert space (RKHS) $\mathscr{H}_F$.

Our analysis is non-trivial even if $G = \mathbb{R}^n$, and even if $n = 1$. If $G = \mathbb{R}^n$, we are concerned in (ii) with finding systems of strongly commuting selfadjoint operators $\{T_i\}$ extending a system of commuting Hermitian operators with common dense domain in $\mathscr{H}_F$.

Why extensions? In science, experimentalists frequently gather spectral data in cases when the observed data is limited, for example limited by the precision of instruments; or on account of a variety of other limiting external factors. (For instance, the human eye can only see a small portion of the electromagnetic spectrum.) Given this fact of life, it is both an art and a science to still produce solid conclusions from restricted or limited data. In a general sense, this section deals with the mathematics of extending some such given partial data-sets obtained from experiments. More specifically, we are concerned with the problems of extending available partial information, obtained, for example, from sampling. In our case, the limited information is a restriction, and the extension in turn is the full positive definite function (in a dual variable); so an extension if available will be an everywhere defined generating function for the exact probability distribution which reflects the data; if it were fully available. Such extensions of local information (in the form of positive definite functions) will in turn furnish us with spectral information. In this form, the problem becomes an operator extension problem, referring to operators in a suitable reproducing kernel Hilbert spaces (RKHS). In our presentation we have stressed hands-on-examples. Extensions are almost never unique, and so we deal with both the question of existence, and if there are extensions, how they relate back to the initial completion problem.

The emphasis here will be the interplay between the two problems. Our aim is a duality theory; and, in the case $G = \mathbb{R}^n$, and $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, we will state our theorems in the language of Fourier duality of abelian groups: With the time frequency duality formulation of Fourier duality for $G = \mathbb{R}^n$ we have that both the time domain and the frequency domain constitute a copy of $\mathbb{R}^n$. We then arrive at a setup such that our extension questions (i) are in time domain, and extensions from (ii) are in frequency domain. Moreover we show that each of the extensions from (i) has a variant in (ii). Specializing to $n = 1$, we arrive of a spectral theoretic characterization of all skew-Hermitian operators with dense domain in a separable Hilbert space, having deficiency-indices $(1, 1)$.

OPERATORS. A systematic study of *densely defined Hermitian operators* with deficiency indices $(1, 1)$, and later (d, d), was initiated by M. Krein [**Kre44**, **Kre46**, **KnKs47**, **Kn49**], and is also part of de Branges' model theory; see [**dBR66a**, **dB66**, **dB68**]. The direct connection between this theme and the problem of extending continuous positive definite functions F when they are only defined on a fixed open subset to $\mathbb{R}^n$ was one of our motivations. One desires continuous p.d. extensions to $\mathbb{R}^n$.

If F is given, we denote the set of such extensions $Ext\,(F)$. If $n = 1$, $Ext\,(F)$ is always non-empty, but for $n = 2$, Rudin gave examples in [**Rud63**, **Rud70**] when $Ext\,(F)$ may be empty. Here we extend these results, and we also cover a number of classes of positive definite functions on locally compact groups in general; so cases when $\mathbb{R}^n$ is replaced with other groups, both Abelian and non-abelian.

Extensions of positive definite continuous functions defined on subsets of Lie groups G was studied in [**Jor91**, **Jor89**, **Jor90**, **JN15**, **JPT15b**]. In our present analysis of connections between extensions of positive definite continuous functions and extension questions for associated operators in Hilbert space, we will be making use of tools from spectral theory, and from the theory of reproducing kernel-Hilbert spaces, such as can be found in e.g., [**Nel69**, **Jr81**, **ABDdS93**, **Aro50**, **JPT15b**, **JT15a**, **JT16d**, **LP11**, **PR16**, **AS57**].

There is a different kind of notion of positivity involving reflections, restrictions, and extensions. It comes up in physics and in stochastic processes [**Jor07**, **IW89**, **Hid80**, **Fal74**, **GJ78**, **GJ85**, **GJ87**], and is somewhat related to our present theme.

While they have several names, "refection positivity" is a popular term [**JO98**, **JO00**, **JNO16**, **OS73**, **OS75**]. In broad terms, the issue is about realizing geometric reflections as "conjugations" in Hilbert space. When the program is successful, for a given unitary representation U of a Lie group G, for example $G=\mathbb{R}$, it is possible to renormalize the Hilbert space on which U is acting. For details, see Chapter 7.

4.4.2. The RKHS $\mathscr{H}_F$. In our theorems and proofs, we shall make use the particular reproducing kernel Hilbert spaces (RKHSs) which allow us to give explicit formulas for our solutions. The general framework of RKHSs were pioneered by Aronszajn in the 1950s [**Aro43**, **Aro50**]; and subsequently they have been used in a host of applications; e.g., [**SZ07**, **SZ09**].

For simplicity we focus on the case $G=\mathbb{R}$. Modifications for other groups will be described in the text.

DEFINITION 4.4.1. Fix $0<a$, let $\Omega=(0,a)$. A function $F:\Omega-\Omega\to\mathbb{C}$ is *positive definite* if

$$\sum\nolimits_i\sum\nolimits_j\overline{c_i}c_jF(x_i-x_j)\geq 0 \tag{4.4.1}$$

for all finite sums with $c_i\in\mathbb{C}$, and all $x_i\in\Omega$. We assume that all the p.d. functions are continuous and bounded.

LEMMA 4.4.2. *F is p.d. if and only if*

$$\int_\Omega\int_\Omega\overline{\varphi(x)}\varphi(y)F(x-y)dxdy\geq 0 \text{ for all } \varphi\in C_c^\infty(\Omega).$$

PROOF. Standard. See, e.g., [**Aro50**]. □

Consider a continuous positive definite function so F is defined on $\Omega-\Omega$. Set

$$F_y\left(x\right):=F\left(x-y\right),\ \forall x,y\in\Omega. \tag{4.4.2}$$

Let $\mathscr{H}_F$ be the *reproducing kernel Hilbert space (RKHS)*, which is the completion of

$$\left\{\sum\nolimits_{\text{finite}}c_jF_{x_j}\mid x_j\in\Omega,\ c_j\in\mathbb{C}\right\} \tag{4.4.3}$$

with respect to the inner product

$$\left\langle\sum\nolimits_i c_iF_{x_i},\sum\nolimits_j d_jF_{y_j}\right\rangle_{\mathscr{H}_F}:=\sum\nolimits_i\sum\nolimits_j\overline{c_i}d_jF\left(x_i-y_j\right). \tag{4.4.4}$$

Below, we introduce an equivalent characterization of the RKHS $\mathscr{H}_F$, which we will be working with in the rest of the present section.

LEMMA 4.4.3. *Fix $\Omega=(0,\alpha)$. Let $\varphi_{n,x}(t)=n\varphi\left(n\left(t-x\right)\right)$, for all $t\in\Omega$; where φ satisfies*

(1) $\operatorname{supp}(\varphi)\subset(-\alpha,\alpha)$;
(2) $\varphi\in C_c^\infty$, $\varphi\geq 0$;
(3) $\int\varphi\left(t\right)dt=1$. *Note that* $\lim_{n\to\infty}\varphi_{n,x}=\delta_x$, *the Dirac measure at* x.

LEMMA 4.4.4. *The RKHS, $\mathscr{H}_F$, is the Hilbert completion of the functions*

$$F_\varphi\left(x\right)=\int_\Omega\varphi\left(y\right)F\left(x-y\right)dy,\ \forall\varphi\in C_c^\infty\left(\Omega\right),x\in\Omega \tag{4.4.5}$$

with respect to the inner product

$$\langle F_\varphi, F_\psi \rangle_{\mathscr{H}_F} = \int_\Omega \int_\Omega \overline{\varphi(x)} \psi(y) F(x-y) \, dx dy, \ \forall \varphi, \psi \in C_c^\infty(\Omega). \tag{4.4.6}$$

In particular,

$$\|F_\varphi\|^2_{\mathscr{H}_F} = \int_\Omega \int_\Omega \overline{\varphi(x)} \varphi(y) F(x-y) \, dx dy, \ \forall \varphi \in C_c^\infty(\Omega) \tag{4.4.7}$$

and

$$\langle F_\varphi, F_\psi \rangle_{\mathscr{H}_F} = \int_\Omega \overline{\varphi(x)} F_\psi(x) \, dx, \ \forall \phi, \psi \in C_c^\infty(\Omega). \tag{4.4.8}$$

PROOF. Indeed, by Lemma 4.4.4, we have

$$\left\| F_{\varphi_{n,x}} - F(\cdot - x) \right\|_{\mathscr{H}_F} \to 0, \text{ as } n \to \infty. \tag{4.4.9}$$

Hence $\{F_\varphi\}_{\varphi \in C_c^\infty(\Omega)}$ spans a dense subspace in $\mathscr{H}_F$. (For more details, see [**Jor86**, **Jor87**, **Jor90**].) □

The two equivalent conditions below will be used to characterize elements in the Hilbert space $\mathscr{H}_F$.

THEOREM 4.4.5. *A continuous function $\xi : \Omega \to \mathbb{C}$ is in $\mathscr{H}_F$ if and only if there exists $A_0 > 0$, such that*

$$\sum_i \sum_j \overline{c_i} c_j \overline{\xi(x_i)} \xi(x_j) \le A_0 \sum_i \sum_j \overline{c_i} c_j F(x_i - x_j) \tag{4.4.10}$$

for all finite system $\{c_i\} \subset \mathbb{C}$ and $\{x_i\} \subset \Omega$.

Equivalently, for all $\psi \in C_c^\infty(\Omega)$,

$$\left| \int_\Omega \psi(y) \overline{\xi(y)} dy \right|^2 \quad \le \quad A_0 \int_\Omega \int_\Omega \overline{\psi(x)} \psi(y) F(x-y) \, dx dy \tag{4.4.11}$$

PROOF. Note that, if $\xi \in \mathscr{H}_F$, then the LHS of (4.4.11) is $|\langle F_\psi, \xi \rangle_{\mathscr{H}_F}|^2$. Indeed,

$$\begin{aligned} \left| \langle F_\psi, \xi \rangle_{\mathscr{H}_F} \right|^2 &= \left| \left\langle \int_\Omega \psi(y) F_y \, dy, \xi \right\rangle_{\mathscr{H}_F} \right|^2 \\ &= \left| \int_\Omega \psi(y) \langle F_y, \xi \rangle_{\mathscr{H}_F} dy \right|^2 \\ &= \left| \int_\Omega \psi(y) \xi(y) \, dy \right|^2 \end{aligned}$$

by the reproducing property of F_ψ. The remaining of the proof is left to the reader. □

4.4.3. The operator $D^{(F)}$. Fix $0 < a$ and a continuous positive definite function F defined on $\Omega - \Omega$, where $\Omega = (0, a)$ as in Definition 4.4.1. Let $\mathscr{H}_F$ be the corresponding RKHS as in (4.4.3).

DEFINITION 4.4.6. Define $D^{(F)}$ on the dense domain $C_c^\infty(\Omega) * F$ by $D^{(F)} F_\psi = F_{\psi'}$, where $\psi' = \frac{d\psi}{dt}$.

One shows that $D^{(F)}$ is well defined by using Schwarz' inequality to prove that $F_\psi = 0$ implies that $F_{\psi'} = 0$.

LEMMA 4.4.7. *$D^{(F)}$ is a well-defined operator with dense domain in $\mathscr{H}_F$. Moreover, it is skew-symmetric and densely defined in $\mathscr{H}_F$.*

PROOF. By Lemma 4.4.4 $\mathrm{dom}\left(D^{(F)}\right)$ is dense in $\mathscr{H}_F$. If $\psi \in C_c^{\infty}(0,a)$ and $|t| < \mathrm{dist}\left(\mathrm{supp}\left(\psi\right), \mathrm{endpoints}\right)$, then

$$\left\|F_{\psi(\cdot+t)}\right\|_{\mathscr{H}_F}^2 = \left\|F_{\psi}\right\|_{\mathscr{H}_F}^2 = \int_0^a \int_0^a \overline{\psi\left(x\right)}\psi\left(y\right)F\left(x-y\right)dxdy \tag{4.4.12}$$

see (4.4.7), so

$$\frac{d}{dt}\left\|F_{\psi(\cdot+t)}\right\|_{\mathscr{H}_F}^2 = 0$$

which is equivalent to

$$\left\langle D^{(F)}F_{\psi}, F_{\psi}\right\rangle_{\mathscr{H}_F} + \left\langle F_{\psi}, D^{(F)}F_{\psi}\right\rangle_{\mathscr{H}_F} = 0. \tag{4.4.13}$$

It follows that $D^{(F)}$ is skew-symmetric.

To show that $D^{(F)}F_{\psi} = F_{\psi'}$ is a well-defined operator on is dense domain in $\mathscr{H}_F$, we proceed as follows:

LEMMA 4.4.8. *The following implication holds:*

$$\left[\psi \in C_c^{\infty}\left(\Omega\right),\ F_{\psi} = 0 \ \mathit{in}\ \mathscr{H}_F\right] \tag{4.4.14}$$

$$\Downarrow$$

$$\left[F_{\psi'} = 0 \ \mathit{in}\ \mathscr{H}_F\right] \tag{4.4.15}$$

PROOF. Substituting (4.4.14) into

$$\left\langle F_{\varphi}, F_{\psi'}\right\rangle_{\mathscr{H}_F} + \left\langle F_{\varphi'}, F_{\psi}\right\rangle_{\mathscr{H}_F} = 0$$

we get

$$\left\langle F_{\varphi}, F_{\psi'}\right\rangle_{\mathscr{H}_F} = 0,\ \forall\varphi \in C_c^{\infty}\left(\Omega\right).$$

Taking $\varphi = \psi'$, yields

$$\left\langle F_{\psi'}, F_{\psi'}\right\rangle = \left\|F_{\psi'}\right\|_{\mathscr{H}_F}^2 = 0$$

which is the desired conclusion (4.4.15). $\square$

This finishes the proof of Lemma Lemma 4.4.7.

$\square$

LEMMA 4.4.9. *Let $\Omega = (\alpha, \beta)$. Suppose F is a real valued positive definite function defined on $\Omega - \Omega$. The operator J on $\mathscr{H}_F$ determined by*

$$JF_{\varphi} = \overline{F_{\varphi(\alpha+\beta-x)}}, \varphi \in C_c^{\infty}(\Omega)$$

is a conjugation, i.e., J is conjugate-linear, J^2 is the identity operator, and

$$\left\langle JF_{\phi}, JF_{\psi}\right\rangle_{\mathscr{H}_F} = \left\langle F_{\psi}, F_{\phi,}\right\rangle_{\mathscr{H}_F}. \tag{4.4.16}$$

Moreover,

$$D^{(F)}J = -JD^{(F)}. \tag{4.4.17}$$

PROOF. Let $a := \alpha + \beta$ and $\phi \in C_c^\infty(\Omega)$. Since F is real valued

$$JF_\phi(x) = \int_\alpha^\beta \overline{\phi(a-y)}\,\overline{F(x-y)}dy = \int_\alpha^\beta \psi(y)\,F(x-y)dy$$

where $\psi(y) := \overline{\phi(a-y)}$ is in $C_c^\infty(\Omega)$. It follows that J maps the domain $\operatorname{dom}\left(D^{(F)}\right)$ of $D^{(F)}$ onto itself. For $\phi, \psi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\langle JF_\phi, F_\psi\rangle_{\mathscr{H}_F} &= \int_\alpha^\beta F_{\phi(a-\cdot)}(x)\psi(x)dx \\ &= \int_\alpha^\beta \int_\alpha^\beta \phi(a-y)F(x-y)\psi(x)dydx.\end{aligned}$$

Making the change of variables $(x,y) \to (a-x, a-y)$ and interchanging the order of integration we see that

$$\begin{aligned}\langle JF_\phi, F_\psi\rangle_{\mathscr{H}_F} &= \int_\alpha^\beta \int_\alpha^\beta \phi(y)F(y-x)\psi(a-x)dydx \\ &= \int_\alpha^\beta \phi(y)F_{\psi(a-\cdot)}(y)dy \\ &= \langle JF_\psi, F_\phi\rangle_{\mathscr{H}_F},\end{aligned}$$

establishing (4.4.16). For $\phi \in C_c^\infty(\Omega)$,

$$JD^{(F)}F_\phi = \overline{F_{\phi'(a-\cdot)}} = -\overline{F_{\frac{d}{dx}(\phi(a-\cdot))}} = -D^{(F)}JF_\phi,$$

hence (4.4.17) holds. □

DEFINITION 4.4.10. Let $\left(D^{(F)}\right)^*$ be the adjoint of $D^{(F)}$. *The deficiency spaces* $DEF^\pm$ consists of $\xi_\pm \in \mathscr{H}_F$, such that $\left(D^{(F)}\right)^* \xi_\pm = \pm\xi_\pm$. That is,

$$DEF^\pm = \left\{\xi_\pm \in \mathscr{H}_F : \langle F_{\psi'}, \xi_\pm\rangle_{\mathscr{H}_F} = \langle F_\psi, \pm\xi_\pm\rangle_{\mathscr{H}_F}, \forall\psi \in C_c^\infty(\Omega)\right\}.$$

COROLLARY 4.4.11. *If F is real valued, then DEF^+ and DEF^- have the same dimension.*

PROOF. This follows from Lemma 4.4.9, see e.g, [**DS88**]. □

LEMMA 4.4.12. *If $\xi \in DEF^\pm$ then $\xi(y) = \text{constant}\, e^{\mp y}$.*

PROOF. Specifically, $\xi \in DEF^+$ if and only if

$$\int_0^a \psi'(y)\,\xi(y)\,dy = \int_0^a \psi(y)\,\xi(y)\,dy,\ \forall\psi \in C_c^\infty(0,a).$$

Equivalently, $y \mapsto \xi(y)$ is a weak solution to $-\xi' = \xi$, i.e., a strong solution in C^1. Thus, $\xi(y) = \text{constant}\ e^{-y}$. The DEF^- case is similar. □

COROLLARY 4.4.13. *Suppose F is real valued. Let $\xi_\pm(y) := e^{\mp y}$, for $y \in \Omega$. Then $\xi_+ \in \mathscr{H}_F$ if and only if $\xi_- \in \mathscr{H}_F$. In the affirmative case $\|\xi_-\|_{\mathscr{H}_F} = e^a\|\xi_+\|_{\mathscr{H}_F}$.*

PROOF. Let J be the conjugation from Lemma 4.4.9. A short calculation:

$$\begin{aligned}\langle J\xi, F_\phi\rangle_{\mathscr{H}_{\mathscr{F}}} &= \left\langle F_{\overline{\phi(a-\cdot)}}, \xi\right\rangle_{\mathscr{H}_{\mathscr{F}}} = \int \phi(a-x)\xi(x)dx \\ &= \int \phi(x)\xi(a-x)dx = \left\langle \overline{\xi(a-\cdot)}, F_\phi\right\rangle_{\mathscr{H}_{\mathscr{F}}}\end{aligned}$$

shows that $(J\xi)(x) = \overline{\xi(a-x)}$, for $\xi \in \mathscr{H}_F$. In particular, $J\xi_- = e^a\xi_+$. Since, $\|J\xi_-\|_{\mathscr{H}_F} = \|\xi_-\|_{\mathscr{H}_F}$, the proof is easily completed. □

COROLLARY 4.4.14. *The deficiency indices of $D^{(F)}$, with its dense domain in $\mathscr{H}_F$ are* $(0,0)$, $(0,1)$, $(1,0)$, *or* $(1,1)$.

The second case in the above corollary happens precisely when $y \mapsto e^{-y} \in \mathscr{H}_F$. We can decide this with the use of (4.4.10)(⇔(4.4.11)).

4.4.4. The extension of F. By Corollary 4.4.14, we conclude that there exists skew-adjoint extension $A^{(F)} \supset D^{(F)}$ in $\mathscr{H}_F$. That is, $\left(A^{(F)}\right)^* = -A^{(F)}$, and

$$\{F_\psi\}_{\psi\in C_c^\infty(0,a)} \subset \mathrm{dom}(A^{(F)}) \subset \mathscr{H}_F.$$

Hence, set $U(t) = e^{tA^{(F)}} : \mathscr{H}_F \to \mathscr{H}_F$, and get the unitary one-parameter group

$$\{U(t) : t \in \mathbb{R}\},\ U(s+t) = U(s)U(t),\ \forall s,t \in \mathbb{R};$$

and if

$$\xi \in dom\left(A^{(F)}\right) = \left\{\xi \in \mathscr{H}_F :\ s.t. \lim_{t\to 0}\frac{U(t)\xi - \xi}{t} \text{ exists}\right\}$$

then

$$A^{(F)}\xi = \lim_{t\to 0}\frac{U(t)\xi - \xi}{t}. \tag{4.4.18}$$

Now use $F_x(\cdot) = F(\cdot - x)$ defined in $(0,a)$; and set

$$F_A(t) := \langle F_0, U(t)F_0\rangle_{\mathscr{H}_F},\ \forall t \in \mathbb{R} \tag{4.4.19}$$

then using (4.4.9), we see that F_A is a continuous positive definite extension of F on $(-a,a)$, i.e., a continuous positive definite function on $\mathbb{R}$, and if $x \in (0,a)$, then we get the following conclusion:

LEMMA 4.4.15. *F_A is a continuous bounded positive definite function of* $\mathbb{R}$ *and*

$$F_A(t) = F(t). \tag{4.4.20}$$

for $t \in (-a,a)$.

PROOF. But $\mathbb{R} \ni t \mapsto F_A(t)$ is bounded and continuous, since $\{U(t)\}$ is a strongly continuous unitary group acting on $\mathscr{H}_F$, and

$$|F_A(t)| = |\langle F_0, U(t)F_0\rangle| \le \|F_0\|_{\mathscr{H}_F}\|U(t)F_0\|_{\mathscr{H}_F} = \|F_0\|^2_{\mathscr{H}_F}$$

where $|\langle F_0, U(t)F_0\rangle| \le \|F_0\|^2_{\mathscr{H}_F} = F(0)$. See the proof of Theorem 4.4.16 and [**Jor89**, **Jor90**, **Jor91**] for the remaining details. □

REMARK. F can be normalized by $F(0) = 1$. Recall that F is defined on $(-a,a) = \Omega - \Omega$ if $\Omega = (0,a)$.

Consider the spectral representation:

$$U(t) = \int_{-\infty}^{\infty} e_t(\lambda)\, P(d\lambda) \tag{4.4.21}$$

where $e_t(\lambda) = e^{i2\pi\lambda t}$; and $P(\cdot)$ is a projection-valued measure on $\mathbb{R}$, $P(B) : \mathscr{H}_F \to \mathscr{H}_F$, $\forall B \in Borel(\mathbb{R})$. Then

$$d\mu(\lambda) = \|P(d\lambda) F_0\|^2_{\mathscr{H}_F}$$

satisfies

$$F_A(t) = \int_{-\infty}^{\infty} e_t(\lambda)\, d\mu(\lambda),\ \forall t \in \mathbb{R}. \tag{4.4.22}$$

Conclusion. The extension F_A from (4.4.19) has nice transform properties, and via (4.4.22) we get

$$\mathscr{H}_{F_A} \simeq L^2(\mathbb{R}, \mu)$$

where $\mathscr{H}_{F_A}$ is the RKHS of F_A.

4.4.5. Enlarging the Hilbert space. Below we describe the dilation-Hilbert space in detail, and prove some lemmas which will then be used in the following sections.

Let $F : \Omega - \Omega \to \mathbb{C}$ be a continuous p.d. function. Let $\mathscr{H}_F$ be the corresponding RKHS and $\xi_x := F(\cdot - x) \in \mathscr{H}_F$.

$$\langle \xi_x, \xi_y \rangle_{\mathscr{H}_F} = F(x-y), \forall x, y \in \Omega. \tag{4.4.23}$$

As usual $F_\varphi = \varphi * F$, $\varphi \in C_c^\infty(\Omega)$. Then

$$\langle F_\varphi, F_\psi \rangle_{\mathscr{H}_F} = \left\langle \xi_0, \pi\left(\varphi^{\#} * \psi\right) \xi_0 \right\rangle = \langle \pi(\phi)\, \xi_0, \pi(\psi)\, \xi_0 \rangle$$

where $\pi(\varphi)\,\xi_0 = F_\varphi$. The following lemma also holds in $\mathbb{R}^n$ with $n > 1$, but we state it for $n = 1$ to illustrate the "enlargement" of $\mathscr{H}_F$ question.

We have introduced three equivalent notations:

$$F_\varphi = \varphi * F = \pi(\varphi)\, F.$$

Recall that if π is a representation of a locally compact topological group G on a Banach space $\mathscr{B}$, then one defines

$$\pi(\varphi)\, v = \int_G \varphi(x)\, \pi(x)\, v\, dx,\ \forall v \in \mathscr{B}$$

where dx denotes the Haar measure on G. Using the left regular representation, one gets $\pi(\varphi)\, F = \varphi * F$, or

$$\pi(\varphi)\, F(x) = \int \varphi(y)\, F(x-y)\, dy = \pi(\varphi)\, \xi_0(x).$$

THEOREM 4.4.16 (Jo-Pedersen-Tian). *The following two conditions are equivalent:*

(i) *F is extendable to a continuous p.d. function $\widetilde{F}$ defined on $\mathbb{R}$, i.e., $\widetilde{F}$ is a continuous p.d. function defined on $\mathbb{R}$ and $F(x) = \widetilde{F}(x)$ for all x in $\Omega - \Omega$.*

(ii) *There is a Hilbert space $\mathscr{K}$, an isometry $W : \mathscr{H}_F \to \mathscr{K}$, and a strongly continuous unitary group $U_t : \mathscr{K} \to \mathscr{K}$, $t \in \mathbb{R}$ such that, if A is the skew-adjoint generator of U_t, i.e.,*

$$\tfrac{1}{t}\left(U_t k - k\right) \to Ak, \forall k \in \operatorname{dom}(A), \tag{4.4.24}$$

then

$$WF_\varphi \in \operatorname{domain}\left(A\right), \forall \varphi \in C_c^\infty(\Omega) \tag{4.4.25}$$

and

$$AWF_\varphi = WF_{\varphi'}, \forall \varphi \in C_c^\infty\left(\Omega\right). \tag{4.4.26}$$

PROOF. $\underline{\Uparrow}$: First, assume there exists $\mathscr{K}$, W, U_t, and A as in (ii). Set

$$\widetilde{F}(t) = \left\langle W\xi_0, U_t W\xi_0\right\rangle, t \in \mathbb{R}. \tag{4.4.27}$$

Then, if $U_t = \int_{\mathbb{R}} e_t(\lambda) P(d\lambda)$, set

$$d\mu(\lambda) = \left\| P(d\lambda) W\xi_0 \right\|_{\mathscr{K}}^2 = \left\langle W\xi_0, P(d\lambda) W\xi_0\right\rangle_{\mathscr{K}}$$

and $\widetilde{F} = \widehat{d\mu}$ is the Bochner transform.

LEMMA 4.4.17. *If $s, t \in \Omega$, and $\varphi \in C_c^\infty(\Omega)$, then*

$$\left\langle WF_\varphi, U_t WF_\varphi\right\rangle = \left\langle \xi_0, \pi\left(\varphi^{\#} * \varphi_t\right)\xi_0\right\rangle_{\mathscr{H}_F}, \tag{4.4.28}$$

where $\varphi_t(\cdot) = \varphi(t - \cdot)$.

Suppose first (4.4.28) has been checked. Let ϕ_ϵ be an approximate identity at $x = 0$. Then

$$\begin{aligned} \widetilde{F}(t) &= \left\langle W\xi_0, U_t W\xi_0\right\rangle \\ &= \lim_{\epsilon \to 0} \left\langle \xi_0, \pi\left(\left(\phi_\epsilon\right)_t\right)\xi_0\right\rangle_{\mathscr{H}_F} \\ &= \left\langle \xi_0, \xi_t\right\rangle = F(t) \end{aligned} \tag{4.4.29}$$

by (4.4.27), (4.4.28), and (4.4.23).

PROOF OF LEMMA 4.4.17. Now (4.4.28) follows from

$$U_t WF_\varphi = WF_{\varphi_t}, \tag{4.4.30}$$

$\varphi \in C_c^\infty(\Omega), t \in \Omega$. Consider now

$$U_{t-s} WF_{\varphi_s} = \begin{cases} U_t WF_\varphi & \text{at } s = 0 \\ WF_{\varphi_t} & \text{at } s = t \end{cases} \tag{4.4.31}$$

and

$$\int_0^t \tfrac{d}{ds} U_{t-s} WF_{\varphi_s} ds = WF_{\varphi_t} - U_t WF_\varphi. \tag{4.4.32}$$

We claim that the left hand side of (4.4.32) equals zero. By (4.4.24) and (4.4.25)

$$\tfrac{d}{ds}\left[U_{t-s} WF_{\varphi_s}\right] = -U_{t-s} AWF_{\varphi_s} + U_{t-s} WF_{\varphi_s'}.$$

But, by (4.4.26) applied to φ_s, we get

$$AWF_{\varphi_s} = WF_{\varphi_s'} \tag{4.4.33}$$

and the desired conclusion (4.4.30) follows. □

$\Downarrow$: Assume (i), let $\widetilde{F} = \widehat{d\mu}$ be a p.d. extension and Bochner transform. Then $\mathscr{H}_{\widetilde{F}} \simeq L^2(\mu)$; and for $\varphi \in C_c^\infty(\Omega)$, set

$$WF_\varphi = \widetilde{F}_\varphi, \tag{4.4.34}$$

then $W : \mathscr{H}_F \to \mathscr{H}_{\widetilde{F}}$ is an isometry.

Proof that (4.4.34) is an isometry . Let $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\left\|\widetilde{F}_\varphi\right\|^2_{\mathscr{H}_{\widetilde{F}}} &= \int\int \overline{\varphi(s)}\varphi(t)\widetilde{F}(t-s)dsdt \\ &= \|F_\varphi\|^2_{\mathscr{H}_F} = \int_{\mathbb{R}} |\widehat{\varphi}(\lambda)|^2 \, d\mu(\lambda)\end{aligned}$$

since $\widetilde{F}$ is an extension of F. □

Now set $U_t : L^2(\mu) \to L^2(\mu)$,

$$(U_t f)(\lambda) = e_t(\lambda) f(\lambda),$$

a unitary group acting in $\mathscr{H}_{\widetilde{F}} \simeq L^2(\mu)$. Using (4.4.23), we get

$$(WF_\varphi)(x) = \int e_x(\lambda)\,\widehat{\varphi}(\lambda)\,d\mu(\lambda), \forall x \in \Omega, \forall \varphi \in C_c^\infty(\Omega). \tag{4.4.35}$$

And therefore (ii) follows. By (4.4.35)

$$\begin{aligned}(WF_{\varphi'})(x) &= \int e_x(\lambda)\, i\lambda\widehat{\varphi}(\lambda)\,d\mu(\lambda) \\ &= \tfrac{d}{dt}\big|_{t=0} U_t W F_\varphi = AWF_\varphi\end{aligned}$$

as claimed. □

4.4.6. $Ext_1(F)$ and $Ext_2(F)$.

Definition 4.4.18. Let G be a locally compact group, and let Ω be an open connected subset of G. Let $F : \Omega^{-1}\cdot\Omega \to \mathbb{C}$ be a continuous positive definite function.

Consider a strongly continuous unitary representation U of G acting in some Hilbert space $\mathscr{K}$, containing the RKHS $\mathscr{H}_F$. We say that $(U, \mathscr{K}) \in Ext(F)$ if and only if there is a vector $k_0 \in \mathscr{K}$ such that

$$F(g) = \langle k_0, U(g)\, k_0\rangle_{\mathscr{K}}, \ \forall g \in \Omega^{-1}\cdot\Omega. \tag{4.4.36}$$

I. The subset of $Ext(F)$ consisting of $(U, \mathscr{H}_F, k_0 = F_e)$ with

$$F(g) = \langle F_e, U(g)\, F_e\rangle_{\mathscr{H}_F}, \ \forall g \in \Omega^{-1}\cdot\Omega \tag{4.4.37}$$

is denoted $Ext_1(F)$; and we set

$$Ext_2(F) := Ext(F)\backslash Ext_1(F);$$

i.e., $Ext_2(F)$ consists of the solutions to problem (4.4.36) for which $\mathscr{K} \supsetneqq \mathscr{H}_F$, i.e., unitary representations realized in an enlargement Hilbert space.

(We write $F_e \in \mathscr{H}_F$ for the vector satisfying $\langle F_e, \xi\rangle_{\mathscr{H}_F} = \xi(e)$, $\forall \xi \in \mathscr{H}_F$, where e is the neutral (unit) element in G, i.e., $e\,g = g$, $\forall g \in G$.)

II. In the special case, where $G = \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ is open and connected, we consider

$$F : \Omega - \Omega \to \mathbb{C}$$

continuous and positive definite. In this case,

$$Ext\,(F) = \Big\{ \mu \in \mathscr{M}_+ \left(\mathbb{R}^n\right) \;\big|\; \widehat{\mu}\,(x) = \int_{\mathbb{R}^n} e^{i\lambda\cdot x} d\mu\,(\lambda) \text{ is a p.d. extension of } F \Big\}. \tag{4.4.38}$$

Remark 4.4.19. Note that (4.4.38) is consistent with (4.4.36): For if $(U, \mathscr{K}, k_0)$ is a unitary representation of $G = \mathbb{R}^n$, such that (4.4.36) holds; then, by a theorem of Stone, there is a projection-valued measure (PVM) $P_U\,(\cdot)$, defined on the Borel subsets of $\mathbb{R}^n$ s.t.

$$U\,(x) = \int_{\mathbb{R}^n} e^{i\lambda\cdot x} P_U\,(d\lambda)\,,\; x \in \mathbb{R}^n. \tag{4.4.39}$$

Setting

$$d\mu\,(\lambda) := \left\| P_U\,(d\lambda)\, k_0 \right\|^2_{\mathscr{K}}\,, \tag{4.4.40}$$

it is then immediate that we have: $\mu \in \mathscr{M}_+\left(\mathbb{R}^n\right)$, and that the finite measure μ satisfies

$$\widehat{\mu}\,(x) = F\,(x)\,,\; \forall x \in \Omega - \Omega. \tag{4.4.41}$$

4.4.7. The case of locally compact abelian groups. We are concerned with extensions of locally defined continuous and positive definite functions F on Lie groups, but some results apply to locally compact groups as well. In the case of locally compact groups, we have stronger theorems, due to the powerful Fourier analysis theory.

We must fix notations:

- G: a given locally compact abelian group, write the operation in G additively;
- dx: denotes the Haar measure of G (unique up to a scalar multiple.)
- $\widehat{G}$: the dual group, i.e., $\widehat{G}$ consists of all continuous homomorphisms: $\lambda : G \to \mathbb{T}$, $\lambda\,(x+y) = \lambda\,(x)\,\lambda\,(y)$, $\forall x, y \in G$; $\lambda\,(-x) = \overline{\lambda\,(x)}$, $\forall x \in G$. Occasionally, we shall write $\langle \lambda, x \rangle$ for $\lambda\,(x)$. Note that $\widehat{G}$ also has its Haar measure.

Theorem 4.4.20 (Pontryagin [**Rud90**]). *$\widehat{\widehat{G}} \simeq G$, and we have the following:*

$$[G \text{ is compact}] \Longleftrightarrow [\widehat{G} \text{ is discrete}]$$

Let $\emptyset \neq \Omega \subset G$ be an open connected subset, and let $F : \Omega - \Omega \to \mathbb{C}$ be a fixed continuous positive definite function. We choose the normalization $F\,(0) = 1$; and introduce the corresponding reproducing kernel Hilbert space (RKHS):

Lemma 4.4.21. *For $\varphi \in C_c\,(\Omega)$, set*

$$F_\varphi\,(\cdot) = \int_\Omega \varphi\,(y)\, F\,(\cdot - y)\, dy, \tag{4.4.42}$$

then $\mathscr{H}_F$ is the Hilbert completion of

$$\left\{ F_\varphi \;\middle|\; \varphi \in C_c\,(\Omega) \right\} / \left\{ F_\varphi \;\middle|\; \|F_\varphi\|_{\mathscr{H}_F} = 0 \right\}$$

with respect to the inner product:

$$\left\langle F_{\varphi},F_{\psi}\right\rangle _{\mathscr{H}_{F}}=\int_{\Omega}\int_{\Omega}\overline{\varphi\left(x\right)}\psi\left(y\right)F\left(x-y\right)dxdy. \tag{4.4.43}$$

Here $C_{c}\left(\Omega\right):=$ *all continuous compactly supported functions in* Ω*.*

LEMMA 4.4.22. *The Hilbert space* $\mathscr{H}_{F}$ *is also a Hilbert space of continuous functions on* Ω *as follows:*

If $\xi:\Omega\rightarrow\mathbb{C}$ *is a fixed continuous function, then* $\xi\in\mathscr{H}_{F}$ *if and only if* $\exists$ $K=K_{\xi}<\infty$ *such that*

$$\left|\int_{\Omega}\overline{\xi\left(x\right)}\varphi\left(x\right)dx\right|^{2}\leq K\int_{\Omega}\int_{\Omega}\overline{\varphi\left(y_{1}\right)}\varphi\left(y_{2}\right)F\left(y_{1}-y_{2}\right)dy_{1}dy_{2}. \tag{4.4.44}$$

When (4.4.44) *holds, then*

$$\left\langle \xi,F_{\varphi}\right\rangle _{\mathscr{H}_{F}}=\int_{\Omega}\overline{\xi\left(x\right)}\varphi\left(x\right)dx,\ \textit{for all }\varphi\in C_{c}\left(\Omega\right).$$

PROOF. We refer to the basics on the theory of RKHSs; e.g., [**Aro50**]. □

LEMMA 4.4.23. *There is a bijective correspondence between all continuous p.d. extensions* $\widetilde{F}$ *to* G *of the given p.d. function* F *on* $\Omega-\Omega$*, on the one hand; and all Borel probability measures* μ *on* $\widehat{G}$*, on the other, i.e., all* $\mu\in\mathscr{M}(\widehat{G})$ *s.t.*

$$F\left(x\right)=\widehat{\mu}\left(x\right),\ \forall x\in\Omega-\Omega \tag{4.4.45}$$

where

$$\widehat{\mu}\left(x\right)=\int_{\widehat{G}}\lambda\left(x\right)d\mu\left(\lambda\right)=\int_{\widehat{G}}\left\langle \lambda,x\right\rangle d\mu\left(\lambda\right),\ \forall x\in G.$$

PROOF. This is an immediate application of Bochner's characterization of the continuous positive definite functions on locally compact abelian groups. □

DEFINITION 4.4.24. Set

$$Ext\left(F\right)=\left\{ \mu\in\mathscr{M}(\widehat{G})\mid s.t.\ (4.4.45)\text{ holds}\right\} .$$

(Note that $Ext\left(F\right)$ is weak $*$-compact and convex [**Rud73**].)

THEOREM 4.4.25.

(1) *Let* F *and* $\mathscr{H}_{F}$ *be as above; and let* $\mu\in\mathscr{M}(\widehat{G})$*; then there is a positive Borel function* h *on* $\widehat{G}$ *s.t.* $h^{-1}\in L^{\infty}(\widehat{G})$*, and* $hd\mu\in Ext\left(F\right)$*, if and only if* $\exists K_{\mu}<\infty$ *such that*

$$\int_{\widehat{G}}\left|\widehat{\varphi}\left(\lambda\right)\right|^{2}d\mu\left(\lambda\right)\leq K_{\mu}\int_{\Omega}\int_{\Omega}\overline{\varphi\left(y_{1}\right)}\varphi\left(y_{2}\right)F\left(y_{1}-y_{2}\right)dy_{1}dy_{2}. \tag{4.4.46}$$

(2) *Assume* $\mu\in Ext\left(F\right)$*, then*

$$\left(fd\mu\right)^{\vee}\in\mathscr{H}_{F},\ \forall f\in L^{2}(\widehat{G},\mu). \tag{4.4.47}$$

PROOF. The assertion in (4.4.46) is immediate from Lemma 4.4.22.

Our conventions for the two transforms used in (4.4.46) and (4.4.47) are as follows:

$$\widehat{\varphi}\left(\lambda\right)=\int_{G}\overline{\left\langle \lambda,x\right\rangle }\varphi\left(x\right)dx. \tag{4.4.48}$$

The transform in (4.4.47) is:

$$(f d\mu)^{\vee} = \int_{\widehat{G}} \langle \lambda, x \rangle f(\lambda)\, d\mu(\lambda). \tag{4.4.49}$$

The remaining computations are left to the reader. □

COROLLARY 4.4.26.

(1) *Let F be as above; then $\mu \in Ext(F)$ if and only if the following operator*
$$T(F_{\varphi}) = \widehat{\varphi},\ \varphi \in C_c(\Omega)$$
is well-defined on $\mathscr{H}_F$, and bounded as follows: $T : \mathscr{H}_F \to L^2(\widehat{G}, \mu)$.

(2) *In this case, the adjoint operator $T^* : L^2(\widehat{G}, \mu) \to \mathscr{H}_F$ is given by*

$$T^*(f) = (f d\mu)^{\vee},\ \forall f \in L^2(\widehat{G}, \mu). \tag{4.4.50}$$

PROOF. If $\mu \in Ext(F)$, then for all $\varphi \in C_c(\Omega)$, and $x \in \Omega$, we have (see (4.4.42))

$$\begin{aligned} F_{\varphi}(x) &= \int_{\Omega} \varphi(y) F(x-y)\, dy \\ &= \int_{\Omega} \varphi(y) \widehat{\mu}(x-y)\, dy \\ &= \int_{\Omega} \varphi(y) \langle \lambda, x-y \rangle\, d\mu(\lambda)\, dy \\ &\overset{\text{(Fubini)}}{=} \int_{\widehat{G}} \langle \lambda, x \rangle \widehat{\varphi}(\lambda)\, d\mu(\lambda). \end{aligned}$$

By Lemma 4.4.22, we note that $(\widehat{\varphi} d\mu)^{\vee} \in \mathscr{H}_F$, see (4.4.49). Hence $\exists K < \infty$ such that the estimate (4.4.46) holds. To see that $T(F_{\varphi}) = \widehat{\varphi}$ is well-defined on $\mathscr{H}_F$, we must check the implication:

$$\left(F_{\varphi} = 0 \text{ in } \mathscr{H}_F\right) \Longrightarrow \left(\widehat{\varphi} = 0 \text{ in } L^2(\widehat{G}, \mu)\right)$$

but this now follows from estimate (4.4.46).

Using the definition of the respective inner products in $\mathscr{H}_F$ and in $L^2(\widehat{G}, \mu)$, we check directly that, if $\varphi \in C_c(\Omega)$, and $f \in L^2(\widehat{G}, \mu)$ then we have:

$$\langle \widehat{\varphi}, f \rangle_{L^2(\mu)} = \left\langle F_{\varphi}, (f d\mu)^{\vee} \right\rangle_{\mathscr{H}_F}. \tag{4.4.51}$$

On the RHS in (4.4.51), we note that, when $\mu \in Ext(F)$, then $\widehat{f d\mu} \in \mathscr{H}_F$. This last conclusion is a consequence of Lemma 4.4.22. Indeed, since μ is finite, $L^2(\widehat{G}, \mu) \subset L^1(\widehat{G}, \mu)$, so $\widehat{f d\mu}$ in (4.4.49) is continuous on G by Riemann-Lebesgue; and so is its restriction to Ω. If μ is further assumed absolutely continuous, then $\widehat{f d\mu} \to 0$ at ∞.

With a direct calculation, using the reproducing property in $\mathscr{H}_F$, and Fubini's theorem, we check directly that the following estimate holds:

$$\left| \int_{\Omega} \overline{\varphi(x)}\, (f d\mu)^{\vee}(x)\, dx \right|^2 \leq \left(\int_{\Omega} \int_{\Omega} \overline{\varphi(y_1)} \varphi(y_2) F(y_1 - y_2)\, dy_1 dy_2 \right) \|f\|^2_{L^2(\mu)}$$

and so Lemma 4.4.22 applies; we get $(f d\mu)^{\vee} \in \mathscr{H}_F$.

It remains to verify the formula (4.4.51) for all $\varphi \in C_c(\Omega)$ and all $f \in L^2(\widehat{G}, \mu)$; but this now follows from the reproducing property in $\mathscr{H}_F$, and Fubini.

Once we have this, both assertions in (1) and (2) in the Corollary follow directly from the definition of the adjoint operator T^* with respect to the two Hilbert spaces in $\mathscr{H}_F \xrightarrow{T} L^2(\widehat{G}, \mu)$. Indeed then (4.4.50) follows. $\square$

We recall a general result on continuity of positive definite functions on any locally compact Lie group:

THEOREM 4.4.27. *If F is p.d. function on a locally compact group G, assumed continuous only in a neighborhood of $e \in G$; then it is automatically continuous everywhere on G.*

PROOF. Since F is positive definite, we may apply the Gelfand-Naimark-Segal (GNS) theorem to get a cyclic unitary representation $(U, \mathscr{H}, v)$, v denoting the cyclic vector, such that $F(g) = \langle v, U(g) v\rangle$, $g \in G$. The stated assertion about continuity for unitary representations is easy to verify; and so it follows for F. $\square$

QUESTION. Suppose $Ext(F) \neq \emptyset$, then what are its extreme points? Equivalently, characterize $ext(Ext(F))$.

Let $\Omega \subset G$, $\Omega \neq \emptyset$, Ω open and connected, and let

$$K_\Omega(\lambda) = \widehat{\chi_\Omega(\lambda)},\ \forall \lambda \in \widehat{G}.$$

THEOREM 4.4.28 (Jo-Pedersen-Tian). *Let $F : \Omega - \Omega \to \mathbb{C}$ be continuous, and positive definite on $\Omega - \Omega$; and assume $Ext(F) \neq \emptyset$. Let $\mu \in Ext(F)$, and let $T_\mu(F_\phi) := \widehat{\varphi}$, defined initially only for $\varphi \in C_c(\Omega)$, be the isometry $T_\mu : \mathscr{H}_F \to L^2(\mu) = L^2(\widehat{G}, \mu)$. Then $Q_\mu := T_\mu T_\mu^*$ is a projection in $L^2(\mu)$* with $K_\Omega(\cdot)$ as kernel:

$$(Q_\mu f)(\lambda) = \int_{\widehat{G}} K_\Omega(\lambda - \xi) f(\xi)\, d\mu(\xi),\ \forall f \in L^2(\widehat{G}, \mu), \forall \lambda \in \widehat{G}. \tag{4.4.52}$$

PROOF. We showed in Theorem 4.4.25 that $T_\mu : \mathscr{H}_F \to L^2(\mu)$ is isometric, and so $Q_\mu := T_\mu T_\mu^*$ is the projection in $L^2(\mu)$. For $f \in L^2(\mu)$, $\lambda \in \widehat{G}$, we have the following computation, where the interchanging of integrals is justified by Fubini's theorem:

$$\begin{aligned}
(Q_\mu f)(\lambda) &= \int_\Omega (f d\mu)^\vee(x) \langle \lambda, x\rangle\, dx \\
&= \int_\Omega \langle \lambda, x\rangle \left(\int_{\widehat{G}} f(\xi) \overline{\langle \xi, x\rangle} d\mu(\xi) \right) dx \\
&\overset{\text{Fubini}}{=} \int_{\widehat{G}} K_\Omega(\lambda - \xi) f(\xi)\, d\mu(\xi)
\end{aligned}$$

which is the desired conclusion (4.4.52). Here, dx denotes the Haar measure on G. $\square$

4.4.8. The case of $G = \mathbb{R}^n$. As a special case of the setting of locally compact Abelian groups from above, the results available for $\mathbb{R}^n$ are more refined. This is also the setting of the more classical studies of extension questions.

Let $\Omega \subset \mathbb{R}^n$ be a fixed open and connected subset; and let $F : \Omega - \Omega \to \mathbb{C}$ be a given continuous and positive definite function defined on

$$\Omega - \Omega := \left\{ x - y \in \mathbb{R}^n \,\middle|\, x, y \in \Omega \right\}. \tag{4.4.53}$$

Let $\mathscr{H}_F$ be the corresponding reproducing kernel Hilbert space (RKHS). We showed that $Ext(F) \neq \emptyset$ if and only if there is a strongly continuous unitary representation $\{U(t)\}_{t\in\mathbb{R}^n}$ acting on $\mathscr{H}_F$ such that

$$\mathbb{R}^n \ni t \mapsto \langle F_0, U(t) F_0 \rangle_{\mathscr{H}_F} \tag{4.4.54}$$

is a p.d. extension of F, extending from (4.4.53) to $\mathbb{R}^n$. Finally, if U is a unitary representation of $G = \mathbb{R}^n$ we denote by $P_U(\cdot)$ the associated projection valued measure (PVM) on $\mathscr{B}(\mathbb{R}^n)$ (= the sigma-algebra of all Borel subsets in $\mathbb{R}^n$).

We have

$$U(t) = \int_{\mathbb{R}^n} e^{it\cdot\lambda} P_U(d\lambda), \; \forall t \in \mathbb{R}^n; \tag{4.4.55}$$

where $t = (t_1, \ldots, t_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $t \cdot \lambda = \sum_{j=1}^n t_j \lambda_j$. Recall, setting

$$d\mu(\cdot) = \|P_U(\cdot) F_0\|^2_{\mathscr{H}_F}, \tag{4.4.56}$$

then the p.d. function on RHS in (4.4.54) satisfies

$$\text{RHS}_{(4.4.54)} = \int_{\mathbb{R}^n} e^{it\cdot\lambda} d\mu(\lambda), \; \forall t \in \mathbb{R}^n. \tag{4.4.57}$$

The purpose of the next theorem is to give an orthogonal splitting of the RKHS $\mathscr{H}_F$ associated to a fixed (Ω, F) when it is assumed that $Ext(F)$ is non-empty. This orthogonal splitting of $\mathscr{H}_F$ depends on a choice of $\mu \in Ext(F)$, and the splitting is into three orthogonal subspaces of $\mathscr{H}_F$, correspond a splitting of spectral types into atomic, absolutely continuous (with respect to Lebesgue measure), and singular.

Theorem 4.4.29 (Jo-Pedersen-Tian). *Let $\Omega \subset \mathbb{R}^n$ be given, $\Omega \neq \emptyset$, open and connected. Suppose F is given p.d. and continuous on $\Omega - \Omega$, and assume $Ext(F) \neq \emptyset$. Let U be the corresponding unitary representations of $G = \mathbb{R}^n$, and let $P_U(\cdot)$ be its associated PVM acting on $\mathscr{H}_F$ (= the RKHS of F.)*

(1) *Then $\mathscr{H}_F$ splits up as an orthogonal sum of three closed and $U(\cdot)$ invariant subspaces*

$$\mathscr{H}_F = \mathscr{H}_F^{(atom)} \oplus \mathscr{H}_F^{(ac)} \oplus \mathscr{H}_F^{(sing)} \tag{4.4.58}$$

with these subspaces characterized as follows:

(a) *The PVM $P_U(\cdot)$ restricted to $\mathscr{H}_F^{(atom)}$ is purely atomic;*

(b) *$P_U(\cdot)$ restricted to $\mathscr{H}_F^{(ac)}$ is absolutely continuous with respect to the Lebesgue measure $d\lambda = d\lambda_1 \cdots d\lambda_n$ on $\mathbb{R}^n$; and*

(c) *$P_U(\cdot)$ is continuous, purely singular, when restricted to $\mathscr{H}_F^{(sing)}$.*

(2) ***Case** $\mathscr{H}_F^{(atom)}$. If $\lambda \in \mathbb{R}^n$ is an atom in $P_U(\cdot)$, i.e., $P_U(\{\lambda\}) \neq 0$, where $\{\lambda\}$ denotes the singleton with λ fixed; then $P_U(\{\lambda\})\mathscr{H}_F$ is one-dimensional, and the function $e_\lambda(x) := e^{i\lambda\cdot x}$, (complex exponential) restricted to Ω, is in $\mathscr{H}_F$. We have:*

$$P_U(\{\lambda\}) \mathscr{H}_F = \mathbb{C} e_\lambda \Big|_\Omega. \tag{4.4.59}$$

***Case** $\mathscr{H}_F^{(ac)}$. If $\xi \in \mathscr{H}_F^{(ac)}$, then it is represented as a continuous function on Ω, and*

$$\langle \xi, F_\varphi \rangle_{\mathscr{H}_F} = \int_\Omega \overline{\xi(x)} \varphi(x) \, dx_{(Lebesgue\ meas.)}, \; \forall \varphi \in C_c(\Omega). \tag{4.4.60}$$

Moreover, there is a $f \in L^2(\mathbb{R}^n, \mu)$ *(where* μ *is given in (*4.4.56*)) such that*

$$\int_\Omega \overline{\xi(x)}\varphi(x)\,dx = \int_{\mathbb{R}^n} \overline{f(\lambda)}\widehat{\varphi}(\lambda)\,d\mu(\lambda), \ \forall\varphi \in C_c(\Omega); \tag{4.4.61}$$

and

$$\xi = (fd\mu)^\vee\Big|_\Omega. \tag{4.4.62}$$

(We say that $(fd\mu)^\vee$ *is the* μ*-extension of* ξ*.)*

Conclusion. Every μ*-extension of* ξ *is continuous on* $\mathbb{R}^n$*, and goes to* 0 *at infinity (in* $\mathbb{R}^n$*,); so the* μ*-extension* $\tilde{\xi}$ *satisfies* $\lim_{|x|\to\infty}\tilde{\xi}(x) = 0$.

Case $\mathscr{H}_F^{(sing)}$***.*** *Vectors* $\xi \in \mathscr{H}_F^{(sing)}$ *are characterized by the following property: The measure*

$$d\mu_\xi(\cdot) := \|P_U(\cdot)\,\xi\|^2_{\mathscr{H}_F} \tag{4.4.63}$$

is continuous and purely singular.

Proof. Most of the proof details are contained in the previous discussion.

For (2), Case $\mathscr{H}_F^{(atom)}$; suppose $\lambda \in (\mathbb{R}^n)$ is an atom, and that $\xi \in \mathscr{H}_F \backslash \{0\}$ satisfies

$$P_U(\{\lambda\})\,\xi = \xi; \tag{4.4.64}$$

then

$$U(t)\,\xi = e^{it\cdot\lambda}\xi, \ \forall t \in \mathbb{R}^n. \tag{4.4.65}$$

Using now (4.4.54)-(4.4.55), we conclude that ξ (as a continuous function on $\mathbb{R}^n$) is a weak solution to the following elliptic system

$$\frac{\partial}{\partial x_j}\xi = \sqrt{-1}\lambda_j\xi \ (\text{on } \Omega), \ 1 \le j \le n. \tag{4.4.66}$$

Hence $\xi = \text{const} \cdot e_\lambda\Big|_\Omega$ as asserted in (2).

Case (2), $\mathscr{H}_F^{(ac)}$ follows from (4.4.62) and the Riemann-Lebesgue theorem applied to $\mathbb{R}^n$; and case $\mathscr{H}_F^{(sing)}$ is immediate. □

Example 4.4.30. Consider the following continuous positive definite function F on $\mathbb{R}$, or on some bounded interval $(-a, a)$, $a > 0$.

$$F(x) = \frac{1}{3}\left(e^{-ix} + \prod_{n=1}^{\infty}\cos\left(\frac{2\pi x}{3^n}\right) + e^{i3x/2}\frac{\sin(x/2)}{(x/2)}\right). \tag{4.4.67}$$

(1) This is the decomposition (4.4.58) of the corresponding RKHSs $\mathscr{H}_F$, all three subspaces $\mathscr{H}_F^{(atom)}$, $\mathscr{H}_F^{(ac)}$, and $\mathscr{H}_F^{(sing)}$ are non-zero; the first one is one-dimensional, and the other two are infinite-dimensional.

(2) The operator

$$D^{(F)}(F_\varphi) := F_{\varphi'} \text{ on } dom(D^{(F)}) = \{F_\varphi \mid \varphi \in C_c^\infty(0, a)\} \tag{4.4.68}$$

is bounded, and so extends by closure to a skew-adjoint operator, i.e., $\overline{D^{(F)}} = -(D^{(F)})^*$.

PROOF. Using infinite convolutions of operators, and results from [**DJ12a**], we conclude that F defined in (4.4.67) is entire analytic, and $F = \widehat{d\mu}$ (Bochner-transform) where

$$d\mu\left(\lambda\right) = \frac{1}{3}\left(\delta_{-1} + \mu_{Cantor} + \chi_{[1,2]}\left(\lambda\right)d\lambda\right). \tag{4.4.69}$$

The measures on the RHS in (4.4.69) are as follows: □

- δ_{-1} is the Dirac mass at -1, i.e., $\delta\left(\lambda+1\right)$.
- μ_{Cantor} = the middle-third Cantor measure μ_c determined as the unique solution in $\mathscr{M}_{+}^{prob}\left(\mathbb{R}\right)$ to

$$\int f\left(\lambda\right)d\mu_c\left(\lambda\right) = \frac{1}{2}\left(\int f\left(\frac{\lambda+1}{3}\right)d\mu_c\left(\lambda\right) + \int f\left(\frac{\lambda-1}{3}\right)d\mu_c\right)$$

  for all $f \in C_c\left(\mathbb{R}\right)$; and the last term
- $\chi_{[1,2]}\left(\lambda\right)d\lambda$ is restriction to the closed interval $[1,2]$ of the Lebesgue measure.

It follows from the literature (e.g. [**DJ12a**]) that μ_c is supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$; and so the three measures on the RHS in (4.4.69) have disjoint compact support, with the three supports positively separately.

The conclusions asserted in the example follow from this, in particular the properties for $D^{(F)}$, in fact

$$\text{spectrum}\left(D^{(F)}\right) \subseteq \{-1\} \cup \left[-\tfrac{1}{2},\tfrac{1}{2}\right] \cup [1,2] \tag{4.4.70}$$

4.4.9. Lie groups.

DEFINITION 4.4.31. Let G be a Lie group. We consider the extension problem for continuous positive definite functions

$$F : \Omega^{-1}\Omega \to \mathbb{C} \tag{4.4.71}$$

where $\Omega \neq \emptyset$, is a connected and open subset in G, i.e., it is assumed that

$$\sum\nolimits_i \sum\nolimits_j \overline{c_i} c_j F\left(x_j^{-1}x_i\right) \geq 0, \tag{4.4.72}$$

for all finite systems $\{c_i\} \subset \mathbb{C}$, and points $\{x_i\} \subset \Omega$. Equivalent,

$$\int_\Omega \overline{\varphi\left(x\right)}\varphi\left(y\right)F\left(y^{-1}x\right)dxdy \geq 0, \tag{4.4.73}$$

for all $\varphi \in C_c\left(\Omega\right)$; where dx denotes a choice of left-invariant Haar measure on G.

LEMMA 4.4.32. *Let F be defined as in (4.4.71)-(4.4.72); and for all $X \in La(G)$ = the Lie algebra of G, set*

$$(\tilde{X}\varphi)\left(g\right) := \frac{d}{dt}\varphi\left(\exp_G\left(-tX\right)g\right) \tag{4.4.74}$$

for all $\varphi \in C_c^\infty\left(\Omega\right)$. Set

$$F_\varphi\left(x\right) := \int_\Omega \varphi\left(y\right)F\left(y^{-1}x\right)dy; \tag{4.4.75}$$

then

$$S_X^{(F)}\left(F_\varphi\right) := F_{\tilde{X}\varphi},\ \varphi \in C_c^\infty\left(\Omega\right) \tag{4.4.76}$$

defines a representation of the Lie algebra $La(G)$ *by skew-Hermitian operators in the RKHS* $\mathscr{H}_F$*, with the operator in (*4.4.76*) defined on the common dense domain* $\{F_\varphi \mid \varphi \in C_c^\infty(\Omega)\} \subset \mathscr{H}_F$.

PROOF. The arguments here follow those of the proof of Lemma 4.4.7 *mutatis mutandis.* □

DEFINITION 4.4.33.

(1) We say that a continuous p.d. function $F : \Omega^{-1}\Omega \to \mathbb{C}$ is extendable if and only if there is a continuous p.d. function $F_{ex} : G \to G$ such that

$$F_{ex}\Big|_{\Omega^{-1}\Omega} = F. \tag{4.4.77}$$

(2) Let $U \in Rep(G, \mathscr{K})$ be a strongly continuous unitary representation of G acting in some Hilbert space $\mathscr{K}$. We say that $U \in Ext(F)$ iff (Def.) there is an isometry $\mathscr{H}_F \hookrightarrow \mathscr{K}$ such that the function

$$G \ni g \mapsto \langle JF_e, U(g) JF_e \rangle_{\mathscr{K}} \tag{4.4.78}$$

satisfies the condition in (1).

THEOREM 4.4.34. *Every extension of some continuous p.d. function* F *on* $\Omega^{-1}\Omega$ *as in (*1*) arises from a unitary representation of* G *as specified in (*2*).*

PROOF. First assume some unitary representation U of G satisfies (2), then (4.4.78) is an extension of F. This follows from the argument in our proof of Lemma 4.4.7.

For the converse; assume some continuous p.d. function F_{ex} on G satisfies (4.4.77). Now apply the GNS-theorem to F_{ex}; and, as a result, we get a cyclic representation $(U, \mathscr{K}, v_0)$ where

- $\mathscr{K}$ is a Hilbert space;
- U is a strongly continuous unitary representation of G acting on $\mathscr{K}$; and
- $v_0 \in \mathscr{K}$ is a cyclic vector, $\|v_0\| = 1$; and

$$F_{ex}(g) = \langle v_0, U(g) v_0 \rangle, \ g \in G. \tag{4.4.79}$$

Defining now $J : \mathscr{H}_F \to \mathscr{K}$ as follows,

$$J(F(\cdot g)) := U(g^{-1})v_0, \ \forall g \in \Omega;$$

and extension by limit, we check that J is isometric and satisfies the condition from (2) in Definition 4.4.33. We omit details as they parallel arguments already contained in Subsection 4.4.2.

□

THEOREM 4.4.35. *Let* Ω*,* G*,* $La(G)$*, and* $F : \Omega^{-1}\Omega \to \mathbb{C}$ *be as in Definition* 4.4.31. *Let* $\tilde{G}$ *be the simply connected universal covering group for* G*. Then* F *has an extension to a p.d. continuous function on* $\tilde{G}$ *if and only if there is a unitary representation* U *of* $\tilde{G}$ *and an isometry* $\mathscr{H}_F \xrightarrow{J} \mathscr{K}$ *such that*

$$JS_X^{(F)} = dU(X) J \tag{4.4.80}$$

holds on $\{F_\varphi \mid \varphi \in C_c^\infty(\Omega)\}$*, for all* $X \in La(G)$*; where*

$$\begin{aligned} dU(X) U(\varphi) v_0 &= U(\tilde{X}\varphi) v_0, \ \text{and} \\ U(\varphi) &= \int_{\tilde{G}} \varphi(g) U(g^{-1}) \, dg. \end{aligned}$$

PROOF. Details are contained in Subsection 4.4.5. □

Assume G is connected. Note that on $C_c^\infty(\Omega)$, the Lie group G acts locally, i.e., by $\varphi \mapsto \varphi_g$ where φ_g denotes translation of φ by some element $g \in G$, such that φ_g is also supported in Ω. Then

$$\|F_\varphi\|_{\mathscr{H}_F} = \left\|F_{\varphi_g}\right\|_{\mathscr{H}_F}; \tag{4.4.81}$$

but only for elements $g \in G$ in a neighborhood of $e \in G$, and with the neighborhood depending on φ.

We recall the following corollaries, and refer to [**Jor86**, **Jor87**] for details.

COROLLARY 4.4.36. *Given*

$$F : \Omega^{-1} \cdot \Omega \to \mathbb{C} \tag{4.4.82}$$

continuous and positive definite, then set

$$L_g(F_\varphi) := F_{\varphi_g},\ \varphi \in C_c^\infty(\Omega), \tag{4.4.83}$$

defining a local representation of G *in* $\mathscr{H}_E$.

COROLLARY 4.4.37. *Given* F*, positive definite and continuous, as in (*4.4.82*), and let* L *be the corresponding local representation of* G *acting on* $\mathscr{H}_F$*. Then* $Ext(F) \neq \emptyset$ *if and only if the local representation (*4.4.83*) extends to a global unitary representation acting in some Hilbert space* $\mathscr{K}$*, containing* $\mathscr{H}_F$ *isometrically.*

4.4.10. Type I v.s. Type II extensions. The material below is adapted primarily from [**JPT15b**, **JPT16**].

In this section, we identify extensions of the initially give p.d. function F which are associated with operator extensions in the RKHS $\mathscr{H}_F$ itself (Type I), and those which require an enlargement of $\mathscr{H}_F$ (Type II). In the case of $G = \mathbb{R}$ (the real line) some of these continuous p.d. extensions arising from the second construction involve a spline-procedure, and a theorem of G. Polya [**Ṕ49**], which leads to p.d. extensions of F that are symmetric, compactly supported in an interval around $x = 0$, and convex on the left and right half-lines. For splines and positive definite functions, we refer to [**GSS83**, **Sch83**].

Part of this is the construction of Polya extensions as follow: Starting with a *convex* p.d. F on $(-a, a)$; we create a new F_{ex} on $\mathbb{R}$, such that $F_{ex}\big|_{\mathbb{R}_+}$ is convex, and $F_{ex}(-x) = F_{ex}(x)$. Polya's theorem [**Ṕ49**] states that F_{ex} is positive definite.

In Figure 4.4.1, the slope of L_+ is chosen to be $F'(a)$; and we take the slope of L_- to be $F'(-a) = -F'(a)$. Recall that F is defined initially only on some fixed interval $(-a, a)$. It then follows by Polya's theorem that each of these spline extensions is continuous and positive definite.

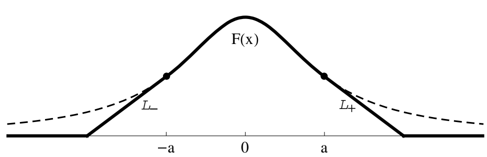


FIGURE 4.4.1. Spline extension of $F : (-a, a) \to \mathbb{R}$

After extending F from $(-a, a)$ by adding one or more line-segments over $\mathbb{R}_+$, and using symmetry by $x = 0$; the lines in the extensions will be such that there is a c, $0 < a < c$, and the extension F_{ex} satisfies $F_{ex}(x) = 0$ for all $|x| \geq c$. See Figure 4.4.2 below.

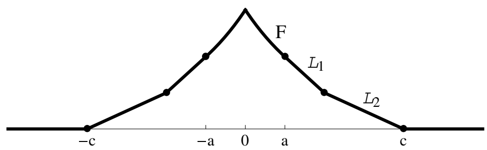


FIGURE 4.4.2. An example of Polya extension of F on $(-a, a)$.

PROPOSITION 4.4.38. *Given $F : (-a, a) \to \mathbb{C}$, and assume F has a Polya extension F_{ex}, then the corresponding measure $\mu_{ex} \in Ext(F)$ has the following form*

$$d\mu_{ex}(\lambda) = \Phi_{ex}(\lambda)\, d\lambda$$

where

$$\Phi_{ex}(\lambda) = \frac{1}{2\pi} \int_{-c}^{c} e^{-i\lambda y} F_{ex}(y)\, dy$$

is entire analytic in λ.

PROOF. An application of Fourier inversion, and the Paley-Wiener theorem. ☐

EXAMPLE 4.4.39 (Cauchy distribution). $F_1(x) = \dfrac{1}{1+x^2}$; $|x| < 1$. F_1 is concave near $x = 0$.

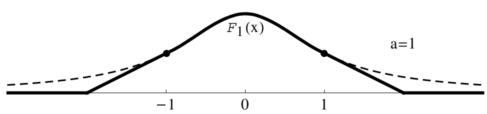


FIGURE 4.4.3. Extension of $F_1(x) = \frac{1}{1+x^2}$; $\Omega = (0, 1)$

EXAMPLE 4.4.40. $F_2(x) = 1 - |x|$; $|x| < \frac{1}{2}$. Consider the following Polya extension:

$$F(x) = \begin{cases} 1 - |x| & \text{if } |x| < \frac{1}{2} \\ \dfrac{1}{3}(2 - |x|) & \text{if } \frac{1}{2} \leq |x| < 2 \\ 0 & \text{if } |x| \geq 2 \end{cases}$$

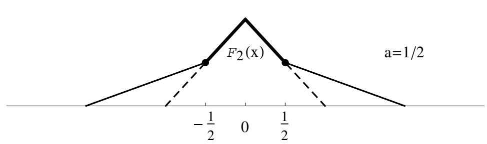


FIGURE 4.4.4. Extension of $F_2(x) = 1 - |x|$; $\Omega = \left(0, \frac{1}{2}\right)$

This is a p.d. spline extension which is convex on $\mathbb{R}_+$. The corresponding measure $\mu \in Ext\,(F)$ has the following form $d\mu\,(\lambda) = \Phi\,(\lambda)\,d\lambda$, where $d\lambda =$ Lebesgue measure on $\mathbb{R}$, and where

$$\Phi\,(\lambda) = \begin{cases} \frac{3}{4\pi} & \text{if } \lambda = 0 \\ \frac{1}{3\pi\lambda^2}\,(3 - 2\cos(\lambda/2) - \cos(2\lambda)) & \text{if } \lambda \neq 0. \end{cases}$$

This solution is in $Ext_2\,(F)$. By contrast, the measure μ_2 in Table 4 satisfies $\mu_2 \in Ext_1(F)$.

EXAMPLE 4.4.41 (Ornstein-Uhlenbeck). $F_3\,(x) = e^{-|x|}$; $|x| < 1$. A p.d. spline extension which is convex on $\mathbb{R}_+$.

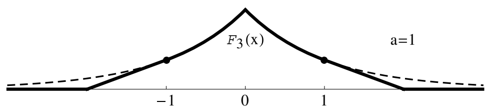


FIGURE 4.4.5. Extension of $F_3\,(x) = e^{-|x|}$; $\Omega = (0, 1)$

EXAMPLE 4.4.42 (Shannon). $F_4\,(x) = \left(\dfrac{\sin \pi x}{\pi x}\right)^2$; $|x| < \frac{1}{2}$. F_4 is concave near $x = 0$.

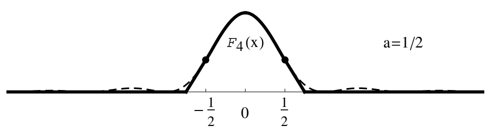


FIGURE 4.4.6. Extension of $F_4\,(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$; $\Omega = \left(0, \frac{1}{2}\right)$

EXAMPLE 4.4.43 (Gaussian distribution). $F_5\,(x) = e^{-x^2/2}$; $|x| < 1$. F_5 is concave in $-1 < x < 1$.

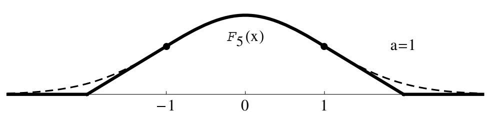


FIGURE 4.4.7. Extension $F_5\,(x) = e^{-x^2/2}$; $\Omega = (0, 1)$

Some spline extensions may not be positive definite. In order for Polya's theorem to be applicable, the spline extended function F_{ex} to $\mathbb{R}$ must be convex on $\mathbb{R}_+$. By construction, our extension to $\mathbb{R}$ is by mirror symmetry around $x = 0$. If we start with a symmetric p.d. function F in $(-a, a)$ which is concave near $x = 0$, then the spline extension does not satisfy the premise in Polya's theorem.

For example, it is easy to check that the two partially defined functions F in Figures 4.4.3 and 4.4.6 are concave near $x = 0$ (just calculate the double derivative F''). The corresponding spline extensions are *not* positive definite.

Polya's theorem only applies when convexity holds on $\mathbb{R}_+$. In that case, the spline extensions will be p.d.. And so Polya's theorem only accounts for those spline extensions F_{ex} which are convex when restricted to $\mathbb{R}_+$. Now there may be p.d. spline extensions that are not convex when restricted to $\mathbb{R}_+$, and we leave open the discovery of those. See Figures 4.4.1-4.4.8.

Of the p.d. functions in Table 2, we note that F_1, F_4, F_5, and F_6 satisfy this: there is a $c > 0$ such that the function in question is concave in the interval $[0, c]$, the value of c varies from one to the next. So these four cases do not yield spline extensions F_{ex} which are convex when restricted to $\mathbb{R}_+$.

We get the nicest spline extensions if we make the derivative $F' = \frac{dF}{dx}$ a spline at the break-points. In Examples 4.4.39-4.4.43, we compute $F'(a)$, see Table 1. We then use symmetry for the left-hand-side of the figure.

TABLE 1. Spline extension at break-points.

$F_1'(1) = -1/2$	$F_4'(1/2) = -16\pi^{-2}$
$F_2'(1/2) = -1$	$F_5'(1) = -e^{-1/2}$
$F_3'(1) = -e^{-1}$	

For each locally defined p.d. function F_i, we then get a deficiency index-problem in the RKHSs $\mathscr{H}_{F_i}$, $i = 1, \ldots, 5$, for the operator $D^{(F_i)} F_\varphi^{(i)} = F_{\varphi'}^{(i)}$, $\forall \varphi \in C_c^\infty(0, a)$. And all the five skew-Hermitian operators in $\mathscr{H}_{F_i}$ will have deficiency indices $(1, 1)$.

The following is an example with deficiency indices $(0, 0)$

EXAMPLE 4.4.44. $F_6(x) = \cos(x)$; $|x| < \frac{\pi}{4}$

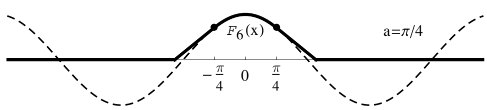


FIGURE 4.4.8. Extension of $F_6(x) = \cos(x)$; $\Omega = \left(0, \frac{\pi}{4}\right)$

LEMMA 4.4.45. *$\mathscr{H}_{F_6}$ is 2-dimensional.*

PROOF. Left for the reader. □

Thus, in all five examples above, $\mathscr{H}_{F_i}$ $(i = 1, \ldots, 5)$ is infinite-dimensional; but $\mathscr{H}_{F_6}$ is 2-dimensional.

In the given five examples, we have p.d. continuous extensions to $\mathbb{R}$ of the following form, $\widehat{d\mu_i}(\cdot)$, $i = 1, \ldots, 5$, where these measures are given in Table 5; also see Figure 4.4.10.

COROLLARY 4.4.46. *In all five examples above, we get isometries as follows*

$$T^{(i)} : \mathscr{H}_{F_i} \to L^2(\mathbb{R}, \mu_i)$$

$$T^{(i)}\left(F_\varphi^{(i)}\right) = \widehat{\varphi}$$

for all $\varphi \in C_c^\infty(\Omega)$, where we note that $\widehat{\varphi} \in L^2(\mathbb{R}, \mu_i)$, $i = 1, \ldots, 5$; and

$$\left\| F_\varphi^{(i)} \right\|_{\mathscr{H}_{F_i}}^2 = \|\widehat{\varphi}\|_{L^2(\mu)}^2 = \int_{\mathbb{R}} |\widehat{\varphi}|^2 \, d\mu_i, \; i = 1, \ldots, 5;$$

$$
\begin{tikzcd}
 & & & \underset{2-\dim}{\mathrm{RKHS}(\cos(x))} \arrow[d] \\
F_{\varphi} \in \mathscr{H}_F \text{ on } \left(-\frac{\pi}{4},\frac{\pi}{4}\right) \arrow[urrr, bend left] & & & \\
\varphi \in C_c^{\infty}\left(-\frac{\pi}{4},\frac{\pi}{4}\right) \arrow[r, "T(F_{\varphi})=\widehat{\varphi}"] & \mathrm{RKHS}\,(\mathrm{Polya}) \arrow[rr, phantom, "\underset{\infty-\dim}{\simeq}"] & & L^2\left(\mathbb{R}, d\mu_{\mathrm{Polya}}\right)
\end{tikzcd}
$$

FIGURE 4.4.9. $\dim\left(\mathrm{RKHS}\left(\cos x \text{ on } \mathbb{R}\right)\right) = 2$; but RKHS (Polya ext. to $\mathbb{R}$) is ∞-dimensional.

but note that $T^{(i)}$ is only isometric into $L^2(\mu_i)$.

For the adjoint operator:

$$\left(T^{(i)}\right)^* : L^2(\mathbb{R},\mu_i) \to \mathscr{H}_{F_i}$$

we have

$$\left(T^{(i)}\right)^* f = (f d\mu_i)^{\vee}, \ \forall f \in L^2(\mathbb{R},\mu_i).$$

Here is an infinite-dimensional example as a version of F_6. Fix some positive p, $0 < p < 1$, and set

$$\prod_{n=1}^{\infty} \cos(2\pi p^n x) = F_p(x)$$

then this is a continuous positive definite function on $\mathbb{R}$, and the law is the corresponding Bernoulli measure μ_p satisfying $F_p = \widehat{d\mu_p}$. Note that some of those measures μ_p are fractal measures.

For fixed $p \in (0,1)$, the measure μ_p is the law of the following random power series

$$X_p(w) := \sum_{n=1}^{\infty} (\pm)\, p^n$$

where $w \in \prod_1^{\infty}\{\pm 1\}$ (= infinite Cartesian product) and where the distribution of each factor is $\left\{-\frac{1}{2},\frac{1}{2}\right\}$, and statically independent. For relevant references on random power series, see [**Neu13**, **Lit99**].

The extensions we generate with the application of Polya's theorem are realized in a bigger Hilbert space. The deficiency indices are computed for the RKHS $\mathscr{H}_F$, i.e., for the "small" p.d. function $F : \Omega - \Omega \to \mathbb{C}$.

EXAMPLE 4.4.47. F_6 on $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ has the obvious extension $\mathbb{R} \ni x \to \cos x$, with a 2-dimensional Hilbert space; but the other p.d. extensions (from Polya) will be in infinite-dimensional Hilbert spaces. See Figure Figure 4.4.9.

We must make a distinction between two classes of p.d. extensions of $F : \Omega - \Omega \to \mathbb{C}$ to continuous p.d. functions on $\mathbb{R}$.

Case 1. There exists a unitary representation $U(t) : \mathscr{H}_F \to \mathscr{H}_F$ such that

$$F(t) = \langle \xi_0, U(t)\,\xi_0 \rangle_{\mathscr{H}_F}, \ t \in \Omega - \Omega \tag{4.4.84}$$

Case 2. (e.g., Polya extension) There exist a Hilbert space $\mathscr{K}$, and an isometry $J : \mathscr{H}_F \to \mathscr{K}$, and a unitary representation $U(t) : \mathscr{K} \to \mathscr{K}$, such that

$$F(t) = \langle J\xi_0, U(t) J\xi_0 \rangle_{\mathscr{K}}, \; t \in \Omega - \Omega \tag{4.4.85}$$

In both cases, $\xi_0 = F(0 - \cdot) \in \mathscr{H}_F$.

In case 1, the unitary representation is realized in $\mathscr{H}_{(F,\Omega-\Omega)}$, while, in case 2, the unitary representation $U(t)$ lives in the expanded Hilbert space $\mathscr{K}$.

Note that the RHS in both (4.4.84) and (4.4.85) is defined for all $t \in \mathbb{R}$.

LEMMA 4.4.48. *Let F_{ex} be one of the Polya extensions if any. Then by the Galfand-Naimark-Segal (GNS) construction applied to $F_{ext} : \mathbb{R} \to \mathbb{R}$, there is a Hilbert space $\mathscr{K}$ and a vector $v_0 \in \mathscr{K}$ and a unitary representation $\{U(t)\}_{t\in\mathbb{R}}$; $U(t) : \mathscr{K} \to \mathscr{K}$, such that*

$$F_{ex}(t) = \langle v_0, U(t) v_0 \rangle_{\mathscr{K}}, \; \forall t \in \mathbb{R}. \tag{4.4.86}$$

Setting $J : \mathscr{H}_F \to \mathscr{K}$, $J\xi_0 = v_0$, then J defines (by extension) an isometry such that

$$U(t) J\xi_0 = J(\text{local translation in } \Omega) \tag{4.4.87}$$

holds locally (i.e., for t sufficiently close to 0.)

Moreover, the function

$$\mathbb{R} \ni t \mapsto U(t) J\xi_0 = U(t) v_0 \tag{4.4.88}$$

is compactly supported.

PROOF. The existence of $\mathscr{K}$, v_0, and $\{U(t)\}_{t\in\mathbb{R}}$ follows from the GNS-construction.

The conclusions in (4.4.87) and (4.4.88) follow from the given data, i.e., $F : \Omega - \Omega \to \mathbb{R}$, and the fact that F_{ex} is a spline-extension, i.e., it is of compact support; but by (4.4.86), this means that (4.4.88) is also compactly supported. $\square$

Example 4.4.40 gives a p.d. F in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ with $D^{(F)}$ of index $(1,1)$ and explicit measures in $Ext_1(F)$ and in $Ext_2(F)$.

We have the following:

Deficiency $(0,0)$**:** The p.d. extension of type 1 is unique; see (4.4.84); but there may still be p.d. extensions of type 2; see (4.4.85).

Deficiency $(1,1)$**:** This is a one-parameter family of extensions of type 1; and some more p.d. extensions are type 2.

So we now divide

$$Ext(F) = \left\{ \mu \in \text{Prob}(\mathbb{R}) \;\middle|\; \widehat{d\mu} \text{ is an extension of } F \right\}$$

up in subsets

$$Ext(F) = Ext_{type1}(F) \cup Ext_{type2}(F);$$

where $Ext_{type2}(F)$ corresponds to the Polya extensions.

Return to a continuous p.d. function $F : (-a, a) \to \mathbb{C}$, we take for the RKHS $\mathscr{H}_F$, and the skew-Hermitian operator

$$D(F_\varphi) = F_{\varphi'}, \; \varphi' = \frac{d\varphi}{dx}$$

TABLE 2. The deficiency indices of $D^{(F)} : F_\varphi \mapsto F_{\varphi'}$ in examples 4.4.39-4.4.44

$F : (-a,a) \to \mathbb{C}$	Indices	The Operator $D^{(F)}$
$F_1(x) = \frac{1}{1+x^2}$, $\lvert x\rvert < 1$	$(0,0)$	$D^{(F)}$ unbounded, skew-adjoint
$F_2(x) = 1 - \lvert x\rvert$, $\lvert x\rvert < \frac{1}{2}$	$(1,1)$	$D^{(F)}$ has unbounded sk. adj. extensions
$F_3(x) = e^{-\lvert x\rvert}$, $\lvert x\rvert < 1$	$(1,1)$	$D^{(F)}$ has unbounded sk. adj. extensions
$F_4(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$, $\lvert x\rvert < \frac{1}{2}$	$(0,0)$	$D^{(F)}$ bounded, skew-adjoint
$F_5(x) = e^{-x^2/2}$, $\lvert x\rvert < 1$	$(0,0)$	$D^{(F)}$ unbounded, skew-adjoint
$F_6(x) = \cos x$, $\lvert x\rvert < \frac{\pi}{4}$	$(0,0)$	$D^{(F)}$ is rank-one, $\dim(\mathscr{H}_{F_6}) = 2$

TABLE 3. Type II extensions. Six cases of p.d. continuous functions F_i defined on a finite interval $(-a,a)$.

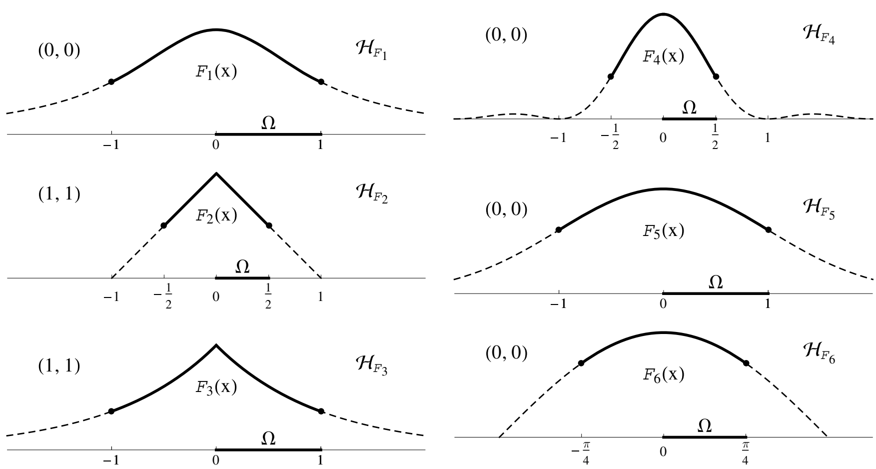


TABLE 4. The canonical isometric embeddings: $\mathscr{H}_{F_i} \hookrightarrow L^2(\mathbb{R}, d\mu_i)$, $i = 1, \ldots, 6$.

$d\mu_1(\lambda) = \frac{1}{2} e^{-\lvert\lambda\rvert} d\lambda$	$d\mu_4(\lambda) = \chi_{(-1,1)}(\lambda)(1 - \lvert\lambda\rvert)\, d\lambda$, cpt. support
$d\mu_2(\lambda) = \left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2 d\lambda$, Shannon	$d\mu_5(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda$, Gaussian
$d\mu_3(\lambda) = \frac{d\lambda}{\pi(1+\lambda^2)}$, Cauchy	$d\mu_6(\lambda) = \frac{1}{2}(\delta_1 + \delta_{-1})$, atomic; two Dirac masses

If $D \subseteq A$, $A^* = -A$ in $\mathscr{H}_F$ then there exists an isometry $J : \mathscr{H}_F \to L^2(\mathbb{R}, \mu)$, where $d\mu(\cdot) = \|P_U(\cdot)\xi_0\|^2$,

$$U_A(t) = e^{tA} = \int_{\mathbb{R}} e^{it\lambda} P_U(d\lambda),$$

$\xi_0 = F(\cdot - 0) \in \mathscr{H}_F$, $J\xi_0 = 1 \in L^2(\mu)$.

4.4.10.1. *Models for operator extensions.* A special case of our extension question for continuous positive definite functions on a fixed finite interval $|x| < a$ in $\mathbb{R}$ is the following: It offers a spectral model representation for ALL Hermitian operators with dense domain in Hilbert space and with deficiency indices $(1, 1)$.

Specifically, on $\mathbb{R}$, all the partially defined continuous p.d. functions extend, and we can make a translation of our p.d. problem into the problem of finding all $(1, 1)$ restrictions selfadjoint operators.

By the Spectral theorem, every selfadjoint operator with simple spectrum has a representation as a multiplication operator M_λ in some $L^2(\mathbb{R}, \mu)$ for some probability measure μ on $\mathbb{R}$. So this accounts for all Hermitian restrictions operators with deficiency indices $(1, 1)$.

Model for restrictions of continuous p.d. functions on $\mathbb{R}$. Let $\mathscr{H}$ be a Hilbert space, A a skew-adjoint operator, $A^* = -A$, which is *unbounded*; let $v_0 \in \mathscr{H}$ satisfying $\|v_0\|_{\mathscr{H}} = 1$. Then we get an associated p.d. continuous function F_A defined on $\mathbb{R}$ as follows:

$$F_A(t) := \left\langle v_0, e^{tA} v_0 \right\rangle = \left\langle v_0, U_A(t) v_0 \right\rangle, \ t \in \mathbb{R}, \tag{4.4.89}$$

where $U_A(t) = e^{tA}$ is a unitary representation of $\mathbb{R}$. Note that $U_A(t)$ is defined by the Spectral Theorem, and (4.4.89) holds for all $t \in \mathbb{R}$.

Let $P_U(\cdot)$ be the projection-valued measure (PVM) of A, then

$$U(t) = \int_{-\infty}^{\infty} e^{i\lambda t} P_U(d\lambda), \ \forall t \in \mathbb{R}. \tag{4.4.90}$$

LEMMA 4.4.49.

(i) *Setting* $d\mu = \|P_U(d\lambda) v_0\|^2$, *we then get*

$$F_A(t) = \widehat{d\mu}(t), \ \forall t \in \mathbb{R} \tag{4.4.91}$$

Moreover, every probability measure μ *on* $\mathbb{R}$ *arises this way.*

(ii) *For Borel functions* f *on* $\mathbb{R}$, *let*

$$f(A) = \int_{\mathbb{R}} f(\lambda) P_U(d\lambda) \tag{4.4.92}$$

be given by functional calculus. We note that

$$v_0 \in dom(f(A)) \Longleftrightarrow f \in L^2(\mu) \tag{4.4.93}$$

where μ *is the measure in part (i). Then*

$$\|f(A) v_0\|^2 = \int_{\mathbb{R}} |f|^2 d\mu. \tag{4.4.94}$$

PROOF. (i) A direct computation using (4.4.89). (ii) This is an application of the Spectral Theorem. □

Now we consider restriction of F_A to, say $(-1, 1)$, i.e.,

$$F(\cdot) = F_A\Big|_{(-1,1)}(\cdot) \tag{4.4.95}$$

LEMMA 4.4.50. *Let $\mathscr{H}_F$ be the RKHS computed for F in (4.4.91); and for $\varphi \in C_c(0,1)$, set $F_\varphi =$ the generating vectors in $\mathscr{H}_F$, as usual. Set*

$$U(\varphi) := \int_0^1 \varphi(y)\, U(-y)\, dy \tag{4.4.96}$$

where $dy =$ Lebesgue measure on $(0,1)$; then

$$F_\varphi(x) = \langle v_0, U(x)\, U(\varphi)\, v_0\rangle, \ \forall x \in (0,1). \tag{4.4.97}$$

PROOF. We have

$$\begin{aligned} F_\varphi(x) &= \int_0^1 \varphi(y)\, F(x-y)\, dy \\ &\overset{\text{(by (4.4.89))}}{=} \int_0^1 \varphi(y) \langle v_0, U_A(x-y)\, v_0\rangle\, dy \\ &= \left\langle v_0, U_A(x) \int_0^1 \varphi(y)\, U_A(-y)\, v_0 dy \right\rangle \\ &\overset{\text{(by (4.4.96))}}{=} \langle v_0, U_A(x)\, U(\varphi)\, v_0\rangle \\ &= \langle v_0, U(\varphi)\, U_A(x)\, v_0\rangle \end{aligned}$$

for all $x \in (0,1)$, and all $\varphi \in C_c(0,1)$. □

COROLLARY 4.4.51. *Let A, $U(t) = e^{tA}$, $v_0 \in \mathscr{H}$, $\varphi \in C_c(0,1)$, and F p.d. on $(0,1)$ be as above; let $\mathscr{H}_F$ be the RKHS of F; then, for the inner product in $\mathscr{H}_F$, we have*

$$\langle F_\varphi, F_\psi\rangle_{\mathscr{H}_F} = \langle U(\varphi)\, v_0, U(\psi)\, v_0\rangle_{\mathscr{H}}, \ \forall \varphi, \psi \in C_c(0,1). \tag{4.4.98}$$

PROOF. Note that

$$\begin{aligned} \langle F_\varphi, F_\psi\rangle_{\mathscr{H}_F} &= \int_0^1 \int_0^1 \overline{\varphi(x)}\psi(y)\, F(x-y)\, dxdy \\ &\overset{\text{(by (4.4.95))}}{=} \int_0^1 \int_0^1 \overline{\varphi(x)}\psi(y) \langle v_0, U_A(x-y)\, v_0\rangle_{\mathscr{H}}\, dxdy \\ &= \int_0^1 \int_0^1 \langle \varphi(x)\, U_A(-x)\, v_0, \psi(y)\, U_A(-y)\, v_0\rangle_{\mathscr{H}}\, dxdy \\ &\overset{\text{(by (4.4.96))}}{=} \langle U(\varphi)\, v_0, U(\psi)\, v_0\rangle_{\mathscr{H}} \end{aligned}$$

□

COROLLARY 4.4.52. *Set $\varphi^{\#}(x) = \overline{\varphi(-x)}$, $x \in \mathbb{R}$, $\varphi \in C_c(\mathbb{R})$, or in this case, $\varphi \in C_c(0,1)$; then we have:*

$$\langle F_\varphi, F_\psi\rangle_{\mathscr{H}_F} = \left\langle v_0, U\left(\varphi^{\#} * \psi\right) v_0\right\rangle_{\mathscr{H}}, \ \forall \varphi, \psi \in C_c(0,1). \tag{4.4.99}$$

PROOF. Immediate from (4.4.98) and Fubini. □

COROLLARY 4.4.53. *Let F and $\varphi \in C_c(0,1)$ be as above; then in the RKHS $\mathscr{H}_F$ we have:*

$$\|F_\varphi\|^2_{\mathscr{H}_F} = \|U(\varphi)\, v_0\|^2_{\mathscr{H}} = \int |\widehat{\varphi}|^2\, d\mu \tag{4.4.100}$$

where μ is the measure in part (i) of Lemma 4.4.49. $\widehat{\varphi} =$ Fourier transform: $\widehat{\varphi}(\lambda) = \int_0^1 e^{-i\lambda x}\varphi(x)\, dx$, $\lambda \in \mathbb{R}$.

PROOF. Immediate from (4.4.99); indeed:

$$\begin{aligned}\|F_\varphi\|^2_{\mathscr{H}_F} &= \int_0^1\int_0^1 \overline{\varphi(x)}\varphi(y)\int_{\mathbb{R}} e_\lambda(x-y)\,d\mu(\lambda)\\ &= \int_{\mathbb{R}} |\widehat{\varphi}(\lambda)|^2\,d\mu(\lambda),\ \forall\varphi\in C_c(0,1).\end{aligned}$$

□

COROLLARY 4.4.54. *Every Borel probability measure μ on $\mathbb{R}$ arises this way.*

PROOF. We shall need to following:

LEMMA 4.4.55. *Let A, $\mathscr{H}$, $\{U_A(t)\}_{t\in\mathbb{R}}$, $v_0\in\mathscr{H}$ be as above; and set*

$$d\mu = d\mu_A(\cdot) = \|P_U(\cdot)v_0\|^2 \tag{4.4.101}$$

as in Lemma 4.4.49. Assume v_0 is cyclic; then $W_\mu f(A)v_0 = f$ defines a unitary isomorphism $W_\mu:\mathscr{H}\to L^2(\mu)$; and

$$W_\mu U_A(t) = e^{it\cdot}W_\mu \tag{4.4.102}$$

where $e^{it\cdot}$ is seen as a multiplication operator in $L^2(\mu)$. More precisely:

$$(W_\mu U(t)\xi)(\lambda) = e^{it\lambda}(W_\mu\xi)(\lambda),\ \forall t,\lambda\in\mathbb{R},\forall\xi\in\mathscr{H}. \tag{4.4.103}$$

(We say that the isometry W_μ intertwines the two unitary one-parameter groups.)

PROOF. Since v_0 is cyclic, it is enough to consider $\xi\in\mathscr{H}$ of the following form: $\xi = f(A)v_0$, with $f\in L^2(\mu)$, see (4.4.93) in Lemma 4.4.49. Then

$$\|\xi\|^2_{\mathscr{H}} = \int_{\mathbb{R}} |f(\lambda)|^2\,d\mu(\lambda),\ \text{so} \tag{4.4.104}$$

$$\|W_\mu\xi\|_{L^2(\mu)} = \|\xi\|_{\mathscr{H}}\ (\Longleftrightarrow (4.4.104))$$

For the adjoint operator $W_\mu^*: L^2(\mathbb{R},\mu)\to\mathscr{H}$, we have

$$W_\mu^* f = f(A)v_0,$$

see (4.4.92)-(4.4.94). Note that $f(A)v_0\in\mathscr{H}$ is well-defined for all $f\in L^2(\mu)$. Also $W_\mu^*W_\mu = I_{\mathscr{H}}$, $W_\mu W_\mu^* = I_{L^2(\mu)}$.

Proof of (4.4.103). Take $\xi = f(A)v_0$, $f\in L^2(\mu)$, and apply the previous lemma, we have

$$W_\mu U(t)\xi = W_\mu U(t) f(A)_0 = W_\mu\left(e^{it\cdot}f(\cdot)\right)(A)v_0 = e^{it\cdot}f(\cdot) = e^{it\cdot}W_\mu\xi;$$

or written differently:

$$W_\mu U(t) = M_{e^{it\cdot}}W_\mu,\ \forall t\in\mathbb{R}$$

where $M_{e^{it\cdot}}$ is the multiplication operator by $e^{it\cdot}$. □

□

REMARK 4.4.56. Deficiency indices $(1,1)$ occur for probability measures μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |\lambda|^2\,d\mu(\lambda) = \infty. \tag{4.4.105}$$

See examples below.

TABLE 5. Application of Theorem 4.4.57 to Table 2.

measure	condition (4.4.105)	deficiency indices
μ_1	$\int_{\mathbb{R}} \lvert\lambda\rvert^2 e^{-\lvert\lambda\rvert} d\lambda < \infty$	$(0,0)$
μ_2	$\int_{\mathbb{R}} \lvert\lambda\rvert^2 \left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2 d\lambda = \infty$	$(1,1)$
μ_3	$\int_{\mathbb{R}} \lvert\lambda\rvert^2 \frac{d\lambda}{\pi(1+\lambda^2)} = \infty$	$(1,1)$
μ_4	$\int_{\mathbb{R}} \lvert\lambda\rvert^2 \chi_{(-1,1)}(\lambda)(1-\lvert\lambda\rvert)\, d\lambda < \infty$	$(0,0)$
μ_5	$\int_{\mathbb{R}} \lvert\lambda\rvert^2 \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda = 1 < \infty$	$(0,0)$

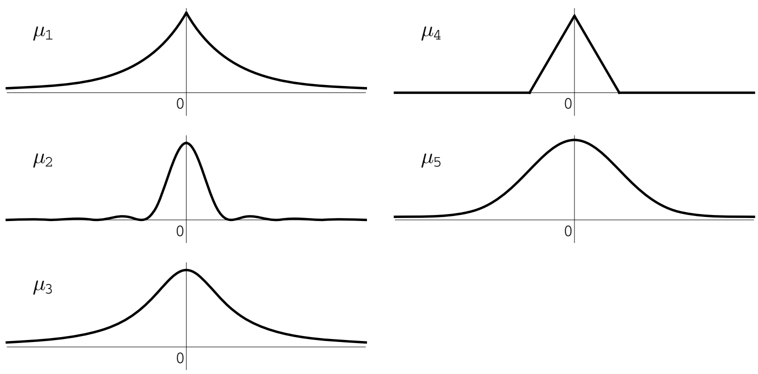


FIGURE 4.4.10. The measures $\mu_i \in Ext(F_i)$ extending p.d. functions F_i in Table 2, $i = 1, 2, \ldots 5$.

SUMMARY. Restrictions with deficiency indices $(1,1)$.

THEOREM 4.4.57 (Jo-Pedersen-Tian). *If μ is a fixed probability measure on $\mathbb{R}$, then the following two conditions are equivalent:*

(1) $\int_{\mathbb{R}} \lambda^2 d\mu(\lambda) = \infty$;
(2) *The set*

$$\left\{ f \in L^2(\mu) \;\middle|\; \lambda f \in L^2(\mu),\ \int_{\mathbb{R}} (\lambda + i) f(\lambda)\, d\mu(\lambda) = 0 \right\}$$

is the <u>dense</u> *domain of a restriction operator $S \subset M_\lambda$ with deficiency indices $(1,1)$, and the deficiency space $DEF_+ = \mathbb{C}\mathbb{1}$, ($\mathbb{1}$ = the constant function 1 in $L^2(\mu)$.)*

A model of ALL deficiency index-$(1,1)$ operators.

LEMMA 4.4.58. *Let μ be a Borel probability measure on $\mathbb{R}$, and denote $L^2(\mathbb{R}, d\mu)$ by $L^2(\mu)$. Then we have TFAE:*

(1) $\int_{\mathbb{R}} \lvert\lambda\rvert^2 d\mu(\lambda) = \infty$;

(2) *The following two subspaces in $L^2(\mu)$ are dense (in the $L^2(\mu)$-norm):*

$$\left\{ f \in L^2(\mu) \,\middle|\, [(\lambda \pm i) f(\lambda)] \in L^2(\mu) \text{ and } \int (\lambda \pm i) f(\lambda)\, d\mu(\lambda) = 0 \right\} \tag{4.4.106}$$

where $i = \sqrt{-1}$.

PROOF. See [**Jr81**]. □

REMARK 4.4.59. If (1) holds, then the two dense subspaces $\mathscr{D}_{\pm} \subset L^2(\mu)$ in (4.4.106) form the dense domain of a restriction S of M_λ in $L^2(\mu)$; and this restriction has deficiency indices $(1,1)$. Moreover, all Hermitian operators having deficiency indices $(1,1)$ arise this way.

LEMMA 4.4.60. *With $i = \sqrt{-1}$, set*

$$dom(S) = \left\{ f \in L^2(\mu) \,\middle|\, \lambda f \in L^2(\mu) \text{ and } \int (\lambda + i) f(\lambda)\, d\mu(\lambda) = 0 \right\} \tag{4.4.107}$$

then $S \subset M_\lambda \subset S^*$; and the deficiency subspaces $DEF_{\pm}$ are as follow:

$$DEF_{+} = \text{the constant function in } L^2(\mu) = \mathbb{C}1 \tag{4.4.108}$$

$$DEF_{-} = span\left\{ \frac{\lambda - i}{\lambda + i} \right\}_{\lambda \in \mathbb{R}} \subseteq L^2(\mu) \tag{4.4.109}$$

where DEF_{-} is also a 1-dimensional subspace in $L^2(\mu)$.

PROOF. Let $f \in dom(S)$, then, by definition,

$$\int_{\mathbb{R}} (\lambda + i) f(\lambda)\, d\mu(\lambda) = 0 \text{ and so}$$

$$\langle 1, (S + iI) f \rangle_{L^2(\mu)} = \int_{\mathbb{R}} (\lambda + i) f(\lambda)\, d\mu(\lambda) = 0 \tag{4.4.110}$$

hence (4.4.108) follows.

Note we have formula (4.4.107) for $dom(S)$. Moreover $dom(S)$ is dense in $L^2(\mu)$ because of (4.4.106) in Lemma 4.4.58.

Now to (4.4.109): Let $f \in dom(S)$; then

$$\begin{aligned} \left\langle \frac{\lambda - i}{\lambda + i}, (S - iI) f \right\rangle_{L^2(\mu)} &= \int_{\mathbb{R}} \left(\frac{\lambda + i}{\lambda - i} \right) (\lambda - i) f(\lambda)\, d\mu(\lambda) \\ &= \int_{\mathbb{R}} (\lambda + i) f(\lambda)\, d\mu(\lambda) = 0 \end{aligned}$$

again using the definition of $dom(S)$ in (4.4.107). □

We have established a representation for <u>all</u> Hermitian operators with dense domain in a Hilbert space, and having deficiency indices $(1,1)$.

To further emphasize to the result we need about deficiency indices $(1,1)$, we have the following:

THEOREM 4.4.61. *Let $\mathscr{H}$ be a separable Hilbert space, and let S be a Hermitian operator with dense domain in $\mathscr{H}$. Suppose the deficiency indices of S are (d,d); and suppose one of the selfadjoint extensions of S has simple spectrum.*

Then the following two conditions are equivalent:

(1) $d = 1$;

(2) *for each of the selfadjoint extensions T of S, we have a unitary equivalence between $(S, \mathscr{H})$ on the one hand, and a system $\left(S_\mu, L^2(\mathbb{R}, \mu)\right)$ on the other, where μ is a Borel probability measure on $\mathbb{R}$. Moreover,*

$$(S_\mu f)(\lambda) = \lambda f(\lambda), \ \forall f \in dom(S_\mu), \forall \lambda \in \mathbb{R}, \ where \tag{4.4.111}$$

$$dom(S_\mu) = \left\{ f \in L^2(\mu) \,\middle|\, \lambda f \in L^2(\mu), \ \int_{\mathbb{R}} (\lambda + i) f(\lambda) \, d\mu(\lambda) = 0 \right\}. \tag{4.4.112}$$

*In case μ satisfies condition (*4.4.111*), then the constant function $\mathbb{1}$ (in $L^2(\mathbb{R}, \mu)$) is in the domain of S_μ^*, and*

$$S_\mu^* \mathbb{1} = i \mathbb{1} \tag{4.4.113}$$

i.e., $\left(S_\mu^* \mathbb{1}\right)(\lambda) = i$, a.a. λ w.r.t. $d\mu$.

Proof. For the implication (2)⇒(1), see Lemma 4.4.60.

(1)⇒(2). Assume that the operator S, acting in $\mathscr{H}$ is Hermitian with deficiency indices $(1, 1)$. This means that each of the two subspaces $DEF_\pm \subset \mathscr{H}$ is one-dimensional, where

$$DEF_\pm = \left\{ h_\pm \in dom(S^*) \,\middle|\, S^* h_\pm = \pm i h_\pm \right\}. \tag{4.4.114}$$

Now pick a selfadjoint extension, say T, extending S. We have

$$S \subseteq T = T^* \subseteq S^* \tag{4.4.115}$$

where "$\subseteq$" in (4.4.115) means "containment of the respective graphs."

Now set $U(t) = e^{itT}$, $t \in \mathbb{R}$, and let $P_U(\cdot)$ be the corresponding projection-valued measure, i.e., we have:

$$U(t) = \int_{\mathbb{R}} e^{it\lambda} P_U(d\lambda), \ \forall t \in \mathbb{R}. \tag{4.4.116}$$

Using the assumption (1), and (4.4.114), it follows that there is a vector $h_+ \in \mathscr{H}$ such that $\|h_+\|_{\mathscr{H}} = 1$, $h_+ \in dom(S^*)$, and $S^* h_+ = i h_+$. Now set

$$d\mu(\lambda) := \|P_U(d\lambda) h_+\|_{\mathscr{H}}^2. \tag{4.4.117}$$

Using (4.4.116), we then verify that there is a unitary (and isometric) isomorphism of $L^2(\mu) \xrightarrow{W} \mathscr{H}$ given by

$$W f = f(T) h_+, \ \forall f \in L^2(\mu); \tag{4.4.118}$$

where $f(T) = \int_{\mathbb{R}} f(T) P_U(d\lambda)$ is the functional calculus applied to the selfadjoint operator T. Hence

$$\begin{aligned} \|Wf\|_{\mathscr{H}}^2 &= \|f(T) h_+\|_{\mathscr{H}}^2 \\ &= \int_{\mathbb{R}} |f(\lambda)|^2 \|P_U(d\lambda) h_+\|^2 \\ &\overset{(\text{by } 4.4.117)}{=} \int_{\mathbb{R}} |f(\lambda)|^2 d\mu(\lambda) = \|f\|_{L^2(\mu)}^2. \end{aligned}$$

To see that W in (4.4.118) is an isometric isomorphism of $L^2(\mu)$ onto $\mathscr{H}$, we use the assumption that T has simple spectrum.

Now set

$$S_{\mu} := W^{*}SW \tag{4.4.119}$$

$$T_{\mu} := W^{*}TW. \tag{4.4.120}$$

We note that T_{μ} is then the multiplication operator M in $L^{2}(\mathbb{R},\mu)$, given by

$$(Mf)(\lambda) = \lambda f(\lambda), \ \forall f \in L^{2}(\mu) \tag{4.4.121}$$

such that $\lambda f \in L^{2}(\mu)$. This assertion is immediate from (4.4.118) and (4.4.117).

To finish the proof, we compute the integral in (4.4.112) in the theorem, and we use the intertwining properties of the isomorphism W from (4.4.118). Indeed, we have

$$\begin{aligned} \int_{\mathbb{R}} (\lambda + i) f(\lambda) \, d\mu(\lambda) &= \langle \mathbb{1}, (M + iI) f \rangle_{L^{2}(\mu)} \\ &= \langle W\mathbb{1}, W(M + iI) f \rangle_{\mathscr{H}} \\ &\overset{(4.4.117)}{=} \langle h_{+}, (T + iI) Wf \rangle_{\mathscr{H}} . \end{aligned} \tag{4.4.122}$$

Hence $Wf \in dom(S) \iff f \in dom(S_{\mu})$, by (4.4.119); and, so for $Wf \in dom(S)$, the RHS in (4.4.122) yields $\langle (S^{*} - iI) h_{+}, Wf \rangle_{\mathscr{H}} = 0$; and the assertion (2) in the theorem follows. □

The case of indices (d,d) where $d > 1$. Let μ be a Borel probability measure on $\mathbb{R}$, and let

$$L^{2}(\mu) := L^{2}(\mathbb{R}, \mathscr{B}, \mu). \tag{4.4.123}$$

The notation $\mathrm{Prob}(\mathbb{R})$ will be used for these measures.

We saw that the restriction/extension problem for continuous positive definite functions F on $\mathbb{R}$ may be translated into a spectral theoretic model in some $L^{2}(\mu)$ for suitable $\mu \in \mathrm{Prob}(\mathbb{R})$. We saw that extension from a finite open ($\neq \emptyset$) interval leads to spectral representation in $L^{2}(\mu)$, and restrictions of

$$(M_{\mu}f)(\lambda) = \lambda f(\lambda), \ f \in L^{2}(\mu) \tag{4.4.124}$$

having deficiency-indices $(1,1)$; hence the case $d = 1$.

Theorem 4.4.62. *Fix $\mu \in Prob(\mathbb{R})$. There is a 1-1 bijective correspondence between the following:*

(1) *certain closed subspaces $\mathscr{L} \subset L^{2}(\mu)$*
(2) *Hermitian restrictions $S_{\mathscr{L}}$ of M_{μ} (see (4.4.124)) such that*

$$DEF_{+}(S_{\mathscr{L}}) = \mathscr{L}. \tag{4.4.125}$$

The closed subspaces in (1) are specified as follows:

(i) $\dim(\mathscr{L}) = d < \infty$
(ii) *the following implication holds:*

$$g \neq 0, \text{ and } g \in \mathscr{L} \Longrightarrow \left([\lambda \mapsto \lambda g(\lambda)] \notin L^{2}(\mu)\right) \tag{4.4.126}$$

Then set

$$dom(S_{\mathscr{L}}) := \left\{ f \in dom(M_{\mu}) \,\Big|\, \int \overline{g(\lambda)}(\lambda + i) f(\lambda) \, d\mu(\lambda), \forall g \in \mathscr{L} \right\} \tag{4.4.127}$$

and set

$$S_{\mathscr{L}} := M_{\mu}\Big|_{dom(S_{\mathscr{L}})} \tag{4.4.128}$$

where $dom(S_{\mathscr{L}})$ is specified as in (4.4.127).

PROOF. Note that the case $d=1$ is contained in the previous theorem.

Proof of (1) $\Rightarrow$ (2). We will be using an idea from [**Jr81**]. With assumptions (i)-(ii), in particular (4.4.126), one checks that $dom\left(S_{\mathscr{L}}\right)$as specified in (4.4.127) is dense in $L^2\left(\mu\right)$. In fact, the converse implication is also true.

Now setting $S_{\mathscr{L}}$ to be the restriction in (4.4.128), we conclude that

$$S_{\mathscr{L}} \subseteq M_\mu \subseteq S_{\mathscr{L}}^*, \text{ where} \tag{4.4.129}$$

$dom\left(S_{\mathscr{L}}^*\right)$ consists of $h \in L^2\left(\mu\right)$ s.t. $\exists C<\infty$, and

$$\left|\int_{\mathbb{R}} \overline{h\left(\lambda\right)} \lambda f\left(\lambda\right) d\mu\left(\lambda\right)\right|^2 \leq C \int_{\mathbb{R}} \left|f\left(\lambda\right)\right|^2 d\mu\left(\lambda\right), \; \forall f \in dom\left(S_{\mathscr{L}}\right).$$

The assertions in (2) now follow from this.

Proof of (2) $\Rightarrow$ (1). Assume that S is a densely defined restriction of M_μ, and let $DEF_+\left(S\right)=$ the $(+)$ deficiency space, i.e.,

$$DEF_+\left(S\right) = \left\{g \in dom\left(S^*\right) \mid S^*g = ig\right\} \tag{4.4.130}$$

Assume $\dim\left(DEF_+\left(S\right)\right) = d$, and $1 \leq d < \infty$. Then set $\mathscr{L} := DEF_+\left(S\right)$. One checks that (1) then holds for this closed subspace in $L^2\left(\mu\right)$.

The fact that (4.4.126) holds for this subspace $\mathscr{L}$ follows from the observation:

$$DEF_+\left(S\right) \cap dom\left(M_\mu\right) = \{0\}$$

for every densely defined restriction S of M_μ. $\square$

Spectral representation of index $(1,1)$ *Hermitian operators.* In this section, we give an explicit answer to the question: How to go from any index $(1,1)$ Hermitian operator to a $\left(\mathscr{H}_F, D^{(F)}\right)$ model; i.e., from a given index $(1,1)$ Hermitian operator with dense domain in a separable Hilbert space $\mathscr{H}$, we build a p.d. continuous function F on $\Omega-\Omega$, where $\Omega=(0,a)$, $a>0$.

So far, we have been concentrating on building transforms going in the other direction. But recall that, for a given continuous p.d. function F on $\Omega-\Omega$, it is often difficult to answer the question of whether the corresponding operator $D^{(F)}$ in the RKHS $\mathscr{H}_F$ has deficiency indices $(1,1)$ or $(0,0)$.

Now this question answers itself once we have an explicit transform going in the opposite direction. Specifically, given any index $(1,1)$ Hermitian operator S in a separable Hilbert space $\mathscr{H}$, we then to find a pair (F,Ω), p.d. function and interval, with the desired properties. There are two steps:

Step 1, writing down explicitly, a p.d. continuous function F on $\Omega-\Omega$, and the associated RKHS $\mathscr{H}_F$ with operator $D^{(F)}$.

Step 2, constructing an intertwining isomorphism $W: \mathscr{H} \to \mathscr{H}_F$, having the following properties: W is an isometric isomorphism, intertwining the pair $(\mathscr{H}, S)$ with $\left(\mathscr{H}_F, D^{(F)}\right)$, i.e., satisfying $WS = D^{(F)}W$; and also intertwining the respective domains and deficiency spaces, in $\mathscr{H}$ and $\mathscr{H}_F$.

Moreover, starting with any $(1,1)$ Hermitian operator, we can even arrange a normalization for the p.d. function F such that $\Omega=(0,1)$ will do the job.

Details. We will have three pairs $(\mathscr{H}, S)$, $\left(L^2\left(\mathbb{R},\mu\right), \text{ restriction of } M_\mu\right)$, and $\left(\mathscr{H}_F, D^{(F)}\right)$, where:

(i) S is a fixed Hermitian operator with dense domain $dom\left(S\right)$ in a separable Hilbert space $\mathscr{H}$, and with deficiency indices $(1,1)$.

(ii) From (i), we will construct a finite Borel measure μ on $\mathbb{R}$ such that an index-$(1,1)$ restriction of $M_\mu : f \mapsto \lambda f(\lambda)$ in $L^2(\mathbb{R},\mu)$, is equivalent to $(\mathscr{H}, S)$.

(iii) Here $F : (-1,1) \to \mathbb{C}$ will be a p.d. continuous function, $\mathscr{H}_F$ the corresponding RKHS; and $D^{(F)}$ the usual operator with dense domain

$$\{F_\varphi \mid \varphi \in C_c^\infty(0,1)\}, \quad \text{and} \quad D^{(F)}(F_\varphi) = \frac{1}{i} F_{\varphi'}, \ \varphi' = \frac{d\varphi}{dx}. \tag{4.4.131}$$

We will accomplish the stated goal with the following system of intertwining operators: See Figure 4.4.11.

But we stress that, at the outset, only (i) is given; the rest (μ, F and $\mathscr{H}_F$) will be constructed. Further, the solutions (μ, F) in Figure 4.4.11 are not unique; rather they depend on choice of selfadjoint extension in (i): Different selfadjoint extensions of S in (i) yield different solutions (μ, F). But the selfadjoint extensions of S in $\mathscr{H}$ are parameterized by von Neumann's theory; see e.g., [**Rud73, DS88**].

$$\begin{tikzcd}
S \text{ in } \mathscr{H} \arrow[rr, "W_\mu"] \arrow[dr, "W:=T_\mu^* W_\mu"'] & & L^2(\mathbb{R},\mu), \text{ restriction of } M_\mu \arrow[dl, bend left, "T_\mu^*"] \\
& D^{(F)} \text{ in } \mathscr{H}_F \arrow[ur, bend left, "T_\mu"] &
\end{tikzcd}$$

FIGURE 4.4.11. A system of intertwining operators.

REMARK. In our analysis of (i)-(iii), we may without loss of generality assume that the following normalizations hold:

(z_1) $\mu(\mathbb{R}) = 1$, so μ is a probability measure;
(z_2) $F(0) = 1$, and the p.d. continuous solution;
(z_3) $F : (-1,1) \to \mathbb{C}$ is defined on $(-1,1)$; so $\Omega := (0,1)$.

REMARK 4.4.63. Further, we may assume that the operator S in $\mathscr{H}$ from (i) has simple spectrum.

THEOREM 4.4.64. *Starting with $(\mathscr{H}, S)$ as in (i), there are solutions (μ, F) to (ii)-(iii), and intertwining operators W_μ, T_μ as in Figure* 4.4.11*, such that*

$$W := T_\mu^* W_\mu \tag{4.4.132}$$

satisfies the intertwining properties for $(\mathscr{H}, S)$ and $\left(\mathscr{H}_F, D^{(F)}\right)$.

PROOF. Since S has indices $(1,1)$, $\dim DEF_\pm(S) = 1$, and S has selfadjoint extensions indexed by partial isometries $DEF_+ \xrightarrow{v} DEF_-$. We now pick $g_+ \in DEF_+$, $\|g_+\| = 1$, and partial isometry v with selfadjoint extension S_v, i.e.,

$$S \subset S_v \subset S_v^* \subset S^*. \tag{4.4.133}$$

Hence $\{U_v(t) : t \in \mathbb{R}\}$ is a strongly continuous unitary representation of $\mathbb{R}$, acting in $\mathscr{H}$, $U_v(t) := e^{itS_v}$, $t \in \mathbb{R}$. Let $P_{S_v}(\cdot)$ be the corresponding projection valued measure (PVM) on $\mathscr{B}(\mathbb{R})$, i.e., we have

$$U_v(t) = \int_{\mathbb{R}} e^{it\lambda} P_{S_v}(d\lambda); \tag{4.4.134}$$

and set

$$d\mu(\lambda) := d\mu_v(\lambda) = \|P_{S_v}(d\lambda)\, g_+\|^2_{\mathscr{H}}. \tag{4.4.135}$$

For $f \in L^2(\mathbb{R}, \mu_v)$, set

$$W_{\mu_v}(f(S_v)\, g_+) = f; \tag{4.4.136}$$

then $W_{\mu_v} : \mathscr{H} \to L^2(\mathbb{R}, \mu_v)$ is isometric onto; and

$$W_{\mu_v^*}(f) = f(S_v)\, g_+, \tag{4.4.137}$$

where

$$f(S_v)\, g_+ = \int_{\mathbb{R}} f(\lambda)\, P_{S_v}(d\lambda)\, g_+. \tag{4.4.138}$$

For justification of these assertions, see e.g., [**Nel69**]. Moreover, W_μ has the intertwining properties sketched in Figure 4.4.11.

Returning to (4.4.134) and (iii) in the theorem, we now set $F := F_\mu|_{(-1,1)}$, where

$$\begin{aligned} F_\mu(t) &:= \langle g_+, U_v(t)\, g_+\rangle \\ &\overset{(4.4.134)}{=} \left\langle g_+, \int_{\mathbb{R}} e^{it\lambda} P_{S_v}(d\lambda)\, g_+ \right\rangle \\ &= \int_{\mathbb{R}} e^{it\lambda} \|P_{S_v}(d\lambda)\, g_+\|^2 \\ &\overset{(4.4.135)}{=} \int_{\mathbb{R}} e^{it\lambda} d\mu_v(\lambda) = \widehat{d\mu_v}(t), \ \forall t \in \mathbb{R}. \end{aligned} \tag{4.4.139}$$

We now show that $F\left(:= F_\mu|_{(-1,1)}\right)$ has the desired properties.

From Corollary 4.4.26, we have the isometry $T_\mu(F_\varphi) = \widehat{\varphi}$, $\varphi \in C_c(0,1)$, with adjoint $T_\mu^*(f) = (f d\mu)^\vee$, see Figure 4.4.11.

The following properties are easily checked:

$$W_\mu(g_+) = \mathbb{1} \in L^2(\mathbb{R}, \mu), \text{ and} \tag{4.4.140}$$

$$T_\mu^*(\mathbb{1}) = F_0 = F(\cdot - 0) \in \mathscr{H}_F, \tag{4.4.141}$$

as well as the intertwining properties stated in the theorem; see Fig. 4.4.11 for a summary.

Proof of (4.4.140) We will show instead that $W_\mu^*(\mathbb{1}) = g_+$. From (4.4.138) we note that if $f \in L^2(\mathbb{R}, \mu)$ satisfies $f = \mathbb{1}$, then $f(S_v) = I_{\mathscr{H}}$. Hence

$$W_\mu^*(\mathbb{1}) \overset{(4.4.137)}{=} \mathbb{1}(S_v)\, g_+ = g_+,$$

which is (4.4.140).

Proof of (4.4.141) For $\varphi \in C_c(0,1)$ we have $\widehat{\varphi} \in L^2(\mathbb{R}, \mu)$, and

$$T_\mu^* T_\mu(F_\varphi) = T_\mu^*(\widehat{\varphi}) \overset{(4.4.50)}{=} (\widehat{\varphi} d\mu)^\vee = F_\varphi.$$

Taking an approximation $(\varphi_n) \subset C_c(0,1)$ to the Dirac unit mass δ_0, we get (4.4.141). ☐

COROLLARY 4.4.65. *The deficiency indices of $D^{(F)}$ in $\mathscr{H}_F$ for $F(x) = e^{-|x|}$, $|x| < 1$, are $(1,1)$.*

PROOF. Let $\mathscr{H}=L^2(\mathbb{R})$ w.r.t. the Lebesgue measure. Take $g_+ := ((\lambda+i)^{-1})^{\vee}(x)$, $x \in \mathbb{R}$; then $g_+ \in \mathscr{H}$ since

$$\int_{\mathbb{R}} |g_+(x)|^2 \, dx \overset{\text{Parseval}}{=} \int_{\mathbb{R}} \left| \frac{1}{\lambda+i} \right|^2 d\lambda = \int_{\mathbb{R}} \frac{1}{1+\lambda^2} d\lambda = \pi.$$

Now for S and S_v in Theorem 4.4.64, we take

$$S_v h = \frac{1}{i}\frac{d}{dx}h, \quad dom(S_v) := \left\{ h \in L^2(\mathbb{R}) \mid h' \in L^2(\mathbb{R}) \right\}, \text{ and} \tag{4.4.142}$$

$$S = S_v \quad \text{restricted to } \{h \in dom(S_v) \mid h(0) = 0\}; \tag{4.4.143}$$

then by [**Jr81**], we know that S has index $(1,1)$, and that $g_+ \in DEF_+(S)$. The corresponding p.d. continuous function F is the restriction to $|t| < 1$ of the p.d. function:

$$\langle g_+, U_v(t)\, g_+ \rangle_{\mathscr{H}} = \int_{\mathbb{R}} \frac{1}{\lambda - i} \frac{e^{it\lambda}}{\lambda+i} d\lambda = \left(\frac{1}{1+\lambda^2} \right)^{\vee}(t) = \pi e^{-|t|}.$$

□

EXAMPLE 4.4.66 (Lüvy-measures (see e.g., [**ST94**])). Let $0 < \alpha \le 2$, $-1 < \beta < 1$, $v > 0$; then the Lüvy-measures μ on $\mathbb{R}$ are indexed by (α, β, ν), so $\mu = \mu_{(\alpha,\beta,\nu)}$. They are absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}$; and for $\alpha = 1$,

$$F_{(\alpha,\beta,\nu)}(x) = \widehat{\mu_{(\alpha,\beta,\nu)}}(x), \ x \in \mathbb{R}, \tag{4.4.144}$$

satisfies

$$F_{(\alpha,\beta,\nu)}(x) = \exp\left(-\nu |x| \cdot \left(1 + \frac{2i\beta}{\pi} - \operatorname{sgn}(x) \ln |x| \right) \right). \tag{4.4.145}$$

The case $\alpha = 2$, $\beta = 0$, reduces to the Gaussian distribution.

The measures $\mu_{(1,\beta,\nu)}$ have infinite variance, i.e.,

$$\int_{\mathbb{R}} \lambda^2 d\mu_{(1,\beta,\nu)} = \infty.$$

As a corollary of Theorem 4.4.64, we therefore conclude that, for the restrictions (see (4.4.144)-(4.4.145)),

$$F^{(res)}_{(1,\beta,\nu)}(x) = F_{(1,\beta,\nu)}(x), \ x \in (-1,1)$$

the associated Hermitian operator $D^{F^{(res)}}$ all have deficiency indices $(1,1)$.

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CHAPTER 5

Harmonic analysis via representations of the Cuntz relations

> If one finds a difficulty in a calculation which is otherwise quite convincing, one should not push the difficulty away; one should rather try to make it the centre of the whole thing.
> — Werner Heisenberg (1901–1976)

Gaussian processes for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases) are considered. These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related Reproducing Kernel Hilbert Spaces.

DEFINITION 5.0.1. A *reproducing kernel Hilbert space* (RKHS), say $\mathscr{H}$, is a Hilbert space of functions on some set $\mathscr{A}$ having the property that the $\mathscr{A}$-evaluations are continuous in the norm of $\mathscr{H}$; specifically for every $a \in \mathscr{A}$, we assume that

$$\mathscr{H} \ni f \longmapsto f(a) \tag{5.0.1}$$

is a continuous linear functional on $\mathscr{H}$. By Riesz, therefore there is a unique $K_a \in \mathscr{H}$ such that

$$f(a) = \langle f, K_a \rangle_{\mathscr{H}}, \quad \forall f \in \mathscr{H}. \tag{5.0.2}$$

Set

$$\widetilde{K}(a,b) = K_a(b), \quad \forall (a,b) \in \mathscr{A} \times \mathscr{A}; \tag{5.0.3}$$

and one checks that, for $\forall N \in \mathbb{N}$, $\forall \{\xi_i\}_{i=1}^{N} \in \mathbb{C}^N$, we have

$$\sum_1^N \sum_1^N \xi_i \overline{\xi}_j \widetilde{K}(a_i, a_j) \geq 0. \tag{5.0.4}$$

A function $\widetilde{K}$ on $\mathscr{A} \times \mathscr{A}$ satisfying (5.0.4) is said to be a *positive definite kernel.*

There theorem of Aronszajn [**Aro50**] states the converse: Every positive definite kernel arises as in (5.0.2)-(5.0.3).

A related result is the following:

LEMMA 5.0.2. *Let $\mathscr{A}$ be a set and $\widetilde{K}$ a scalar valued function on $\mathscr{A} \times \mathscr{A}$; then $\widetilde{K}$ is positive definite if and only if there is a pair $(l, \mathscr{K})$ where $\mathscr{K}$ is a Hilbert space, and l is a function*

$$l : \mathscr{A} \longrightarrow \mathscr{K} \tag{5.0.5}$$

such that

$$\widetilde{K}(a,b) = \langle l(a), l(b) \rangle_{\mathscr{K}}, \quad \forall (a,b) \in \mathscr{A} \times \mathscr{A}. \tag{5.0.6}$$

Examples of solutions to (5.0.5)-(5.0.6) include Gaussian processes, i.e., $\mathscr{K} = L^2(\Omega, \mathscr{F}, \mathbb{P})$ a probability space, $l(a)$, for $a \in \mathscr{A}$, a Gaussian random variable with distribution $N(0, \widetilde{K}(a,a))$, and covariance

$$\mathbb{E}_{\mathbb{P}}(l(a)\,\overline{l(b)}) = \widetilde{K}(a,b)\,,\ (a,b) \in \mathscr{A} \times \mathscr{A}. \tag{5.0.7}$$

5.1. From frequency band filters to signals and to wavelet expansions

In this section, we present some of the parallels between multiresolution theory and the theory of orthonormal Fourier series on fractals. It is based primarily on [**DJ08b**, **BJ02**, **Jor06**] by Jorgensen et al.

A popular approach to wavelet constructions is based on a so-called scaling identity, or scaling equation, see (5.1.4). A solution to this equation is a function on $\mathbb{R}^d$ for some d. The equation is related to a subdivision scheme that is used in numerical analysis and in computer graphics. In that language, it arises from a fixed scaling matrix, assumed expansive, a system of masking coefficients, and a certain subdivision algorithm. An iteration of the scaling produces a succession of subdivisions into smaller and smaller frequency bands. In signal processing, the coefficients in the equation refer to "frequency response". There are various refinements, however, of this setup: two such refinements are *multi-wavelets* and *singular systems*.

If the masking coefficients are turned into a generating function, called a *low-pass filter* m_0, then the scaling identity takes a form which admits solutions with an infinite product representation. Various regularity assumptions are usually placed on the function m_0. The first requirement is usually that the solution, i.e., the scaling function, is in $L^2(\mathbb{R}^d)$, but other Hilbert spaces of functions on $\mathbb{R}^d$ are also considered. If the number of masking coefficients is finite, then m_0 is a Fourier polynomial. (For the Daubechies wavelet, there are four coefficients, and $d = 1$.) In general, however, m_0 might be a fairly singular function. In favorable cases, the associated infinite product will be the Fourier transform of the scaling function. This function, sometimes called the *father function*, is the starting point of most wavelet constructions, the multiresolution schemes. The function m_0 is a function of one or more frequency variables, and convergence of the associated infinite product dictates requirements on m_0 for small frequencies, hence low-pass. The term "low-pass" suggests a filter which lets low-frequency signals pass with high probability. A complete system, of which m_0 is a part, and which is built from appropriately selected frequency bands, offers an effective tool for wavelet analysis and for signal processing. Such a system gives rise to operators F_i, and their duals F_i^*, that are the starting point for a class of algorithms called pyramid algorithms. They are basic to both signal processing and the analysis of wavelet packets. (In operator theory, F_i^* is usually denoted S_i, and S_i^* is set equal to F_i. The reason is that it is the operator F_i^* that is isometric.) In the more traditional approaches, m_0 is a Fourier polynomial, or at least a Lipschitz-class function on a suitable torus, and the low-pass signal analysis is then relatively well understood. But a variety of applications, for example to multi-wavelets, dictate filters m_0 that are no better than continuous, or perhaps only measurable. Then the standard tools break down, and probabilistic and operator theoretic methods are forced on us. This is the setting which is the focus of the present paper.

Recent developments in wavelet analysis have brought together ideas from engineering and from computational mathematics, as well as fundamentals from representation theory.

By now, the subject draws on ideas from a variety of directions. Of these directions, we single out quadrature-mirror filters from signal/image processing, see Figure 1.4.1. High-pass/low-pass signal-processing algorithms have now been adopted by pure mathematicians, although they historically first were intended for speech signals. Perhaps unexpectedly, essentially the same quadrature relations were rediscovered in operator algebra theory, and they are now used in relatively painless constructions of varieties of wavelet bases. The connection to signal processing is rarely stressed in the math literature. Yet, the flow of ideas between signal processing and wavelet mathematics is a success story that deserves to be told. Without these recent synergistic trends, we would perhaps only know isolated examples of wavelets. Thus, mathematicians have borrowed from engineers; and the engineers may be happy to know that what they do is used in mathematics.

Multiresolution wavelets and IFSs

As mentioned in previous chapters, the IFSs include dynamical systems defined from a finite set of affine and contractive mappings in $\mathbb{R}^d$, or from the branches of inverses of complex polynomials, or of rational mappings in the complex plane.

In terms of signal processing, what the two have in common, wavelets and IFSs, is that large scale data may be compressed into a few functions or parameters. In the case of IFSs, only a few matrix entries are needed, and a finite set of vectors in $\mathbb{R}^d$ must be prescribed. This can be turned into effective codes for large images. Similarly discrete wavelet algorithms can be applied to digital images and to data mining [**Mal98**, **Jor06**]. The efficiency in these application lies in the same fact: The wavelets may be represented and determined by a small set of parameters; a choice of scaling matrix and of masking coefficients, i.e., the coefficients (a_k) in the scaling identity.

It turns out that there is a Perron-Frobenius operator in wavelet and fractal theory which encodes orthogonality relations. We call it "the transfer operator", or "the Perron-Frobenius-Ruelle operator", but other names may reasonably be associated with it, see e.g., [**BJ02**]. In the context of wavelets, the operator was studied in [**Law91**] . It will be specified by a chosen filter function m_0, and we denote the associated operator by R_{m_0}.

We are interested in the question: what spectral measures can be constructed, other than the Lebesgue measure on an interval? It has been shown in previous chapters that a surprising answer will be offered by affine iterated function systems.

Setting. Let A be a $d \times d$ expansive integer matrix. We say that a matrix is *expansive* if all its eigenvalues have absolute value strictly greater than 1.

Let B be a finite set of points $\mathbb{Z}^d$, of cardinality $|B| =: N$. For each $b \in B$, we define the following affine maps on $\mathbb{R}^d$,

$$\tau_b(x) = A^{-1}(x + b), \quad x \in \mathbb{R}^d. \tag{5.1.1}$$

The family of functions $(\tau_b)_{b\in B}$ is then an *affine iterated function system.*

For a general affine iterated system $(\tau_b)_{b\in B}$, the equation defining the attractor (see e.g., [**Hut95**, **Hut81**]) $K =: X_B$ can be rewritten

$$AX_B = \bigcup_{b\in B} (X_B + b).$$

Actually, the attractor has the following representation:

$$X_B = \left\{ \sum_{k=1}^{\infty} A^{-k} b_k \mid b_k \in B \right\}.$$

If we assume that there is no overlap between the sets $X_B + b$, $b \in B$, we can construct the function $\phi_B := \chi_{X_B}$ and it satisfies

$$\phi_B(A^{-1}x) = \sum_{b \in B} \phi_B(x - b), \quad x \in \mathbb{R}^d. \tag{5.1.2}$$

Note that is precisely a form of the scaling equation in multiresolution wavelet theory! However the role of the scaling function will not be played here by the attractor of the iterated function system, but by an associated invariant measure. There is a direction of research which exploits equation (5.1.2), which enables one to construct Haar type wavelets on fractal measures. This was pursued in [**DJ06a**, **DMP08**] where wavelets were constructed on Cantor sets and Sierpinski gaskets.

Multiresolutions

The purpose of multiresolution theory (see [**Mal89**, **Mal98**]) is to construct wavelets, that is orthonormal bases for $L^2(\mathbb{R}^d)$ of the form

$$\left\{ |\det A|^{j/2} \psi_i(A^j \cdot - k) \mid j \in \mathbb{Z}, k \in \mathbb{Z}^d, i \in \{1, \dots, L\} \right\}, \tag{5.1.3}$$

where $\psi_1, \dots, \psi_L$ are some functions in $L^2(\mathbb{R}^d)$ which will be called *wavelets*.

DEFINITION 5.1.1. A *multiresolution* is a sequence of subspaces $(V_n)_{n \in \mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ with the following properties:

(1) $V_n \subset V_{n+1}$, for all $n \in \mathbb{Z}$;
(2) $\bigcup_{n \in \mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R}^d)$;
(3) $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;
(4) $f \in V_n$ if and only if $f(A \cdot) \in V_{n+1}$;
(5) There exists a function $\varphi \in V_0$ such that

$$\left\{ \varphi(\cdot - k) \mid k \in \mathbb{Z}^d \right\}$$

is an ONB for V_0.

Given a multiresolution, then wavelets can be constructed by considering the *detail space* $W_0 := V_1 \ominus V_0$. Here one can find vectors $\psi_1, \dots, \psi_L$ such that their translations $\{\psi_i(\cdot - k) \mid k \in \mathbb{Z}^d, i \in \{1, \dots, L\}\}$ form an orthonormal basis for W_0. Then, applying the dilation, all the detail spaces $V_{n+1} \ominus V_n$, and the orthonormal basis in ((5.1.3)) is obtained from the properties of the multiresolution.

The scaling function φ in Definition 5.1.1 satisfies an important equation, called the *scaling equation*:

$$\frac{1}{\sqrt{|\det A|}} \varphi(A^{-1}x) = \sum_{k \in \mathbb{Z}} a_k \varphi(x - k), \quad x \in \mathbb{R}^d, \tag{5.1.4}$$

where $(a_k)_{k \in \mathbb{Z}}$ is a sequence of complex numbers.

The role of the scaling function in the harmonic analysis of fractal measures is played by an invariant measure.

Specifically, let $(\tau_i)_{i=1}^N$ be a contractive iterated function system on a complete metric space X. Let $p_1, \dots, p_N \in [0,1]$ be a list of probabilities. Then there is a unique probability measure μ_p on X such that

$$\mu_p(E) = \sum_{i=1}^{N} p_i \mu_p(\tau_i^{-1}(E)), \quad \text{for all Borel subsets } E. \tag{5.1.5}$$

Moreover, the measure μ_p is supported on the attractor of the iterated function system $(\tau_i)_{i=1}^N$.

EXAMPLE 5.1.2. Consider the IFS associated to the middle third Cantor set. The invariant measure in this case is the Hausdorff measure with the Hausdorff dimension $\frac{\ln 2}{\ln 3}$.

REMARK. It should be stressed that the affine fractals X constructed by IFS-iterations, i.e., from the Cantor-Hutchinson schemes are non-linear objects. As attractors for IFSs, they are "chaotic"; see e.g., [**Bar06**]. This is similarly true for Julia sets from complex dynamics. They too are non-linear, and they carry no group structure; so there is no available Haar measure.

In [**DJ07d**], it is shown that when an IFS is given, then the measures μ on X are induced by infinite product measures corresponding to a choice of probability distribution $(p_i)_{i=1}^N$. The choice of $p_i = 1/N$ is motivated by the search for spectra for our fractals X. A theorem in [**DJ07b**] states that for non-uniform weights the Hadamard is not satisfied, and presumably the measures μ_B never have spectra.

We can rewrite the invariance equation for continuous compactly supported functions on $\mathbb{R}^d$ as follows:

$$\int f \, d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu_B, \quad (f \in C_c(\mathbb{R}^d)). \tag{5.1.6}$$

or

$$\int f(Ax) \, d\mu_B(x) = \frac{1}{N} \sum_{b \in B} \int f(x+b) \, d\mu_B(x), \quad (f \in C_c(\mathbb{R}^d). \tag{5.1.7}$$

Note that both the scaling equation (5.1.4) and the invariance equation (5.1.7) express the dilation of an object in terms of the sum of translated copies of the same object. The resemblance will be more apparent when we take the Fourier transform of both equations.

Applying the Fourier transform to the scaling equation (5.1.4) one obtains:

$$\hat{\varphi}(A^T x) = m_0((A^T)^{-1}x)\hat{\varphi}((A^T)^{-1}x), \quad x \in \mathbb{R}^d, \tag{5.1.8}$$

where A^T is the transpose of the matrix A, and

$$m_0(x) := \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k \cdot x}, \quad x \in \mathbb{R}^d. \tag{5.1.9}$$

The $\mathbb{Z}^d$-periodic function m_0 is called the *low-pass filter* in wavelet theory.

For the invariant measure μ_B, the invariance equation implies

$$\hat{\mu}_B(x) = m_B((A^T)^{-1}x)\hat{\mu}_B((A^T)^{-1}x), \quad x \in \mathbb{R}^d, \tag{5.1.10}$$

where

$$m_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x}, \quad x \in \mathbb{R}^d. \tag{5.1.11}$$

We will call the function m_B the *low-pass filter* of the affine IFS $(\tau_b)_{b \in B}$.

We mentioned above that the wavelets are constructed from a multiresolution. The multiresolution is constructed from the scaling function, which in turn is built from the low-pass filter m_0. If we iterate formally (5.1.8), and impose the condition $m_0(0) = 1$ (from which the name "low-pass" comes from), we get the following:

PROPOSITION 5.1.3. *The infinite product formula for $\hat{\varphi}$ is*

$$\hat{\varphi}(x) = \prod_{k=1}^{\infty} m_0((A^T)^{-n}x), \quad x \in \mathbb{R}^d. \tag{5.1.12}$$

The infinite product is uniformly convergent on compact subsets if m_0 is assumed to be Lipschitz.

In the same fashion, the Fourier transform of the invariant measure, $\hat{\mu}_B$ has an infinite product formula in terms of the low-pass filter m_B (which satisfies $m_B(0) = 1$):

PROPOSITION 5.1.4. *The infinite product formula for $\hat{\mu}_B$ is*

$$\hat{\mu}_B(x) = \prod_{k=1}^{\infty} m_B((A^T)^{-n}x), \quad x \in \mathbb{R}^d. \tag{5.1.13}$$

The infinite product is uniformly convergent on compact subsets because m_B is a trigonometric polynomial.

The quadrature mirror filter condition and Hadamard triples

We saw that the scaling function φ can be obtained from the low-pass filter m_0 by an infinite product formula (5.1.12). But we want the translates of the scaling function to be orthogonal. Combining the orthogonality condition with the scaling equation (5.1.8), we see that m_0 must satisfy the *quadrature mirror filter (QMF)* condition:

$$\sum_{l \in \mathcal{L}} |m_0((A^T)^{-1}(x + l)|^2 = 1, \quad x \in \mathbb{R}^d, \tag{5.1.14}$$

where $\mathcal{L}$ is a complete set of representatives for $\mathbb{Z}^d/A^T\mathbb{Z}^d$.

For the moment it is not clear what a QMF condition should be for the fractal measure μ_B. One of the reasons is that we do not have a candidate for the spectrum Λ yet. However, we will see that the crucial notion of Hadamard triples introduced by Jorgensen and Pedersen in [**JP98a**] can be interpreted as a QMF condition.

DEFINITION 5.1.5. Let A be a $d \times d$ integer matrix. Let B, L be two finite subsets of $\mathbb{Z}^d$ of the same cardinality $|B| = |L| =: N$. Then (A, B, L) is called a *Hadamard triple* if the matrix

$$\frac{1}{\sqrt{N}} \left(e^{2\pi i A^{-1} b \cdot l} \right)_{b \in B, l \in L} \tag{5.1.15}$$

is unitary.

REMARK. The unitarity condition in (5.1.15) may be understood as follows; $d = 1$. Suppose $B = \{0, b_1, \ldots, b_{N-1}\}$. Then unitarity in (5.1.15) holds for some $L \subset \mathbb{R}$ if and only if the complex numbers

$$\{e^{2\pi i (A^T)^{-1}(l_i - l_k)}\}_{l_i \neq l_k \in L} \subset \mathbb{T}$$

are roots of the polynomial $1 + z^{b_1} + z^{b_2} + \cdots + z^{b_{N-1}}$.

To illustrate the restriction placed on the given pair (A, B), take the example when $B = \{0, 2, 3\}$. In that case the associated polynomial equation $1 + z^2 + z^3 = 0$ has no solutions with $|z| = 1$, and so there is no set $L \subset \mathbb{R}$ which produces a complex Hadamard matrix.

In the case of the middle-third-Cantor example, $\tau_0(x) = x/3$ and $\tau_2(x) = (x + 2)/3$; so $A = 3$, and $B = \{0, 2\}$. Up to a translation in $\mathbb{R}$, the possibilities for the set L are $L = \{0, \frac{3}{4}(n + \frac{1}{2})\}$, for some $n \in \mathbb{Z}$. Since none of these solutions $l = \frac{3}{4}(n + \frac{1}{2})$ are in $\mathbb{Z}$, the unitarity condition (5.1.15) is not satisfied.

EXAMPLE 5.1.6. We will not use the example of the middle third Cantor set. It was proved in [**JP98a**] that this fractal measure does not admit more than 2 mutually orthogonal exponential functions (also see Section 2.2). However, it was shown that a modification of this does provide an example of a fractal spectral measure. Take $A = 4$, $B = \{0, 2\}$, so $\tau_0 x = x/4$, $\tau_2 x = (x + 2)/4$. The attractor of this iterated function system is the Cantor set obtained by dividing the unit interval in four equal pieces and keeping the first and the third piece, and iterating this process.

Then one can pick $L = \{0, 1\}$ to obtain the Hadamard triple (A, B, L). The unitary matrix in (5.1.15) is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; and the measure μ_B has spectrum

$$\Lambda = \left\{ \sum_{k=0}^{n} 4^k l_k \,|\, l_k \in L,\ n \in \mathbb{N} \right\}.$$

This was the first example of a (non-atomic) spectral measure which is singular with respect to the Lebesgue measure.

It is clear that we have now a QMF condition for our fractal setting.

DEFINITION 5.1.7. We say that the iterated function system

$$\sigma_l(x) = (A^T)^{-1}(x + l), \quad x \in \mathbb{R}^d, l \in L$$

is *dual* to the IFS $(\tau_b)_{b \in B}$ if (A, B, L) is a Hadamard triple.

With the Hadamard triple we have a first candidate for a spectrum of μ_B:

$$\Lambda_0 := \left\{ \sum_{k=0}^{n} (A^T)^k l_k \,|\, l_k \in L, n \in \mathbb{N} \right\}. \tag{5.1.16}$$

5.2. Stochastic processes via representations of the Cuntz relations

In the study of representations of $\mathscr{O}_N$ on a Hilbert space $\mathscr{H}$, an identification of suitably closed invariant subspaces of $\mathscr{H}$ plays a central role. Here we refer to a representation in the form of a system operators S_i and their adjoints S_i^* satisfying the Cuntz relations. Of the possibilities for subspaces, invariance under the S_i^* operators is more interesting: i.e., invariance under a system of generalized backwards shifts.

In many cases, these invariant subspaces have small dimension, and they help us define new isomorphism invariants for the representations under discussion. For example, a *permutative representation* is one with the property that the vectors in some choice of ONB are permuted by the S_i^* operators. Moreover, in important applications to quantum statistical mechanics, certain subspaces of states that are invariant under the adjoints S_i^*, are often finite-dimensional. They are called *finitely correlated states*. And they are one of the main features of interest in statistical mechanics, see e.g., [**FNW92**, **FNW94**, **BJ97**, **Mat98**, **Ohn07**, **BJKW00**]. They are analogues of "attractors" in classical (commutative) symbolic dynamics.

There is a new notion of Martin boundary for representations of the Cuntz algebras. It bridges two ideas which have been studied extensively in the literature, but so far have not been connected in a systematic fashion. In summary, they are:

(i) *the non-commutativity of the Cuntz algebras*, and the subtleties of their representations [**Gli60**, **Gli61**], on the one hand; and
(ii) *symbolic representations* of Markov chains and their classical Martin boundaries, on the other (see, e.g., [**JT15a**, **SBM07**, **Kor08**, **Tak11**]).

5.2.1. Preliminaries. We begin with a technical lemma regarding projections in Hilbert space.

Let $\mathscr{H}$ be a Hilbert space. By an orthogonal projection P on $\mathscr{H}$, we mean an operator satisfying $P = P^* = P^2$. There is a bijective correspondence between projections P (we shall assume that P is orthogonal even if not stated) on the one hand, and closed subspaces $\mathscr{F} = \mathscr{F}_P$ in $\mathscr{H}$ on the other, given by $\mathscr{F} = P\mathscr{H}$; see e.g., [**JT17b**].

We shall use the following

LEMMA 5.2.1. *Let P and Q be projections, and let $\mathscr{F}_P$ and $\mathscr{F}_Q$ be the corresponding closed subspaces, then TFAE:*

(1) $P = PQ$;
(2) $P = QP$;
(3) $\mathscr{F}_P \subseteq \mathscr{F}_Q$;
(4) $\|Ph\| \leq \|Qh\|$, $\forall h \in \mathscr{H}$;
(5) $\langle h, Ph\rangle \leq \langle h, Qh\rangle$, $\forall h \in \mathscr{H}$.

When the conditions hold we say that $P \leq Q$.

PROOF. This is standard in operator theory. We refer to [**JT17b**] for details. $\square$

DEFINITION 5.2.2.

(1) Let $\mathscr{H}$ be a Hilbert space, and V an operator in $\mathscr{H}$. If $P := V^*V$ is a projection, we say that V is a *partial isometry*. In that case, $Q = VV^*$ is also a projection: We say that P is the initial projection of V, and that Q is the final projection.
(2) If $\mathfrak{A}$ is a C^*-algebra, and V, P, Q are as above. If V is in $\mathfrak{A}$, then we say that the two projections P and Q are $\mathfrak{A}$-*equivalent.*

LEMMA 5.2.3. *Let $\{P_k\}_{k\in\mathbb{N}}$ be monotone, i.e.,*

$$P_1 \leq P_2 \leq \cdots, \tag{5.2.1}$$

then the limit

$$P_\infty := \lim_{k\to\infty} P_k \tag{5.2.2}$$

(*in the strong operator topology of $\mathscr{B}(\mathscr{H})$*) *exists, and P_∞ is the projection onto the closed span of the subspaces $\{\mathscr{F}_{P_k}\}_{k\in\mathbb{N}}$.*

The analogous conclusion holds for monotone decreasing sequence of projections

$$\cdots \leq Q_{n+1} \leq Q_n \leq \cdots \leq Q_2 \leq Q_1. \tag{5.2.3}$$

In this case

$$Q_\infty = \lim_{k\to\infty} Q_k \tag{5.2.4}$$

is the projection onto $\bigcap_k \mathscr{F}_{Q_k}$.

5.2.2. A projection valued random variable. The theme here falls at the crossroads of *representation theory* and the study of *fractal measures* and their stochastic processes.

The past two decades has seen a burst of research dealing with representations of classes of *infinite* C^*-algebras, which includes the Cuntz algebras [**Cun77**], $\mathscr{O}_N$ (see (5.2.9)) as well as other graph-C^*-algebras [**FGJ$^+$17**, **FGJ$^+$18b**]. A source of motivation for our present work includes more recent research which includes both pure and applied mathematics: branching laws for endomorphisms, subshifts, endomorphisms from measurable partitions, Markov measures and topological Markov chains, wavelets and multiresolutions, signal processing and filters, iterated function systems (IFSs) and fractals, complex projective spaces, quasicrystals, orbit equivalence, and substitution dynamical systems, and tiling systems [**AJ15**, **JT15a**, **AJL17**, **JT17b**, **AJL18**, **Mal98**].

A projection P is said to be *infinite* iff (Def.) it contains proper subprojections, say Q, $Q \lneqq P$, such that P and Q are *equivalent*; (see Definition 5.2.2 (2)). The Cuntz algebras $\mathscr{O}_N$ contain infinite projections.

The questions considered here for *representations of the Cuntz algebras* are of independent interest as part of *non-commutative harmonic analysis*, i.e., the study of representations of non-abelian groups and C^*-algebras. A basic question in representation theory is that of determining parameters for the *equivalence classes of representations*, where "*equivalence*" refers to unitary equivalence. Since analysis and synthesis of representations must entail direct integral decompositions, a minimal requirement for a list of parameters for the equivalence classes of representations, is that it be Borel. When such a choice is possible, we say that there is a *Borel cross section* for the representations under consideration.

A pioneering paper by J. Glimm [**Gli60**] showed that there are infinite C^*-algebras whose representations do not have Borel cross sections. (Loosely speaking, the representations do not admit classification.) It is known that the Cuntz algebras, and C^*-algebras of higher-rank graphs, fall in this class. Hence, the approach to representations must narrow to suitable and amenable classes of representations which arise naturally in *applications*, and which do admit *Borel cross sections*.

A leading theme in the present section is a formulation of a *boundary theory* for representations of the Cuntz algebra. This in turn ties in with multiresolutions and with iterated function system (IFS) measures. A boundary theory for the latter has recently been suggested in various special cases.

A multiresolution approach to the study of representations of the Cuntz algebras was initiated by P. Jorgensen and O. Brattelli [**BJKR01**, **BJKR02**, **BJ02**, **BJOk04**]; and it includes such applications as construction of new *multiresolution wavelets*, and of wavelet algorithms from multi-band wavelet filters. And yet other applications studied by the first named author and D. Dutkay lead to the study of such classes of representations as *monic*, and *permutative* [**DJ14a**, **FGJ$^+$18a**];

and their use in fractal analysis. The introduction of these classes begins with the fact that every representation of the Cuntz algebra corresponds in a canonical fashion to a certain *projection valued measure*. We begin with these projection valued measures.

Let N be a positive integer, and let A be an alphabet with $|A| = N$; set

$$\Omega_N := A^{\mathbb{N}} = \underbrace{A \times A \times A \times \cdots\cdots\cdots}_{\aleph_0-\text{infinite Cartesian product}}. \tag{5.2.5}$$

Points in Ω_N are denoted $\omega := (x_1, x_2, \cdots)$, and we set

$$\pi_n(\omega) := x_n, \quad \forall \omega \in \Omega_N. \tag{5.2.6}$$

When $k \in \mathbb{N}$ is fixed, and $\omega = (x_i) \in \Omega_N$, we set

$$\omega|_k = (x_1, x_2, \cdots, x_k) = \text{the } k\text{-truncated (\textit{finite}) word.} \tag{5.2.7}$$

Let $\mathscr{H}$ be a separable Hilbert space, $\dim \mathscr{H} = \aleph_0$, and let $\mathfrak{M}$ be a commutative family of orthogonal projections in $\mathscr{H}$.

By an $\mathfrak{M}$-valued random variable X, we mean a measurable function

$$X : \Omega_N \longrightarrow \mathfrak{M}. \tag{5.2.8}$$

See, e.g., [**AJ15**, **AJL17**, **AJL18**].

Let $\mathscr{O}_N$ denote the Cuntz algebra with N generators, i.e., the C^*-algebra on symbols $\{s_i\}_{i=1}^N$, satisfying the following two relations:

$$s_i^* s_j = \delta_{ij} \mathbb{1}, \quad \text{and} \quad \sum_{i=1}^N s_i s_i^* = \mathbb{1}, \tag{5.2.9}$$

where $\mathbb{1}$ denotes the unit element in $\mathscr{O}_N$.

By a representation of $\mathscr{O}_N$ we mean a function $s_i \mapsto S_i = \pi(s_i)$ such that

$$S_i^* S_j = \delta_{ij} I, \quad \text{and} \quad \sum_{i=1}^N S_i S_i^* = I \tag{5.2.10}$$

where δ_{ij} denotes the Kronecker delta, and I denotes the identity operator in $\mathscr{H}$; we say that $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$ if (5.2.10) holds.

The following lemma is basic and will be used throughout.

LEMMA 5.2.4. *Let $\pi = (S_i)_{i=1}^N$ be a representation of $\mathscr{O}_N$ acting in a fixed Hilbert space $\mathscr{H}$, i.e., $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$. For finite words $f = (x_1, \cdots, x_n)$ in the alphabet $A = \{1, 2, \cdots, N\}$, set*

$$P_f := S_{x_1} S_{x_2} \cdots S_{x_k} S_{x_k}^* S_{x_{k-1}}^* \cdots S_{x_1}^* \tag{5.2.11}$$

with the conventions:

$$P_i := S_i S_i^*, \quad \textit{and} \quad P_\emptyset = 0. \tag{5.2.12}$$

(1) *Then as f varies over all finite non-empty words, the projections $\{P_f\}$* form an abelian family.
(2) *Moreover,*

$$\sum_{i=1}^N P_{(fi)} = P_f, \tag{5.2.13}$$

and in particular,

$$P_{(fg)} \leq P_f \tag{5.2.14}$$

for any pair of finite non-empty words f and g. Here (fg) denotes concatenation of the two words.

PROOF. This is an application of (5.2.10), and the details are left for the reader. □

THEOREM 5.2.5. *Let N, $\mathscr{H}$, $\mathscr{O}_N$, and $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$ be as above.*

(1) *Then there is a unique random variable X (projection-valued, see (5.2.8)) such that*

$$X(\omega) = \lim_{k\to\infty} S_{\omega|_k} S^*_{\omega|_k},\ \omega \in \Omega_N, \tag{5.2.15}$$

where

$$S_{\omega|_k} = S_{x_1} S_{x_2} \cdots S_{x_k} \tag{5.2.16}$$

and $\omega|_k$ is the corresponding truncated word as in (5.2.7).

(2) *Moreover, the following relations hold:*
If $a \in A$, and $\omega = (x_1, x_2, x_3, x_4, \cdots) \in \Omega_N$, then

$$S_a X(\omega) S^*_a = X(a\omega), \tag{5.2.17}$$

and

$$S^*_a X(\omega) S_a = \delta_{a,\pi_1(\omega)} X(\sigma(\omega)), \tag{5.2.18}$$

where

$$\sigma(\omega) = (x_2, x_3, x_4, \cdots),\ \text{and}\quad a\omega = (a, x_1, x_2, x_3, x_4, \cdots).$$

(3) *Finally, we have:*

$$X(\omega) X(\omega') = \delta_{\omega,\omega'} X(\omega), \tag{5.2.19}$$

for all $\omega, \omega' \in \Omega_N$.

The reader is referred to the original paper [**JT18a**]. There are a number of technical steps involved, and it would take us too far afield if we were to include them here. However, the underlying main ideas can be gleaned from the discussion above.

5.2.3. A projection valued *path-space measure*. Recall, when N is a fixed integer, at least 2, the corresponding Cuntz algebra $\mathscr{O}_N$ has a rich family of representations (see, e.g., [**Gli60**, **Gli61**, **Cun77**, **BJ02**, **BJOk04**]). They are studied in the previous two sections, with the use of the associated projection-valued measures. As noted in Section 5.2.3, some of the $\mathscr{O}_N$ representations correspond to iterated function systems (IFSs), where the iteration of branching laws is given by a system of N prescribed endomorphisms in a measure space. One reason the use of IFSs is powerful is that the framework allows one to make precise iteration of *self-similarity* in Cantor-dynamics, and, more generally, in *non-reversible dynamics*, as well as the corresponding "chaos-limits." (See [**Hut81**, **Hut95**, **DJ14a**, **AJL17**].) The setting of IFS-systems includes a rich class of fractals, e.g., those corresponding to affine IFSs, and others to complex dynamics.

Two themes are addressed in this section: (i) We present the correspondence between *representations of the Cuntz algebra $\mathscr{O}_N$*, on one hand, and IFSs with N

generating endomorphisms, on the other. (ii) Our focus will be a use of the $\mathscr{O}_N$ representations *in a realization of generalized Martin boundaries* for the IFSs under consideration. For this purpose, it will be convenient to first fix an alphabet A, of size N. We then consider kernels indexed by both finite words in A, as well as by infinite words; see Section 5.2.3 for details.

In Theorem 5.2.29 below, we show that such a boundary theory may be derived from the random variables Y which we introduced in Section 5.2.2. In broad outline, our boundary representations will be obtained as limits of kernels indexed initially by finite words in the alphabet A; – the limit referring to finite vs infinite words in the symbolic representations. This theme will be expanded further in Section 5.3 below.

The present section concludes with a number of explicit examples.

Let (M, d) be a compact metric space, $N \in \mathbb{N}$ fixed, $N \geq 2$,

$$p_1, \cdots, p_N, \ , p_i > 0, \ \sum_{i=1}^{N} p_i = 1, \text{ fixed.} \tag{5.2.20}$$

Let $\tau_i : M \to M$, $1 \leq i \leq N$, be a system of strict contractions in (M, d). Let $\Omega_N = \{1, 2, \cdots, N\}^{\mathbb{N}}$, and let

$$\mathbb{P} = \times_1^\infty p = \underbrace{p \times p \times p \cdots\cdots}_{\aleph_0 \text{ product measure}} \tag{5.2.21}$$

(see [**Kak43**, **Hid80**].)

In this section, we construct random variables Y with values in M (some measure space $(M, \mathscr{B}_M)$), so $Y : \Omega \to M$, such that the corresponding distribution $\mu := \mathbb{P} \circ Y^{-1}$ satisfies

$$\mu = \sum_{i=1}^{N} p_i \mu \circ \tau_i^{-1}.$$

Here $\mathbb{P}$ is the infinite -product measure (5.2.21).

EXAMPLE 5.2.6 (A Julia construction). Although the early analysis of many of the iterated function systems (IFSs) focused on iteration of systems of affine maps in some ambient $\mathbb{R}^d$ (see, e.g., [**Hut81**]), there is also a rich literature dealing with complex dynamics, and iteration of conformal maps, see e.g., [**Mil06**]. Also in these cases, there are IFS measures, see Theorem 5.2.7 2. In the simplest cases these Julia iteration limits arise from an iteration of branches of the inverse of complex polynomials. The corresponding IFS limits are typically Julia sets; named after Gaston Julia. Examples are included in Figure 5.2.1.

THEOREM 5.2.7. *For points $\omega = (i_1, i_2, i_3, \cdots) \in \Omega_N$ and $k \in \mathbb{N}$, set*

$$\omega\big|_k = (i_1, i_2, \cdots, i_k), \text{ and} \tag{5.2.22}$$

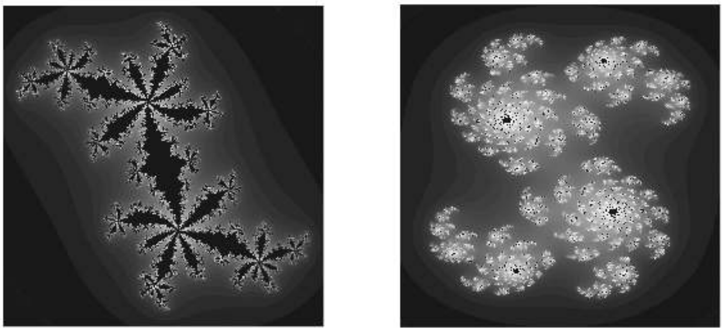
$$\tau_{\omega|_k} = \tau_{i_1} \circ \tau_{i_2} \circ \cdots \circ \tau_{i_k}. \tag{5.2.23}$$

Then $\bigcap_{k=1}^\infty \tau_{\omega|_k}(M)$ is a singleton, say $\{x(\omega)\}$. Set $Y(\omega) = x(\omega)$, i.e.,

$$\{Y(\omega)\} = \bigcap_{k=1}^{\infty} \tau_{\omega|_k}(M); \tag{5.2.24}$$

then:

(1) *$Y : \Omega_N \to M$ is an (M, d)-valued random variable.*



(A) $c = 0.125 + 0.625i$ (B) $c = 0.375 - 0.125i$

FIGURE 5.2.1. $\mathbb{C} \ni z \to z^2 + c$ ($c \in \mathbb{C} \setminus \{0\}$ fixed), $\tau_{\pm} : z \to \pm\sqrt{z - c}$.

(2) *The distribution of* Y, *i.e., the measure*

$$\mu = \mathbb{P} \circ Y^{-1} \tag{5.2.25}$$

is the unique Borel probability measure on (M, d) *satisfying:*

$$\mu = \sum_{i=1}^{N} p_i \mu \circ \tau_i^{-1}; \tag{5.2.26}$$

equivalently,

$$\int_M f d\mu = \sum_{i=1}^{N} p_i \int_M (f \circ \tau_i)\, d\mu, \tag{5.2.27}$$

holds for all Borel functions f *on* M.

(3) *The support* $M_\mu = supp\,(\mu)$ *is the minimal closed set (IFS),* $\neq \emptyset$, *satisfying*

$$M_\mu = \bigcup_{i=1}^{\infty} \tau_i (M_\mu)\,. \tag{5.2.28}$$

PROOF. We shall make use of standard facts from the theory of iterated function systems (IFS).

*Proof of (*5.2.24*).* We use that when $\omega \in \Omega_N$ is fixed then the sets $\tau_{\omega|_k}(M)$ is a monotone family of compact subsets

$$\tau_{\omega|_{k+1}}(M) \subset \tau_{\omega|_k}(M)\,, \tag{5.2.29}$$

and since τ_i is strictly contractive for all i, we get

$$\lim_{k\to\infty} diameter\left(\tau_{\omega|_k}(M)\right) = 0, \tag{5.2.30}$$

and so (5.2.24) follows; i.e., the intersection $\bigcap_{i=1}^{\infty}$ is a singleton depending only on ω.

Monotonicity: This conclusion again follows from the assumptions placed on $\{\tau_i\}_{i=1}^{N}$, but we shall specify the respective σ-algebras, the one on Ω_N and the one on M.

$$E(\omega|_1) \supset E(\omega|_2) \supset \cdots \supset E(\omega|_k) \supset E(\omega|_{k+1}) \supset \cdots$$

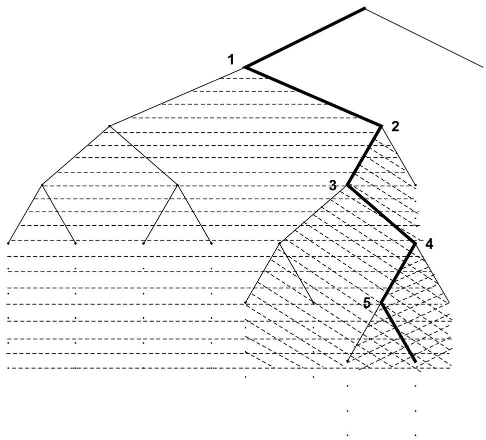


FIGURE 5.2.2. $\{\omega\} = \bigcap_{k=1}^{\infty} E(\omega|_k)$. Monotone families of *tail sets*. Let Ω_N be the set of all infinite words, i.e., the infinite Cartesian product. Start with a fixed infinite word ω, so ω in Ω_N (highlighted in 5.2.2.) For every positive k, we truncate ω, thus forming a finite word $\omega|_k$. Then the set $E(\omega|_k)$ is the set of all infinite words that begin with $\omega|_k$, but unrestricted after k. The intersection in k of all these sets $E(\omega|_k)$ is then the singleton $\{\omega\}$.

The σ-algebra of subsets of Ω_N will be generated by cylinder sets: If $f = (i_1, i_2, \cdots, i_k)$ is a *finite word*, the corresponding cylinder set $E(f) \subset \Omega_N$ is

$$E(f) = \{\omega \in \Omega_N \mid \omega_j = i_j,\ 1 \leq j \leq k\}. \tag{5.2.31}$$

On M, we pick the Borel σ-algebra determined from the fixed metric d on M. The measure $\mathbb{P} = \mathbb{P}_p$ is specified by its values on cylinder sets; i.e, set

$$\mathbb{P}(E(f)) = p_{i_1} p_{i_2} \cdots p_{i_k} =: p_f \tag{5.2.32}$$

where the numbers $p_1, \cdots, p_N$ are as in (5.2.20).

*Proof of (*5.2.26*).* The argument is based on the following: On Ω_N, introduce the *shifts* $\hat{\tau}_b(i_1, i_2, i_3, \cdots) = (b, i_1, i_2, i_3, \cdots)$, $b \in \{1, 2, \cdots, N\}$, and let Y be as in (5.2.24)-(5.2.25). Then

$$\tau_b Y = Y \hat{\tau}_b, \tag{5.2.33}$$

or equivalently,

$$\begin{tikzcd} \Omega_N \arrow[r, "Y"] \arrow[d, "\hat{\tau}_b"'] & M \arrow[d, "\tau_b"] \\ \Omega_N \arrow[r, "Y"'] & M \end{tikzcd}$$

$$\tau_b(Y(\omega)) = Y(\hat{\tau}_b(\omega)),\ \forall \omega \in \Omega_N. \tag{5.2.34}$$

Now (5.2.34) is immediate from (5.2.24).

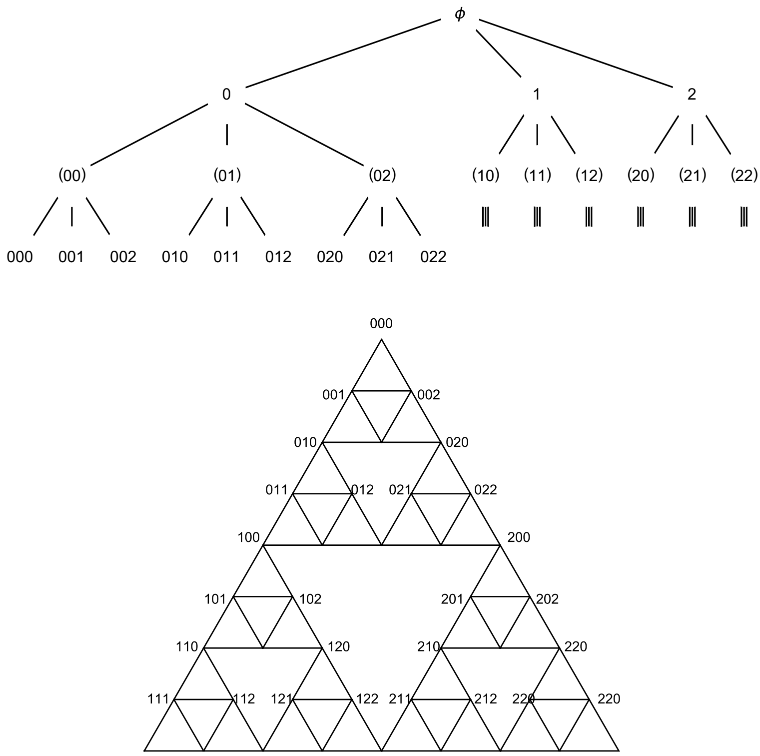


FIGURE 5.2.3. *Encoding of words* into IFS. Infinite words $\omega \in \Omega$ $\longrightarrow$ singletons in the Sierpinski gasket.

We now show (5.2.27), equivalently (5.2.26). Let f be a Borel function on M, then

$$\begin{aligned}
\int_M f\, d\mu &= \int_{\Omega_N} (f \circ Y)\, d\mathbb{P} && \text{(by (5.2.25))} \\
&= \sum_{i=1}^{N} p_i \int_{\Omega_N} f \circ Y \circ \hat{\tau}_i\, d\mathbb{P} && \begin{pmatrix} \text{since } \mathbb{P} \text{ is the product} \\ \text{measure } \times_1^\infty p \text{, see (5.2.32)} \end{pmatrix} \\
&= \sum_{i=1}^{N} p_i \int_{\Omega_N} f \circ \tau_i \circ Y\, d\mathbb{P} && \text{(by (5.2.33))} \\
&= \sum_{i=1}^{N} p_i \int_M f \circ \tau_i\, d\mu && \text{(by (5.2.25))}
\end{aligned}$$

which is the desired conclusion. □

Using $\Omega_N = \{1, 2, \cdots, N\}^{\mathbb{N}}$ for encoding iterated function systems (IFS).

EXAMPLE 5.2.8 (*Sierpinski gasket*). $M = [0, 1] \times [0, 1]$ with the usual metric,

$$\tau_0(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \tau_1(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right), \quad \tau_2(x, y) = \left(\frac{x}{2}, \frac{y+1}{2}\right),$$

and so the Sierpinski gasket M_{Si} satisfies

$$M_{Si} = \tau_0 (M_{Si}) \bigcup \tau_1 (M_{Si}) \bigcup \tau_2 (M_{Si}) .$$

See Figure 5.2.3.

Let $\mathscr{H}$ be a separable Hilbert space, and fix $N \geq 2$, and $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$. We shall be concerned with two tools directly related to the study of representations of $\mathscr{O}_N$ on $\mathscr{H}$.

With $\pi(s_i) = S_i$, $1 \leq i \leq N$ fixed, set $\beta = \beta_\pi \in End(\mathscr{B}(\mathscr{H}))$, *endomorphism*, and $\mathbb{Q} = \mathbb{Q}_\pi$, a canonical *projection-valued path-space measure*. Before giving the precise details, we shall need a few facts about the path space,

$$\Omega_N = \{1, 2, \cdots, N\}^{\mathbb{N}} . \tag{5.2.35}$$

This version of path-space is chosen for simplicity: We have taken as alphabet the set $A := \{1, 2, \cdots, N\}$, but the fixed alphabet could be any finite set A with $|A| = N$; and so $\Omega_N = A^{\mathbb{N}} = \underbrace{A \times A \times \cdots}_{\aleph_0}$, the infinite Cartesian product.

Definition 5.2.9. For $T \in \mathscr{B}(\mathscr{H})$, set

$$\beta_\pi (T) = \sum_{i=1}^{N} S_i T S_i^* . \tag{5.2.36}$$

Then $\beta_\pi \in End(\mathscr{B}(\mathscr{H}))$, i.e.,

$$\beta_\pi (TT') = \beta_\pi (T) \beta_\pi (T') , \ \forall T, T' \in \mathscr{B}(\mathscr{H}) ; \tag{5.2.37}$$

$$\beta_\pi (T^*) = \beta_\pi (T)^* , \text{ and} \tag{5.2.38}$$

$$\beta_\pi (I) = I. \tag{5.2.39}$$

Given a representation $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, then the corresponding endomorphism,

$$\beta_\pi : \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{H}) \tag{5.2.40}$$

plays an important role in representation theory. For example, for decompositions of π, by Schur, we will need the commutant $\{\pi\}'$, defined as follows:

$$\{\pi\}' := \{T \in \mathscr{B}(\mathscr{H}) \,;\, T\,\pi(A) = \pi(A)\,T, \ \forall A \in \mathscr{O}_N\} . \tag{5.2.41}$$

Lemma 5.2.10. *Let π and β_π be as in (5.2.36) and (5.2.40); then*

$$\{\pi\}' = \{T \in \mathscr{B}(\mathscr{H}) \,;\, \beta_\pi (T) = T\} \, (= Fix(\beta_\pi) .) \tag{5.2.42}$$

Proof. We have the following bi-implications:

$$\begin{array}{rcl} \beta_\pi (T) & = & T \\ & \Updownarrow & \\ S_i^* \beta_\pi (T) & = & S_i^* T, \quad 1 \leq i \leq N \\ & \Updownarrow & \text{(by (5.2.36))} \\ T S_i^* & = & S_i^* T, \quad 1 \leq i \leq N \\ & \Updownarrow & \\ T & \in & \{\pi\}' . \end{array}$$

$\square$

In applications to *statistical mechanics*, given $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, it is important to determine the closed subspaces $\mathscr{K} \subset \mathscr{H}$, invariant under the operators S_i^*, $1 \leq i \leq N$.

Notation: When $\mathscr{K}$ is a closed subspace, we shall denote the corresponding projection by $P\,(= P_{\mathscr{K}})$.

LEMMA 5.2.11. *Let $(\pi, \mathscr{H}, \mathscr{K}$ (with projection P)) be as above; then TFAE:*

(1) $S_i^* \mathscr{K} \subseteq \mathscr{K}$, $1 \leq i \leq N$;
(2) $P S_i^* P = S_i^* P$, $1 \leq i \leq N$;
(3) $P \leq \beta_\pi(P)$, *in the order of projections (see Section* 5.2.1*); and*
(4) $P \leq \beta_\pi(P) \leq \cdots \leq \beta_\pi^k(P) \leq \beta_\pi^{k+1}(P) \leq \cdots$.

PROOF. The argument is the same as that used in the proof of Lemma 5.2.10. ☐

LEMMA 5.2.12. *Let $(\pi, \mathscr{H}, \mathscr{K}$ (with projection P)) be as in Lemma* 5.2.11, *and set*

$$Q = \bigvee_{k=1}^{\infty} \beta_\pi^k(P); \tag{5.2.43}$$

then $Q \in \{\pi\}'$, and Q is the smallest projection in $\{\pi\}'$ satisfying $P \leq Q$.

PROOF. The conclusion is immediate from the formula:

$$S_i^* \beta_\pi^{k+1}(P) = \beta_\pi^k(P) S_i^*, \; 1 \leq i \leq N.$$

Assuming (5.2.43), we then get

$$S_i^* Q = Q S_i^*, \; 1 \leq i \leq N,$$

and by taking adjoints

$$QS_i = S_i Q;$$

so $Q \in \{\pi\}'$. The remaining parts of the proof are immediate. ☐

DEFINITION 5.2.13. We shall use the standard σ-algebra $\mathscr{C}$ of subsets of Ω_N (the path-space). The σ-algebra is generated by *cylinder sets E_f*. Here $f = (i_1, i_2, \cdots, i_k)$ is a finite word, $|f| = k$; and

$$E_f := \{\omega \in \Omega_N \mid \omega_j = i_j, \; 1 \leq j \leq k\}; \tag{5.2.44}$$

is one of the *basic cylinder sets* (see Fig 5.2.4). The σ-algebra $\mathscr{C}$ is the smallest σ-algebra containing the sets E_f as f varies over all finite words in the fixed alphabet A.

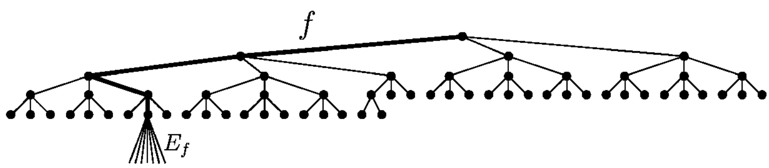


FIGURE 5.2.4. A basic cylinder set.

DEFINITION 5.2.14 (Operators on path-space). We recall the shift operators on Ω_N, as follows: If $\omega = (i_1, i_2, i_3, \cdots) \in \Omega_N$, set

$$\sigma(\omega) := (i_2, i_3, i_4, \cdots), \text{ and} \tag{5.2.45}$$

$$\hat{\tau}_j(\omega) := (j, i_1, i_2, i_3, \cdots). \tag{5.2.46}$$

If $E \subset \Omega_N$ is a subset, and f is a finite word, we set

$$\sigma(E) = \{\sigma(\omega) \mid \omega \in E\}, \tag{5.2.47}$$

$$\hat{\tau}_j(E) = \{\hat{\tau}_j(\omega) \mid \omega \in E\}, \tag{5.2.48}$$

$$\sigma^{-1}(E) = \{\omega \in \Omega_N \mid \sigma(\omega) \in E\}, \text{ and} \tag{5.2.49}$$

$$fE = \{\underbrace{f\omega}_{\text{concatination of words.}} \mid \omega \in E\} \tag{5.2.50}$$

Note

$$\sigma^{-1}(E) = \bigcup_{j=1}^{N} \hat{\tau}_j(E). \tag{5.2.51}$$

LEMMA 5.2.15.

(1) *The sample space $\Omega_N = \{1, 2, \cdots, N\}^{\mathbb{N}}$ is a compact metric space when equipped with the metric d_N as follows: Given $\omega, \omega' \in \Omega_N$, and set*

$$k := \sup\{j \in \mathbb{N} \mid \omega_i = \omega_i', \ 1 \le i \le j\}$$

with the convention that $k = \infty$ if and only if $\omega = \omega'$. Then

$$d_N(\omega, \omega') = N^{-k}. \tag{5.2.52}$$

(2) *The shift maps $\{\hat{\tau}_j\}_{j=1}^{N}$ in (5.2.46) are contractive as follows:*

$$d_N(\hat{\tau}_j(\omega), \hat{\tau}_j(\omega')) \le N^{-1} d_N(\omega, \omega')$$

for all $1 \le j \le N$, $\forall \omega, \omega' \in \Omega_N$.

PROOF. Uses standard facts about infinite products, and is left to the reader. ☐

When discussing measures $\mathbb{Q}$ on $(\Omega_N, \mathscr{C})$, we refer to σ-additivity; i.e., if $\{E_j\}_{j\in\mathbb{N}}$, $E_j \in \mathscr{C}$, $E_i \cap E_j = \emptyset$, $i \neq j$, is given, we require that

$$\mathbb{Q}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{Q}(E_j).$$

We say that a measure $\mathbb{Q}$ on $(\Omega_N, \mathscr{C})$ is *projection valued*, i.e., $\mathbb{Q}(E)$ is a projection in $\mathscr{H}$, for all $E \in \mathscr{C}$, and

$$\mathbb{Q}(\Omega_N) = I, \quad \text{and} \quad \mathbb{Q}(\emptyset) = 0. \tag{5.2.53}$$

Let $\mathbb{Q}$ be the projection valued measure on $(M, \mathscr{C})$ taking values in $\mathscr{B}(\mathscr{H})$ for a fixed Hilbert space $\mathscr{H}$, and let $\psi \in \mathscr{H}$; we then get a *scalar valued* measure

$$\mu_\psi(E) := \langle \psi, \mathbb{Q}(E)\psi \rangle_{\mathscr{H}} = \|\mathbb{Q}(E)\psi\|_{\mathscr{H}}^2, \ \forall E \in \mathscr{C}.$$

Conversely, $\mathbb{Q}(\cdot)$ is determined by these measures.

In the discussion below, the projection valued measures will depend on a prescribed (fixed) representation $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$:

THEOREM 5.2.16. *Given $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, then there is a unique projection valued measure $\mathbb{Q} = \mathbb{Q}_\pi$ on $(\Omega_N, \mathscr{C})$ which is specified on the basic cylinder sets E_f, $f = (i_1, \cdots, i_k)$, as follows:*

$$\mathbb{Q}_\pi(E_f) = S_f S_f^* = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*. \tag{5.2.54}$$

The measure satisfies the following properties:

$$\beta_\pi(\mathbb{Q}_\pi(E)) = \mathbb{Q}_\pi\left(\sigma^{-1}(E)\right), \ \forall E \in \mathscr{C}; \tag{5.2.55}$$

*(see (*5.2.36*) for the definition of β_π.)*

$$S_i \mathbb{Q}_\pi(E) = \mathbb{Q}(iE) S_i, \ \forall i \in A, \ \forall E \in \mathscr{C}; \tag{5.2.56}$$

and

$$S_i^* \mathbb{Q}_\pi(E) = \delta_{i, \pi_1(E)} \mathbb{Q}_\pi(\sigma(E)) S_i^*. \tag{5.2.57}$$

PROOF. We begin with $\mathbb{Q}_\pi$ defined initially only on the basic cylinder sets E_f, $f \in \{\text{finite words}\}$; see (5.2.54). To show that it extends to the full σ-algebra $\mathscr{C}$, we make use of *Kolmogorov's consistency principle* (see [**Kol83**, **Hid80**]). Specifically, we must check from (5.2.54) that

$$\mathbb{Q}_\pi(E_f) = \sum_{i=1}^{N} \mathbb{Q}_\pi\left(E_{(fi)}\right) \tag{5.2.58}$$

where E_f is one of basic cylinder-sets. But (5.2.58) is immediate from:

$$S_f S_f^* = \sum_{i=1}^{N} S_f S_i S_i^* S_f^* = \sum_{i=1}^{N} S_{(fi)} S_{(fi)}^*.$$

The Kolmogorov extension also implies that the values $\mathbb{Q}_\pi(E)$, $E \in \mathscr{C}$, are determined by those on E_f, f finite; this is a standard inductive limit argument; see e.g., [**Hid80**, **Kol83**, **Tum08**, **HJr94**, **MO86**, **Tju72**].

Hence, to verify that these three conditions (5.2.55)-(5.2.57) in the theorem, we may restrict the checking to the case when E has the form E_f, for some finite word $f = (i_1, \cdots, i_k)$ fixed.

The argument for (5.2.55) is:

$$\sum_{i=1}^{N} S_i S_f S_f^* S_i^* = \sum_{i=1}^{N} S_{(if)} S_{(if)}^* = \mathbb{Q}_\pi\left(\sigma^{-1}(E_f)\right).$$

The argument for (5.2.56) is:

$$S_i\left(S_f S_f^*\right) = \left(S_{(if)} S_{(if)}^*\right) S_i;$$

and finally the argument for (5.2.57) is:

$$S_j^* S_f S_f^* = \delta_{j i_1} S_{(i_2, \cdots, i_k)} S_{(i_2, \cdots, i_k)}^* S_j^*.$$

When these identities are combined with the *Kolmogorov consistency* / inductive limit arguments [**Kol83**, **Hid80**], the conclusions of the theorem now follow. We turn to the details of this in Section 5.2.3 below. □

DEFINITION 5.2.17. A projection-valued measure $\mathbb{Q}$ on $(\Omega_N, \mathscr{C})$, taking values in $\mathscr{B}(\mathscr{H})$, is said to be *orthogonal* iff (Def.)

$$\mathbb{Q}(E \cap E') = \mathbb{Q}(E)\,\mathbb{Q}(E') \tag{5.2.59}$$

for all sets E and E' in $\mathscr{C}$.

REMARK 5.2.18. The condition in (5.2.59) is called *orthogonality* because of the following: If (5.2.59) is satisfied, and if $E \cap E' = \emptyset$ where E and E' are picked from $\mathscr{C}$ (the σ-algebra), then

$$\mathbb{Q}_\pi(E)\,\mathscr{H} \perp \mathbb{Q}_\pi(E')\,\mathscr{H}. \tag{5.2.60}$$

To see this, compute the inner products of vectors $h, h' \in \mathscr{H}$:

$$\begin{aligned}\langle \mathbb{Q}_\pi(E)\,h, \mathbb{Q}_\pi(E')\,h'\rangle &= \langle h, \mathbb{Q}_\pi(E)\,\mathbb{Q}_\pi(E')\,h'\rangle \\ &= \langle h, \mathbb{Q}_\pi(\underbrace{E \cap E'}_{=\emptyset})h'\rangle = \langle h, 0h'\rangle = 0,\end{aligned}$$

which is the orthogonality (5.2.60).

The Kolmogorov consistency construction

Fix $N > 1$, and a Hilbert space $\mathscr{H}$. Let $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, $\pi(s_i) = S_i$, $1 \leq i \leq N$. Let $\Omega = \Omega_N := (\mathbb{Z}_N)^{\mathbb{N}}$ (= the infinite Cartesian product). For $\omega \in \Omega$, and $k \in \mathbb{N}$, set $\omega|_k = (\omega_1, \omega_2, \cdots, \omega_k)$, the truncated word. Let $C(\Omega) :=$ all continuous functions on Ω. Set

$$\mathscr{F}_k = \{F \in C(\Omega) \mid F(\omega) = F(\omega|_k)\}, \tag{5.2.61}$$

i.e., $\mathscr{F}_k$ consists of functions depending on only the first k coordinates. $\mathscr{F}_0 =$ the constant functions on Ω. Finally, we set

$$\mathscr{F}_\infty := \bigcup_{k=0}^{\infty} \mathscr{F}_k. \tag{5.2.62}$$

LEMMA 5.2.19. *With the notation from above, $\mathscr{F}_\infty$ is a dense subalgebra in $C(\Omega)$, i.e., dense in the uniform norm on $C(\Omega)$.*

PROOF SKETCH. The conclusion follows from the Stone-Weierstrass theorem [**Hel69**]: We only need to show that $\mathscr{F}_\infty$ is an algebra, contains the constant function $\mathbb{1}$, and separates points in Ω.

But the properties are immediate from (5.2.61). Indeed, if $\omega, \omega' \in \Omega$, and $\omega \neq \omega'$. Pick k such that $\omega_k \neq \omega'_k$; then take $F = \pi_k$ (= the coordinate projection); it is in $\mathscr{F}_k$, and satisfies $F(\omega) \neq F(\omega')$. $\square$

PROOF OF THEOREM (5.2.16) CONTINUED. We now turn to the projection-valued measure $\mathbb{Q}_\pi$, defined initially only for $\mathscr{F}_\infty$. We define $\mathbb{Q}_\pi$ as a positive linear functional, taking values in the projections in $\mathscr{H}$; see Fig 5.2.5.

<table>
<tr><td colspan="8">$\mathscr{H}$</td></tr>
<tr><td colspan="4">P_0</td><td colspan="4">P_1</td></tr>
<tr><td colspan="2">P_{00}</td><td colspan="2">P_{01}</td><td colspan="2">P_{10}</td><td colspan="2">P_{11}</td></tr>
<tr><td>P_{000}</td><td>P_{001}</td><td>P_{001}</td><td>P_{011}</td><td>P_{100}</td><td>P_{101}</td><td>P_{110}</td><td>P_{111}</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td></tr>
</table>

$$P\left(i_1, i_2, \cdots, i_k\right) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1} = S_I S_I^*$$

FIGURE 5.2.5. Multiresolution as a nested family of projections.

In detail: If $F \in \mathscr{F}_k$, set

$$\mathbb{Q}_\pi^{(k)}\left(F\right) = \sum_{\substack{I=(i_1,\cdots,i_k)\\ \in \mathbb{Z}_N^k}} F\left(I\right) S_I S_I^*. \tag{5.2.63}$$

To show that $\mathbb{Q}_\pi^{(k)}$, as defined in (5.2.63) is positive, we need to check that

$$\mathbb{Q}_\pi^{(k)}\left(F^2\right) \geq 0. \tag{5.2.64}$$

(Recall, we have restricted the checking to real valued functions, but this can easily be modified to apply to the complex valued case. In that case, we must consider $\mathbb{Q}_\pi^{(k)}\left(\left|F\right|^2\right)$ in (5.2.64).)

For $I, J \in \left(\mathbb{Z}_N\right)^k$, we have

$$S_I S_I^* S_J S_J^* = \delta_{I,J} S_I S_I^* \tag{5.2.65}$$

where $\delta_{I,J} = \prod_{l=1}^k \delta_{i_l, j_l}$. Now, combining (5.2.63) and (5.2.65), we get

$$\begin{aligned}
\mathbb{Q}_\pi^{(k)}\left(F\right)^2 &= \sum_I \sum_J F\left(F\right) F\left(J\right) S_I S_I^* S_J S_J^* \\
&= \sum_I F\left(I\right)^2 S_I S_I^* = \mathbb{Q}_\pi^{(k)}\left(F^2\right),
\end{aligned}$$

and the desired positivity (5.2.64) follows.

To get the desired *Kolmogorov extension* (see [**Kol83**, **Hid80**]), we only need to check *consistency*: Let $F \in \mathscr{F}_k \subseteq \mathscr{F}_{k+1}$, i.e., F is considered as a function on $\left(\mathbb{Z}_N\right)^{k+1}$, but constant in the last variable i_{k+1}.

We now have:

$$\mathbb{Q}_\pi^{(k+1)}\left(F\right) = \mathbb{Q}_\pi^{(k)}\left(F\right). \tag{5.2.66}$$

Indeed,

$$\begin{aligned}
\mathrm{LHS}_{(5.2.66)} &= \sum_{I \in (\mathbb{Z}_N)^k} \sum_{j \in \mathbb{Z}_N} F\left(I\right) S_I S_j S_j^* S_I^* \\
&= \sum_{I \in (\mathbb{Z}_N)^k} F\left(I\right) S_I S_I^* = \mathrm{RHS}_{(5.2.66)},
\end{aligned}$$

since $\sum_j S_j S_j^* = I$ by (5.2.10).

Now Kolmogorov consistency, and an application of the Riesz representation theorem (see [**Hel69**]), yields the final conclusion: The projection valued measure

$\mathbb{Q}_\pi$ arises as a projective limit of the individual measures $\left(\mathbb{Q}_\pi^{(k)}(\cdot), \mathscr{F}_k\right)$ introduced above in (5.2.63). □

REMARK 5.2.20. Consider the family $\{\mathscr{F}_k\}_{k\in\mathbb{N}_0}$ in (5.2.61). By abuse of notation, we may also consider this *as a family of σ-algebras*, i.e.,

$$\mathscr{F}_k = (\text{the } \sigma\text{-algebra generated by } \{\pi_1, \pi_2, \cdots, \pi_k\}\ ), \tag{5.2.67}$$

see (5.2.6). Moreover, $\mathscr{C} = \bigvee_k \mathscr{F}_k$, where we use the lattice operation for σ-algebras.

From (5.2.66), we obtain the projection valued measure $\mathbb{Q}_\pi$ as a solution to the problem

$$\mathbb{Q}_\pi^{(k)}(\cdot\cdot) = \mathbb{Q}_\pi(\cdot\cdot \mid \mathscr{F}_k) \tag{5.2.68}$$

where " $\mid \mathscr{F}_k$" refers to conditional expectation.

Hence the solution $\mathbb{Q}_\pi(\cdot)$ may be viewed as a *martingale limit*: We have for all $k, l \in \mathbb{N}$, $k < l$:

$$\mathbb{Q}_\pi(\cdot\cdot \mid \mathscr{F}_k) = \mathbb{Q}_\pi(\cdot\cdot \mid \mathscr{F}_l \mid \mathscr{F}_k)\,; \tag{5.2.69}$$

and for all measurable functions F on $(\Omega, \mathscr{C})$, we have

$$\mathbb{Q}_\pi(F) = \lim_{k\to\infty} \mathbb{Q}_\pi^{(k)}(F) = \lim_{k\to\infty} \mathbb{Q}_\pi(F \mid \mathscr{F}_k)$$

where

$$\mathbb{Q}_\pi(F) := \int_\Omega F(\omega)\, \mathbb{Q}_\pi(d\omega)\,.$$

REMARK 5.2.21. Let $E \subset \Omega$, and assume $E \in \mathscr{F}_k = \sigma\text{-algebra}(\{\pi_i\}_{i=1}^k)$:

Let $j \in \mathbb{Z}_N$. Then $Ej := \bigcup_{e\in E}(ej) \in \mathscr{F}_{k+1}$, and

$$\mathbb{Q}_\pi(E) = \sum_{j\in\mathbb{Z}_N} \mathbb{Q}_\pi(Ej)\,; \tag{5.2.70}$$

but, in general,

$$\sum_{i\in\mathbb{Z}_N} \mathbb{Q}_\pi(iE) \neq \mathbb{Q}_\pi(E)\,. \tag{5.2.71}$$

Note in general,

$$\bigcup_{i\in\mathbb{Z}_N} iE = \sigma^{-1}(E)\,, \tag{5.2.72}$$

(as a disjoint union on the left hand side) where σ is the *shift* in Ω; see (5.2.45) and (5.2.50). So the assertion in (5.2.71) above is that, in general, we may have:

$$\mathbb{Q}_\pi\left(\sigma^{-1}E\right) \neq \mathbb{Q}_\pi(E)\,.$$

COROLLARY 5.2.22. *Let $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, and let $\mathbb{Q}_\pi$ be the corresponding projection valued measure introduced in Theorem* 5.2.16. *Then $\mathbb{Q}_\pi$ is orthogonal, i.e., (*5.2.59*) holds.*

PROOF. Because of the Kolmogorov-consistency construction, it is enough to verify the orthogonality (5.2.59) for $\mathbb{Q}_\pi$ in the special case when the two sets have the form E_f, E_g, where f and g are *finite* words in the alphabet, say $f = (i_1, i_2, \cdots, i_k)$ and $g = (j_1, j_2, \cdots, j_l)$ where k and l denote the respective word lengths. We say that containment holds for the two words if and only if one of the two contains the other in the following manner: say $f \subseteq g$, if $k \leq l$ and $i_1 = j_1, \cdots, i_k = j_k$. In this case $g = (fh)$ where h is the tail end in the word g. (There is a symmetric condition when instead $g \subseteq f$.)

When $f \subseteq g$, then

$$E_f \cap E_g = E_g. \tag{5.2.73}$$

Hence we must verify that, in this case,

$$\mathbb{Q}_\pi (E_g) = \mathbb{Q}_\pi (E_f) \mathbb{Q}_\pi (E_g). \tag{5.2.74}$$

But using $g = (fh)$ for some finite word h, we get for the RHS in (5.2.74):

$$\begin{aligned} \mathrm{RHS}_{(5.2.74)} &= S_f \left(S_f^* S_f\right) S_h S_h^* S_f^* = S_f S_h S_h^* S_f^* \\ &= S_g S_g^* \quad (\text{since } g = fh) \\ &= \mathbb{Q}_\pi (E_g) = \mathrm{LHS}_{(5.2.74)} \end{aligned}$$

and the desired conclusion follows.

The remaining case is, if none of the possible containment holds, i.e., f not contained in g, and g not contained in f. In this case, $E_f \cap E_g = \emptyset$, and so both sides in equation (5.2.74) are zero.

Having verified that $\mathbb{Q}_\pi$ satisfies condition (5.2.59) for basic cylinder-sets, it now follows that it must also hold for all pairs of sets $E, E' \in \mathscr{C}$. This is an application of the Kolmogorov extension principle. The proof of the theorem is concluded. □

Corollary 5.2.23. *Let the setting be as above, $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, and let $\mathbb{Q}_\pi (\cdot)$ be the corresponding projection valued measure.*

(1) *For $\omega \in \Omega_N$, and $k \in \mathbb{N}$, set $Z_k(\omega) = \omega_k$ $(\in A_N \simeq \{1, 2, \cdots, N\})$, then the following projection-valued Markov property holds: Let $k > 1$, then*

$$\mathbb{P}rob^{(\pi)} (Z_{k+1} = j \mid Z_k = i) = \beta_\pi^{k-1} \left(S_i S_j S_j^* S_i^*\right), \tag{5.2.75}$$

where β_π is the endomorphism in Definition 5.2.9 (eq. (5.2.36)).

(2) *If $\psi \in \mathscr{H}$, $\|\psi\| = 1$, let $\mu_\psi (\cdot) := \langle \psi, \mathbb{Q}_\pi (\cdot) \psi \rangle_{\mathscr{H}}$ be the corresponding scalar valued measure. Then the associated transition probabilities are*

$$\begin{aligned} \mathbb{P}rob^{(\mu_\psi)} (Z_{k+1} = j \mid Z_k = i) &= \frac{\left\langle \psi, \beta_\pi^{k-1} \left(S_i S_j S_j^* S_i^*\right) \psi \right\rangle_{\mathscr{H}}}{\left\langle \psi, \beta_\pi^{k-1} \left(S_i S_i^*\right) \psi \right\rangle_{\mathscr{H}}} \\ &= \frac{\left\| \beta_\pi^{k-1} \left(S_j^* S_i^*\right) \psi \right\|_{\mathscr{H}}^2}{\left\| \beta_\pi^{k-1} \left(S_i^*\right) \psi \right\|_{\mathscr{H}}^2}. \end{aligned} \tag{5.2.76}$$

(3) *The Markov property holds for the process in (2) if and only if β_π-invariance holds, in the following sense:*

$$\langle \psi, \beta_\pi (\mathbb{Q}_\pi (\cdot)) \psi \rangle_{\mathscr{H}} = \langle \psi, \mathbb{Q}_\pi (\cdot) \psi \rangle_{\mathscr{H}} = \mu_\psi (\cdot).$$

Proof. For (5.2.75), we have

$$\begin{aligned} \mathrm{Prob}^{(\pi)} (Z_{k+1} = j \mid Z_k = i) &= \sum_{I \in \mathbb{Z}_N^{k-1}} \mathbb{Q}_\pi (E(Iij)) \\ &= \sum_{I \in \mathbb{Z}_N^{k-1}} S_I S_i S_j S_j^* S_i^* S_I^* \underset{(\text{by } (5.2.36))}{=} \beta_\pi^{k-1} \left(S_i S_j S_j^* S_i^*\right). \end{aligned}$$

Parts (2) and (3) follow immediately from this. □

Monic representations

Let $\pi \in Rep(\mathscr{O}_N, \mathscr{H})$, and let $\mathbb{Q}_\pi$ be the corresponding projection valued measure. Let $\mathfrak{M}_\pi$ be the abelian $*$-algebra generated by $\mathbb{Q}_\pi$, i.e., the operators

$$\int_{\Omega_N} f(\omega)\, \mathbb{Q}_\pi(d\omega) \tag{5.2.77}$$

where f ranges over the measurable functions on $(\Omega_N, \mathscr{C})$.

Following [**DJ14a**], we make the following:

DEFINITION 5.2.24. We say that π is *monic* iff (Def.) there is a vector $\psi_0 \in \mathscr{H}$, $\|\psi_0\| = 1$, such that

$$[\mathfrak{M}_\pi \psi_0] = \mathscr{H}, \tag{5.2.78}$$

i.e., ψ_0 is $\mathfrak{M}_\pi$-cyclic.

Starting with $\mathbb{Q}_\pi$ and (5.2.78), we use the construction outlined before Theorem 5.2.16, to get a scalar measure via:

$$\mu_0(E) = \langle \psi_0, \mathbb{Q}_\pi(E)\psi_0 \rangle_{\mathscr{H}}, \; E \in \mathscr{C}. \tag{5.2.79}$$

Using [**DJ14a**], we then get a *random variable* $Y : \Omega_N \to M$ for a measure space $(M, \mathscr{B})$ such that the measure $\mu := \mu_0 \circ Y^{-1}$ satisfies the conditions listed below:

It was proved in [**DJ14a**] that a representation $\pi \, (\in Rep(\mathscr{O}_N, \mathscr{H}))$ is monic if and only if it is unitarily equivalent to one realized in $L^2(M, \mu)$ as follows for some measure space (M, μ):

There are endomorphisms $\left(\{\tau_i\}_{i=1}^N, \sigma\right)$, such that $\sigma \circ \tau_i = id_M$, $\mu \circ \tau_i^{-1} \ll \mu$, and $L^2(\mu)$-function f_i on M, such that

$$\frac{d\left(\mu \circ \tau_i^{-1}\right)}{d\mu} = |f_i|^2, \; 1 \le i \le N, \tag{5.2.80}$$

$$f_i \neq 0 \quad a.e.\ \mu \text{ in } \tau_i(M). \tag{5.2.81}$$

Then the isometries S_i are as follows:

$$S_i^{(\mu)} \varphi = f_i(\varphi \circ \sigma), \; 1 \le i \le N, \tag{5.2.82}$$

i.e., $\left\{S_i^{(\mu)}\right\}_{i=1}^N \in Rep\left(\mathscr{O}_N, L^2(\mu)\right)$; see (5.2.10).

Symbol space representations as groups

In the study of iterated function systems (IFSs), and more generally, in symbolic dynamics, we consider a fixed finite alphabet A, as well as words in A. Both *finite* as well as *infinite* words are needed. For many purposes, it is helpful to give A in the form of a cyclic group $\mathbb{Z}/N\mathbb{Z} \simeq \{0, 1, 2, \cdots, N-1\}$. In this case both the finite words Ω_N^*, as well as infinite words $\Omega_N := A^{\mathbb{N}}$ become groups. In the representation below, we identify Ω_N^*, and Ω_N, as a pair of abelian groups in duality. Since Ω_N^* (finite words) is discrete, we get Ω_N realized as a compact abelian group.

LEMMA 5.2.25. *Let $N \in \mathbb{N}$, $N \ge 2$, be fixed, and let Ω_N^*, resp. Ω_N, denote the finite, resp., infinite words in $\mathbb{Z}_N \simeq \mathbb{Z}/N\mathbb{Z}$.*

(1) *If $x = (x_j)_{j=1}^\infty \in \Omega_N$, and $y = (y_j)_{j=1}^{finite} \in \Omega_N^*$, are fixed, then set*

$$\langle x, y \rangle := \prod_{k=1}^\infty \exp\left(i2\pi \left(\frac{x_k y_k}{N^k}\right)\right), \tag{5.2.83}$$

so we have

$$\langle x + x', y\rangle = \langle x, y\rangle \langle x', y\rangle, \text{ and} \tag{5.2.84}$$

$$\langle x, y + y'\rangle = \langle x, y\rangle \langle x, y'\rangle, \tag{5.2.85}$$

for all $x, x' \in \Omega_N$, and $y, y' \in \Omega_N^*$.

(2) *In the category of abelian groups, we get*

$$dual\,(\Omega_N) = \Omega_N^*, \text{ and} \tag{5.2.86}$$

$$dual\,(\Omega_N^*) = \Omega_N, \text{ where} \tag{5.2.87}$$

"dual" refers to Pontryagin duality. Note $\Omega_N^* = \bigcup_{k=1}^{\infty} N^{-k}\mathbb{Z}$; and

$$\mathbb{Z} \subset N^{-1}\mathbb{Z} \subset N^{-2}\mathbb{Z} \subset \cdots \subset N^{-k}\mathbb{Z} \subset N^{-(k+1)}\mathbb{Z} \subset \cdots. \tag{5.2.88}$$

(3) *The Haar measure on Ω_N is the infinite product norm on $(\mathbb{Z}_N)^{\mathbb{N}}$ with weights $\left(\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}\right)$ on each factor.*

PROOF. The lemma follows from results in the literature (see [**DHJ15**, **DJ15a**]), and is left to the reader.

Identify a finite word $y = (y_1, \cdots, y_k) \in \Omega_N^*$ $(y_j \in \mathbb{Z}_N = \{0, 1, \cdots, N-1\})$ with

$$\begin{aligned}\tilde{y} &= \frac{y_1 N^{k-1} + \cdots + y_{k-1}N + y_k}{N^k} \\ &= y_1/N + \cdots + y_k/N^k \in N^{-k}\mathbb{Z};\end{aligned} \tag{5.2.89}$$

see (5.2.88). Set

$$S_y = S_{y_1} S_{y_2} \cdots S_{y_k}; \tag{5.2.90}$$

and if $x = (x_j)_{j=1}^{\infty} \in \Omega_N$ $(x_j \in \mathbb{Z}_N)$, define an automorphism action $\alpha(x)$ of $\mathscr{O}_N$ by its values on generators S_y as follows:

$$\alpha(x) S_y = \langle x, y\rangle S_y; \tag{5.2.91}$$

called the gauge-action.

In particular,

$$\alpha(x)\left(S_y S_y^*\right) = S_y S_y^*. \tag{5.2.92}$$

The abelian $*$-subalgebra $\mathfrak{M}_N$ in $\mathscr{O}_N$ generated by the projections $\left\{S_y S_y^*\right\}_{y \in \Omega_N^*}$ (Ω_N^* = finite words) is $\mathfrak{M}_N = \{M \in \mathscr{O}_N \mid \alpha(x) M = M, \forall x \in \Omega_N\}$. ∎

REMARK 5.2.26. It follows from Lemma 5.2.25 that the projection valued measures from Theorems 5.2.5 and 5.2.16 may be realized on the compact group Ω_N.

For the study of Markov chains, the following extension of the lemma will be useful:

LEMMA 5.2.27. *Let M be a fixed $N \times N$ matrix over $\mathbb{Z}$, and assume its eigenvalues λ_j satisfy $|\lambda_j| > 1$.*

From the nested chain of groups we then obtain inductive, and projective limits, in the form of discrete groups Ω_M^*, and compact dual Ω_M.

Case 1 (inductive)

$$\mathbb{Z}^N / M^{k+1}\mathbb{Z}^N \hookrightarrow \mathbb{Z}^N / M^k \mathbb{Z}^N \tag{5.2.93}$$

and the dual projective group formed from the groups

$$\left(M^T\right)^k \mathbb{Z}^N \tag{5.2.94}$$

where M^T denotes the transposed matrix:

$$\Omega_M^* = \bigcup_{k=1}^{\infty} M^{-k}(\mathbb{Z}_N);$$

and note $\mathbb{Z}_N \subset M^{-1}\mathbb{Z}_N \subset M^{-2}\mathbb{Z}_N \subset \cdots \subset M^{-k}\mathbb{Z}_N \subset M^{-(k+1)}\mathbb{Z}_N \subset \cdots$.

When M is fixed, and pair $x = (x_j)$ and $y = (y_j)$ are infinite, resp., finite, words in $\mathbb{Z}^N / M\mathbb{Z}^N$, then the Pontryagin duality is then

$$\langle x, y\rangle_M := \prod_{k=1}^{\infty} \exp\left(i2\pi \left(M^T\right)^{-k} x_j \cdot y_j\right). \tag{5.2.95}$$

PROOF. See, e.g., [**BJKR01**, **BJKR02**, **BJOk04**].

Note that if x and $y \in \mathbb{Z}^N$, and $k \in \mathbb{N}$, then in the quotient group we have

$$\left(M^T\right)^{-k} x \cdot y = \left(M^T\right)^{-(k+1)} x \cdot My.$$

□

5.2.4. *Boundaries of representations.* Let M be a compact Hausdorff space, with Borel σ-algebra $\mathscr{B}$, and let A be a finite alphabet, $|A| = N$. Let $\{\tau_i\}_{i\in A}$ be a system of endomorphisms. For every $\omega \in \Omega_N$ $\left(= A^{\mathbb{N}}\right)$, and $k \in \mathbb{N}$, set $\omega|_k = (\omega_1, \cdots, \omega_k)$ (= the truncated finite word), and set

$$\tau_{\omega|_k} = \tau_{\omega_1} \circ \cdots \circ \tau_{\omega_k}. \tag{5.2.96}$$

DEFINITION 5.2.28. We say that $\{\tau_i\}_{i\in A}$ is *tight* iff (Def.)

$$\bigcap_{k=1}^{\infty} \tau_{\omega|_k}(M) = \{Y(\omega)\} \tag{5.2.97}$$

is a singleton for $\forall\omega \in \Omega_N$; and we define $Y : \Omega_N \to M$ by eq. (5.2.97).

THEOREM 5.2.29. *Let $\left(M, \{\tau_i\}_{i\in A}\right)$ be as above, assume <u>tight</u>. Let π be a representation of $\mathscr{O}_N$ on some Hilbert space $\mathscr{H}$, and let $\mathbb{Q}_\pi$ be the corresponding projection-valued measure. Assume $\mathbb{Q}_\pi$ has one-dimensional range; see Corollary* 5.2.22*; set*

$$\mu := \mathbb{Q}_\pi \circ Y^{-1}. \tag{5.2.98}$$

Then for all $\omega \in \Omega_N = A^{\mathbb{N}}$, we have

$$\mu \circ \tau_{\omega|_k}^{-1} \xrightarrow[k\to\infty]{} \delta_{Y(\omega)}, \tag{5.2.99}$$

i.e., for all $f \in C(M)$, we have

$$\lim_{k\to\infty} \int_M f \circ \tau_{\omega|_k} d\mu = f(Y(\omega)). \tag{5.2.100}$$

PROOF. Let $\varepsilon > 0$. Since f is uniformly continuous, there is a neighborhood O_ω of $Y(\omega)$ such that

$$|f(y) - f(y')| < \varepsilon \quad \text{for } \forall y, y' \in O_\omega. \tag{5.2.101}$$

Since by assumption $\mu(M) = 1$, we conclude from (5.2.101) and (5.2.97), that for $\forall k, l \geq k_0$, we have

$$\left|f \circ \tau_{\omega|_k} - f \circ \tau_{\omega|_l}\right| \leq \varepsilon; \tag{5.2.102}$$

as a uniform estimate on M. Since

$$\int_M f \circ \tau_{\omega|_k} d\mu = \int_M f \, d\left(\mu \circ \tau_{\omega|_k}^{-1}\right), \tag{5.2.103}$$

a second application of (5.2.97) now yields:

$$\lim_{k\to\infty} \int_M f \circ \tau_{\omega|_k} d\mu = f\left(Y\left(\omega\right)\right)$$

which is the desired conclusion. □

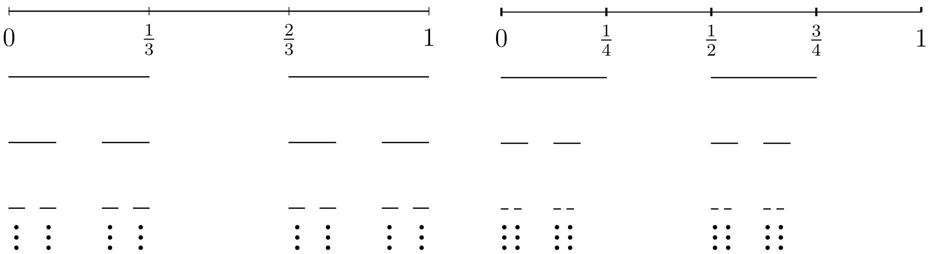
5.2.5. Three examples. Below we give three examples of IFS-measures, as in Theorem 5.2.7: (i) *the Lebesgue measure* restricted to the unit interval $[0,1]$, (ii) *the middle-third Cantor measure* μ_3, and (iii) *the* $1/4$*-Cantor measure* μ_4 with two gaps. Their respective properties follow from Theorem 5.2.7, and are summarized in Table 1. Also see Figures 5.2.6, 5.2.7, and 5.2.9.

The difference in the graphs of the *cumulative distributions* in Ex 2 and Ex 3, is explained by the following: In Ex 3, we have *two omitted intervals in each iteration step*, as opposed to just one in Ex 2, the Middle-third Cantor construction. See Fig 5.2.8.

In each of the three examples in Table 1, we give the initial step in the IFS iteration. Each IFS-limit yields a measure, and a support set. The second and the third examples are the fractal limits known as the Cantor measure μ_3, and the Cantor measure μ_4. The details of the iteration steps are outlined in the subsequent figures and algorithms. Figures 5.2.7 and 5.2.8 deal with the associated cumulative distribution $F\left(x\right) := \mu\left([0,x]\right)$.

TABLE 1. Three inequivalent examples, each with $\Omega_N = A^{\mathbb{N}}$, $|A| = 2$, and infinite product measure $\times_1^\infty \left(\frac{1}{2}, \frac{1}{2}\right)$. See also Fig 5.2.9.

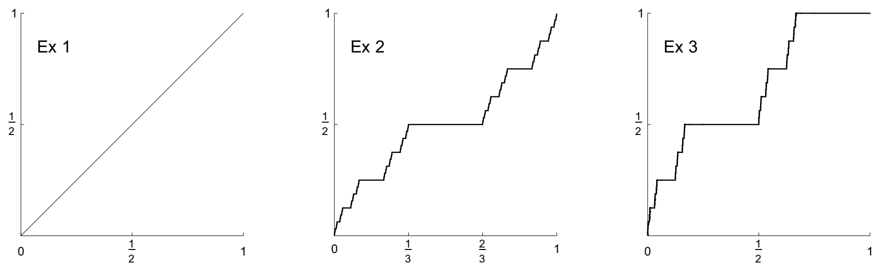
$\{\tau_i\}_{i=1}^2$	σ	$(p_i)_{i=1}^2$	Scaling dimension (SD) of the IFS-measure (μ, M_μ)
$\tau_0\left(x\right) = \frac{x}{2}$, $\tau_1\left(x\right) = \frac{x+1}{2}$	$\sigma\left(x\right) = 2x \bmod 1$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\mu = \lambda =$ Lebesgue measure, SD $= 1$
$\tau_0\left(x\right) = \frac{x}{3}$, $\tau_1\left(x\right) = \frac{x+2}{3}$	$\sigma\left(x\right) = 3x \bmod 1$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\mu = \mu_3 =$ middle-third Cantor measure, SD $= \frac{\ln 2}{\ln 3}$
$\tau_0\left(x\right) = \frac{x}{4}$, $\tau_1\left(x\right) = \frac{x+2}{4}$	$\sigma\left(x\right) = 4x \bmod 1$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\mu = \mu_4 =$ the $1/4$-Cantor measure, SD $= \frac{1}{2}$



(A) The middle-third Cantor set.

(B) The 1/4-Cantor set.

FIGURE 5.2.6. Examples of Cantor sets.



$F_\lambda(x) = \lambda([0,x])$; points of increase = the support of the normalized λ, so the interval $[0,1]$.

$F_{1/3}(x) = \mu_3([0,x])$; points of increase = the support of μ_3, so the middle third Cantor set $C_{1/3}$ (the Devil's staircase).

$F_{1/4}(x) = \mu_4([0,x])$; points of increase = the support of μ_4, so the double-gap Cantor set $C_{1/4}$.

FIGURE 5.2.7. The three *cumulative distributions*. The three support sets, $[0,1]$, $C_{1/3}$, and $C_{1/4}$ are IFSs, and they are also presented in detail inside Table 1 above.

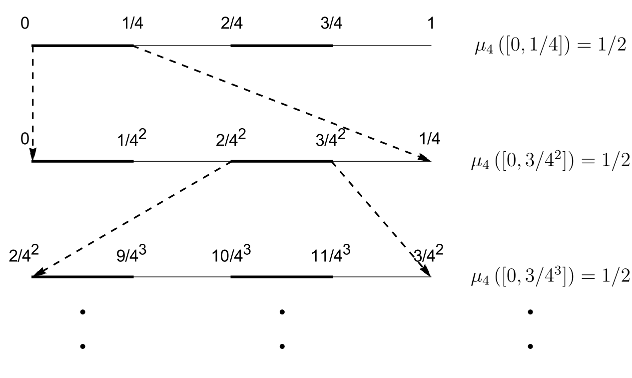


FIGURE 5.2.8. Illustration of $F_{1/4}(x) = \mu_4([0,x])$ in Ex 3. Note that $\inf\{F_{1/4}^{-1}(1/2)\} = \frac{1}{4} - \left(\sum_{n=2}^{\infty} \frac{1}{4^n}\right) = \frac{1}{6}$, and $\inf\{F_{1/4}^{-1}(1)\} = \frac{2}{3}$.

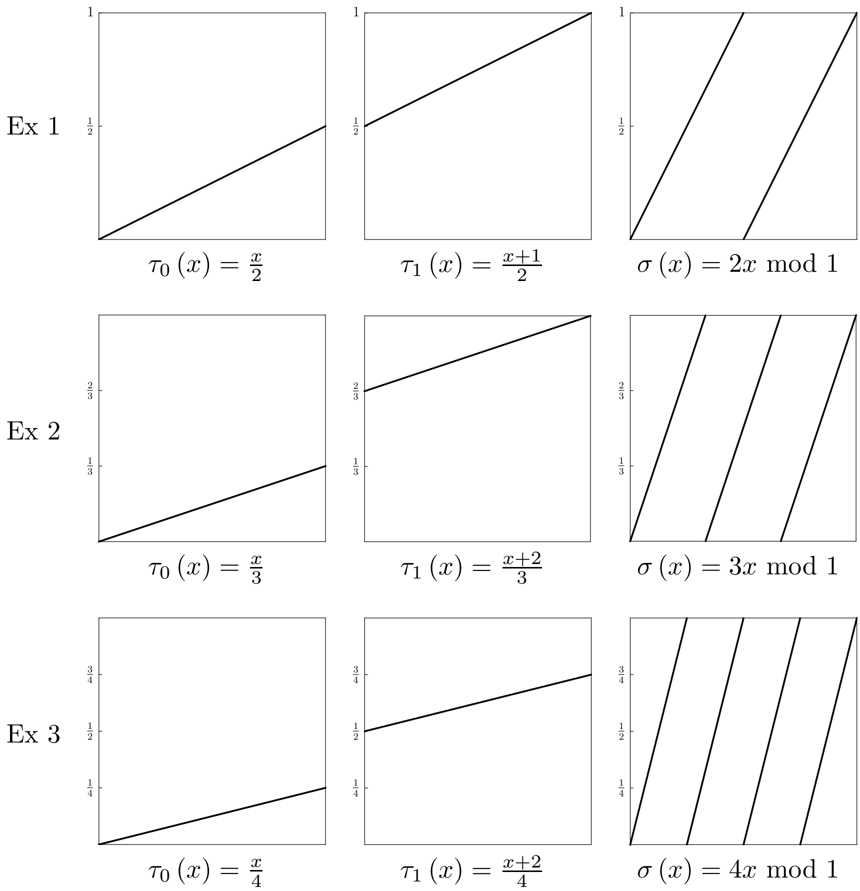


FIGURE 5.2.9. The *endomorphisms* in the three examples.

Bit-representation of the respective IFSs in each of the three examples

In the three examples from Table 1, the associated random variable Y (from Theorem 5.2.7, eq (5.2.24)) is as follows:

$$\text{Ex 1} \qquad Y_\lambda(\varepsilon_i) = \frac{1}{2}\sum_{i=1}^{\infty}\frac{\varepsilon_i}{2^i},$$

$$\text{Ex 2} \qquad Y_{\mu_3}(\varepsilon_i) = \sum_{i=1}^{\infty}\frac{\varepsilon_i}{3^i}, \quad \text{and}$$

$$\text{Ex 3} \qquad Y_{\mu_4}(\varepsilon_i) = \sum_{i=1}^{\infty}\frac{\varepsilon_i}{4^i}, \quad \varepsilon_i \in \{0,2\},\ (\varepsilon_i) \in \Omega_2.$$

Boundary representation for the two measures μ_3 and μ_4, (see Theorem 5.2.7, and Figs 5.2.6-5.2.7.)

DEFINITION 5.2.30. Let μ be a (singular) measure with support contained in the interval $I = [0,1] \simeq \partial\mathbb{D}$, the boundary of the disk $\mathbb{D} = \{z \in \mathbb{Z}\ ;\ |z| < 1\}$.

A function $K : \mathbb{D} \times I \to \mathbb{C}$ is said to be a *boundary representation* iff (Def.) the following four axioms hold:

(1) $K(\cdot, x)$ is analytic in $\mathbb{D}$ for all $x \in I$;
(2) $K(z, \cdot) \in L^2(\mu)$, $\forall z \in \mathbb{D}$;
(3) Setting, for $f \in L^2(I, \mu)$,

$$(Kf)(z) = \int_0^1 f(x) K(z, x) \, d\mu(x), \tag{5.2.104}$$

then $Kf \in H_2(\mathbb{D})$, the Hardy-space; and
(4) The following limit exists in the $L^2(\mu)$-norm:

$$\lim_{\substack{r \nearrow 1 \\ r<1}} (Kf)(r\, e(x)) = f(x),$$

where $e(x) := e^{i2\pi x}$, $x \in I$.

We say that K is *self-reproducing* if there is a kernel $K^{\mathbb{C}} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ satisfying

$$\lim_{r \nearrow 1} K^{\mathbb{C}}(z, r\, e(x)) = K(z, x), \ \forall z \in \mathbb{D}, \ x \in I; \tag{5.2.105}$$

and

$$\int_0^1 K(z, x) \overline{K(w, x)} d\mu(x) = K^{\mathbb{C}}(z, w), \ \forall (z, w) \in \mathbb{D} \times \mathbb{D}. \tag{5.2.106}$$

Remark 5.2.31. When $\left(K, K^{\mathbb{C}}\right)$ satisfy the two conditions (5.2.105)-(5.2.106), it is immediate that $K^{\mathbb{C}}$ is then a *positive definite kernel* on $\mathbb{D} \times \mathbb{D}$. We shall denote the corresponding *reproducing kernel Hilbert space* RKHS by $\mathscr{H}\left(K^{\mathbb{C}}\right)$.

Furthermore, the assignment

$$T_\mu : \underbrace{K^{\mathbb{C}}(\cdot, z)}_{\text{as a function on } \mathbb{D}} \longmapsto \underbrace{K(z, \cdot)}_{\text{in } L^2(I,\mu)} \tag{5.2.107}$$

extends by linearity, and norm-closure to an *isometry*:

$$T_\mu : \mathscr{H}\left(K^{\mathbb{C}}\right) \longrightarrow L^2(\mu); \tag{5.2.108}$$

with "isometry" relative to the respective Hilbert norms in (5.2.108).

Moreover, the adjoint operator

$$T_\mu^* : L^2(\mu) \longrightarrow \mathscr{H}\left(K^{\mathbb{C}}\right) \tag{5.2.109}$$

is the original operator Kf specified in (5.2.104), i.e., for $\forall f \in L^2(\mu)$, we have:

$$\left(T_\mu^* f\right)(z) = \int_0^1 f(x) K(z, x) \, d\mu(x), \ \forall z \in \mathbb{D}. \tag{5.2.110}$$

Corollary 5.2.32. *If the measure μ (as above) has a self-reproducing kernel $K^{\mathbb{C}}$, then the corresponding operator K (see (5.2.104)) satisfies*

$$KT_\mu = I_{\mathscr{H}(K^{\mathbb{C}})} \tag{5.2.111}$$

where the subscript refers to the identity operator in the RKHS $\mathscr{H}\left(K^{\mathbb{C}}\right)$.

Proposition 5.2.33. *Each of the measures μ_3 and μ_4 from Fig 5.2.7 has a boundary representation.*

PROOF. We shall refer the reader to the two papers [**JP98a**] and [**HJW18b**]. In the case of μ_4, the construction is as follows:

$$K_4(z,x) = \prod_{n=0}^{\infty}\left(1 + \left(\overline{e(x)}z\right)^{4^n}\right), \tag{5.2.112}$$

and we refer to [**JP98a**] for details.

In the case of μ_3, let b be the *inner function* corresponding to μ_3 via the Herglotz-formula; then

$$K_3(z,x) = \frac{1 - b(z)\overline{b(e(x))}}{1 - z\overline{e(x)}}. \tag{5.2.113}$$

For the proof details, showing that K_3 in (5.2.113) satisfies conditions (1)-(4), readers are referred to [**HJW18b**]. ☐

COROLLARY 5.2.34. *The two kernels K_4 and K_3 are self-reproducing.*

5.3. Representations in a *universal Hilbert space*

Our starting point is a compact Hausdorff space M and continuous maps $\sigma\colon M \to M$, $\tau_i\colon M \to M$, $i = 1, \ldots, N$, such that

$$\sigma \circ \tau_i = id_M. \tag{5.3.1}$$

It follows from (5.3.1) that σ is onto, and that each τ_i is one-to-one. We will be especially interested in the case when there are distinct branches $\tau_i\colon M \to M$ such that

$$\bigcup_{i=1}^{N} \tau_i(M) = M. \tag{5.3.2}$$

For such systems, we show that there is a *universal* representation of $\mathscr{O}_N$ in a Hilbert space $\mathscr{H}(M)$ which is functorial, is naturally defined, and contains every representation of $\mathscr{O}_N$.

The elements in *the universal Hilbert space $\mathscr{H}(M)$* are equivalence classes of pairs (φ, μ) where φ is a Borel function on M and where μ is a positive Borel measure on M. We will set $\varphi\sqrt{d\mu} := \text{class}(\varphi,\mu)$ for reasons which we spell out below.

While our present methods do adapt to the more general framework when the space M of (5.3.1)-(5.3.2) is not assumed compact, but only σ-compact, we will still restrict the discussion here to the compact case. This is for the sake of simplicity of the technical arguments. But we encourage the reader to follow our proofs below, and to formulate for him/herself the corresponding results when M is not necessarily assumed compact. Moreover, if M is not compact, then there is a variety of special cases to take into consideration, various abstract notions of "escape to infinity". We leave this wider discussion for a later investigation, and we only note here that our methods allow us to relax the compactness restriction on M.

There is a classical construction in operator theory which lets us realize point transformations in Hilbert space. It is called the Koopman representation; see, for example, [**Mac89**, p. 135]. But this approach only applies if the existence of invariant, or quasi-invariant, measures is assumed. In general such measures are not available. We propose a different way of realizing families of point transformations in Hilbert space in a general context where no such assumptions are made. Our Hilbert spaces are motivated by S. Kakutani [**Kak48**], L. Schwartz, and E. Nelson [**Nel69**], among others. The reader is also referred to an updated presentation of the measure-class Hilbert spaces due to Tsirelson [**Tsi03**] and Arveson [**Arv03b**, Chapter 14].

We say that $(\varphi, \mu) \sim (\psi, \nu)$ if there is a third positive Borel measure λ on M such that $\mu \ll \lambda$, $\nu \ll \lambda$, and

$$\varphi\sqrt{\frac{d\mu}{d\lambda}} = \psi\sqrt{\frac{d\nu}{d\lambda}}, \quad \lambda \text{ a.e. on } M, \tag{5.3.3}$$

where $\ll$ denotes relative absolute continuity, and where $d\mu/d\lambda$ denotes the usual Radon-Nikodym derivative, i.e., $d\mu/d\lambda \in L^1(\lambda)$, and $d\mu = (d\mu/d\lambda)\, d\lambda$.

One checks that $\sim$ for pairs (φ, μ), i.e., (function, measure), indeed defines an *equivalence relation.* Notation: $\operatorname{class}(\varphi, \mu) =: \varphi\sqrt{d\mu}$.

We shall review some basic properties of the Hilbert space $\mathscr{H}(M)$. This space is called the Hilbert space of σ-functions, or square densities, and it was studied for different reasons in earlier papers of L. Schwartz, E. Nelson [**Nel69**], and W. Arveson [**Arv03a**].

THEOREM 5.3.1. *Isometries $S_i \colon \mathscr{H}(M) \to \mathscr{H}(M)$ are defined by*

$$S_i \colon\ (\varphi, \mu) \longmapsto \left(\varphi \circ \sigma, \mu \circ \tau_i^{-1}\right), \tag{5.3.4}$$

or equivalently, $S_i \colon \varphi\sqrt{d\mu} \mapsto \varphi \circ \sigma\sqrt{d\mu \circ \tau_i^{-1}}$, and these operators satisfy the Cuntz relations.

PROOF. Note that, at the outset, it is not even clear *a priori* that S_i in (5.3.4) defines a transformation of $\mathscr{H}(M)$. To verify this, we will need to show that if two equivalent pairs are substituted on the left-hand side in (5.3.4), then they produce equivalent pairs as output, on the right-hand side. Recalling the definition (5.3.3) of the equivalence relation $\sim$, there is no obvious or intuitive reason for why this should be so.

Before turning to the proof, we shall need some preliminaries and lemmas. $\square$

To stress the intrinsic transformation rules of $\mathscr{H}(M)$, the vectors in $\mathscr{H}(M)$ are usually denoted $\varphi\sqrt{d\mu}$ rather than (φ, μ). This suggestive notation motivates the definition of the *inner product* of $\mathscr{H}(M)$. If $\varphi\sqrt{d\mu}$ and $\psi\sqrt{d\nu}$ are in $\mathscr{H}(M)$, we define their Hilbert inner product by

$$\left\langle \varphi\sqrt{d\mu}, \psi\sqrt{d\nu} \right\rangle := \int_M \bar{\varphi}\psi\sqrt{\frac{d\mu}{d\lambda}}\sqrt{\frac{d\nu}{d\lambda}} d\lambda, \tag{5.3.5}$$

where λ is some positive Borel measure, chosen such that $\mu \ll \lambda$ and $\nu \ll \lambda$. For example, we could take $\lambda = \mu + \nu$. To be in $\mathscr{H}(M)$, $\varphi\sqrt{d\mu}$ must satisfy

$$\left\| \varphi\sqrt{d\mu} \right\|^2 = \int_M |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda = \int_M |\varphi|^2\, d\mu < \infty. \tag{5.3.6}$$

5.3.1. Isometries in $\mathscr{H}(M)$. In this preliminary section we prove three general facts about the process of *inducing operators in the Hilbert space $\mathscr{H}(M)$* from underlying point transformations in M. The starting point is a given continuous mapping $\sigma \colon M \to M$, mapping onto M. We will be concerned with the special case when M is a compact Hausdorff space, and when there is one or more continuous branches $\tau_i \colon M \to M$ of the inverse, i.e., when

$$\sigma \circ \tau_i = id_M. \tag{5.3.7}$$

Recall that *elements in $\mathscr{H}(M)$ are equivalence classes* of pairs (φ, μ) where φ is a Borel function on M, μ is a positive Borel measure on M, and $\int_M |\varphi|^2 \, d\mu < \infty$. An equivalence class will be denoted $\varphi\sqrt{d\mu}$, and we show that there are isometries

$$S_i \colon \varphi\sqrt{d\mu} \longmapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau_i^{-1}}, \tag{5.3.8}$$

with orthogonal ranges in the Hilbert space $\mathscr{H}(M)$. Moreover, we calculate an explicit formula for the adjoint co-isometries S_i^*.

LEMMA 5.3.2. *Let M be a compact Hausdorff space, and let the mapping $\sigma \colon M \to M$ be onto. Suppose $\tau \colon M \to M$ satisfies $\sigma \circ \tau = id_M$. Assume that both σ and τ are continuous. Let $\mathscr{H} = \mathscr{H}(M)$ be the Hilbert space of classes (φ, μ) where φ is a Borel function on M and μ is a positive Borel measure such that $\int |\varphi|^2 \, d\mu < \infty$. The equivalence relation is defined in the usual way: two pairs (φ, μ) and (ψ, ν) are said to be equivalent, written $(\varphi, \mu) \sim (\psi, \nu)$, if for some positive measure λ, $\mu \ll \lambda$, $\nu \ll \lambda$, we have the following identity:*

$$\varphi\sqrt{\frac{d\mu}{d\lambda}} = \psi\sqrt{\frac{d\nu}{d\lambda}} \quad \text{(a.e. } \lambda). \tag{5.3.9}$$

Then there is an isometry $S \colon \mathscr{H} \to \mathscr{H}$ which is well defined by the assignment

$$S\left((\varphi, \mu)\right) := \left(\varphi \circ \sigma, \mu \circ \tau^{-1}\right), \tag{5.3.10}$$

or

$$S \colon \varphi\sqrt{d\mu} \longmapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau^{-1}},$$

where $\mu \circ \tau^{-1}(E) := \mu\left(\tau^{-1}(E)\right)$, and $\tau^{-1}(E) := \{x \in M \mid \tau(x) \in E\}$, for $E \in \mathscr{B}(M)$.

PROOF. We leave the verification of the following four facts to the reader; see also [**Nel69**].

(1) If $\varphi\sqrt{\frac{d\mu}{d\lambda}} = \psi\sqrt{\frac{d\nu}{d\lambda}}$ for some λ such that $\mu \ll \lambda$, $\nu \ll \lambda$, and if some other measure λ' satisfies $\mu \ll \lambda'$, $\nu \ll \lambda'$, then
$$\varphi\sqrt{\frac{d\mu}{d\lambda'}} = \psi\sqrt{\frac{d\nu}{d\lambda'}} \quad \text{(a.e. } \lambda').$$

(2) The "vectors" in $\mathscr{H}$ are equivalence classes of pairs (φ, μ) as described in the statement of the lemma. For two elements (φ, μ) and (ψ, ν) in $\mathscr{H}$, define the sum by
$$(\varphi, \mu) + (\psi, \nu) := \left(\phi\sqrt{\frac{d\mu}{d\lambda}} + \psi\sqrt{\frac{d\nu}{d\lambda}}, \lambda\right), \tag{5.3.11}$$
where λ is a positive Borel measure satisfying $\mu \ll \lambda$, $\nu \ll \lambda$. The sum in (5.3.11) is also written $\varphi\sqrt{d\mu} + \psi\sqrt{d\nu}$. The definition of the sum (5.3.11) passes through the equivalence relation $\sim$, i.e., we get an equivalent result on the right-hand side in (5.3.11) if equivalent pairs are used as input on the left-hand side. A similar conclusion holds for the definition (5.3.12) below of the inner product $\langle \cdot, \cdot \rangle$ in the Hilbert space $\mathscr{H}$.

(3) Scalar multiplication, $c \in \mathbb{C}$, is defined by $c(\varphi, \mu) := (c\varphi, \mu)$, and *the Hilbert space inner product* is

$$\left\langle \varphi\sqrt{d\mu}, \psi\sqrt{d\nu} \right\rangle = \langle (\varphi, \mu), (\psi, \nu) \rangle := \int_M \overline{\varphi}\psi \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda \tag{5.3.12}$$

where $\mu \ll \lambda$, $\nu \ll \lambda$.

(4) It is known, see [**Nel69**], that $\mathscr{H}$ is a Hilbert space. In particular, it is *complete*: if a sequence (φ_n, μ_n) in $\mathscr{H}$ satisfies

$$\lim_{n,m\to\infty} \|(\varphi_n, \mu_n) - (\varphi_m, \mu_m)\|^2 = 0,$$

then there is a pair (φ, μ) with

$$\int_M |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda = \int_M |\varphi|^2 d\mu < \infty, \tag{5.3.13}$$

where

$$\lambda := \sum_{n=1}^{\infty} 2^{-n} \mu_n (M)^{-1} \mu_n, \tag{5.3.14}$$

and $\|(\varphi, \mu) - (\varphi_n, \mu_n)\|^2 \underset{n\to\infty}{\longrightarrow} 0$.

Assuming that the expression in (5.3.10) defines an operator S in $\mathscr{H}$, it follows from (5.3.11) that S is linear. To see this, let (φ, μ), (ψ, ν), and λ be as stated in the conditions below (5.3.11). Then $\mu \circ \tau^{-1} \ll \lambda \circ \tau^{-1}$, and $\nu \circ \tau^{-1} \ll \lambda \circ \tau^{-1}$, and a calculation shows that the following formula holds for the transformation of the Radon-Nikodym derivatives: setting

$$\frac{d\mu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}} = k_\mu, \tag{5.3.15}$$

we have

$$k_\mu \circ \tau = \frac{d\mu}{d\lambda} \quad (\text{a.e.}\, \lambda). \tag{5.3.16}$$

Similarly, $k_\nu := \dfrac{d\nu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$ satisfies

$$k_\nu \circ \tau = \frac{d\nu}{d\lambda} \quad (\text{a.e.}\, \lambda). \tag{5.3.17}$$

The argument above yields:

LEMMA 5.3.3. *Let τ and σ be endomorphisms in M such that $\sigma \circ \tau = id_M$. Let μ, λ be a pair of positive measures with $\mu \ll \lambda$, and set $L := d\mu/d\lambda$; then*

$$\frac{d\left(\mu \circ \tau^{-1}\right)}{d\left(\lambda \circ \tau^{-1}\right)} = L \circ \sigma, \tag{5.3.18}$$

*i.e., composition on the RHS in (*5.3.18*).*

To show that S is linear, we must calculate the sum

$$\left(\varphi \circ \sigma, \mu \circ \tau^{-1}\right) + \left(\psi \circ \sigma, \nu \circ \tau^{-1}\right), \tag{5.3.19}$$

or, in expanded notation, we must verify that
(5.3.20)

$$\left(\varphi \circ \sigma\sqrt{k_\mu} + \psi \circ \sigma\sqrt{k_\nu}, \lambda \circ \tau^{-1}\right) \sim \left(\left(\varphi\sqrt{\frac{d\mu}{d\lambda}} + \psi\sqrt{\frac{d\nu}{d\lambda}}\right) \circ \sigma, \lambda \circ \tau^{-1}\right).$$

We get this class identity by an application of (5.3.16) as follows:

$$k_\mu(x) = k_\mu(\tau(\sigma(x))) = \left(\sqrt{\frac{d\mu}{d\lambda}} \circ \sigma\right)\Bigg|_{\tau(M)} (x) \quad (\text{a.e. } \lambda \circ \tau^{-1}).$$

Similarly, for the other measure, we get

$$k_\nu = \left(\sqrt{\frac{d\nu}{d\lambda}} \circ \sigma\right)\Bigg|_{\tau(M)} \quad (\text{a.e. } \lambda \circ \tau^{-1}). \tag{5.3.21}$$

Assuming again that S in (5.3.10) is well defined, we now show that it is isometric, i.e., that $\|S(\varphi,\mu)\|^2 = \|(\varphi,\mu)\|^2$, referring to the norm of $\mathscr{H}$. In view of (5.3.11) and (5.3.20), it is enough to show that

$$\int_M |\varphi \circ \sigma|^2 k_\mu d\lambda \circ \tau^{-1} = \int_M |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda. \tag{5.3.22}$$

But, using (5.3.16), we get

$$\begin{aligned}\int_M |\varphi \circ \sigma|^2 k_\mu d\lambda \circ \tau^{-1} &= \int_M |\varphi \circ \sigma \circ \tau|^2 k_\mu \circ \tau \, d\lambda \\ &\underset{(5.3.16)}{=} \int_M |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda,\end{aligned}$$

which is the desired formula (5.3.22).

It remains to prove that S is well defined, i.e., that the following implication holds:

$$(\varphi,\mu) \sim (\psi,\nu) \Longrightarrow \left(\varphi \circ \sigma, \mu \circ \tau^{-1}\right) \sim \left(\psi \circ \sigma, \nu \circ \tau^{-1}\right). \tag{5.3.23}$$

To do this, we go through a sequence of implications which again uses the fundamental transformation rules (5.3.16) and (5.3.21).

□

LEMMA 5.3.4. *Pick some λ such that $\mu \ll \lambda$ and $\nu \ll \lambda$. We then have the following implication:*

$$\varphi\sqrt{\frac{d\mu}{d\lambda}} = \psi\sqrt{\frac{d\nu}{d\lambda}} \ (\text{a.e. } \lambda) \Longrightarrow (\varphi \circ \sigma)\sqrt{k_\mu} = (\psi \circ \sigma)\sqrt{k_\nu} \ (\text{a.e. } \lambda \circ \tau^{-1}), \tag{5.3.24}$$

where $k_\mu = \dfrac{d\mu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$ *and* $k_\nu = \dfrac{d\nu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$. *(The desired conclusion (*5.3.23*) follows from this.)*

PROOF. We now turn to the proof of the implication (5.3.24). We pick a third measure λ as described, and assume the identity

$$\varphi\sqrt{\frac{d\mu}{d\lambda}} = \psi\sqrt{\frac{d\nu}{d\lambda}} \quad \text{a.e. } \lambda.$$

Let f be a bounded Borel function on M. In the following calculations, all integrals are over the full space M, but the measures change as we make transformations, and we use the definition of the Radon-Nikodym formula. First note that

$$\begin{aligned}\int f\, k_\mu \left(\frac{d\nu}{d\lambda} \circ \sigma\right) d\lambda \circ \tau^{-1} &= \int f \left(\frac{d\nu}{d\lambda} \circ \sigma\right) d\mu \circ \tau^{-1} \\ &= \int f \circ \tau \frac{d\nu}{d\lambda} d\mu = \int f \circ \tau \frac{d\nu}{d\lambda}\frac{d\mu}{d\lambda} d\lambda.\end{aligned}$$

But by symmetry, we also have

$$\int f\,k_\nu\left(\frac{d\mu}{d\lambda}\circ\sigma\right)d\lambda\circ\tau^{-1}=\int f\circ\tau\frac{d\nu}{d\lambda}\frac{d\mu}{d\lambda}d\lambda.$$

Putting the last two formulas together, we arrive at the following identity:

$$\int_M f\,k_\mu\left(\frac{d\nu}{d\lambda}\circ\sigma\right)d\lambda\circ\tau^{-1}=\int_M f\,k_\nu\left(\frac{d\mu}{d\lambda}\circ\sigma\right)d\lambda\circ\tau^{-1}.$$

Since the function f is arbitrary, we get

$$k_\mu\left(\frac{d\nu}{d\lambda}\circ\sigma\right)=k_\nu\left(\frac{d\mu}{d\lambda}\circ\sigma\right)\quad \text{a.e. }\lambda\circ\tau^{-1}$$

and, of course,

$$\sqrt{k_\mu}\sqrt{\frac{d\nu}{d\lambda}}\circ\sigma=\sqrt{k_\nu}\sqrt{\frac{d\mu}{d\lambda}}\circ\sigma\quad \text{a.e. }\lambda\circ\tau^{-1}.$$

Using now the identity

$$\varphi\sqrt{\frac{d\mu}{d\lambda}}=\psi\sqrt{\frac{d\nu}{d\lambda}}\quad \text{a.e. }\lambda,$$

we arrive at the formula

$$\varphi\circ\sigma\sqrt{k_\mu}\sqrt{\frac{d\mu}{d\lambda}}\circ\sigma\sqrt{\frac{d\nu}{d\lambda}}\circ\sigma=\psi\circ\sigma\sqrt{k_\nu}\sqrt{\frac{d\mu}{d\lambda}}\circ\sigma\sqrt{\frac{d\nu}{d\lambda}}\circ\sigma,$$

and by cancellation,

$$\varphi\circ\sigma\sqrt{k_\mu}=\psi\circ\sigma\sqrt{k_\nu}\quad \text{a.e. }\lambda\circ\tau^{-1}.$$

This completes the proof of the implication (5.3.24), and therefore also of (5.3.23). This means that, if the linear operator S is defined as in (5.3.10), then the result is independent of which element is chosen *in the equivalence class* represented by the pair (φ,μ). Putting together the steps in the proof, we conclude that $S\colon\mathscr{H}\to\mathscr{H}$ is an *isometry*, and that it has the properties which are stated in the lemma.

Combining the lemmas, the proof of Theorem 5.3.1 is now completed. □

LEMMA 5.3.5. *Let M be a compact Hausdorff space, and let σ be as in the statement of Lemma* 5.3.2, *i.e., $\sigma\colon M\to M$ is onto and continuous. Suppose σ has two distinct branches of the inverse, i.e., $\tau_i\colon M\to M$, $i=1,2$, continuous, and satisfying $\sigma\circ\tau_i=id_M$, $i=1,2$. Let $S_i\colon\mathscr{H}\to\mathscr{H}$ be the corresponding isometries, i.e.,*

$$S_i\left((\varphi,\mu)\right):=\left(\varphi\circ\sigma,\mu\circ\tau_i^{-1}\right),\tag{5.3.25}$$

or

$$S_i\colon\varphi\sqrt{d\mu}\longmapsto\varphi\circ\sigma\ \sqrt{d\mu\circ\tau_i^{-1}}.$$

Then the two isometries have orthogonal ranges, i.e.,

$$\langle\, S_1\left((\varphi,\mu)\right),S_2\left((\psi,\nu)\right)\rangle=0\tag{5.3.26}$$

for all pairs of vectors in $\mathscr{H}$, i.e., all $(\varphi,\mu)\in\mathscr{H}$ and $(\psi,\nu)\in\mathscr{H}$.

Proof. Note that in the statement (5.3.26) of the conclusion, we use $\langle \cdot, \cdot \rangle$ to denote the inner product of the Hilbert space $\mathscr{H}$, as it was defined in (5.3.12).

With the two measures μ and ν given, then the expression in (5.3.26) involves the transformed measures $\mu \circ \tau_1^{-1}$ and $\nu \circ \tau_2^{-1}$. Now pick some measure λ such that $\mu \circ \tau_1^{-1} \ll \lambda$ and $\nu \circ \tau_2^{-1} \ll \lambda$. Then the expression in (5.3.26) is

$$\int_M \overline{\varphi \circ \sigma}\, \psi \circ \sigma \sqrt{\frac{d\mu \circ \tau_1^{-1}}{d\lambda}} \sqrt{\frac{d\nu \circ \tau_2^{-1}}{d\lambda}} d\lambda. \tag{5.3.27}$$

But $\frac{d\mu \circ \tau_1^{-1}}{d\lambda}$ is supported on $\tau_1 (M)$, while $\frac{d\nu \circ \tau_2^{-1}}{d\lambda}$ is supported on $\tau_2 (M)$. Since $\tau_1 (M) \cap \tau_2 (M) = \emptyset$ by the choice of distinct branches for the inverse of σ, we conclude that the integral in (5.3.27) vanishes. $\square$

Corollary 5.3.6. *Let M be a compact Hausdorff space, and let $N \in \mathbb{N}$, $N \geq 2$, be given. Let $\sigma \colon M \to M$ be continuous and onto. Suppose there are N distinct branches of the inverse, i.e., continuous $\tau_i \colon M \to M$, $i = 1, \ldots, N$, such that*

$$\sigma \circ \tau_i = id_M. \tag{5.3.28}$$

Suppose there is a positive Borel measure μ such that $\mu (M) = 1$, and

$$\mu \circ \tau_i^{-1} \ll \mu, \quad i = 1, \ldots, N. \tag{5.3.29}$$

Then the isometries

$$S_i \varphi := \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau_i^{-1}}{d\mu}} \tag{5.3.30}$$

satisfy

$$\sum_{i=1}^{N} S_i S_i^* = I_{L^2(\mu)} \tag{5.3.31}$$

if and only if

$$\bigcup_{i=1}^{N} \tau_i (M) = M. \tag{5.3.32}$$

Proof. We already know from Lemma 5.3.5 that the isometries $S_i \colon L^2 (\mu) \to L^2 (\mu)$ are mutually orthogonal, i.e., that

$$S_i^* S_j = \delta_{i,j} I_{L^2(\mu)}. \tag{5.3.33}$$

It follows that the terms in the sum (5.3.31) are *commuting projections.* Hence

$$\sum_{i=1}^{N} S_i S_i^* \leq I_{L^2(\mu)}. \tag{5.3.34}$$

Moreover, we conclude that (5.3.31) holds if and only if

$$\sum_{i=1}^{N} \|S_i^* \varphi\|^2 = \|\varphi\|^2, \quad \varphi \in L^2 (\mu). \tag{5.3.35}$$

Setting $p_i := \frac{d\mu \circ \tau_i^{-1}}{d\mu}$, we get

$$S_i^* \varphi = \varphi \circ \tau_i \, (p_i \circ \tau_i)^{-1/2}. \tag{5.3.36}$$

It follows that

$$\begin{aligned} \|S_i^* \varphi\|^2 &= \int_M |\varphi \circ \tau_i|^2 (p_i \circ \tau_i)^{-1} \, d\mu \\ &= \int_{\tau_i(M)} |\varphi|^2 p_i^{-1} d\mu \circ \tau_i^{-1} = \int_{\tau_i(M)} |\varphi|^2 \, d\mu. \end{aligned}$$

Recall that the branches τ_i of the inverse are distinct, and so the sets $\tau_i(M)$ are non-overlapping. The equivalence (5.3.31)$\Leftrightarrow$(5.3.32) now follows directly from the previous calculation. $\square$

5.3.2. Distributions. Consider the following setting, generalizing that of the three examples in Section 5.2.5: Let $(\Omega_N, \mathscr{C}, \mathbb{P})$ be a probability space, and $(M, \mathscr{B})$ be a measure space; see Section 5.2.4 for definitions.

Let $\mathscr{H}(M)$ be *the Hilbert space of equivalence classes*, see Lemma 5.3.2 above. As shown in [**Nel69**], if μ is a fixed positive σ-finite measure on $(M, \mathscr{B})$, then the subspace $\left\{ f\sqrt{d\mu} \mid f \in L^2(\mu) \right\}$ in $\mathscr{H}(M)$ is closed, denoted $\mathscr{H}(\mu)$; and

$$L^2(\mu) \ni f \longmapsto f\sqrt{d\mu} \in \mathscr{H}(\mu) \tag{5.3.37}$$

is an *isometric isomorphism*; called the canonical isomorphism.

KEY PROPERTIES. The following is known; see e.g. [**Nel69**]: For two σ-finite positive measures μ_1, μ_2 on $(M, \mathscr{B})$, we have the following three equivalences:

$$\mu_1 \ll \mu_2 \Longleftrightarrow \mathscr{H}(\mu_1) \subseteq \mathscr{H}(\mu_2), \tag{5.3.38}$$

$$\begin{pmatrix} \mu_1 \text{ and } \mu_2 \text{ are} \\ \text{mutually singular} \end{pmatrix} \Longleftrightarrow \mathscr{H}(\mu_1) \perp \mathscr{H}(\mu_2), \text{ and} \tag{5.3.39}$$

$$\begin{pmatrix} \mu_1 \text{ and } \mu_2 \text{ are} \\ \text{equivalent} \end{pmatrix} \Longleftrightarrow \mathscr{H}(\mu_1) = \mathscr{H}(\mu_2). \tag{5.3.40}$$

COROLLARY 5.3.7. *Let $Y_i : \Omega_N \to M$, $i = 1, 2$, be two random variables; i.e., the two are measurable functions w.r.t. the respective σ-algebras $\mathscr{C}$ and $\mathscr{B}$. The corresponding distributions*

$$\mu_i := \mathbb{P} \circ Y_i^{-1}, \quad i = 1, 2 \tag{5.3.41}$$

are measures on $(M, \mathscr{B})$; and

$$T_i f := f \circ Y_i, \quad i = 1, 2, \tag{5.3.42}$$

(see Fig 5.3.1) *are isometries*

$$L^2(\mu_i) \simeq \mathscr{H}(\mu_i) \xrightarrow{\;T_i\;} L^2(\mathbb{P}), \quad i = 1, 2. \tag{5.3.43}$$

$$\begin{tikzcd} \Omega_N \arrow[rr, "Y_i"] \arrow[dr, "f \circ Y_i"'] & & M \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd}$$

FIGURE 5.3.1.

Hence the three conditions in (5.3.38)*,* (5.3.39) *and* (5.3.40) *are statements about the two random variables.*

For the operators $T_2^* T_1$, see Fig 5.3.2, *we have the following: For $f \in L^2(\mu_1)$, and $x \in M$:*

$$(T_2^* T_1)(f)(x) = \mathbb{E}_{Y_2 = x}(f \circ Y_1 \mid \mathscr{F}_{Y_2}). \tag{5.3.44}$$

$$\begin{tikzcd} \mathscr{H}(\mu_1) \arrow[rr, "T_2^* T_1"] \arrow[dr, "T_1"'] & & \mathscr{H}(\mu_2) \arrow[dl, "T_2"] \\ & L^2(\mathbb{P}) & \end{tikzcd}$$

FIGURE 5.3.2.

PROOF. For $f \in L^2(\mu_1)$ and $g \in L^2(\mu_2)$, we have:

$$\begin{aligned} \langle T_2^* T_1 f, g \rangle_{\mathscr{H}(\mu_2)} &= \langle T_1 f, T_2 g \rangle_{L^2(\mathbb{P})} \\ &= \mathbb{E}[(f \circ Y_1)(g \circ Y_2)] \\ &= \mathbb{E}[\mathbb{E}(f \circ Y_1 \mid \mathscr{F}_{Y_2})(g \circ Y_2)] \\ &= \int_M \mathbb{E}_{(Y_2 = x)}(f \circ Y_1 \mid \mathscr{F}_{Y_2})\, g(x)\, d\mu_2(x), \end{aligned}$$

and the desired formula (5.3.44) follows from this, and (5.3.37). □

5.3.3. Fractional calculus. In recent papers [**FS15**, **FHH17**], a number of authors have studied *gradient operators* computed with respect to singular measures. The purpose of this subsection is to display some operator theoretic properties of these gradients ∇_μ , and to connect them to our boundary analysis.

In order to add clarity, we shall consider singular measures μ supported on compact subsets of the real line $\mathbb{R}$, but the ideas extend to more general measure spaces. For particular examples, readers are referred to the three examples in Section 5.2.5 above.

Let $I = [0, 1]$ be the unit-interval with the Borel σ-algebra. By $\mathscr{H}(I)$ we shall denote the Hilbert space of equivalence classes. When μ is a fixed positive measure, we considered the isometric isomorphism $T_\mu : L^2(\mu) \simeq \mathscr{H}(\mu)$ in (5.3.37).

In Proposition 5.3.9 below, we shall identity the gradient ∇_μ with the adjoint operator T_μ^* , referring to the respective inner products from (5.3.37).

DEFINITION 5.3.8. Let F be a function on $\mathbb{R}$ of bounded variation, and let dF be the corresponding Stieltjes measure, with variation measure $|dF|$ defined in the usual way. If $|dF| \ll \mu$, then the Radon-Nikodym derivative

$$RN_\mu(dF) =: \nabla_\mu F \tag{5.3.45}$$

is well defined; we have:

$$(dF)(B) = \int_B (\nabla_\mu F)\, d\mu, \ \forall B \in \mathscr{B}, \tag{5.3.46}$$

where $\mathscr{B}$ is the Borel σ-algebra. For the case of $(I, \mathscr{B})$, (5.3.46) is equivalent to

$$F(x) = \int_0^x dF = \int_0^x \nabla_\mu F\, d\mu, \ \forall x \in [0, 1] \tag{5.3.47}$$

(We shall adopt the normalization $F(0) = 0$.)

PROPOSITION 5.3.9. *If $T_\mu : L^2(\mu) \to \mathscr{H}(\mu)$ is as in (5.3.37), then the adjoint operator T_μ^* agrees with ∇_μ.*

PROOF. In view of Corollary 5.3.7 in Section 5.3.2, the desired conclusion will follow if we check that, when F is of bounded variation with $|dF| \ll \mu$, and if $\varphi \in L^2(\mu)$, then

$$\langle \underbrace{\varphi\sqrt{d\mu}}_{T_\mu\varphi}, dF \rangle_{\mathscr{H}(\mu)} = \langle \varphi, \underbrace{\nabla_\mu F}_{T_\mu^* F} \rangle_{L^2(\mu)}. \tag{5.3.48}$$

But, using our analysis from Sections 5.3.1-5.3.2 above, the verification of (5.3.48) is equivalent to:

$$\mathrm{LHS}_{(5.3.48)} = \underbrace{\int_I \varphi \underbrace{(\nabla_\mu F)}_{\text{Radon-Nikodym der}} d\mu}_{L^2(\mu)\,-\text{inner product}} = \mathrm{RHS}_{(5.3.48)};$$

and the conclusion follows. □

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CHAPTER 6

Positive definite functions and kernel analysis

"Nowadays group theoretical methods—especially those involving characters and representations, pervade all branches of quantum mechanics."
— George Whitelaw Mackey (1916–2006)

Reproducing Kernel Hilbert Spaces appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process, the present chapter will set the stage by discussing Reproducing Kernel Hilbert Spaces that appear in the context of positive definite functions, and the associated harmonic analysis.

We consider the question of spectral measures from the perspective of positive definite functions. Since the measures are spectral, the corresponding positive definite functions have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of RKHSs.

6.1. Positive definite kernels and harmonic analysis in $L^2(\mu)$ when μ is a gap IFS fractal measure

The study of positive definite (p.d.) functions and p.d. kernels is motivated by diverse themes in analysis and operator theory, in white noise analysis, applications of reproducing kernel (RKHS) theory, extensions by Laurent Schwartz, and in reflection positivity from quantum physics. Below we make more precise some parallels between, on the one hand, the standard case from Case 1, of continuous positive definite functions f on $\mathbb{R}$, the setting of Bochner's theorem, including generalizations to non-abelian locally compact groups. Hence we obtain representation formulas for positive definite tempered distributions in the sense of L. Schwartz [**Sch64a**, **Sch64b**]. The parallels between Bochner's theorem (for continuous p.d. functions), and the generalization to Bochner/Schwartz representations for positive definite tempered distributions will be made clear. In the first case, we have the Bochner representation via finite positive measures μ; and in the second case, instead via tempered positive measures. This parallel also helps make precise the respective reproducing kernel Hilbert spaces (RKHSs). This further leads to a more unified approach to the treatment of the stationary-increment Gaussian processes [**AJL11**, **AJ12**, **AJ15**]. A key argument will rely on the existence of a unitary representation U of $(\mathbb{R},+)$, acting on the particular RKHS under discussion. In fact, the same idea (with suitable modifications) will also work in the wider context of locally compact groups. In the abelian case, we shall make use of the Stone representation for U in the form of orthogonal projection valued measures; and in more general settings, the Stone-Naimark-Ambrose-Godement (SNAG) representation [**Sto32**].

THEOREM 6.1.1.

(a) *Let f be a continuous positive definite (p.d.) function on $\mathbb{R}$ (a p.d. tempered distribution* [**Sch64a**, **Sch64b**]*); then there is a unique finite positive Borel measure μ on $\mathbb{R}$ (resp., a unique tempered measure on $\mathbb{R}$) such that $f=\widehat{\mu}$.*

(b) *Given f as above, let $\mathscr{H}_f$ denote the corresponding reproducing kernel Hilbert space, i.e., the Hilbert completion of $\{\varphi * f\}_{\varphi\in C_c(\Omega)}$ (resp. $\varphi\in\mathcal{S}$) w.r.t*

$$\|\varphi * f\|^2_{\mathscr{H}_f}=\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\overline{\varphi(y)}f(x-y)\,dxdy$$

resp., $\langle f(x-y),\varphi*\overline{\varphi}\rangle$; action in the sense of distributions. Then there is a unique isometric transform

$$\mathscr{H}_f\xrightarrow{T_f}L^2(\mathbb{R},\mathscr{B},\mu),\quad T_f(\varphi * f)=\widehat{\varphi},\ \textit{i.e.},$$
$$\|\varphi * f\|^2_{\mathscr{H}_f}=\int_{\mathbb{R}}|\widehat{\varphi}|^2\,d\mu=\|T_f\varphi\|^2_{L^2(\mu)}.$$

(c) *If μ is tempered, e.g., if $\int_{\mathbb{R}}\frac{d\mu(\lambda)}{1+\lambda^2}<\infty$, then*

$$\|\varphi * f\|^2_{\mathscr{H}_f}=\int\left(|\widehat{\varphi}|^2+\left|\widehat{(D_x\varphi)}\right|^2\right)\frac{d\mu(\lambda)}{1+\lambda^2};$$

where $D_x\varphi=\frac{d\varphi}{dx}$, and where "$\widehat{\cdot}$" denotes the standard Fourier transform on $\mathbb{R}$.

COROLLARY 6.1.2. *For every tempered positive definite measure μ (see Theorem 6.1.1) there is a unique Gaussian process $X=X^{(\mu)}$ indexed by $x\in\mathbb{R}$, with mean zero, and variance*

$$r^{(\mu)}(x)=\mathbb{E}\left(\left|X^{(\mu)}_x\right|^2\right)=\int_{\mathbb{R}}\left|1-e^{i\lambda x}\right|^2\frac{d\mu(\lambda)}{\lambda^2},$$

and in addition,

$$\mathbb{E}\left(X^{(\mu)}_x\overline{X^{(\mu)}_y}\right)=\frac{r^{(\mu)}(|x|)+r^{(\mu)}(|y|)-r^{(\mu)}(|x-y|)}{2},$$

and

$$\mathbb{E}\left(\left|X^{(\mu)}_x-X^{(\mu)}_y\right|^2\right)=r^{(\mu)}(|x-y|).$$

PROOF. This family of stationary increment Gaussian processes were studied in [**AJL11**], and so we omit details here. The idea is to apply the transform T_μ from Theorem 6.1.1 (b) to the associated Gaussian process.

Setting $\varphi_x = \varphi = \begin{cases} \chi_{[0,x]}(\cdot) & \text{if } x \geq 0 \\ -\chi_{[0,x]}(\cdot) & \text{if } x < 0 \end{cases}$, we get

$$\begin{aligned} r^{(\mu)}(x) &= \int_{\mathbb{R}} |\widehat{\varphi}(\lambda)|^2 \, d\mu(\lambda) \quad \text{(see Thm. 6.1.1 (b))} \\ &= \int_{\mathbb{R}} \left|1 - e^{i\lambda x}\right|^2 \frac{d\mu(\lambda)}{\lambda^2}, \; x \in \mathbb{R}, \end{aligned}$$

as claimed. □

We shall leave the proof of Theorem 6.1.1 to the reader, and turn to some applications.

Dirac combs

In Theorem 6.1.1, we made a distinction between the two cases: that of (i) continuous p.d. functions, and (ii) the case of positive definite tempered distributions. The two cases are connected with the studies of Aronszajn [**Aro50**], in case (i); and of Schwartz [**Sch64b**], in case (ii). In the present section, we illustrate this distinction in detail.

The conclusions of Theorem 6.1.1 are made precise in the following:

PROPOSITION 6.1.3 (The Dirac comb [**BJV16**, **GP16**, **KL13**]). *Set*

$$\mu := \sum_{n \in \mathbb{Z}} \delta_n \tag{6.1.1}$$

*where δ_n in (*6.1.1*) denotes the Dirac distribution. Then $f = \widehat{\mu}$ is the tempered Schwartz distribution, written formally as*

$$f(x) = \sum_{n \in \mathbb{Z}} e^{inx}, \; x \in \mathbb{R}. \tag{6.1.2}$$

In this case the Hilbert completion $\mathscr{H}_f$ from Theorem 6.1.1 *is the Hilbert space of all 2π-periodic functions h on $\mathbb{R}$ subject to the condition*

$$\|h\|^2_{\mathscr{H}_f} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 \, dx < \infty. \tag{6.1.3}$$

PROOF. A positive measure μ on $\mathbb{R}$ is said to be *tempered* if and only if $\exists M \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^{2M}} < \infty. \tag{6.1.4}$$

The measure μ in (6.1.1) is clearly tempered, and in particular it is σ-finite. Specifically if $B \in \mathscr{B}_{\mathbb{R}}$ (the Borel σ-algebra), then

$$\mu(B) = \#(B \cap \mathbb{Z}). \tag{6.1.5}$$

For M in (6.1.4) we may take $M = 1$.

We now turn to the Hilbert completion $\mathscr{H}_f$ where f is as in (6.1.2). For all test-function $\varphi \in \mathcal{S}$, we have:

$$(\varphi * f)(x) = \sum_{n \in \mathbb{Z}} \widehat{\varphi}(n) e^{-inx} \tag{6.1.6}$$

where the interpretation of (6.1.6) is in the sense of tempered Schwartz distributions. Moreover,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi\left(x\right)\overline{\varphi\left(y\right)}f\left(x-y\right)dxdy=\sum_{n\in\mathbb{Z}}\left|\widehat{\varphi}\left(n\right)\right|^{2}. \tag{6.1.7}$$

Now, combining (6.1.6) and (6.1.7), we get that $\mathscr{H}_f$ is the Hilbert space described before (6.1.3). To see this, we apply the Plancherel-Fourier theorem, i.e., for $\forall\left(c_n\right)\in l^2$, the function $h\left(x\right)=\sum_{n\in\mathbb{Z}}c_ne^{inx}$ is well defined, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|h\left(x\right)\right|^{2}dx=\sum_{n\in\mathbb{Z}}\left|c_n\right|^{2}. \tag{6.1.8}$$

Comparing now with (6.1.6), the desired conclusion follows. □

The case of IFS-Cantor measures

Let $\nu=\nu_4$ be the scale 4-Cantor fractal measure (see [**JP93b**,**JP98a**]) specified by the IFS-identity:

$$\frac{1}{2}\int\left(h\left(\frac{x}{4}\right)+h\left(\frac{x+2}{4}\right)\right)d\nu_4\left(x\right)=\int h\left(x\right)d\nu_4\left(x\right) \tag{6.1.9}$$

for all h. Introduce the transform

$$\widehat{\nu}\left(\xi\right):=\int_{\mathbb{R}}e^{i\xi x}d\nu\left(x\right), \tag{6.1.10}$$

and (6.1.9) is equivalent to

$$\widehat{\nu_4}\left(\xi\right)=\frac{1+e^{i\xi/2}}{2}\widehat{\nu_4}\left(\xi/4\right),\ \forall\xi\in\mathbb{R}. \tag{6.1.11}$$

Note that, as a consequence, the support of this cantor measure ν_4 is then precisely the scale-4 Cantor set from Figure 1.3.2. Recall that $L^2\left(\nu_4\right)$ has an orthonormal basis (ONB) of functions $e_\lambda\left(x\right):=e^{i\lambda x}$. One may take for example

$$\begin{aligned}\Lambda_4&:=\left\{0,1,4,5,16,17,20,21,64,65,\cdots\right\}\\&=\left\{\sum\nolimits_0^{\text{finite}}b_j4^j\ ;\ b_j\in\left\{0,1\right\}\right\}.\end{aligned} \tag{6.1.12}$$

Using now the same ideas from the present section, we get the following:

PROPOSITION 6.1.4. *Let (ν_4,Λ_4) be as above; see (6.1.9)-(6.1.12), and set*

$$\mu_4:=\sum_{\lambda\in\Lambda_4}\delta_\lambda,$$

and

$$f_4\left(x\right):=\sum_{\lambda\in\Lambda_4}e^{i\lambda x},\ x\in\mathbb{R},$$

realized as a tempered p.d. distribution. Let $\mathscr{H}_{f_4}$ be the associated Hilbert space from Theorem 6.1.1. Then there is a natural isometric isomorphism between the two Hilbert spaces $\mathscr{H}_{f_4}$ and $L^2\left(\nu_4\right)$.

PROOF. The details are the same as those of the proof of Proposition 6.1.3. The key step is use of the fact from [**JP98a**] that $\left\{e_\lambda\ ;\ \lambda\in\Lambda_4\right\}$ is an ONB in the Hilbert space $L^2\left(\nu_4\right)$ defined from the Cantor measure ν_4. □

Summary: Correspondences and applications

Continuous p.d. functions on $\mathbb{R}$

LEMMA. *Let f be a continuous function on $\mathbb{R}$. Then the following are equivalent:*

(1) *f is p.d., i.e., $\forall\varphi \in C_c(\mathbb{R})$, we have*

(6.1.13)
$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\overline{\varphi(y)}f(x-y)\,dxdy \geq 0.$$

(2) *$\forall\{x_j\}_{j=1}^n \subset \mathbb{R}$, $\forall\{c_j\}_{j=1}^n \subset \mathbb{C}$, and $\forall n \in \mathbb{N}$, we have*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\overline{c}_k f(x_j - x_k) \geq 0. \tag{6.1.14}$$

p.d. tempered distributions on $\mathbb{R}$

LEMMA. *Let f be a tempered distribution on $\mathbb{R}$. Then f is p.d. if and only if*

(6.1.15)
$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\overline{\varphi(y)}f(x-y)\,dxdy \geq 0$$

hold, for all $\varphi \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space.

Equivalently,

(6.1.16)
$$\langle f(x-y), \varphi\otimes\overline{\varphi}\rangle \geq 0,\ \forall\varphi \in \mathcal{S}.$$

Here $\langle\cdot,\cdot\rangle$ denotes distribution action.

RKHS

Bochner's theorem

$\exists!$ positive finite measure μ on $\mathbb{R}$ such that
$$f(x) = \int_{\mathbb{R}} e^{ix\lambda}d\mu(\lambda).$$

Let $\mathscr{H}_f$ be the RKHS of f.

- Then

(6.1.17)
$$\|\varphi * f\|_{\mathscr{H}_f}^2 = \int_{\mathbb{R}} |\widehat{\varphi}(\lambda)|^2\,d\mu(\lambda)$$

where $\widehat{\varphi}$ = the Fourier transform.

- f admits the factorization

$$f(x_1 - x_2) = \langle f(\cdot - x_1), f(\cdot - x_2)\rangle_{\mathscr{H}_f}$$

$\forall x_1, x_2 \in \mathbb{R}$, with
$$\mathbb{R} \ni x \longrightarrow f(\cdot - x) \in \mathscr{H}_f.$$

Bochner/Schwartz

$\exists$ positive tempered measure μ on $\mathbb{R}$ such that
$$f = \widehat{\mu}$$
where $\widehat{\mu}$ is in the sense of distribution.

Let $\mathscr{H}_f$ denote the corresponding RKHS.

- For all $\varphi \in \mathcal{S}$, we have

(6.1.18)
$$\|\varphi * f\|_{\mathscr{H}_f}^2 = \langle f(x-y), \varphi\otimes\overline{\varphi}\rangle,$$

distribution action.

- $\mathcal{S} \ni \varphi \longmapsto \varphi * f \in \mathscr{H}_f$, where

$$(\varphi * f)(\cdot) = \int \varphi(y) f(\cdot - y)\,dy.$$

Applications

Now applied to Bochner's theorem.

Set $\mathscr{H}_f$ = RKHS of f, and $w_0 = f(\cdot - 0)$. Then
$$U_t w_0 = w_t = f(\cdot - t),\ t \in \mathbb{R}$$
defines a strongly continuous unitary representation of $\mathbb{R}$.

On white noise space:
$$\mathbb{E}\left(e^{i\langle\varphi,\cdot\rangle}\right) = e^{-\frac{1}{2}\int|\widehat{\varphi}|^2 d\mu}$$
where $\mathbb{E}(\cdots)$ = expectation w.r.t the Gaussian path-space measure.

(The proof for the special case when f is assumed p.d. and continuous carries over with some changes to the case when f is a p.d. tempered distribution.)

NOTE. In both cases, we have the following representation for vectors in the RKHS $\mathscr{H}_f$:

$$\langle\varphi * f, \psi * f\rangle_{\mathscr{H}_f} = \left\langle\varphi * \overline{\psi}, f\right\rangle,\ \forall\varphi,\psi \in \mathcal{S}; \tag{6.1.19}$$

where $\varphi * f :=$ the standard convolution w.r.t. Lebesgue measure.

6.2. Positive definite kernels and harmonic analysis in $L^2(\mu)$ when μ is a general singular measure in a finite interval

We saw in Chapter 2, that there are families of singular measures μ on the circle that admit a Fourier duality corresponding to associated sets of lacunary Fourier frequencies [**JP98a**, **DJ11a**, **Str00**]. The latter sets of frequencies in turn index certain closed subspaces of H^2 that can be shown to have boundary representations, referring now instead to $L^2(\mu)$ boundary values. By "lacunary" we refer to Fourier series having asymptotically an infinite sequence of gaps between non-zero coefficients, the successive gaps growing at a geometric rate.

In the present section, we consider the following two questions: which positive matrices (or kernels) does the Hardy space contain that reproduce themselves by boundary functions with respect to a given measure, and with respect to which measures will a positive matrix reproduce itself by boundary functions? The material below is based primarily on ideas in the paper [**HJW16**] by Jorgensen et al.

The boundary representations considered in the present section go beyond that of spectral measures from spectral pairs. The results in turn are based on a new kernel analysis. We make use of frames and of the structure theorem of Wold for isometries in Hilbert spaces. The frame expansions are constructive in that we generate them from the Kaczmarz algorithm, a procedure originally used to solve systems of linear equations.

The boundary value problems we consider here are motivated by two cases considered earlier. One is Fatou's theorem for the Hardy space H^2 on the disk $\mathbb{D}$, yielding an isomorphism between H^2 on the one hand, and L^2 of the boundary circle $\mathbb{T}$ on the other, with the L^2 referring to the Haar (normalized Lebesgue) measure on $\mathbb{T}$. In particular, this theorem shows that every f in H^2 has a non-tangential limit a.e. with respect to Lebesgue measure on $\mathbb{T}$, and that the L^2 norm of the limit function agrees with the H^2 norm of f. Now because of a more general duality theory, it is natural to ask for boundary representations specified by certain lacunary subspaces in H^2.

The classical Hardy space H^2 consists of those holomorphic functions f defined on $\mathbb{D}$ satisfying

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^1 \left| f\left(re^{2\pi ix}\right)\right|^2 dx < \infty. \tag{6.2.1}$$

It is well-known that an equivalent description of H^2 is as the space of holomorphic functions on $\mathbb{D}$ with square-summable coefficients:

$$H^2 = \left\{ \sum_{n=0}^{\infty} c_n z^n \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},$$

where the norm is then equivalently given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 .$$

In addition, for each $f \in H^2$, there exists a (unique) function $f^* \in L^2(\mathbb{T})$, which we shall call the Lebesgue boundary function of f, such that

$$\lim_{r\to 1^-}\int_0^1 \left|f\left(re^{2\pi ix}\right)-f^*\left(e^{2\pi ix}\right)\right|^2 dx = 0. \tag{6.2.2}$$

In fact, $\lim_{r\to 1^-} f\left(re^{2\pi ix}\right) = f^*\left(e^{2\pi ix}\right)$ pointwise for almost every x. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two members of H^2, the inner product of f and g in H^2 can be described in two ways:

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty} a_n\overline{b_n} = \int_0^1 f^*\left(e^{2\pi ix}\right)\overline{g^*\left(e^{2\pi ix}\right)}\,dx. \tag{6.2.3}$$

Because the point-evaluation functionals on the Hardy space are bounded, the Hardy space is a reproducing kernel Hilbert space. Its kernel is the classical Szegő kernel $k(z,w) =: k_z$, defined by

$$k_z(w) := \frac{1}{1-\overline{z}w}.$$

We then have

$$f(z) = \langle f, k_z\rangle_{H^2} = \int_0^1 f^*\left(e^{2\pi ix}\right)\overline{k_z^*\left(e^{2\pi ix}\right)}\,dx$$

for all $f \in H^2$. In particular,

$$k_z(w) := \int_0^1 k_z^*\left(e^{2\pi ix}\right)\overline{k_w^*\left(e^{2\pi ix}\right)}\,dx. \tag{6.2.4}$$

Equation (6.2.4) shows that the Szegő kernel reproduces itself with respect to what is, by some definition, its boundary. The measure on the circle used to define k_z^* in (6.2.2) is Lebesgue measure, as is the measure in (6.2.4). The intent of this section is to show that among the functions in the Hardy space, there are a host of other kernels that reproduce with respect to their boundaries. However, these boundary functions will not be taken with respect to Lebesgue measure, but with respect to a given singular measure, and the integration of these boundary functions will also be done with respect to this measure.

Definition 6.2.1. A positive matrix (in the sense of E. H. Moore), also called a positive definite kernel, on a domain E is a function $K(z,w) : E\times E \to \mathbb{C}$ such that for all finite sequences $\zeta_1, \zeta_2, \ldots, \zeta_n \in E$, the matrix

$$(K(\zeta_j, \zeta_i))_{ij}$$

is positive semidefinite. We will usually write $K_z(w)$ instead of $K(z,w)$, to emphasize that each fixed z yields a function in w. Given a positive matrix $K_z(w)$, we will use the bare notation K to refer to the set $\{K_z : z \in E\}$ of functions from E to $\mathbb{C}$ comprising it, though sometimes we will use K to refer to the positive matrix itself as a function from $E\times E$ to $\mathbb{C}$.

Definition 6.2.2.

(1) Let μ be a nonnegative Borel measure on $[0,1)$. We define $\mathcal{K}(\mu)$ to be the set of positive matrices K on $\mathbb{D}$ such that for each fixed $z \in \mathbb{D}$, K_z possesses an $L^2(\mu)$-boundary K_z^*, and $K_z(w)$ reproduces itself with respect to integration of these $L^2(\mu)$-boundaries, i.e.

$$K_z(w) = \int_0^1 K_z^*(x)\,\overline{K_w^*(x)}\,d\mu(x) \tag{6.2.5}$$

for all $z, w \in \mathbb{D}$.

(2) Let K be a positive matrix on $\mathbb{D}$. We define $\mathcal{M}(K)$ to be the set of nonnegative Borel measures μ on $[0,1)$ such that for each fixed $z \in \mathbb{D}$, K_z possesses an $L^2(\mu)$-boundary K_z^*, and $K_z(w)$ reproduces itself with respect to integration of these $L^2(\mu)$-boundaries.

DEFINITION 6.2.3. A sequence $\{x_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathbb{H}$ is called a frame [**DS52**] if there exist positive constants A and B such that

$$A\left\|\phi\right\|^2 \leq \sum_{n=0}^{\infty}\left|\langle\phi, x_n\rangle\right|^2 \leq B\left\|\phi\right\|^2 \tag{6.2.6}$$

for all $\phi \in \mathbb{H}$. If $\{x_n\}_{n=0}^{\infty}$ satisfies (possibly only) the right-hand inequality in (6.2.6), it is called a Bessel sequence. If $A = B$, the frame is called tight, and if $A = B = 1$, it is called a Parseval frame.

DEFINITION 6.2.4. Let $\{x_n\}_{n=0}^{\infty}$ be a frame in a Hilbert space $\mathbb{H}$. A frame $\{y_n\}_{n=0}^{\infty}$ in $\mathbb{H}$ is a dual frame of $\{x_n\}_{n=0}^{\infty}$ if

$$\sum_{n=0}^{\infty}\langle\phi, x_n\rangle\, y_n = \phi \text{ for all } \phi \in \mathbb{H}. \tag{6.2.7}$$

If (6.2.7) is satisfied, then it is necessarily also true that

$$\sum_{n=0}^{\infty}\langle\phi, y_n\rangle\, x_n = \phi \text{ for all } \phi \in \mathbb{H}.$$

Thus, frame duality is symmetric. A given frame will generally have many dual frames, but every frame possesses a unique canonical dual frame. A Parseval frame is its own canonical dual.

The quaternary Cantor measure μ_4 is the restriction of the $\frac{1}{2}$-dimensional Hausdorff measure to the quaternary Cantor set. Likewise, the ternary Cantor measure μ_3 is the restriction of the $\frac{\ln(2)}{\ln(3)}$-dimensional Hausdorff measure to the ternary Cantor set. In [**JP98a**], Jorgensen and Pedersen showed that the quaternary Cantor measure is spectral. That is, there exists a set $\Gamma \subset \mathbb{Z}$ such that the set of complex exponentials $\left\{e^{2\pi i\lambda x}\right\}_{\lambda\in\Gamma}$ is an orthonormal basis of $L^2(\mu_4)$. From this, Dutkay and Jorgensen [**DJ11a**] constructed a positive matrix G_Γ inside H^2 that reproduces itself both in H^2 and with respect to $L^2(\mu_4)$-boundary integration. Thus $G_\Gamma \in \mathcal{K}(\mu_4)$.

It was also shown that μ_3 is not spectral. Thus, it is not possible to construct a positive matrix for μ_3 in the same way as for μ_4. However, it is sufficient for μ_3 to possess an exponential frame:

PROPOSITION 6.2.5 ([**HJW16**]). *If there exists a sequence $\{n_j\}_{j=0}^{\infty}$ of nonnegative integers such that $\{e^{2\pi i n_j x} : j \geq 0\} \subset L^2(\mu)$ is a frame, then $\mathcal{K}(\mu)$ is nonempty.*

It is still unknown whether μ_3 possesses an exponential frame. Despite this seeming impediment, we will show not only that $\mathcal{K}(\mu_3)$ is nonempty, but that it contains infinitely many members within H^2. In fact, we will show this for all singular probability measures on $[0,1)$.

6.2.1. Kernels in $\mathcal{K}(\mu)$ that are also H^2 kernels. We first show that for μ a singular probability measure $\mathcal{K}(\mu)$ has a rich variety of inhabitants, we consider when projections of the Szegő kernel onto appropriate subspaces of H^2 will be elements in $\mathcal{K}(\mu)$. For the measure μ, there is a canonical subspace of H^2 identified with μ–it is the image of $L^2(\mu)$ under the Normalized Cauchy transform, which also is a de Branges-Rovnyak space. This subspace will give rise to many kernels in $\mathcal{K}(\mu)$.

Definition 6.2.6. A function $b \in H^\infty(\mathbb{D})$ (the space of bounded holomorphic functions on $\mathbb{D}$) is said to be inner if the radial limits $b^*(e^{2\pi i x}) := \lim_{r\to 1^-} b(re^{2\pi i x})$ exist for almost all $x \in [0,1)$ with respect to Lebesgue measure and $|b^*(e^{2\pi i x})| = 1$ for almost all x.

There is a one-to-one correspondence between the nonconstant inner functions b and the finite nonnegative singular measures μ on $[0,1)$ given by the Herglotz representation:

$$\operatorname{Re}\left(\frac{1+b(z)}{1-b(z)}\right) = \int_0^1 \frac{1-|z|^2}{|e^{2\pi i x}-z|^2}\, d\mu(x). \tag{6.2.8}$$

In other words, on the RHS in (6.2.8), we have the Poisson transform of the given positive measure μ. For a singular measure μ and an inner function b related in this way, we will say that μ is the measure "corresponding" to b, or that b is the inner function "corresponding" to μ.

Let S denote the forward shift on H^2, i.e. $Sf(z) = zf(z)$. Beurling's Theorem says that the nontrivial invariant subspaces for S are those subspaces of H^2 of the form bH^2, where b is an inner function. The nontrivial invariant subspaces of the backward shift S^* $\left(S^*f(z) = \frac{f(z)-f(0)}{z}\right)$ are then of the form $H^2 \ominus bH^2$, where b is inner. For each $b \in H^\infty$, there is a de Branges-Rovnyak space $\mathcal{H}(b)$ [**dBR66b**, **ADV09**], which is by definition the range of the operator $A = (I - T_b T_{\overline{b}})^{1/2} : H^2 \to H^2$ along with the Hilbert space structure that makes A a partial isometry from H^2 to $\mathcal{H}(b)$. (Here T_b is the Toeplitz operator with symbol b.) Here, we are only concerned with the situation in which b is inner, and in that case we have $\mathcal{H}(b) = H^2 \ominus bH^2$ with the norm inherited from H^2. For a complete treatment, see Sarason's book [**Sar94**].

Definition 6.2.7. For a finite nonnegative Borel measure μ on $[0,1)$, we define the normalized Cauchy transform V_μ from $L^1(\mu)$ to the set of functions on $\mathbb{C}\setminus\mathbb{T}$ by

$$V_\mu f(z) = \frac{\displaystyle\int_0^1 \frac{f(x)}{1-ze^{-2\pi i x}}\, d\mu(x)}{\displaystyle\int_0^1 \frac{1}{1-ze^{-2\pi i x}}\, d\mu(x)}. \tag{6.2.9}$$

If μ is a singular probability measure on $[0,1)$ with corresponding inner function b, then V_μ is an isometry of $L^2(\mu)$ onto $\mathcal{H}(b)$ [**Cla72**, **Sar90**, **Pol93**, **Her16**].

The unnormalized Cauchy transform shall here be denoted C_μ (in [**Sar94**], K_μ):

$$C_\mu f(z) = \int_0^1 \frac{f(x)}{1-ze^{-2\pi i x}}\, d\mu(x). \tag{6.2.10}$$

Define $e_\lambda(x) := e^{2\pi i\lambda x}$. In [**Her16**], it was proved that if μ is a singular probability measure, then the sequence $\{g_n\}_{n=0}^{\infty}$ defined by

$$g_0 = e_0, \qquad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i \rangle\, g_i. \tag{6.2.11}$$

is a Parseval frame in $L^2(\mu)$ satisfying

$$\sum_{n=0}^{\infty} \langle f, g_n \rangle\, e_n = f \tag{6.2.12}$$

in norm for all $f \in L^2(\mu)$. Equations (6.2.11) and (6.2.12) are referred to as the Kaczmarz algorithm [**Kac37**, **KM01**]. Equation (6.2.12) can be interpreted as a Fourier expansion of $f \in L^2(\mu)$; see also [**Pol93**, **Str06**].

There exists a sequence $\{\alpha_n\}$ of scalars (depending on μ) such that

$$g_n = \sum_{i=0}^{n} \overline{\alpha_{n-i}} e_i \tag{6.2.13}$$

for all $n \in \mathbb{N}_0$. This sequence is obtained by inverting a lower triangular banded matrix whose jk-th entry is $\hat{\mu}(j-k)$. For an explicit expression, see [**Her16**]. The following was also proved:

THEOREM 6.2.8 ([**Her16**]). *If μ is a singular probability measure, then for all $f \in L^2(\mu)$,*

$$V_\mu f(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle\, z^n. \tag{6.2.14}$$

The following is proven in [**Pol93**]; we give an alternate proof here using Theorem 6.2.8.

THEOREM 6.2.9. *If μ is a singular probability measure and $f \in L^2(\mu)$, then f is an $L^2(\mu)$-boundary function of $V_\mu f(z)$. Consequently, for any $F \in \mathcal{H}(b)$, $V_\mu^{-1} F = F^*$.*

PROOF. Since the sum in (6.2.12) is summable in $L^2(\mu)$, it is Abel summable, and hence by (6.2.14) we have that

$$\lim_{r\to 1^-} V_\mu f(re^{2\pi i x}) = \lim_{r\to 1^-} \sum_{n=0}^{\infty} \langle f, g_n\rangle r^n e_n = \sum_{n=0}^{\infty} \langle f, g_n\rangle e_n = f$$

in the $L^2(\mu)$ norm. Hence, f is an $L^2(\mu)$-boundary function of $V_\mu f(z)$.

Now if $F \in \mathcal{H}(b)$, then by bijectivity of V_μ, there exists a $f \in L^2(\mu)$ such that $V_\mu f(z) = F(z)$. Then f is an $L^2(\mu)$-boundary of $V_\mu f(z) = F(z)$, and since an $L^2(\mu)$-boundary is unique, we have $F^* = f$. Hence, $V_\mu^{-1} F = F^*$. □

COROLLARY 6.2.10. *If μ is a singular probability measure with corresponding inner function b, then for any $f(z), j(z) \in \mathcal{H}(b)$, we have*

$$\langle f, j \rangle_{\mathcal{H}(b)} = \langle f^*, j^* \rangle_\mu\,, \tag{6.2.15}$$

where f^* and j^* are the $L^2(\mu)$-boundary functions of f and j, respectively.

PROOF. Since V_μ is an isometry from $L^2(\mu)$ to $\mathcal{H}(b)$, Proposition 6.2.9 implies

$$\langle f, j\rangle_{\mathcal{H}(b)} = \langle V_\mu^{-1} f, V_\mu^{-1} j\rangle_\mu = \langle f^*, j^*\rangle_\mu.$$

□

Thus, for inner functions b with $b(0) = 0$, functions in $\mathcal{H}(b)$ not only have Lebesgue boundaries, but also $L^2(\mu)$-boundaries, and the norm of $\mathcal{H}(b)$ is equal to boundary integration with respect to either boundary/measure pair. As an ordinary subspace of H^2, $\mathcal{H}(b)$ is of course a reproducing kernel Hilbert space. Let $k_z(w) \in H^2$ denote the Szegő kernel of H^2. It is known (see [**Sar94**]) that the kernel of $\mathcal{H}(b)$ is given by

$$k_z^b(w) = \left(1 - \overline{b(z)} b(w)\right) k_z(w).$$

Using (6.2.14), we give the following alternative form:

THEOREM 6.2.11. *Let μ be a singular probability measure with corresponding inner function b and associated sequence $\{g_n\}_{n=0}^\infty \subset L^2(\mu)$ defined by (6.2.11). Then*

$$k_z^b(w) = \sum_{m=0}^\infty \sum_{n=0}^\infty \langle g_n, g_m\rangle_\mu \overline{z}^n w^m. \tag{6.2.16}$$

PROOF. We can combine eq 6.2.8 with a result in [**Her16**] (which uses ideas in [**KM01**]) to obtain that the inner function b satisfies

$$b(z) = 1 - \frac{1}{C_\mu 1(z)} = 1 - \sum_{n=0}^\infty \alpha_n z^n = -\sum_{n=1}^\infty \alpha_n z^n. \tag{6.2.17}$$

Since the sequence $\{g_n\}_{n=0}^\infty$ is Bessel, $\sum_{n=0}^\infty \overline{z}^n g_n$ converges in $L^2(\mu)$ for all $z \in \mathbb{D}$. Observe that for a fixed $z \in \mathbb{D}$,

$$\begin{aligned}
\sum_{n=0}^\infty \overline{z}^n g_n &= \sum_{n=0}^\infty \overline{z}^n \left(\sum_{j=0}^n \overline{\alpha_{n-j}} e_j\right) \\
&= \sum_{j=0}^\infty \sum_{n=0}^\infty \overline{z}^{n+j} \overline{\alpha_n} e_j \\
&= \left(\sum_{n=0}^\infty \overline{\alpha_n z^n}\right) \left(\sum_{j=0}^\infty \overline{z}^j e_j\right) \\
&= \left(1 - \overline{b(z)}\right) k_z^*. \qquad (6.2.18)
\end{aligned}$$

The rearrangement of summation above is justified, because

$$\sum_{j=0}^\infty \sum_{n=0}^\infty \left\| \overline{z}^{n+j} \overline{\alpha_n} e_j \right\| \leq \sum_{j=0}^\infty |z|^j \sqrt{\sum_{n=0}^\infty |z^2|^n} \sqrt{\sum_{n=0}^\infty |\alpha_n|^2} < \infty$$

which shows that the sum converges absolutely. Recall from Theorem 6.2.8 that for $f \in L^2(\mu)$, $V_\mu f(w) = \sum_{n=0}^{\infty} \langle f, g_n \rangle w^n$. Therefore, we have

$$\begin{aligned} V_\mu \left[\sum_{n=0}^{\infty} \overline{z}^n g_n \right](w) &= \sum_{m=0}^{\infty} \left\langle \sum_{n=0}^{\infty} \overline{z}^n g_n, g_m \right\rangle w^m \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle g_n, g_m \rangle \overline{z}^n w^m. \end{aligned}$$

On the other hand, in [**Sar94**] it is computed via the Herglotz representation that

$$C_\mu k_z^*(w) = \left(1 - \overline{b(z)}\right)^{-1} (1 - b(w))^{-1} k_z^b(w).$$

Therefore (by V_μ's original definition, but in accordance with (6.2.9), (6.2.10), and (6.2.17)),

$$\begin{aligned} V_\mu \left[\left(1 - \overline{b(z)}\right) k_z^* \right](w) &:= (1 - b(w))\, C_\mu \left[\left(1 - \overline{b(z)}\right) k_z^* \right](w) \\ &= (1 - b(w)) \left(1 - \overline{b(z)}\right) C_\mu k_z(w) = k_z^b(w). \end{aligned}$$

Equation (6.2.16) now follows from Equation (6.2.18). □

THEOREM 6.2.12. *If μ is a singular probability measure on $[0,1)$ with corresponding inner function b, then $k^b \in \mathcal{K}(\mu)$, and $\mu \in \mathcal{M}(k_z^b)$.*

PROOF. k_z^b is a reproducing kernel of $\mathcal{H}(b)$ with respect to the H^2 norm. By Corollary 6.2.10, it reproduces itself with respect to $L^2(\mu)$-boundary. □

REMARK. It should be noted that Proposition 6.2.9 and Corollary 6.2.10 are previously known. See, for example, Clark's influential paper [**Cla72**], Poltoratskii [**Pol93**], and Sarason's book [**Sar94**]. Theorem 6.2.12 is thus simply a formality. However, it can be proven another way, by combining Theorem 6.2.11 with Theorem 6.2.17, which is to come.

COROLLARY 6.2.13. *If $V \subseteq \mathcal{H}(b)$ is a closed subspace and P_V is the orthogonal projection onto V, then $P_V k_z^b \in \mathcal{K}(\mu)$.*

Since the ternary Cantor measure μ_3 is singular, Theorem 6.2.12 shows that $\mathcal{K}(\mu_3)$ is nonempty, despite μ_3 being nonspectral. Corollary 6.2.13 shows that $\mathcal{K}(\mu_3)$ contains other members as well. We shall see that there are many more kernels in $\mathcal{K}(\mu_3)$, including some that lie outside $\mathcal{H}(b)$.

6.2.2. Wold decompositions. Let b be an inner function, and let μ be its corresponding singular measure. Since the Toeplitz operator $T_b : H^2 \to H^2$ is an isometry, and $\mathcal{H}(b)$ is a wandering subspace for T_b, the Wold Decomposition Theorem [**Wol54**] implies

$$H^2 = \bigoplus_{n=0}^{\infty} T_b^n \mathcal{H}(b).$$

Although the Wold Decomposition Theorem is well-known [**MP88**, **LS97**, **Ste99**], we offer the following alternative proof for the present situation:

THEOREM 6.2.14. *Let μ be a finite singular measure on $[0,1)$, and let b be the inner function corresponding to μ via the Herglotz representation. Then for any $f \in H^2$, there exists a unique sequence of functions $\{\phi_n\}_{n=0}^{\infty} \subset \mathcal{H}(b)$ such that*

$$f = \sum_{n=0}^{\infty} \phi_n \cdot b^n.$$

PROOF. We know that $k_z^b(w) = \frac{1-\overline{b(z)}b(w)}{1-\overline{z}w}$ is the kernel of $\mathcal{H}(b)$. Thus, $K_z(w) = \overline{b^n(z)}b^n(w)k_z^b(w) \in b^n\mathcal{H}(b)$ for each n. (Indeed, it is easy to see it is the kernel of $b^n\mathcal{H}(b)$.) Now, let

$$L = \overline{\operatorname{span}}\left\{b^n \cdot \phi : n \in \mathbb{N}_0, \phi \in \mathcal{H}(b)\right\}.$$

For each $k \in \mathbb{N}$, we have that

$$\sum_{n=0}^{k-1} \overline{b^n\,(z)}b^n\,(w)\,k_z^b\,(w) = \frac{1-\overline{b^k\,(z)}b^k\,(w)}{1-\overline{z}w} \in L.$$

Now, observe that

$$\begin{aligned}
\left\|\frac{1-\overline{b^k\,(z)}b^k\,(w)}{1-\overline{z}w} - \frac{1}{1-\overline{z}w}\right\|_{H^2}^2 &= \int_{[0,1)} \frac{\left|b^k(z){b^*}^k(e^{2\pi i x})\right|^2}{\left|1-\overline{z}e^{2\pi i x}\right|^2}\,dx \\
&= \int_0^1 \frac{|b^k\,(z)|^2}{\left|1-\overline{z}e^{2\pi i x}\right|^2}\,dx \\
&\leq |b(z)|^{2k}\,C,
\end{aligned}$$

where $C = \frac{1}{1-|z|} > 0$. Since b is inner, for each fixed $z \in \mathbb{D}$,

$$\lim_{k\to\infty} \frac{1-\overline{b^k\,(z)}b^k\,(w)}{1-\overline{z}w} = \frac{1}{1-\overline{z}w}$$

in the H^2-norm. Thus, $\frac{1}{1-\overline{z}w} \in L$ for each fixed $z \in \mathbb{D}$. Since $k_z(w) = \frac{1}{1-\overline{z}w}$ is the kernel of H^2, this implies $L = H^2$.

Since T_b is an isometry, and $\mathcal{H}(b)$ is the orthogonal complement of the range of T_b, it follows readily that $b^n\mathcal{H}(b) \perp b^k\mathcal{H}(b)$ for all $n \neq k$ and thus

$$f = \sum_{n=0}^{\infty} \phi_n \cdot b^n,$$

where ϕ_n is the unique member of $\mathcal{H}(b)$ such that $\phi_n \cdot b^n$ is the orthogonal projection of f onto $b^n\mathcal{H}(b)$. $\square$

It is easy to show that for $f \in \mathcal{H}(b)$, $(bf)^* = f^*$, and so every element of $b^n\mathcal{H}(b)$ has an $L^2(\mu)$-boundary. Therefore, if the Wold decomposition of a function $f \in H^2$ is a finite sum, it has an $L^2(\mu)$-boundary. Thus, the Wold Decomposition shows, among other things, that the set of functions in H^2 possessing $L^2(\mu)$-boundary is dense.

PROPOSITION 6.2.15. *Let μ be a singular probability measure with corresponding inner function b. Suppose $V_0, V_1, \ldots, V_N$ are mutually orthogonal closed subspaces of $\mathcal{H}(b)$. Let $k_z^{(n)}(w)$ denote the kernel of V_n. Then the space $W = \bigoplus_{n=0}^N b^n V_n$ is a reproducing kernel Hilbert space with respect to the norm of $L^2(\mu)$-boundary integration, and its kernel is $K_z := \sum_{n=0}^N \overline{b^n(z)} b^n k_z^{(n)}$. Consequently, $K_z \in \mathcal{K}(\mu)$, and $\mu \in \mathcal{M}(K)$.*

PROOF. For any $f \in W$, we may write $f = f_0 + bf_1 + b^2 f_2 + \ldots + b^N f_N$, where $f_n \in V_n$. Then observe that by mutual orthogonality of the spaces

$$\mathcal{H}(b),\ b\mathcal{H}(b),\ b^2\mathcal{H}(b), \ldots, b^N\mathcal{H}(b) \subset H^2,$$

we have

$$\|f\|_{H^2}^2 = \sum_{n=0}^N \|b^n f_n\|_{H^2}^2 = \sum_{n=0}^N \|f_n\|_{H^2}^2 = \sum_{n=0}^N \|f_n\|_{\mathcal{H}(b)}^2 = \sum_{n=0}^N \|f_n^*\|_\mu^2.$$

By mutual orthogonality of the spaces $V_0, V_1, \ldots, V_N$ in $\mathcal{H}(b)$, the f_n are orthogonal in $\mathcal{H}(b)$, and hence by Corollary 6.2.10 the f_n^* are orthogonal in $L^2(\mu)$. Hence,

$$\begin{aligned}\sum_{n=0}^N \|f_n^*\|_\mu^2 &= \sum_{n=0}^N \left\|(b^n f)^*\right\|_\mu^2 \\ &= \left\|\sum_{n=0}^N (b^n f_n)^*\right\|_\mu^2 = \left\|\left(\sum_{n=0}^N b^n f_n\right)^*\right\|_\mu^2 = \|f^*\|_\mu^2 .\end{aligned}$$

This shows that the H^2 norm and the $L^2(\mu)$-boundary norm are equal on W. Hence, the inner products are equal as well by the polarization identity. The proof is completed by noting that by orthogonality,

$$\begin{aligned}\left\langle f, \sum_{n=0}^N \overline{b^n(z)} b^n k_z^{(n)} \right\rangle_{H^2} &= \left\langle \sum_{m=0}^N b^m f_m, \sum_{n=0}^N \overline{b^n(z)} b^n k_z^{(n)} \right\rangle_{H^2} \\ &= \sum_{n=0}^N b^n(z) \left\langle f_n, k_z^{(n)} \right\rangle_{H^2} = f(z)\end{aligned}$$

□

6.2.3. Kernels in $\mathcal{K}(\mu)$ that are not H^2 kernels.

We have seen that for a singular probability measure μ, there are many kernels in $\mathcal{K}(\mu)$, obtained by projecting the Szegő kernel onto appropriate subspaces of H^2. We now turn to showing that there are many kernels in $\mathcal{K}(\mu)$ which are not obtained in this way, and in fact the kernels will generate subspaces of H^2 for which the norm defined by the kernel is not identical to the norm in H^2. The following definition will be convenient in our subsequent discussions:

DEFINITION 6.2.16. Given a Hilbert space $\mathbb{H}$ and two sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ in $\mathbb{H}$, if we have

$$\sum_{n=0}^\infty \langle f, x_n \rangle y_n = f \tag{6.2.19}$$

with convergence in norm for all $f \in \mathbb{H}$, then $\{x_n\}_{n=0}^{\infty}$ is said to be dextrodual to $\{y_n\}_{n=0}^{\infty}$ (or, "a dextrodual of $\{y_n\}_{n=0}^{\infty}$"), and $\{y_n\}_{n=0}^{\infty}$ is said to be levodual to $\{x_n\}_{n=0}^{\infty}$.

In the parlance of frame theory, if S_y is the synthesis operator of $\{y_n\}$ and A_x is the analysis operator of $\{x_n\}$, then $\{x_n\}$ is dextrodual to $\{y_n\}$ if $S_y A_x = I$. However, a sequence does not need to be a frame to have a dextrodual. For example, $\{e_n\}_{n=0}^{\infty}$ is not even Bessel in $L^2(\mu)$ for μ a singular measure, but (6.2.12) shows that the Parseval frame $\{g_n\}_{n=0}^{\infty}$ is dextrodual to $\{e_n\}_{n=0}^{\infty}$.

THEOREM 6.2.17. *Let μ be a Borel measure on $[0,1)$. Let $\{h_n\} \subset L^2(\mu)$ be a Bessel sequence that is dextrodual to $\{e_n\}$. Then for each fixed $z \in \mathbb{D}$,*

$$K_z(w) := \sum_m \sum_n \langle h_n, h_m \rangle_\mu \, \overline{z}^n w^m$$

is a well-defined function on $\mathbb{D}$. $K_z(w) \in H^2$ and possesses an $L^2(\mu)$-boundary function K_z^*. Moreover,

$$K_z(w) = \langle K_z^*, K_w^* \rangle_\mu \,,$$

and thus $K \in \mathcal{K}(\mu)$.

PROOF. Fix $z \in \mathbb{D}$. Let $N \in \mathbb{N}_0$, and suppose $n > m \geq N$. Then since $\{h_n\}$ is Bessel, we have

$$\left\| \sum_{k=0}^{n} \overline{z}^k h_k - \sum_{k=0}^{m} \overline{z}^k h_k \right\|_\mu = \left\| \sum_{k=m+1}^{n} \overline{z}^k h_k \right\|_\mu \leq B \sqrt{\sum_{k=m+1}^{n} |z|^{2k}} \leq B \sqrt{\sum_{k=N}^{\infty} |z|^{2k}}.$$

As $N \to \infty$, the right side goes to 0, which shows that the sequence $\left\{\sum_{k=0}^{n} \overline{z}^k h_k\right\}_n$ is Cauchy and hence convergent in $L^2(\mu)$. By continuity of the inner product in $L^2(\mu)$, we then have

$$\begin{aligned} K_z(w) &:= \sum_m \sum_n \langle h_n, h_m \rangle \, \overline{z}^n w^m \\ &= \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle w^m. \end{aligned}$$

Observe that since $\{h_n\}$ is Bessel,

$$\sum_{m=0}^{\infty} \left| \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle \right|^2 \leq B' \left\| \sum_n \overline{z}^n h_n \right\|_\mu^2 < \infty,$$

which shows that $K_z(w) \in H^2$. Define $K_z^* \in L^2(\mu)$ by $K_z^* = \sum_n \overline{z}^n h_n$. Because $\{h_n\}$ is dextrodual to $\{e_n\}$, we have

$$K_z^* := \sum_n \overline{z}^n h_n = \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m.$$

A summable series in a normed linear space is Abel summable. Hence, for all $0 < r \leq 1$, we have that

$$\sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m$$

converges in $L^2(\mu)$, and

$$\begin{aligned}
& \lim_{r\to 1^-} \left\| \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m - \sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m \right\|_\mu \\
= \quad & \lim_{r\to 1^-} \left\| \sum_n \overline{z}^n h_n - \sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m \right\|_\mu = 0.
\end{aligned}$$

Since

$$K_z(re^{2\pi ix}) = \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle r^m e^{2\pi imx},$$

the above shows that for each $0 < r < 1$, $K_z(re^{2\pi ix}) \in L^2(\mu)$ with respect to the variable x, and K_z^* is an $L^2(\mu)$-boundary function of $K_z(w)$. We compute that

$$\langle K_z^*, K_w^* \rangle = \left\langle \sum_n \overline{z}^n h_n, \sum_m \overline{w}^m h_m \right\rangle = \sum_m \sum_n \langle h_n, h_m \rangle \overline{z}^n w^m = K_z(w).$$

□

In the sequel, unless otherwise stated, we shall denote the usual Lebesgue measure on the interval by λ.

6.2.4. The set $\mathcal{M}(K)$. Starting with a singular probability measure μ, we have seen large classes of positive matrices $K_z(w)$ that reproduce with respect to $L^2(\mu)$-boundary integration. Reproducing in this way potentially has desirable application, but it may happen in practice that we are more tied to a particular positive matrix than we are a measure. Thus, it is natural for us to ask a question in the opposite direction: Given a positive matrix $K \subset H^2(\mathbb{D})$, for which Borel measures μ does $K_z(w)$ reproduce with respect to $L^2(\mu)$-boundary integration? In other words, which measures are in $\mathcal{M}(K)$? For a given K, it is *a priori* possible that $\mathcal{M}(K) = \varnothing$, though we know of no examples yet. As we have seen, though, this is thankfully not always the case, and the following results give us some more insight.

THEOREM 6.2.18. *Let V be a closed subspace of H^2, and let K be the reproducing kernel of V. If*

$$\overline{\cup_{n=0}^\infty S^{*n} V} \neq H^2,$$

then there exists a singular measure $\mu \in \mathcal{M}(K)$. Indeed, to each inner function b with $b(0) = 0$ there corresponds a distinct such measure.

PROOF. $\overline{\cup_{n=0}^\infty S^{*n} V}$ is the smallest closed S^*-invariant subspace containing V. Every proper closed S^*-invariant subspace of H^2 is a de Branges-Rovnyak space $\mathcal{H}(u)$ for some inner function u. Let b be an inner function such that $b(0) = 0$, and let μ be the singular probability measure corresponding to ub. Then by Corollary 6.2.10, the H^2 norm on $\mathcal{H}(ub)$ is equal to the norm of $L^2(\mu)$-boundary integration. Thus since $V \subset \mathcal{H}(u) \subset \mathcal{H}(ub)$ and K reproduces with respect to the H^2 norm in $\mathcal{H}(ub)$, it reproduces with respect to the $L^2(\mu)$-boundary norm. Hence, $\mu \in \mathcal{M}(K)$. □

LEMMA 6.2.19. *Let ν and μ be finite Borel measures on $[0, 1)$, and suppose $\nu = \nu_a + \nu_s$ is the Lebesgue decomposition of ν with respect to μ. If $\mu, \nu \in \mathcal{M}(K)$*

and $\frac{d\nu_a}{d\mu}$ is bounded, then the affine hull of ν and μ intersected with the set of nonnegative Borel measures is contained in $\mathcal{M}(K)$.

For a nonconstant inner function b, let μ_n denote the unique singular measure on $[0,1)$ corresponding to b^n via the Poisson integral. Note that $\mathcal{H}(b) \subset \mathcal{H}(b^n)$.

PROPOSITION 6.2.20. *If K is a positive matrix in H^2 such that $\mu = \mu_1 \in \mathcal{M}(K)$ and $K \subseteq \mathcal{H}(b)$, then $\mu_n \in \mathcal{M}(K)$ for all $n \geq 1$.*

PROOF. Let $n \in \mathbb{N}$. We have $\{K_z : z \in \mathbb{D}\} \subseteq \mathcal{H}(b) \subseteq \mathcal{H}(b^n)$, and since functions in $\mathcal{H}(b^n)$ have $L^2(\mu_n)$-boundaries, each K_z has an $L^2(\mu_n)$-boundary K^*_{z,μ_n}. Recall that the norms on $\mathcal{H}(b)$ and $\mathcal{H}(b^n)$ are both equal to the H^2 norm and hence equal to each other. We therefore have

$$\begin{aligned} K_z(w) &= \langle K_z, K_w\rangle_{\mathcal{H}(b)} \\ &= \langle K_z, K_w\rangle_{\mathcal{H}(b^n)} = \int_0^1 K^*_{z,\mu_n} \overline{K^*_{w,\mu_n}}\, d\mu_n. \end{aligned}$$

Thus $\mu_n \in \mathcal{M}(K)$. $\square$

Given that $\mathcal{H}(b)$ is so (relatively) well understood, it is a perhaps more interesting question to ask what happens when a positive matrix lies outside of $\mathcal{H}(b)$. Given a positive matrix $K_z(w)$ and an inner function b, for which n, if any, is $\mu_n \in \mathcal{M}(K)$? We propose to begin a study of this question here. We begin by revealing the relationship between μ's family of Clark measures and the measures μ_n.

LEMMA 6.2.21. *Let $b : \mathbb{D} \to \mathbb{D}$, and let $n \in \mathbb{N}$. Then for all $z \in \mathbb{D}$,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \frac{1 + e^{-2\pi i j/n} b(z)}{1 - e^{-2\pi i j/n} b(z)} = \frac{1 + b^n(z)}{1 - b^n(z)}.$$

PROOF. For $z \in \mathbb{D}$ such that $b(z) = 0$, the equality is obvious. So suppose $z \in \mathbb{D}$ is such that $b(z) \neq 0$. We have

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{1 + e^{-2\pi i j/n} b(z)}{1 - e^{-2\pi i j/n} b(z)} &= \sum_{j=0}^{n-1} \frac{e^{2\pi i j/n} + b(z)}{e^{2\pi i j/n} - b(z)} \\ &= \sum_{j=0}^{n-1} \frac{e^{2\pi i j/n}}{e^{2\pi i j/n} - b(z)} + \sum_{j=0}^{n-1} \frac{b(z)}{e^{2\pi i j/n} - b(z)} \\ &= \sum_{j=0}^{n-1} \frac{1}{1 - e^{-2\pi i j/n} b(z)} - \sum_{j=0}^{n-1} \frac{1}{1 - \frac{e^{2\pi i j/n}}{b(z)}}. \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{1}{1 - e^{-2\pi i j/n} b(z)} &= \sum_{j=0}^{n-1}\sum_{l=0}^{\infty} (e^{-2\pi i j/n} b(z))^l \\ &= \sum_{l=0}^{\infty} b^l(z) \begin{cases} 0 & \text{if } l \neq 0 \bmod n \\ n & \text{if } l = 0 \bmod n \end{cases} \\ &= \frac{n}{1 - b^n(z)}. \end{aligned}$$

A similar computation shows that

$$\sum_{j=0}^{n-1} \frac{1}{1-\frac{e^{2\pi ij/n}}{b(z)}} = \frac{n}{1-\frac{1}{b^n(z)}}.$$

Hence,

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1} \frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)} &= \frac{1}{1-b^n(z)} - \frac{1}{1-\frac{1}{b^n(z)}} \\ &= \frac{1+b^n(z)}{1-b^n(z)}.\end{aligned}$$

□

LEMMA 6.2.22. *Given an inner function b, if μ_n is the singular measure associated to b^n, then we have*

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \sigma_{e^{2\pi ij/n}},$$

where σ_α is the singular measure corresponding to the inner function $\overline{\alpha}b$.

PROOF. By Lemma 6.2.21, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{1+b^n(z)}{1-b^n(z)}\right) &= \mathrm{Re}\left(\frac{1}{n}\sum_{j=0}^{n-1} \frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)}\right) \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{Re}\left(\frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)}\right) \\ &= \frac{1}{n}\sum_{j=0}^{n} \int_0^1 \frac{1+|z|^2}{|z-\xi|^2}\, d\sigma_{e^{2\pi ij/n}}(\xi) \\ &= \int_0^1 \frac{1+|z|^2}{|z-\xi|^2}\, d\left[\frac{1}{n}\sum_{j=0}^{n-1} \sigma_{e^{2\pi ij/n}}\right](\xi)\end{aligned}$$

This shows that $\frac{1}{n}\sum_{j=0}^{n-1} \sigma_{e^{2\pi ij/n}}$ is the singular measure corresponding to the inner function b^n via the Herglotz representation theorem, which completes the proof. □

THEOREM 6.2.23. *Let $K_z(w)$ be a positive matrix and let b be an inner function. Let m, n, and q be positive integers such that $n = qm$. Let*

$$\rho = \frac{q}{(q-1)n}\sum_{\substack{j=0 \\ q\nmid j}}^{n-1} \sigma_{e^{2\pi ij/m}}.$$

If two of the measures μ_m, μ_n, and ρ are in $\mathcal{M}(K)$, then so is the third.

PROOF. By Lemma 6.2.22, we have

$$\begin{aligned}\mu_n &= \frac{1}{n}\left(\sum_{\substack{j=0\\ q|j}}^{n-1}\sigma_{e^{2\pi ij/(qm)}} + \sum_{\substack{j=0\\ q\nmid j}}^{n-1}\sigma_{e^{2\pi ij/n}}\right)\\ &= \frac{1}{n}\left(\sum_{j=0}^{m-1}\sigma_{e^{2\pi ij/m}} + \sum_{\substack{j=0\\ q\nmid j}}^{n-1}\sigma_{e^{2\pi ij/n}}\right)\\ &= \frac{1}{q}\mu_m + \frac{q-1}{q}\rho.\end{aligned}$$

So, each of the measures μ_n, μ_m, and ρ is in the affine hull of the other two.

Recall that the Clark measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ are mutually singular [**Pol93**]. It follows that μ_m and ρ, since they are sums of Clark measures that do not share a common Clark measure, are mutually singular. Hence, if $\rho = \rho_a + \rho_s$ is the Lebesgue decomposition of ρ with respect to μ_m, we must have $\rho_a = 0$, and hence $\frac{d\rho_a}{d\mu_m} = 0$.

So the Radon-Nikodym derivative of the part of ρ absolutely continuous to μ_m is bounded. Furthermore, it is clear that μ_m and ρ are absolutely continuous with respect to μ_n with respective Radon-Nikodym derivatives $\frac{d\mu_m}{d\mu_n} \equiv \frac{1}{q}$ and $\frac{d\rho}{d\mu_n} \equiv \frac{q-1}{q}$. Therefore, by Lemma 6.2.19, if two of the three measures are in $\mathcal{M}(K)$, so is the third. $\square$

6.2.5. A matrix characterization. Spectral measures give rise to a natural harmonic analysis on the unit disc via a boundary representation of a positive matrix arising from a spectrum of the measure. Now for a positive matrix in the Hardy space of the unit disc we consider which measures, if any, yield a boundary representation of the positive matrix. We introduce a potential characterization of those measures via a matrix identity and show that the characterization holds in several important special cases. The reader is referred to the original paper [**HJW18b**].

Kernels from a coefficient matrix. Let $C = (c_{mn})$ be a bi-infinite matrix, where $m, n \geq 0$. We consider the formal power series

$$K_C(w,z) = \sum_n \sum_m c_{mn}\overline{w}^m z^n. \tag{6.2.20}$$

We shall assume that $c_{nm} = \overline{c}_{mn}$; we shall make additional assumptions on C as needed. For example, if we assume that $\{c_{mn}\}$ is a bounded sequence, then the formal power series K_C converges absolutely on $\mathbb{D}\times\mathbb{D}$, and thus K_C is holomorphic on $\mathbb{D}$ in z and antiholomorphic on $\mathbb{D}$ in w. For the remainder of the present section, we shall assume the coefficient sequence is bounded.

Moreover, we wish K_C to be a positive matrix on $\mathbb{D}\times\mathbb{D}$, so we assume that C has this property. When the matrix C defines a bounded linear operator on $\ell^2(\mathbb{N}_0)$, then K_C is a positive matrix if and only if C is a positive operator. Indeed, for $z \in \mathbb{D}$, we denote by $\vec{z}$ the element of $\ell^2(\mathbb{N}_0)$ where $(\vec{z})_n = (z^n)_n$. Then, for

$z_1, \ldots, z_N \in \mathbb{D}$ and $\xi_1, \ldots, \xi_N \in \mathbb{C}$,

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \xi_j \overline{\xi}_k K_C(z_k, z_j) = \sum_{j=1}^{N}\sum_{k=1}^{N} \xi_j \overline{\xi}_k \langle C\vec{z}_j, \vec{z}_k \rangle_{\ell^2} = \left\langle C \sum_{j=1}^{N} \xi_j \vec{z}_j, \sum_{k=1}^{N} \xi_k \vec{z}_k \right\rangle_{\ell^2}$$

which is nonnegative if and only if C is a positive selfadjoint operator on $\ell^2(\mathbb{N}_0)$. Assuming that C is a bounded linear operator on $\ell^2(\mathbb{N}_0)$ has the additional virtue that for every $w \in \mathbb{D}$, $K_C(w, \cdot) \in H^2(\mathbb{D})$, since the coefficient sequence $(\overline{C\vec{w}})_m$ is square-summable. We have established the following:

LEMMA 6.2.24. *If $C = (c_{mn})_{mn}$ is a bounded, positive, selfadjoint operator on $\ell^2(\mathbb{N}_0)$, then the kernel K_C as given in Equation (*6.2.20*) is a positive matrix such that for each $w \in \mathbb{D}$, $K_C(w, \cdot) \in H^2(\mathbb{D})$. Moreover, for $w, z \in \mathbb{D}$,*

$$K_C(w, z) = \langle C\vec{z}, \vec{w} \rangle_{\ell^2}. \tag{6.2.21}$$

For a given C which defines a positive matrix as in (6.2.20), we wish to determine which Borel measures on $\mathbb{T}$, if any, are in $\mathcal{M}(K_C)$. We shall approach the question via the following meta-theorem:

THEOREM 6.2.25. *A measure μ is in $\mathcal{M}(K_C)$ if and only if the matrix equation $C = CMC$ is satisfied, where the matrix $M = (\hat{\mu}(n - m))_{mn}$.*

We describe this as a meta-theorem for several reasons. First, even if C is a bounded operator on $\ell^2(\mathbb{N}_0)$, the expression CMC may not be well-defined. Indeed, *a priori* this product is only defined when C and M are bounded operators on $\ell^2(\mathbb{N}_0)$; we may have only one or neither of these matrices with that property. Second, the matrix equality does not *a priori* assure that the kernel functions $K_C(w, \cdot)$ have μ-boundaries. Our goal is to establish the meta-theorem for two special cases: i) for diagonal matrices C, and ii) for C and μ for which M which are bounded operators on $\ell^2(\mathbb{N}_0)$. We have a description of which μ has the property that M is bounded [**Cas00**, **DHSW11**], see also [**Lai12**]:

LEMMA 6.2.26. *The matrix $M = (\hat{\mu}(n-m))_{mn}$ is a bounded operator on $\ell^2(\mathbb{N}_0)$ if and only if $\mu << \lambda$ and the Radon-Nikodym derivative $\dfrac{d\mu}{d\lambda} \in L^\infty(\lambda)$.*

Lebesgue measure: kernels in $H^2(\mathbb{D})$ with equal norms. We assume that the coefficient matrix C defines a bounded, positive, selfadjoint operator on $\ell^2(\mathbb{N}_0)$ and consider initially the special case of Lebesgue measure.

THEOREM 6.2.27. *Suppose $C = (c_{mn})$ defines a bounded, positive, selfadjoint operator on $\ell^2(\mathbb{N}_0)$. The following are equivalent:*

(1) *$\lambda \in \mathcal{M}(K_C)$;*
(2) *the coefficient matrix C is a projection;*
(3) *the norm induced by K_C is equal to the Hardy space norm in the following sense: for all $\xi_1, \ldots, \xi_N \in \mathbb{C}$ and $w_1, \ldots, w_N \in \mathbb{D}$,*

$$\left\| \sum_{j=1}^{N} \xi_j K_C(w_j, \cdot) \right\|_{K_C} = \left\| \sum_{j=1}^{N} \xi_j K_C(w_j, \cdot) \right\|_{H^2};$$

(4) *there exists a subspace M of the Hardy space such that the Parseval frame $g_n = P_M z^n$ is such that $c_{mn} = \langle g_n, g_m \rangle$;*

(5) *there exists a subspace M of the Hardy space such that the projection of the Szegü kernel onto M is K_C.*

The equivalence of 1 and 2 would follow immediately from our meta-theorem. We will establish the meta-theorem for absolutely continuous measures with bounded Radon-Nikodym derivative in the next section–we present here a proof that uses only the equality of norms.

PROOF. (1)$\Longleftrightarrow$(3) If $\lambda \in \mathcal{M}(K_C)$, then

$$\begin{aligned}
\left\| \sum_{j=1}^{N} \xi_j K_C(w_j, \cdot) \right\|_{H^2}^2 &= \int_0^1 \left(\sum_{j=1}^{N} \xi_j K_C^\star(w_j, \cdot) \right) \left(\overline{\sum_{k=1}^{N} \xi_k K_C^\star(w_k, \cdot)} \right) d\lambda \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_j \overline{\xi}_k \int_0^1 K_C^\star(w_j, \cdot) \overline{K_C^\star(w_k, \cdot)} d\lambda \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_j \overline{\xi}_k K_C(w_j, w_k) \\
&= \left\| \sum_{j=1}^{N} \xi_j K_C(w_j, \cdot) \right\|_{K_C}^2 .
\end{aligned}$$

Conversely, if the norms are equal, we have by the polarization identity

$$\begin{aligned}
K_C(w, z) &= \langle K_C(w, \cdot), K_C(z, \cdot) \rangle_{K_C} \\
&= \langle K_C(w, \cdot), K_C(z, \cdot) \rangle_{H^2} \\
&= \int_0^1 K_C^\star(w, \cdot) \overline{K_C^\star(z, \cdot)} d\lambda.
\end{aligned}$$

(2)$\Longleftrightarrow$(3) Consider the following calculations:

$$\begin{aligned}
\left\| \sum_{j=1}^{N} \xi_j K_C(w_j, \cdot) \right\|_{K_C}^2 &= \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_j \overline{\xi}_k K_C(w_j, w_k) \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_j \overline{\xi}_k \langle C \vec{w}_k, \vec{w}_j \rangle_{\ell^2} \\
&= \left\langle C \left(\sum_{k=1}^{N} \overline{\xi}_k \vec{w}_k \right), \sum_{j=1}^{N} \overline{\xi}_j \vec{w}_j \right\rangle_{\ell^2} \qquad (6.2.22)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\sum_{j=1}^{N}\xi_j K_C(w_j,\cdot)\right\|_{H^2}^2 &= \left\|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}c_{mn}\sum_{j=1}^{N}\xi_j\overline{w}_j^m\right)z^n\right\|_{H^2}^2\\
&= \left\|\sum_{n=0}^{\infty}\overline{\left(\sum_{m=0}^{\infty}c_{nm}\sum_{j=1}^{N}\overline{\xi}_j w_j^m\right)}z^n\right\|_{H^2}^2\\
&= \left\langle C\left(\sum_{j=1}^{N}\overline{\xi}_j\vec{w}_j\right), C\left(\sum_{k=1}^{N}\overline{\xi}_k\vec{w}_k\right)\right\rangle_{\ell^2}. \qquad (6.2.23)
\end{aligned}$$

It follows that if C is projection, then the inner-products in Equations (6.2.22) and (6.2.23) are equal. Conversely, if the norms are equal, then by the polarization identity, we have that $C^2 = C$; since C is assumed selfadjoint, C is a projection.

(3)$\Longleftrightarrow$(5) If the norms are equal, then the RKHS generated by K_C is a closed subspace M of $H^2(\mathbb{D})$ (with equal norm), and hence the projection P_M of the Szegü kernel is the reproducing kernel for M, as is K_C. Conversely, if K_C is the projection of the Szegü kernel onto M, then the norms are equal.

(2)$\Longleftrightarrow$(4) If C is a projection, then we can define $\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ by

$$\Phi f(z) = \sum_m (C\vec{f})_m z^m, \quad \text{where} \quad f(z) = \sum_n f_n z^n \quad \text{and} \quad \vec{f} = (f_n)_n.$$

It is readily verified that Φ is a projection on $H^2(\mathbb{D})$. We have

$$c_{mn} = \langle \Phi z^n, z^m\rangle = \langle \Phi z^n, \Phi z^m\rangle.$$

Conversely, if $g_n = P_M z^n$, then $\{g_n\}$ is a Parseval frame, thus its Grammian matrix is a projection. $\square$

Diagonal coefficient matrices. We consider the special case of when C is a diagonal matrix. Let $\Gamma \subset \mathbb{N}_0$ and consider $K_\Gamma(w,z) = \sum_{\gamma\in\Gamma}(z\overline{w})^\gamma$. We will see that either a) there are many absolutely continuous measures in $\mathcal{M}(K_\Gamma)$, or b) only Lebesgue measure is in $\mathcal{M}(K_\Gamma)$. The determining factor of which possibility occurs is the difference set of Γ.

The kernels K_4 and K_3

Two specific kernels that fall into this category that we wish to understand are the kernels K_3 and K_4. Recall that a spectrum for μ_4 is

$$\Gamma_4 := \left\{\sum_{j=0}^{N} l_j 4^j \mid l_j \in \{0,1\}\right\} = \{0,1,4,5,16,17,20,21,\dots\}.$$

Then

$$K_4(w,z) := \sum_{\gamma\in\Gamma_4}(\overline{w}z)^\gamma = \prod_{j=0}^{\infty}\left(1 + (\overline{w}z)^{4^j}\right).$$

An introduction to K_4 appears in [**DJ11a**], where it is shown that $\mu_4 \in \mathcal{M}(K_4)$. We show in Corollary 6.2.35 below that there are many (absolutely continuous) measures in $\mathcal{M}(K_4)$.

We also consider the kernel K_3, defined analogously to K_4:

$$K_3(w,z) := \prod_{j=0}^{\infty}\left(1+(\overline{w}z)^{3^j}\right) = \sum_{n\in\Gamma_3}(\overline{w}z)^n,$$

where

$$\Gamma_3 = \left\{\sum_{j=0}^{N} l_j 3^j \mid l_j \in \{0,1\}\right\} = \{0,1,3,4,9,10,12,13,\dots\}.$$

We shall show in Corollary 6.2.33 below that $\mathcal{M}(K_3)$ contains only Lebesgue measure.

Note that K_Γ corresponds to the diagonal coefficient matrix C with $c_{mm}=1$ if and only if $m\in\Gamma$, and $c_{mm}=0$ otherwise. Therefore C is a projection, and hence as a consequence of Theorem 6.2.27, we have:

COROLLARY 6.2.28. *For any* $\Gamma\subset\mathbb{N}_0$, $\lambda\in\mathcal{M}(K_\Gamma)$.

We shall also consider diagonal matrices C which are not projections–in fact we can consider diagonal matrices C which are not bounded operators on $\ell^2(\mathbb{N}_0)$. For example, the Bergmann kernel is given by

$$K_B(w,z) = \sum_{n=0}^{\infty}(n+1)(\overline{w}z)^n.$$

We shall show in Corollary 6.2.30 below that there are no representing measures for C which have distinct nonzero diagonal entries.

The meta-theorem for diagonal coefficient matrices

For two matrices $A=(a_{mn})$ and $B=(b_{mn})$, we say that AB is *defined in the matrix sense* if for every $m,n\in\mathbb{N}_0$, the sum $\sum_{k=0}^{\infty} = a_{mk}b_{kn}$ converges. Note that this holds if A,B are bounded operators on $\ell^2(\mathbb{N}_0)$. We say ABC is defined in the matrix sense if AB, BC, $(AB)C$, and $A(BC)$ are defined in the matrix sense and $(AB)C = A(BC)$.

THEOREM 6.2.29. *Suppose C is diagonal matrix such that $c_{nn}\geq 0$ and for every $0<r<1$, $\sum c_{nn}r^n < +\infty$. Let μ be a Borel probability measure on $[0,1)$ with $M=(\widehat{\mu}(n-m))_{mn}$. Then the following hold:*

(1) $\sum_{n=0}^{\infty} c_{nn}\overline{w}^n e^{2\pi inx}$ *converges in* $L^2(\mu)$.
(2) CMC *is defined in the matrix sense.*
(3) $K_C(w,z)$ *reproduces itself with respect to $L^2(\mu)$ boundaries if and only if the equation $C=CMC$ holds.*

PROOF. For the first part, we have for any $|w|<1$,

$$\sum_{n=0}^{\infty}\left\|c_{nn}\overline{w}^n e^{2\pi inx}\right\|_\mu = \sum_{n=0}^{\infty}|c_{nn}\overline{w}^n| < \infty.$$

Thus, $\sum_{n=0}^{\infty} c_{nn}\overline{w}^n e^{2\pi inx}$ is absolutely summable in $L^2(\mu)$ and thus converges in $L^2(\mu)$.

For the second part, observe that

$$(CM)_{mn} = \sum_{k=0}^{\infty} c_{mk}\hat{\mu}(n-k) = c_{mm}\hat{\mu}(n-m)$$
$$(MC)_{mn} = \sum_{k=0}^{\infty} \hat{\mu}(k-m)c_{kn} = \hat{\mu}(n-m)c_{nn}.$$

Likewise,

$$(C(MC))_{mn} = \sum_{k=0}^{\infty} c_{mk}\hat{\mu}(n-k)c_{nn} = c_{mm}c_{nn}\hat{\mu}(n-m) = c_{mm}M_{mn}c_{nn}$$
$$((CM)C)_{mn} = \sum_{k=0}^{\infty} c_{mm}\hat{\mu}(k-m)c_{kn} = c_{mm}\hat{\mu}(n-m)c_{nn} = c_{mm}M_{mn}c_{nn}.$$

This shows that CM, MC, $C(MC)$, and $(CM)C$ are defined in the matrix sense, and that $C(MC) = (CM)C$.

Now, suppose $K_C(w,z)$ reproduces itself with respect to $L^2(\mu)$ boundary. The first part, together with Abel summability, shows that

$$K_C^\star(w,x) = \sum_{m=0}^{\infty} c_{mm}\overline{w}^n e^{2\pi i m x}.$$

By continuity of the inner product in $L^2(\mu)$, we have

$$\begin{aligned}
\int_0^1 K_C^\star(w,x)\overline{K_C^\star(z,x)}\,d\mu(x) &= \sum_{n=0}^{\infty} \overline{c_{nn}} \left(\int_0^1 K_C^\star(w,x)e^{-2\pi i n x}\,d\mu(x)\right) z^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nn}c_{mm}\left(\int_0^1 e^{2\pi i m x}e^{-2\pi i n x}\,d\mu(x)\right)\overline{w}^m z^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nn}c_{mm}\hat{\mu}(n-m)\overline{w}^m z^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mm}M_{mn}c_{nn}\overline{w}^m z^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (CMC)_{mn}\overline{w}^m z^n
\end{aligned}$$

Therefore, Equation (6.2.5) holds if and only if

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn}\overline{w}^m z^n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (CMC)_{mn}\overline{w}^m z^n$$

holds, which by uniqueness of Taylor coefficients, holds if and only if $C = CMC$. $\square$

COROLLARY 6.2.30. *Suppose C is a diagonal matrix which satisfies the hypotheses of Theorem* 6.2.29 *and which has two distinct nonzero diagonal entries. Then $\mathcal{M}(K_C) = \emptyset$.*

PROOF. Suppose $\mu \in \mathcal{M}(K_C)$. Then we must have $c_{mm} = (CMC)_{mm} = c_{mm}M_{mm}c_{mm} = \|\mu\|c_{mm}^2$ for all $m \in \mathbb{N}_0$. Thus, for any nonzero diagonal entry $c_{mm} = \|\mu\|$. $\square$

DEFINITION 6.2.31. For a set $A \subset \mathbb{R}$, the difference set is

$$\mathcal{D}(A) = \{x - y \mid x, y \in A\}.$$

COROLLARY 6.2.32. *If $\Gamma \subset \mathbb{N}_0$ and μ is a probability measure, $\mu \in \mathcal{M}(K_\Gamma)$ if and only if*

$$\hat{\mu}(n) = 0 \tag{6.2.24}$$

for all $n \in \mathcal{D}(\Gamma) \setminus \{0\}$.

PROOF. We verify $C = CMC$ holds if and only if Equation (6.2.24) holds. We have $c_{mn} = 1$ if $m = n \in \Gamma$ and 0 otherwise. Thus by the calculation in Theorem 6.2.29,

$$(CMC)_{mn} = M_{mn}$$

if $m, n \in \Gamma$ and

$$(CMC)_{mn} = 0$$

otherwise. Thus, $C = CMC$ holds if and only if $M_{mn} = 0$ whenever $m, n \in \Gamma$ with $m \neq n$. The result now follows since

$$M_{mn} = \hat{\mu}(n - m).$$

□

The kernels K_3 and K_4, continued

Recall, we denote the usual Lebesgue measure on the interval by λ.

COROLLARY 6.2.33. *Suppose $\Gamma \subset \mathbb{N}_0$ is such that $\mathcal{D}(\Gamma) = \mathbb{Z}$. Then $\mathcal{M}(K_\Gamma) = \{\lambda\}$. In particular, $\mathcal{M}(K_3) = \{\lambda\}$.*

PROOF. By Corollary 6.2.28, we have $\lambda \in \mathcal{M}(K_\Gamma)$. Now, suppose $\mu \in \mathcal{M}(K_\Gamma)$. We must have that the matrix equation $C = CMC$ is satisfied. Thus, for $m \neq n \in \Gamma$, we have $0 = c_{mn} = c_{mm}M_{mn}c_{nn} = \hat{\mu}(n - m)$. Since the difference set of Γ is $\mathbb{Z}$, it follows that $\hat{\mu}(k) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$, whence μ must be Lebesgue measure. The claim for K_3 is a consequence of Lemma 6.2.34. □

LEMMA 6.2.34. *The difference set $\mathcal{D}(\Gamma_3) = \mathbb{Z}$.*

PROOF. We prove that $\mathcal{D}(\Gamma_3)$ is invariant under the iterated functions $\varphi_0(x) = 3x$, $\varphi_1(x) = 3x + 1$, and $\varphi_2(x) = 3x - 1$. Indeed, suppose that $n \in \mathcal{D}(\Gamma_3)$, then $n = \eta_1 - \eta_2$ for $\eta_k \in \Gamma_3$. Since Γ_3 is invariant under φ_0 and φ_1, we have

$$\begin{aligned}
\varphi_0(n) &= \varphi_0(\eta_1) - \varphi_0(\eta_2) \in \mathcal{D}(\Gamma_3) \\
\varphi_1(n) &= \varphi_1(\eta_1) - \varphi_0(\eta_2) \in \mathcal{D}(\Gamma_3) \\
\varphi_2(n) &= \varphi_0(\eta_1) - \varphi_1(\eta_2) \in \mathcal{D}(\Gamma_3).
\end{aligned}$$

Clearly $\{-1, 0, 1\} \subset \mathcal{D}(\Gamma_3)$, so since it is invariant under φ_0, φ_1, φ_2, our claim is established. □

A consequence of Corollary 6.2.33 is that Γ_3 is *not* a spectrum of any measure. Likewise, any $\Gamma \subsetneq \mathbb{Z}$ whose difference set is $\mathbb{Z}$ is not a spectrum of any measure.

COROLLARY 6.2.35. *If $\Gamma \subset \mathbb{N}_0$ is such that $\mathcal{D}(\Gamma) \neq \mathbb{Z}$, then there exist absolutely continuous measures in $\mathcal{M}(K_\Gamma)$. In particular, $\mathcal{M}(K_4)$ contains many absolutely continuous measures.*

Proof. We define μ by its Radon-Nikodym derivative: choose

$$\frac{d\mu}{d\lambda}(\theta) = 1 + \sum_{n \notin \mathcal{D}(\Gamma)} b_n \cos(2\pi n\theta)$$

subject to the constraint that $\sum_{n \notin \mathcal{D}(\Gamma)} |b_n| < 1$. It follows that μ is a probability measure such that

$$\widehat{\frac{d\mu}{d\lambda}}(n) = 0$$

for $n \in \mathcal{D}(\Gamma) \setminus \{0\}$, and so satisfies Corollary 6.2.32.

Now, we claim that $\mathcal{D}(\Gamma_4) \neq \mathbb{Z}$. Indeed, if we define

$$\Gamma_4' = \left\{ \sum_{j=0}^{N} l_j 4^j \mid l_j \in \{0, 2\} \right\},$$

then we claim that $\Gamma_4' \cap \mathcal{D}(\Gamma_4) = \{0\}$. To establish this, suppose we have

$$\sum_{j=0}^{N_1} l_j 4^j = \sum_{j=0}^{N_2} p_j 4^j - \sum_{j=0}^{N_3} q_j 4^j$$

with $l_j \in \{0, 2\}$ and $p_j, q_j \in \{0, 1\}$. We may assume $N_1 = N_2 = N_3$ by padding with 0's if necessary. Thus, we have

$$\sum_{j=0}^{N} p_j 4^j = \sum_{j=0}^{N} l_j 4^j + \sum_{j=0}^{N} q_j 4^j = \sum_{j=0}^{N} (l_j + q_j) 4^j,$$

where $l_j + q_j \in \{0, 1, 2, 3\}$. Since the base 4 expansion is unique, we must have that $p_j = l_j + q_j$ for all j, which can only occur when $l_j = 0$ for all j. □

Absolutely continuous measures. We proceed now to prove the Meta-Theorem in the case that the Grammian matrix M is a bounded operator on $\ell^2(\mathbb{N}_0)$. As mentioned previously, this occurs when the measure μ is absolutely continuous with bounded Radon-Nikodym derivative.

Theorem 6.2.36. *Suppose C is a bounded, positive, selfadjoint operator on $\ell^2(\mathbb{N})$, $\mu << \lambda$, and $\frac{d\mu}{d\lambda} \in L^\infty(\mathbb{T})$. Then $\mu \in \mathcal{M}(K_C)$ if and only if $C = CMC$, where M is the Grammian matrix of $\{e_n\}_{n=0}^\infty \subset L^2(\mu)$, i.e. $M_{mn} = \hat{\mu}(n-m)$.*

Proof. Since $\frac{d\mu}{d\lambda} \in L^\infty(\mathbb{T})$, the sequence $\{e_n\}_{n=0}^\infty \subset L^2(\mu)$ is a Bessel sequence. Thus, M is a bounded operator on $\ell^2(\mathbb{N})$, and the matrix product CMC is defined. Moreover, for every $w \in \mathbb{D}$, we have that since

$$\left(C\vec{w}\right)_n \in \ell^2(\mathbb{N})$$

the series

$$\sum_{n=0}^\infty \left(\sum_{m=0}^\infty c_{mn} \overline{w}^m \right) e^{2\pi int}$$

converges in $L^2(\mu)$. Thus, we have that for every $w \in \mathbb{D}$, the $L^2(\mu)$ boundary is given by

$$K_C^\star(w, t) = \sum_{n=0}^\infty \left(\sum_{m=0}^\infty c_{mn} \overline{w}^m \right) e^{2\pi int}$$

by Abel summation. We calculate

$$\begin{aligned}
&\int_0^1 K_C^\star(w,t)\overline{K_C^\star(z,t)}d\mu(t) \\
=\;& \sum_{n=0}^\infty \left(\sum_{m=0}^\infty c_{mn}\overline{w}^m\right)\int_0^1 e^{2\pi int}\overline{K_C^\star(z,t)}d\mu(t) \\
=\;& \sum_{n=0}^\infty \left(\sum_{m=0}^\infty c_{mn}\overline{w}^m\right)\sum_{k=0}^\infty\left(\sum_{l=0}^\infty \overline{c}_{lk}z^l\right)\int_0^1 e^{2\pi int}e^{-2\pi ikt}d\mu(t) \\
=\;& \sum_{n=0}^\infty \left(\sum_{m=0}^\infty c_{mn}\overline{w}^m\right)\sum_{k=0}^\infty\left(\sum_{l=0}^\infty c_{kl}z^l\right)M_{nk}. \qquad (6.2.25)
\end{aligned}$$

We have by the Cauchy-Schwarz inequality:

$$\begin{aligned}
\sum_l\sum_k |c_{kl}M_{nk}z^l| &\leq \sum_l |z^l|\sqrt{\sum_k |c_{kl}|^2}\sqrt{\sum_k |M_{nk}|^2} \\
&\leq \|C\|\,\|M\|\sum_l |z^l| < \infty.
\end{aligned}$$

Therefore,

$$(6.2.25) = \sum_{n=0}^\infty\left(\sum_{m=0}^\infty c_{mn}\overline{w}^m\right)\sum_{l=0}^\infty (MC)_{nl}\,z^l. \tag{6.2.26}$$

Again by the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\sum_m\sum_n\sum_l |c_{mn}\,(MC)_{nl}\,z^l\overline{w}^n| \\
\leq\;& \sum_m\sum_l |z^l\overline{w}^n|\sqrt{\sum_n |c_{mn}|^2}\sqrt{\sum_n |\,(MC)_{nl}|^2} \\
\leq\;& \|C\|\|MC\|\sum_m\sum_l |z^l\overline{w}^n| < \infty.
\end{aligned}$$

Whence

$$(6.2.26) = \sum_{l=0}^\infty\sum_{m=0}^\infty (CMC)_{ml}\overline{w}^m z^l. \tag{6.2.27}$$

Consequently, Equation (6.2.5) if and only if

$$\sum_{n=0}^\infty\sum_{m=0}^\infty c_{mn}\overline{w}^m z^n = \sum_{n=0}^\infty\sum_{m=0}^\infty (CMC)_{mn}\overline{w}^m z^n. \tag{6.2.28}$$

Equation (6.2.28) holds if and only if $C = CMC$ by the uniqueness of Taylor series coefficients. □

Preservation of norms of subspaces of $L^2(\lambda)$. The proofs of Theorems 6.2.29 and 6.2.36 suggest that the property that K_C reproduces itself with respect to some μ on the boundary is related to the following question: given a closed subspace $V \subset L^2(\lambda)$, for which measures ν does the following norm preservation identity hold for all $f \in V$:

$$\int |f|^2 d\lambda = \int |f|^2 d\nu?$$

Of course, this is ill-defined, because for $f \in L^2(\lambda)$, the question of whether $f \in L^2(\nu)$ and subsequently norm equality may depend on the representative. However, this ambiguity can be made precise using the boundary behavior of kernels as in Theorems 6.2.39 and 6.2.42.

DEFINITION 6.2.37. Suppose $V \subset L^2_+(\lambda)$ is a closed subspace and let $\widetilde{V} \subset H^2(\mathbb{D})$ be the space consisting of all functions whose $L^2(\lambda)$ boundaries are in V. We say the measure μ preserves the norm of V if for every $f \in V$, with $f(x) = \sum_{n=0}^{\infty} a_n e^{2\pi i n x}$, the corresponding function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ has a $L^2(\mu)$ boundary $F^\star$ and

$$\|f\|_\lambda = \|F^\star\|_\mu.$$

LEMMA 6.2.38. *Suppose $\Gamma \subset \mathbb{N}_0$ and μ satisfies Equation 6.2.24, then for every f in the subspace generated by K_Γ, f possesses a $L^2(\mu)$-boundary, and the norm of the $L^2(\mu)$-boundary agrees with the $H^2(\mathbb{D})$ norm of f.*

PROOF. Consider $\Gamma_1 \subset \Gamma$ of finite cardinality, and $f(z) = \sum_{\gamma\in\Gamma_1} a_\gamma z^\gamma$. Let $f^\star(e^{2\pi i\theta}) = \sum_{\gamma\in\Gamma_1} a_\gamma e^{2\pi i\gamma\theta}$. We claim that $f^\star$ is the $L^2(\mu)$-boundary of f and that

$$\|f^\star\|_\mu = \|f\|_{H^2}.$$

Indeed, we have that the $L^2(\mu)$-boundary of the function z^γ is $e^{2\pi i\gamma\theta}$ by uniform convergence, and thus by linearity $f^\star$ is the $L^2(\mu)$-boundary of f. Moreover,

$$\begin{aligned}
\|f^\star\|_\mu^2 &= \int_0^1 \left(\sum_{\gamma\in\Gamma_1} a_\gamma e^{2\pi i\gamma\theta}\right)\left(\sum_{\gamma'\in\Gamma_1} \overline{a_{\gamma'}} e^{-2\pi i\gamma'\theta}\right) d\mu \\
&= \sum_{\gamma\in\Gamma_1}\sum_{\gamma'\in\Gamma_1} a_\gamma \overline{a_{\gamma'}} \int_0^1 e^{2\pi i(\gamma-\gamma')} d\mu \\
&= \sum_{\gamma\in\Gamma_1} |a_\gamma|^2 \\
&= \|f\|_{H^2}^2. \qquad (6.2.29)
\end{aligned}$$

Now, for $f(z) = \sum_{\gamma\in\Gamma} a_\gamma z^\gamma$, the series

$$f^\star(e^{2\pi i\theta}) = \sum_{\gamma\in\Gamma} a_\gamma e^{2\pi i\gamma\theta}$$

converges in $L^2(\mu)$, and $f^\star$ is the $L^2(\mu)$-boundary for f by Abel summability. The equality of norms follows from taking limits in Equation (6.2.29). $\square$

THEOREM 6.2.39. *Let $\Gamma \subset \mathbb{N}_0$ and let V be the closed span of $\{e^{2\pi i\gamma\theta}\}_{\gamma\in\Gamma}$ in $L^2(\lambda)$. The measure μ preserves the norm of V if and only if $C = CMC$, where C is the diagonal matrix $c_{nn} = 1$ if $n \in \Gamma$ and 0 otherwise, and $M = (\hat{\mu}(n-m))_{mn}$.*

PROOF. We have that $K_C(= K_\Gamma)$ is the reproducing kernel for the space $\widetilde{V}$, and thus, if μ preserves the norm of V, we have by the polarization identity that $\mu \in \mathcal{M}(K_C)$. By Theorem 6.2.29 we must have $C = CMC$.

Conversely, if $C = CMC$, then $\mu \in \mathcal{M}(K_C)$, and thus μ preserves the norms of finite linear combinations $\sum_{j=1}^{N} b_j K_C^{\star}(w_j, \cdot)$:

$$\begin{aligned}
\|\sum_{j=1}^{N} b_j K_C^{\star}(w_j,\cdot)\|_{\mu}^2 &= \sum_{j=1}^{N}\sum_{k=1}^{N} b_j \overline{b_k} \int_0^1 K_C^{\star}(w_j,\cdot)\overline{K_C^{\star}(w_k,\cdot)}d\mu \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N} b_j \overline{b_k} K_C(w_j, w_k) \\
&= \|\sum_{j=1}^{N} b_j K_C(w_j,\cdot)\|_{H^2}^2 \\
&= \|\sum_{j=1}^{N} b_j K_C^{\star}(w_j,\cdot)\|_{\lambda}^2.
\end{aligned}$$

We see by the proof of Corollary 6.2.32 that if $C = CMC$, then μ satisfies Equation (6.2.24). By Lemma 6.2.38, every element of the space spanned by K_C possesses an $L^2(\mu)$ boundary, and by density, μ then preserves the norms of all elements of V. $\square$

LEMMA 6.2.40. *Suppose C is a projection on $\ell^2(\mathbb{N}_0)$ and $\mu << \lambda$ is such that $\frac{d\mu}{d\lambda} \in L^{\infty}(\lambda)$. Let $N = (\hat{\mu}(m-n))_{mn}$. If $C = CNC$, then for every sequence $(a_n)_n$ in the range of C*

$$\|\sum a_n e_n\|_{\mu} = \|\sum a_n e_n\|_{\lambda}.$$

(Note that $N = M^T$ in our previous notation.)

PROOF. Our hypotheses yield that the series $\sum_n a_n e^{2\pi i n\theta}$ converges in $L^2(\mu)$. We have

$$\begin{aligned}
\|\sum a_n e^{2\pi i n\theta}\|_{\mu}^2 &= \sum_{n,m} a_n \overline{a_m} \int_0^1 e^{2\pi i(n-m)\theta} d\mu(\theta) \\
&= \langle N(a_n)_n, (a_n)_n\rangle \\
&= \langle CNC(a_n)_n, (a_n)_n\rangle \\
&= \langle C(a_n)_n, (a_n)_n\rangle \\
&= \|\sum a_n e^{2\pi i n\theta}\|_{\lambda}^2.
\end{aligned}$$

$\square$

LEMMA 6.2.41. *Suppose V is a subspace of $L^2_+(\lambda)$. Let C be the projection on $\ell^2(\mathbb{N}_0)$ such that $f = \sum_n a_n e_n \in V$ if and only if $(a_n)_n$ is in the range of C. Then the reproducing kernel of $\widetilde{V}$ is K_{C^T}.*

PROOF. First note that for $w \in \mathbb{D}$,

$$K_{C^T}(w,z) = \sum_n \sum_m (C^T)_{mn} \overline{w}^m z^n = \sum_n \left(C\vec{\overline{w}}\right)_n z^n$$

is such that the coefficients are in the range of C. Thus, $K_C(w,\cdot) \in \widetilde{V}$. Now, for $f(z) = \sum_n a_n z^n \in \widetilde{V}$, we have that

$$\langle f, K_{C^T}\rangle_{H^2} = \langle (a_n)_n, C\vec{\overline{w}}\rangle_{\ell^2} = \langle (a_n)_n, \vec{\overline{w}}\rangle_{\ell^2} = \sum_n a_n w^n = f(w),$$

and thus K_{C^T} is the kernel as claimed. ◻

THEOREM 6.2.42. *Suppose V is a subspace of $L^2_+(\lambda)$, and let $\mu << \lambda$ with $\frac{d\mu}{d\lambda} \in L^\infty(\lambda)$. Then μ preserves the norm of V if and only if $C^T = C^T M C^T$, where C is the projection on $\ell^2(\mathbb{N}_0)$ with the property that K_{C^T} is the reproducing kernel of $\widetilde{V} \subset H^2(\mathbb{D})$ and $M = (\hat{\mu}(n-m))_{mn}$.*

PROOF. For $w_1, w_2 \in \mathbb{D}$, $K_{C^T}(w_j, z) = \sum_n (C\vec{\overline{w}}_j)_n z^n$. By our assumptions, the $L^2(\mu)$ boundary of $K_{C^T}(w_j,\cdot)$ is $\sum_n (C\vec{\overline{w}}_j)_n e^{2\pi i n\theta}$.

Suppose that μ preserves the norm of V. We have by the Polarization Identity:

$$\begin{aligned}
\int_0^1 K^\star_{C^T}(w_1,\theta)\overline{K^\star_{C^T}(w_2,\theta)}d\mu(\theta) &= \langle K^\star_{C^T}(w_1,\cdot), K^\star_{C^T}(w_2,\cdot)\rangle_\mu \\
&= \left\langle \sum_n (C\vec{\overline{w}}_1)_n e^{2\pi i n\theta}, \sum_n (C\vec{\overline{w}}_2)_n e^{2\pi i n\theta} \right\rangle_\lambda \\
&= \langle K_{C^T}(w_1,\cdot), K_C(w_2,\cdot)\rangle_{H^2} \\
&= K_{C^T}(w_1, w_2).
\end{aligned}$$

Therefore, $\mu \in \mathcal{M}(K_{C^T})$, and hence by Theorem 6.2.36, $C^T = C^T M C^T$.

Conversely, if $C^T = C^T M C^T$, then $C = CNC$. Therefore, for every finite linear combination $\sum_{j=1}^N \xi_j K^\star_{C^T}(w_j,\cdot) \in V$, we have by Lemma 6.2.40 that

$$\|\sum_{j=1}^N \xi_j K^\star_{C^T}(w_j,\cdot)\|_\mu = \|\sum_{j=1}^N \xi_j K^\star_{C^T}(w_j,\cdot)\|_\lambda.$$

By density, μ preserves the norm of V. ◻

6.2.6. A characterization via the Abel product. In this section, we prove a characterization of those representing measures via a matrix identity by introducing a new operator product called the Abel Product. The reader is referred to the original paper [**HJW18a**].

Given a sequence of vectors $\{x_n\}_{n=0}^\infty$ in a Hilbert space $\mathbb{H}$, we define the synthesis operator of $\{x_n\}$, $S_x : \mathbb{C}^\mathbb{N} \to \mathbb{H}$ by

$$S_x[(c_n)_n] = \sum_{n=0}^\infty c_n x_n$$

where the convergence on the right is in the norm of $\mathbb{H}$ and the analysis operator of $\{x_n\}$, $A_x : \mathbb{H} \to \mathbb{C}^\mathbb{N}$ by

$$A_x[f] = (\langle f, x_n\rangle_\mathbb{H})_{n=0}^\infty.$$

(It is understood that the domain of S_x is not $\mathbb{C}^\mathbb{N}$ itself, but rather a subset thereof on which the series converges. Depending on the situation, S_x and A_x are understood to have smaller domains and codomains than those above.)

Since C is a positive matrix, it is the Gramian of some sequence of vectors. That is to say, there exists some sequence of vectors $\{x_n\}_{n=0}^{\infty}$ in some Hilbert space $\mathbb{H}$ such that

$$C = (\langle x_m, x_n \rangle_{\mathbb{H}})_{mn}.$$

Observe that

$$A_x S_x[(c_n)_n] = A_x \left(\sum_{n=0}^{\infty} c_n x_n \right) = \left(\sum_{n=0}^{\infty} c_n \langle x_m, x_n \rangle_{\mathbb{H}} \right)_m.$$

Thus, the composition $A_x S_x = (\langle x_n, x_m \rangle)_{mn}$ as an operator on sequences is the transpose of the Gramian of the $\{x_n\}_{n=0}^{\infty}$. It follows that C can be realized by some sequence $\{x_n\} \subset \mathbb{H}$ as $C = (A_x S_x)^T$.

We denote by $S_{\overline{e}}$ and $A_{\overline{e}}$ the synthesis and analysis operators for $\{\overline{e_n}\} \subset L^2(\mu)$, respectively. Note that $\overline{e_n} = e_{-n}$. It is easily seen that $(A_{\overline{e}} S_{\overline{e}})^T = A_e S_e$, and therefore the matrix $M = (\hat{\mu}(n-m))_{mn}$, which is the Grammian matrix of the $\{e_n\}_{n=0}^{\infty} \subset L^2(\mu)$, can be factored as

$$M = A_{\overline{e}} S_{\overline{e}} \tag{6.2.30}$$

as a matrix. We can also formalize the factorization in Equation (6.2.30) as follows.

LEMMA 6.2.43. *The mappings*

$$S_e : \ell^1 \to L^2(\mu) : (c_n)_n \mapsto \sum_{n=0}^{\infty} c_n e_n$$

and

$$A_e : L^2(\mu) \to \ell^{\infty} : f \mapsto (\langle f, e_n \rangle_{\mu})_n$$

are bounded operators. Likewise for $S_{\overline{e}}$ and $A_{\overline{e}}$. Consequently, the matrix M defines a bounded operator from ℓ^1 to ℓ^{∞} and $M = A_{\overline{e}} S_{\overline{e}}$.

PROOF. The mapping S_e is well-defined and bounded by absolute summability, while the mapping A_e is well-defined and bounded by the Cauchy-Schwarz inequality. It follows that the composition is bounded, and the matrix M represents the composition $A_{\overline{e}} S_{\overline{e}}$ as argued above. $\square$

DEFINITION 6.2.44.

(1) By D_s we shall mean the operator from ℓ^{∞} to ℓ^1 given by the diagonal matrix whose diagonal is the vector $\vec{s}$, where $0 < s < 1$. Thus, $D_s[(x_n)_n] = (s^n x_n)_n$.

(2) We shall define

$$V := span\{\vec{v} : v \in \mathbb{D}\}.$$

Note that V is a proper, dense subspace of ℓ^2.

(3) Let X and Y be inner product spaces. Let S be a (possibly unclosed) subspace of ℓ^1. Let $T_1 : X \to \ell^{\infty}$ and $T_2 : S \to Y$ be (possibly unbounded) linear operators. Suppose that $D_s T_1 X \subseteq S$ for all $0 < s < 1$. If there exists a bounded linear operator $A : X \to Y$ such that

$$\lim_{s \to 1^-} \langle T_2 D_s T_1 x, y \rangle_Y = \langle Ax, y \rangle_Y \tag{6.2.31}$$

for all $x \in X$ and $y \in Y$, then we say that A is the Abel product of T_2 and T_1, and we denote this product by $T_2 \circledast T_1$. If X and/or Y are subspaces of larger spaces, then the existence of the limit may depend on the X

and Y, and in fact may not exist for other subspaces. We indicate this dependence as $T_2 \circledast T_1 = A\restriction_{X,Y}$.

Abel products:

(1) The name "Abel product" is inspired by Abel summation: if T_2 and T_1 are matrices whose Abel product exists on the finite span of the standard basis vectors $\{\delta_n\}$ in ℓ^2, then

$$\lim_{s\to 1^-} \langle T_2 D_s T_1 \delta_n, \delta_m \rangle_{\ell^2} = \lim_{s\to 1^-} \sum_{k=0}^{\infty} (T_2)_{mk} s^k (T_1)_{kn}$$

which is the Abel sum of the ordinary matrix product of T_2 and T_1.

(2) The Abel product extends the ordinary operator (matrix) product as follows: if T_2 and T_1 are bounded operators, then $T_2 \circledast T_1 = T_2 \circ T_1$.

(3) If X and Y are complete spaces, the existence of the limit in (6.2.31) for all $x \in X$ and $y \in Y$ is by the Uniform Boundedness Principle enough to imply the existence of A.

(4) The same technique in [**AMP92**, **AFMP94**] for dealing with infinite matrices which are unbounded operators on $\ell^2(\mathbb{N}_0)$: in those papers the authors "pre-condition" an unbounded operator with the same diagonal matrix as here, but the authors in [**AMP92**, **AFMP94**] use the diagonal matrix to effectively perform a variable substitution, and do not consider the limit as we do here.

Our main result will establish a characterization of when μ is a representing measure for K_C. Theorem 6.2.50 says that $\mu \in \mathcal{M}(K_C)$ if and only if $C = (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)$. If in the special case that M is a bounded operator from ℓ^2 to ℓ^2, i.e. both $A_{\overline{e}}$ and $S_{\overline{e}}$ are bounded (which occurs when $\{e_n\} \subset L^2(\mu)$ is a Bessel sequence, then we have the following consequence of Theorem 6.2.50:

$$\begin{aligned} C &= (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C) \\ &= (C \circ A_{\overline{e}})(S_{\overline{e}} \circ C) \\ &= C(A_{\overline{e}} \circ S_{\overline{e}})C \\ &= CMC. \end{aligned}$$

Therefore, the Abel product will allow us to rigorously extend this heuristic to the case when M is not a bounded operator.

THEOREM 6.2.45. *Suppose $C : \ell^2 \to \ell^2$ is a bounded positive operator representable by an infinite matrix, and form the positive matrix $K_C(w,z) := \langle C\vec{z}, \vec{w} \rangle$. Suppose $K_C(w,z)$ has weak $L^2(\mu)$-boundaries in the sense that for each fixed $w \in \mathbb{D}$, there exists a function $K_w^\star \in L^2(\mu)$ such that for every $h(x) \in L^2(\mu)$,*

$$\lim_{s\to 1^-} \left\langle K_C(w, se^{2\pi i x}), h(x) \right\rangle_\mu = \langle K_w^\star(x), h(x) \rangle_\mu .$$

Suppose further that $K_C(w,z)$ reproduces itself with respect to these weak boundaries, i.e.

$$K_C(w,z) = \langle K_w^\star, K_z^\star \rangle_\mu .$$

Then there exists a bounded linear operator $L : \ell^2 \to L^2(\mu)$ such that

$$K_w^\star = L\vec{\overline{w}} \tag{6.2.32}$$

for all $w \in \mathbb{D}$, and

$$\lim_{s\to 1^-} \left\langle S_e D_s C^T v, h(x)\right\rangle_\mu = \langle Lv, h(x)\rangle_\mu \tag{6.2.33}$$

for all $v \in V$ and $h \in L^2(\mu)$. Consequently, $L \restriction_{V,L^2(\mu)} = S_e \circledast C^T$.

Proof. Observe that

$$\begin{aligned}
\langle K_w^\star(x), h(x)\rangle_\mu &= \lim_{s\to 1^-} \left\langle K(w, se^{2\pi i x}), h(x)\right\rangle_\mu \\
&= \lim_{s\to 1^-} \left\langle \sum_{n=0}^\infty \sum_{m=0}^\infty c_{mn}\overline{w}^m s^n e^{2\pi i n x}, h(x)\right\rangle_\mu \\
&= \lim_{s\to 1^-} \left\langle \sum_{n=0}^\infty \left(C^T\vec{\overline{w}}\right)_n s^n e^{2\pi i n x}, h(x)\right\rangle_\mu \\
&= \lim_{s\to 1^-} \left\langle \sum_{n=0}^\infty \left(D_s C^T\vec{\overline{w}}\right)_n e^{2\pi i n x}, h(x)\right\rangle_\mu \\
&= \lim_{s\to 1^-} \left\langle S_e D_s C^T \vec{\overline{w}}, h(x)\right\rangle_\mu . \qquad (6.2.34)
\end{aligned}$$

Define

$$L : V \to L^2(\mu) : (\sum_{j=0}^{N-1} \alpha_j \vec{w_j}) = \sum_{j=0}^{N-1} \alpha_j K^\star_{\overline{w_j}}. \tag{6.2.35}$$

We claim that L is well-defined. Indeed, suppose that $\sum_{j=0}^{N-1} \alpha_j \vec{w_j} = \vec{0}$ for some distinct $\vec{w_j}$'s. Then in particular, the first N entries of $\sum_{j=0}^{N-1} \alpha_j \vec{w_j}$ are equal to 0. However, this is impossible, because the $N \times N$ Vandermonde matrix

$$V = ((\vec{w_m})_n)_{mn}$$

is nonsingular, since the $\vec{w_j}$ are distinct. Thus, it has linearly independent rows. By construction, then, L is linear, and (6.2.32) and therefore (6.2.33) hold.

We next claim that L is bounded on V, and hence can be extended to all of ℓ^2. Because $K_C(w,z)$ reproduces itself with respect to its boundaries,

$$\begin{aligned}\left\| L\left(\sum_{j=0}^{N}\alpha_j\vec{w_j}\right)\right\|_{\mu}^{2} &= \left\|\sum_{j=0}^{N}\alpha_j K^{\star}_{\overline{w_j}}\right\|_{\mu}^{2}\\ &= \sum_{j=0}^{N}\sum_{k=0}^{N}\alpha_j\overline{\alpha_k}K(\overline{w_j},\overline{w_k})\\ &= \sum_{j=0}^{N}\sum_{k=0}^{N}\alpha_j\overline{\alpha_k}\left\langle C\overline{\vec{w_k}},\overline{\vec{w_j}}\right\rangle_{\ell^2}\\ &= \left\langle C\left(\sum_{k=0}^{N}\overline{\alpha_k\vec{w_k}}\right),\sum_{j=0}^{N}\overline{\alpha_j w_j}\right\rangle_{\ell^2}\\ &\le \|C\|\left\|\sum_{j=0}^{N}\overline{\alpha_j\vec{w_j}}\right\|_{\ell^2}^{2}\\ &= \|C\|\left\|\sum_{j=0}^{N}\alpha_j\vec{w_j}\right\|_{\ell^2}^{2}.\end{aligned}$$

It now follows from Equations (6.2.34) and (6.2.35) that $L\restriction_{V,L^2(\mu)} = S_e \circledast C^T$. $\square$

LEMMA 6.2.46. *For $v\in\ell^2$, $s\in(0,1)$, we have*

$$S_{\overline{e}}D_sCv = \overline{S_eD_sC^T\overline{v}}. \tag{6.2.36}$$

PROOF. We calculate

$$\begin{aligned}S_{\overline{e}}D_sCv &= S_{\overline{e}}D_s\left(\sum_{n=0}^{\infty}c_{mn}v_n\right)_m\\ &= \sum_{m=0}^{\infty}s^m\left(\sum_{n=0}^{\infty}c_{mn}v_n\right)\overline{e_m}\end{aligned}$$

whereas

$$\begin{aligned}S_eD_sC^T\overline{v} &= S_eD_s\left(\sum_{n=0}^{\infty}c_{nm}\overline{v_n}\right)_m\\ &= \sum_{m=0}^{\infty}s^m\left(\sum_{n=0}^{\infty}c_{nm}\overline{v_n}\right)e_m.\end{aligned}$$

$\square$

COROLLARY 6.2.47. *Under the hypotheses of Theorem* 6.2.45*, there exists a bounded linear operator $\tilde{L}:\ell^2\to L^2(\mu)$ such that*

$$\lim_{s\to 1^-}\left\langle S_{\overline{e}}D_sCv,h\right\rangle_{\mu} = \left\langle \tilde{L}v,h\right\rangle_{\mu}$$

for all $v\in V$ and $h\in L^2(\mu)$. Consequently, $S_{\overline{e}}\circledast C = \tilde{L}\restriction_{V,L^2(\mu)}$.

PROOF. Define $\tilde{L} : \ell^2 \to L^2(\mu)$ by

$$\tilde{L}v := \overline{L\overline{v}}.$$

It is easy to see that $\tilde{L}$ is linear and has the same bound as L. Let $h \in L^2(\mu)$. We have:

$$\begin{aligned}
\lim_{s\to 1^-} \langle S_{\overline{e}} D_s C v, h\rangle_\mu &= \lim_{s\to 1^-} \int_0^1 [S_{\overline{e}} D_s C v](x)\overline{h(x)}\, d\mu(x) \\
&= \lim_{s\to 1^-} \int_0^1 \overline{[S_e D_s C^T \overline{v}](x) h(x)}\, d\mu(x) \\
&= \int_0^1 \left([\overline{L\overline{v}}](x)\right) \overline{h(x)}\, d\mu(x) = \left\langle \tilde{L}v, h\right\rangle_\mu .
\end{aligned}$$

□

LEMMA 6.2.48. *Under the hypotheses of Theorem* 6.2.45, *for all* $w, z \in \mathbb{D}$,

$$\left\langle S_e \circledast C^T \vec{\overline{w}}, S_e \circledast C^T \vec{\overline{z}}\right\rangle = \langle S_{\overline{e}} \circledast C\vec{z}, S_{\overline{e}} \circledast C\vec{w}\rangle .$$

PROOF. By Theorem 6.2.45 and Corollary 6.2.47, both $S_e \circledast C^T$ and $S_{\overline{e}} \circledast C$ exist on V. We have by Lemma 6.2.46:

$$\begin{aligned}
\left\langle S_e \circledast C^T \vec{\overline{w}}, S_e \circledast C^T \vec{\overline{z}}\right\rangle_\mu &= \lim_{s\to 1^-} \lim_{r\to 1^-} \int_0^1 [S_e D_r C^T \vec{\overline{w}}](x)\overline{[S_e D_s C^T \vec{\overline{z}}](x)}\, d\mu(x) \\
&= \lim_{s\to 1^-} \lim_{r\to 1^-} \int_0^1 \overline{[S_{\overline{e}} D_r C \vec{w}](x)} [S_{\overline{e}} D_s C \vec{z}](x)\, d\mu(x) \\
&= \langle S_{\overline{e}} \circledast C\vec{z}, S_{\overline{e}} \circledast C\vec{w}\rangle_\mu .
\end{aligned}$$

□

LEMMA 6.2.49. *Suppose there exists a bounded operator* $L : \ell^2 \to L^2(\mu)$ *such that* $L \restriction_{V,L^2(\mu)} = S_e \circledast C^T$. *Then for all* $h \in L^2(\mu)$ *and* $v \in V$,

$$\lim_{r\to 1^-} \left\langle C^T D_r A_e h, v\right\rangle_{\ell^2} = \langle L^* h, v\rangle_{\ell^2} .$$

Consequently, $C^T \circledast A_e = L^* \restriction_{L^2(\mu),V}$.

PROOF. We calculate:

$$\begin{aligned}
\lim_{r\to 1^-} \left\langle C^T D_r A_e h, v\right\rangle_{\ell^2} &= \lim_{r\to 1^-} \left\langle \left(\sum_{n=0}^{\infty} c_{nm} r^n \langle h, e_n\rangle_\mu\right)_m, v\right\rangle_{\ell^2} \\
&= \lim_{r\to 1^-} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{v_m} c_{nm} r^n \langle h, e_n\rangle_\mu \\
&= \lim_{r\to 1^-} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \overline{v_m} c_{nm} r^n \langle h, e_n\rangle_\mu \qquad \text{[by abs. summability]} \\
&= \lim_{r\to 1^-} \left\langle h, \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} v_m c_{mn} r^n e_n\right\rangle_\mu \\
&= \lim_{r\to 1^-} \left\langle h, S_e D_r C^T v\right\rangle_\mu \\
&= \langle h, Lv\rangle_\mu = \langle L^* h, v\rangle_{\ell^2} .
\end{aligned}$$

□

We are now in a position to prove our main result, which is a characterization of when a Borel measure μ is a representing measure for a positive matrix K_C on $\mathbb{D}$. Here we formalize the heuristic $C = CMC$ using the Abel product.

THEOREM 6.2.50. *Suppose C is a positive bounded operator on ℓ^2 and μ is a Borel measure on $[0,1)$. Then $\mu \in \mathcal{M}(K_C)$ if and only if $C \restriction_{V,\ell^2} = (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)$.*

PROOF. If $\mu \in \mathcal{M}(K_C)$, then by definition, $K_C(w,z)$ has weak boundaries and reproduces with respect to those boundaries, and so Theorem 6.2.45 applies. Hence, for any $v = \sum_{j=0}^{M-1} \alpha_j \vec{z_j} \in V$ and $w = \sum_{k=0}^{M-1} \beta_k \vec{w_k} \in V$,

$$\begin{aligned}
&\left\langle C\left(\sum_{j=0}^{M-1} \alpha_j \vec{z_j}\right), \sum_{k=0}^{N-1} \beta_k \vec{w_k}\right\rangle_{\ell^2} \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle C\vec{z_j}, \vec{w_k}\right\rangle_{\ell^2} = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} K_C(w_k, z_j) \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle K^\star_{w_k}, K^\star_{z_j}\right\rangle_\mu = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle L\vec{\overline{w_k}}, L\vec{\overline{z_j}}\right\rangle_\mu \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle S_e \circledast C^T \vec{\overline{w_k}}, S_e \circledast C^T \vec{\overline{z_j}}\right\rangle_\mu \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle S_{\overline{e}} \circledast C\vec{z_j}, S_{\overline{e}} \circledast C\vec{w_k}\right\rangle_\mu \quad \text{[by Lem. 6.2.48]} \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} \alpha_j \overline{\beta_k} \left\langle (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)\vec{z_j}, \vec{w_k}\right\rangle_{\ell^2} \quad \text{[by Lem. 6.2.49]} \\
&= \left\langle (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)\left(\sum_{j=0}^{M-1} \alpha_j \vec{z_j}\right), \sum_{k=0}^{N-1} \beta_k \vec{w_k}\right\rangle_{\ell^2}.
\end{aligned}$$

Since V is dense in ℓ^2, by continuity of the inner product the above holds not just for $w \in V$ but for all $w \in \ell^2$, and hence $C \restriction_{V,\ell^2} = (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)$.

Conversely, suppose $C \restriction_{V,\ell^2} = (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)$. Since $S_{\overline{e}} \circledast C$ is assumed to exist boundedly on V, there exists a bounded extension $\tilde{L} : \ell^2 \to L^2(\mu)$ of $S_{\overline{e}} \circledast C$. Lemma 6.2.49 applies to show that $(C^T \circledast A_{\overline{e}}) = \tilde{L}^* \restriction_{L^2(\mu),V}$, and $S_e \circledast C^T$ exists by the proof of Corollary 6.2.47. Let $h(x) \in L^2(\mu)$. We have

$$\begin{aligned}
\lim_{s\to 1^-} \left\langle K_C(w, se^{2\pi i x}), h(x)\right\rangle_\mu &= \lim_{s\to 1^-} \left\langle \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn} \overline{w}^m s^n e^{2\pi i n x}, h(x)\right\rangle_\mu \\
&= \lim_{s\to 1^-} \left\langle S_e D_s C^T \vec{\overline{w}}, h\right\rangle_\mu \\
&= \left\langle (S_e \circledast C^T)\vec{\overline{w}}, h\right\rangle_\mu,
\end{aligned}$$

which shows that $K_C(w,z)$ possesses weak $L^2(\mu)$ boundaries $K_w^\star = (S_e \circledast C^T)\vec{\overline{w}}$. Then observe that by Lemma 6.2.48

$$\begin{aligned}
\langle K_w^\star, K_z^\star\rangle_\mu &= \left\langle (S_e \circledast C^T)\vec{\overline{w}}, (S_e \circledast C^T)\vec{\overline{z}}\right\rangle_\mu \\
&= \langle (S_{\overline{e}} \circledast C)\vec{z}, (S_{\overline{e}} \circledast C)\vec{w}\rangle_\mu && \text{[by Lemma 6.2.48]} \\
&= \left\langle \tilde{L}\vec{z}, \tilde{L}\vec{w}\right\rangle_\mu && \text{[by Corollary 6.2.47]} \\
&= \left\langle \tilde{L}^*\tilde{L}\vec{z}, \vec{w}\right\rangle_{\ell^2} \\
&= \langle (C \circledast A_{\overline{e}})(S_{\overline{e}} \circledast C)\vec{z}, \vec{w}\rangle_{\ell^2} \\
&= \langle C\vec{z}, \vec{w}\rangle_{\ell^2} && \text{[by assumption]} \\
&= K_C(w,z).
\end{aligned}$$

Thus, $\mu \in \mathcal{M}(K_C)$. □

EXAMPLE 6.2.51. Here are a few examples to illustrate the Abel product. First, consider the matrices

$$B = \begin{pmatrix} 1 & 1 & 1 & \dots \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 1 & -1 & \dots \end{pmatrix}^T.$$

We view A as being a bounded operator from $\mathbb{C} = \ell^2(\{0\})$ to $\ell^\infty(\mathbb{N}_0)$, and B a bounded operator from $\ell^1(\mathbb{N}_0)$ to $\mathbb{C}$. Note that the neither the matrix product nor the composition between B and A exist. However, for $x, y \in \mathbb{C}$, we have

$$\langle BD_rAx, y\rangle = \sum_{n=0}^{\infty}(-r)^n x\overline{y} \to \frac{1}{2}\langle x, y\rangle.$$

Thus, the Abel product exists, and $(B \circledast A)x = \dfrac{1}{2}x$.

6.3. Positive definite kernels and the associated Gaussian processes

The material below is based primarily on ideas in [**JT17a**].

The aim of the present section is two-fold: One is an extension of the traditional setting for reproducing kernel Hilbert space (RKHS) theory; — in more detail, an extension of the more traditional context of Aronszajn [**Aro43**, **Aro50**] to a measurable category which will adapt much better to a host of applications; especially to problems from probability theory, from stochastic processes, see e.g. [**AJ12**, **JT16b**], from mathematical physics [**HKL$^+$17**, **RAKK05**], and to measurable dynamics; and for the latter, especially to the context of reversible processes (see, e.g., [**TB13**, **CXY15**, **Sko13**, **Her12**, **DJ11a**, **Rob11**], and also [**BP17**, **PSS16**]). For applications to random processes, a kernel in the sense of Aronszajn will typically represent a covariance kernel. The applications include a new spectral theoretic analysis of (i) transient Markov processes, and of (ii) generalized Gaussian fields and their Ito-integrals (see e.g., [**IM74**]).

In the standard approach to RKHSs of Aronszajn, one starts with a positive definite (p.d) function, K on $M \times M$ (often called a p.d. kernel) where M is a given set. The term "reproducing" refers to the fact that for every f in $\mathscr{H}$, the values $f(x)$ can be reproduced from the inner Hilbert-product in $\mathscr{H}$. With a standard construction, starting with M and K, one then arrives at a Hilbert space $\mathscr{H}$ of functions on M, the so called reproducing kernel Hilbert space (RKHS). It depends on the pair (M, K) of course; so is denoted $\mathscr{H}(K)$ when the kernel is not given from the context. *A priori*, the set M is not given any additional structure, but a key point is that both K and the functions f in the RKHS $\mathscr{H}(K)$ are defined everywhere on X. If for example, M is a complex domain, in interesting applications, then the functions in $\mathscr{H}(K)$ will be analytic, or in the case of the familiar RKHS of Bargmann, the functions in $\mathscr{H}(K)$ will be entire analytic. If M has a topology, and if K is assumed continuous, then the functions in $\mathscr{H}(K)$ will then also be continuous.

But up to now, many of the applications have focused on Hilbert spaces of regular functions. If for example, a kernel represents a Green's function for an elliptic partial differential operator (PDO), then the associated RKHS will consist of functions which have some degree of smoothness.

The starting point of the present section, however, is a fixed measure space $(M, \mathscr{B}, \mu)$ where μ is assumed σ-finite. Set

$$\mathscr{B}_{fin} := \{A \in \mathscr{B}\,;\, \mu(A) < \infty\}.$$

We shall then consider positive definite functions (kernels) K on $\mathscr{B}_{fin} \times \mathscr{B}_{fin}$.

The Aronszajn approach has serious limitations: Often functions will be defined only almost everywhere with respect to some measure which is prescribed on the set M; for example, if M represents time, in one or more dimensions, the prescribed measure μ is often Lebesgue measure. For fractal random fields, μ may be a fractal measure. For this reason, and others (to be outlined inside the paper), it is useful to instead let M be a measure space, say $(M, \mathscr{B}, \mu)$. If M is a locally compact Hausdorff space, then $\mathscr{B}$ is the corresponding sigma-algebra of Borel sets, and ν is a fixed positive measure, and assumed to be a regular measure on $(M, \mathscr{B})$. The modification in the resulting new definition of p.d. kernels K in this context is subtle. Here we just mention that, for a p.d. system $(M, \mathscr{B}, \mu)$ and kernel K in the measurable category, the associated RKHS $\mathscr{H}$ will now instead be a Hilbert space of measurable functions on X, more precisely, measurable with respect to $\mathscr{B}$, and locally in $L^2(\mu)$. We shall say that $\mathscr{H}$ is contained in $L^2_{loc}(\mu)$. The p.d. kernel K itself will be a random family of signed measures on $(M, \mathscr{B})$.

DEFINITION 6.3.1. A function K on $\mathscr{B}_{fin} \times B_{fin}$ (mapping into $\mathbb{R}$) is said to be *positive definite* iff (Def.), for $\forall n \in \mathbb{N}$, $\forall \{\alpha_i\}_1^n$, $\alpha_i \in \mathbb{R}$, and all $\{A_i\}_1^n$, $A_i \in \mathscr{B}_{fin}$, we have

$$\sum_i \sum_j \alpha_i \alpha_j K(A_i, A_j) \geq 0. \tag{6.3.1}$$

When a positive definite kernel K is given, we shall denote the corresponding reproducing kernel Hilbert space (RKHS) by $\mathscr{H}(K)$.

Setting. Fix $(M, \mathscr{B}, \mu)$, μ σ-finite measure; and set

$$K = K_\mu, \quad K(A, B) := \mu(A \cap B), \tag{6.3.2}$$

for $\forall A, B \in \mathscr{B}_{fin}$, set $\mathscr{H} = \mathrm{RKHS}(K_\mu) =$ the RKHS of the kernel K_μ defined in (6.3.2).

THEOREM 6.3.2. *Consider functions F on $\mathscr{B}_{fin}$; then*

$$F \in \mathscr{H} \Longleftrightarrow \exists f \in L^2(\mu) \text{ s.t. } F(A) = \int_A f\, d\mu,\ \forall A \in \mathscr{B}_{fin}. \tag{6.3.3}$$

PROOF. Step 1. Define T initially by

$$L^2(\mu) \ni \chi_A \xmapsto{\ T\ } K_\mu(\cdot, A),$$

then T extends by linearity and closure to an isometry $T : L^2(\mu) \to \mathscr{H}$. The operator T is in fact *onto* $\mathscr{H}$. More specifically, set $\varphi = \sum c_i \chi_{A_i}$ then

$$\begin{aligned}
\|T\varphi\|_{\mathscr{H}}^2 &= \left\langle \sum c_i K(\cdot, A_i), \sum c_j K(\cdot, A_j) \right\rangle_{\mathscr{H}} \\
&= \sum_i \sum_j c_i c_j K(A_i \cap A_j) \\
&= \sum \sum c_i c_j \mu(A_i \cap A_j) \\
&= \int |\varphi|^2 \, d = \|\varphi\|_{L^2(\mu)}^2 .
\end{aligned}$$

Note the operator T is in fact *onto* $\mathscr{H}$.

Step 2. Fix $F \in \mathscr{H}$; for $\forall A \in \mathscr{B}_{fin}$, set

$$F(A) = \int_A (T^* F) \, d\mu. \tag{6.3.4}$$

Hence when $F \in \mathscr{H}$ is given, then $f = T^* F \in L^2(\mu)$.

$$\begin{tikzcd}
L^2(\mu) \arrow[r, bend left, "T"] & \mathscr{H} \arrow[l, bend left, "T^*"]
\end{tikzcd}$$

Note that

$$\begin{aligned}
F(A) &= \langle K(\cdot, A), F \rangle_{\mathscr{H}} \quad \text{by the RKHS propertity} \\
&= \langle T(\chi_A), F \rangle_{\mathscr{H}} \\
&= \langle \chi_A, T^*(F) \rangle_{L^2(\mu)} \\
&= \int_A T^*(F) \, d\mu,
\end{aligned}$$

which is (6.3.4). ☐

Fix μ and let $\mathscr{H} = \mathscr{H}(K_\mu)$ be the corresponding RKHS. Now use T^* to define $d/d\mu$, where

$$T^*(F) = \frac{dF}{d\mu}, \quad \forall F \in \mathscr{H}.$$

More generally, we define the Malliavin-Ito derivative

$$D\psi_n \left(X_{h_1}^{(\mu)}, \cdots, X_{h_n}^{(\mu)} \right) = \sum_j \frac{\partial \psi_n}{\partial x_j} \left(X_{h_1}^{(\mu)}, \cdots, X_{h_n}^{(\mu)} \right) h_j,$$

$\forall n$, $\forall \{h_j\}$, $h_j \in L^2(\mu)$.

Application of the isometry: The generalized Wiener-process

Here we consider the following kernel K on $\mathscr{B}_{fin} \times \mathscr{B}_{fin}$: Set

$$K(A, B) = \mu(A \cap B), \quad A, B \in \mathscr{B}_{fin}, \tag{6.3.5}$$

where the measure space $(M, \mathscr{B}, \mu)$ is specified as above.

PROPOSITION 6.3.3.

(1) $K = K^{(\mu)}$ *in* (6.3.5) *is positive definite.*

(2) $K^{(\mu)}$ *is the covariance kernel for the stationary Wiener process* $X = X^{(\mu)}$ *indexed by* $\mathscr{B}_{fin}$, *i.e., Gaussian, mean zero, and*

$$\mathbb{E}\left(X_A^{(\mu)} X_B^{(\mu)}\right) = K^{(\mu)}(A,B) = \mu(A \cap B). \tag{6.3.6}$$

(3) *If* $f \in L^2(\mu)$, *and* $X_f^{(\mu)} = \int_X f(x)\, dX_x^{(\mu)}$ *denotes the corresponding Ito-integral, then*

$$\mathbb{E}\left(\left|X_f^{(\mu)}\right|^2\right) = \int_X |f|^2\, d\mu;$$

in particular,if $f = \sum_i \alpha_i \chi_{A_i}$, *then*

$$\sum_i \sum_j \alpha_i \alpha_j K^{(\mu)}(A_i, A_j) = \int_X \left|\sum_i \alpha_i \chi_{A_i}\right|^2 d\mu.$$

(4) *The RKHS* $\mathscr{H}\left(K^{(\mu)}\right)$ *of the positive definite kernel in* (6.3.5) *consists of functions* F *on* $\mathscr{B}_{fin}$ *represented by* $f \in L^2(\nu)$ *via*

$$F(A) = \int_A f d\mu, \quad A \in \mathscr{B}_{fin}; \tag{6.3.7}$$

and

$$\|F\|^2_{\mathscr{H}(K)} = \|f\|^2_{L^2(\mu)} = \int_M |f|^2\, d\mu. \tag{6.3.8}$$

PROOF. We use a completion argument, and the fact that $f \longrightarrow X_f^{(\mu)}$ is isometric, $L^2(\mu) \longrightarrow L^2(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathscr{F})$ such that

$$\mathbb{E}_{\mathbb{P}}(X_A^{(\mu)} X_B^{(\mu)}) = K_\mu(A,B), \quad \forall A, B \in \mathscr{B}_{fin}. \tag{6.3.9}$$

$$\begin{tikzcd} & \mathscr{H}(K_\mu) \arrow[d] \\ L^2(\mu) \arrow[ur, bend left] \arrow[r] & L^2(\Omega, \mathscr{F}, \mathbb{P}) \end{tikzcd}$$

The details can be found at various places in the literature; see e.g., [**AJ12**,**JT16e**]. ☐

Summary. Once we have the generalized Brownian motion $X^{(\mu)}$ defined for a fixed σ-finite measure μ on $(M, \mathscr{B})$ where $(M, \mathscr{B})$ is a given measure space, we then also have an Ito-integral process [**IM74**]

$$X_f^{(\mu)} = \int_M f(x)\, dX_x^{(\mu)} \tag{6.3.10}$$

defined, $\forall f \in L^2(\mu)$ such that we get a process indexed by $L^2(\mu)$:

$$\mathbb{E}\left(X_{f_1}^{(\mu)} X_{f_2}^{(\mu)}\right) = \int_M f_1 f_2\, d\mu = \langle f_1, f_2 \rangle_{L^2(\mu)}, \quad \forall f_1, f_2 \in L^2(\mu). \tag{6.3.11}$$

$X^{(\mu)}$ Gaussian, $\mathbb{E}(X_A^{(\mu)}) = 0$	$\mathbb{E}(X_A^{(\mu)} X_B^{(\mu)}) = \mu(A \cap B)$, $\forall A, B \in \mathscr{B}_{fin}$
For $f \in L^2(\mu)$, set $F(A) = \int_A f\, d\mu$, $A \in \mathscr{B}_{fin}$	$X_f^{(\mu)} = \int_M f(y)\, dX_y^{(\mu)}$, extending Ito integral

An important property of the generalized Ito integral (6.3.10). Consider σ-finite measures μ on $(M, \mathscr{B})$ and functions f such that $f \in L^2(\mu)$. For pairs (f, μ) and (f', μ'), we recall the equivalence relation "$\sim$" from Lemma 5.3.4.

THEOREM 6.3.4 (Alpay-Jo [**AJ15**]). *If $(f, \mu) \sim (f', \mu')$, then $X_f^{(\mu)} = X_{f'}^{(\mu')}$; in fact, the implication goes both ways.*

PROOF. We refer to [**AJ15**], but the reader will be able to fill in the details on the basis of the discussion in Section 5.3 above. □

REMARK 6.3.5. We also point to applications of the theorem (in [**AJ15**]) to stochastic calculus.

For example, we have the following: Let $(M, \mathscr{B}, \mu)$ be a fixed σ-finite measure space, and let $X^{(\mu)}$ be the corresponding real-valued Gaussian process, see Proposition 6.3.3 above. Suppose further that the measure μ is *non-atomic*; then the *quadratic variation* of $X^{(\mu)}$ coincides with the measure μ.

Specifically, if $B \in \mathscr{B}$, and $0 < \mu(B) < \infty$, consider partitions:

$$\text{PART}(B) = \left\{ \{A_i\}_{i \in \mathbb{N}} \mid A_i \in \mathscr{B},\ \cup_i A_i = B,\ A_i \cap A_j = \emptyset,\ \text{for } i \neq j \right\}. \tag{6.3.12}$$

We further consider PART(B) as a *net* where the limit with respect to the net is define by

$$\lim \Big[\max \{ \mu(A_i) \mid \{A_i\} \in \text{PART}(B) \} \Big] = 0; \tag{6.3.13}$$

i.e., refinement of μ-mesh of elements in $\mathscr{B}$ from partitions.

PROPOSITION 6.3.6 (Alpay-Jo). *Subject to the conditions above, we have:*

$$\lim_{PART(B)} \left(\sum_i \left(X_{A_i}^{(\mu)} \right)^2 \right) = \mu(B). \tag{6.3.14}$$

REMARK 6.3.7. The convergence in (6.3.14) is in $L^2\left(\Omega, Cyl, \mathbb{P}^{(\mu)}\right)$ where $\mathbb{P}^{(\mu)}$ is the Wiener-measure from (6.3.23)–(6.3.25). On the RHS in (6.3.14), we have the constant random variable on (Ω, Cyl), i.e., as a function on Ω, it is $\mu(B)$.

Realization of the process $X^{(\mu)}$

As seen from the above discussion, there is a close connection between generalized Gaussian fields, and associated Ito-integrals [**IM74**], on the one hand, and the class of positive definite kernels considered here, on the other.

The path-space Ω. Starting with a σ-finite measure space $(M, \mathscr{B}, \mu)$, we may choose $\Omega = \mathbb{R}^{\mathscr{B}_{fin}}$ = the infinite Cartesian product, so that

$$\omega \in \Omega, \quad X_A^{(\mu)}(\omega) = \omega(A),\ \forall A \in \mathscr{B}_{fin}. \tag{6.3.15}$$

Let $\mathscr{C}$:= the cylinder sigma-algebra of subsets of Ω; and $\mathbb{P}$ be the Gaussian probability measure on Ω, defined on $\mathscr{C}$, and indexed by K.

In details, if $F = \{A_i\}_1^n$ is a finite system, $A_i \in \mathscr{B}_{fin}$, set

$$\mathscr{C}_F := \prod_1^n A_i \times \prod_{F^c} \dot{\mathbb{R}}, \tag{6.3.16}$$

(a cylinder subset); and let $\mathscr{C}$ be the sigma-algebra of all subsets of Ω which is generated by the cylinder sets.

To construct $\mathbb{P}$ as a probability measure, and defined on $\mathscr{C}$, we first specify its finite-dimensional joint distributions,

$$\begin{aligned}\mathbb{P}\left(\cdot \mid \mathscr{C}_F\right) := &\text{ the Gaussian on } \mathbb{R}^n \text{ which has 0 mean, and} \\ &\text{covariance matrix } \left(K\left(A_i, A_j\right)\right)_{i,j=1}^n . \end{aligned} \tag{6.3.17}$$

By Kolmogorov's consistency property, we then get a unique probability measure $\mathbb{P}$ on $(\Omega, \mathscr{B})$ which satisfies (6.3.17).

By construction, the expectation $\mathbb{E}$, defined from $\mathbb{P}$, satisfies $\mathbb{E}\left(X_A\right) = 0$,

$$\mathbb{E}\left(X_A^{(\mu)} X_B^{(\mu)}\right) = K\left(A, B\right). \tag{6.3.18}$$

Specifically, each X_A is Gaussian with distribution $N\left(0, K\left(A, A\right)\right)$; and the joint distribution of $\left(X_{A_1}, X_{A_2}, \cdots, X_{A_n}\right)$ is the Gaussian from (6.3.17).

Now, let

$$\begin{aligned}\mathscr{A} := \{A_i\} = &\text{ a countable partition of } X,\ A_i \in \mathscr{B}_{fin} \\ &A_i \cap A_j = \emptyset, i \neq j,\end{aligned} \tag{6.3.19}$$

and let $\mathscr{C}_{\mathscr{A}} :=$ the sigma-subalgebra of $\mathscr{C}$ which is generated by $\{X_{A_i}\}$, $A_i \in \mathscr{A}$. Introducing conditional expectations, we get an inductive system of Ito-integrals, indexed by the set of partitions $\mathscr{A}$ (as above) where we use the usual ordering of partitions $\mathscr{A} \leq \mathscr{A}'$ given by refinement: If $\varphi = \sum_i \alpha_i \chi_{A_i}$,

$$\mathbb{E}\left(\left|\int \varphi \, dX_x\right|^2 \mid \mathscr{C}_{\mathscr{A}}\right) = \sum\nolimits_i \sum\nolimits_j \alpha_i \alpha_j K\left(A_i, A_j\right) \tag{6.3.20}$$

where the "|" stands for conditional expectation.

Passing to the limit, over the set of all partitions, we get a necessary and sufficient condition for when the Ito-integral $\int_X \varphi\left(x\right) dX_x$ is well defined, and is in $L^2\left(\Omega, \mathscr{C}, \mathbb{P}\right)$, i.e., when $\mathbb{E}\left(\left|\int \varphi \, dX_x\right|^2\right) < \infty$.

Remark. In (6.3.17) we define the measure $\mathbb{P}^{(\mu)}$ on Ω, with respect to the cylinder subsets. More specifically, given $A_1, \cdots, A_n \in \mathscr{B}_{fin}$, and $\varepsilon_i > 0$, $y_i \in \mathbb{R}$, let

$$Cyl_{\{A_i\}_1^n} = \left\{\omega \in \Omega \mid \left|\omega\left(A_i\right) - y_i\right| < \varepsilon_i\right\}. \tag{6.3.21}$$

Set

$$J_i = \left\{x \in \mathbb{R} \mid \left|x - y_i\right| < \varepsilon_i\right\} \tag{6.3.22}$$

and we may define $\mathbb{P}^{(\mu)}$ first on cylinder sets, as follows:

$$\mathbb{P}^{(\mu)}\left(Cyl\left(J_i\right)\right) := \int_{J_1}\int_{J_2}\cdots\int_{J_n} Gauss^{\{A_i\}}\left(x_1 x_2 \cdots x_n\right) dx_1 dx_2 \cdots dx_n \tag{6.3.23}$$

where

$$Gauss^{\{A_i\}}\left(x_1 \cdots x_n\right) = \det G^{1/2} e^{-\frac{1}{2} x G^{(A_i)} x} \tag{6.3.24}$$

$$G^{(\mu)} = \mu\left(A_i \cap A_j\right)^{-1} \text{ inverse matrix.} \tag{6.3.25}$$

One checks that generalized Kolmogorov consistency holds for (6.3.22)-(6.3.25), and so the Kolmogorov inductive/projective limit then shows there is then a unique measure $\mathbb{P}^{(\mu)}$ on (Ω, Cyl) such that (6.3.23) holds. This in turn yields (6.3.18).

For example, let $n = 2$, $A_1, A_2 \in \mathscr{B}_{fin}$, and assume $\mu(A_1 \cap A_2) \neq 0$; then

$$\begin{aligned} G &= \begin{pmatrix} \mu(A_1) & \mu(A_1 \cap A_2) \\ \mu(A_1 \cap A_2) & \mu(A_2) \end{pmatrix}^{-1} \\ &= \frac{1}{\mu(A_1)\mu(A_2) - \mu(A_1 \cap A_2)^2} \begin{pmatrix} \mu(A_2) & -\mu(A_1 \cap A_2) \\ -\mu(A_1 \cap A_2) & \mu(A_1) \end{pmatrix} \end{aligned}$$

and the cylinder function (6.3.24) applies to $n = 2$, where

$$\mathbb{E}\left(X_{A_1}^{(\mu)} X_{A_2}^{(\mu)}\right) = \int_{\mathbb{R}^2} x_1 x_2 \det G \, e^{-\frac{1}{2}xGx} dx_1 dx_2 = \mu(A \cap B).$$

We also get a Karhunen–Loüve representation:

THEOREM 6.3.8. *Fix $(M, \mathscr{B}, \mu)$, σ-finite as usual, and let $X^{(\mu)}$ be the Gaussian process such that $\mathbb{E}\left(X_A^{(\mu)} X_B^{(\mu)}\right) = \mu(A \cap B)$. For $\forall \{f_n\}_1^\infty$ ONB in $L^2(\mu)$, $\forall \{Z_n\}_1^\infty$ i.i.d. $N(0,1)$ random variables, we have*

$$X_A^{(\mu)} = \sum_1^\infty \left(\int_A f_n d\mu\right) Z_n, \tag{6.3.26}$$

or, equivalently,

$$X_h^{(\mu)} = \sum_1^\infty \langle h, f_n \rangle_{L^2(\mu)} Z_n. \tag{6.3.27}$$

PROOF. This is an immediate application of the new RKHS $\mathscr{H} = \mathscr{H}(K_\mu)$, i.e., the RKHS of the positive kernel from (6.3.5).

Using the isometry $\mathscr{H} \ni K(\cdot, A) \longmapsto X_A \in L^2(\Omega, \mathbb{P})$, we get the function expansion in $\mathscr{H}$,

$$F(A) = \sum_1^\infty \left\langle F, \int f_n d\mu \right\rangle_{\mathscr{H}(K)} \int_A f_n d\mu$$

where $F(A) = \int (T^* F) d\mu$, and $F_n(A) = \int_M f_n d\mu$ is an ONB in $\mathscr{H}$, and so

$$\mathbb{E}\left(X_{f_n}^{(\mu)} X_{f_m}^{(\mu)}\right) = \langle f_n, f_m \rangle_{L^2(\mu)} = \delta_{n,m}.$$

For more details, see [**JT18b**].

□

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CHAPTER 7

Representations of *Lie groups.* Non-commutative harmonic analysis

We were [initially] entirely in Heisenberg's footsteps. He had the idea that one should take matrices, although he did not know that his dynamical quantities were matrices.... And when one had such a programme of formulating everything in matrix language, it takes some effort to get rid of matrices. Though it seemed quite natural for me to represent perturbation theory in the algebraic way, this was not a particularly new way.

— Max Born (1882–1970)

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of reflection positivity has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

The general theme in the present chapter, non-commutative harmonic analysis, is vast, and we have selected here only three areas which connect directly to the main subjects in the first 6 chapters of the book.

In two of the three Chapter 7 sections (so 7.2-7.3) we discuss some aspects of the theory of reflection positivity. But even the current and active research in the subject is both diverse and extensive. Here we have divided it into three subareas: (i) quantum field theory; (ii) the interplay between reflection positivity and the theory of unitary representations of Lie groups (Section 7.2); and (iii) connections to spectral theory. In our treatment in 7.2-7.3 below we omit (i), and we only include a very rough sketch of (ii); as we shall concentrate on (iii). The latter has more direct connections to the earlier 6 chapters in the book. For the main parts of the subject areas (i)-(ii) we shall merely refer readers to the cited literature.

The subject in Section 7.1 below is the study of fundamental domains, and it is a non-commutative variant of questions from chapters 1-6 dealing with translation tilings.

7.1. Fundamental domains as non-commutative tiling constructions

The material in the present section is based primarily on ideas in [**LPT01**, **DHJP13**, **DJS12**].

While this book emphasizes orthogonality, stressing fractal measures, many of the problems have a history beginning with the case when the measure μ under consideration is a restriction of Lebesgue measure on $\mathbb{R}^d$. The original problem was motivated by von Neumann's desire to use his Spectral Theorem on partial differential operators (PDOs), much like the Fourier methods had been used boundary value problems in ordinary differential equations.

If functions in $L^2(\Omega)$ are translated locally in the d different coordinate directions, we will expect that the issue of selfadjoint extension operators should be related to the matching of phases on the boundary of Ω, and therefore related to the *tiling* of $\mathbb{R}^d$ by translations of Ω; i.e., with translations of Ω which cover $\mathbb{R}^d$, and which do not overlap on sets of positive Lebesgue measure. The global motion by continuous translation in the d coordinate directions will be determined uniquely by the spectral theorem if we can find commuting selfadjoint extensions of the d partial derivative operators

$$i\frac{\partial}{\partial x_j},\quad j=1,\dots,d,$$

defined on the dense domain $\mathcal{D}$ of differentiable functions on Ω which vanish on the boundary. We can take $\mathcal{D}=C_c^\infty(\Omega)$. These d operators are commuting and formally Hermitian, but not selfadjoint. In fact when $d>1$, each of the operators has deficiency indices (∞,∞). So in each of the d coordinate directions, $i\,\partial/\partial x_j|_{\mathcal{D}}$ has an infinite variety of selfadjoint extensions. But experimentation with examples shows that "most" choices of Ω will yield non-commuting selfadjoint extensions. Each operator individually does have selfadjoint extensions, and the question is if they can be chosen to be mutually commuting. By this we mean that the corresponding projection-valued spectral measures commute.

The spectral representation for every selfadjoint extension $H_j\supset i\,\partial/\partial x_j|_{\mathcal{D}}$ has the form $H_j=\int_{\mathbb{R}}\lambda\,E_j(d\lambda)$, $j=1,\dots,d$, where $E_j\colon\mathcal{B}(\mathbb{R})\to\operatorname{Proj}\left(L^2(\Omega)\right)$ denotes the Borel subsets of $\mathbb{R}$. We say that a family of d selfadjoint extensions $H_1,\dots,H_d$ of the respective $i\,\partial/\partial x_j$ operators is commuting if $E_j(A_j)\,E_k(A_k)=E_k(A_k)\,E_j(A_j)$ for all $A_j,A_k\in\mathcal{B}(\mathbb{R})$, and $j\neq k$.

When commuting extensions exist, we form the product measure

$$E=E_1\times\cdots\times E_d$$

on $\mathcal{B}\left(\mathbb{R}^d\right)$, and set

$$U(t)=\int_{\mathbb{R}^d}e^{i\lambda\cdot t}\,E(d\lambda),$$

where $t,\lambda\in\mathbb{R}^d$ and $\lambda\cdot t=\sum_{j=1}^d\lambda_j t_j$. Then clearly

$$U(t)\,U(t')=U(t+t'),\quad t,t'\in\mathbb{R}^d,$$

i.e., U is a unitary representation of $\mathbb{R}^d$ acting on $\mathcal{H}=L^2(\Omega)$.

It might be natural to expect that an open spectral set Ω will have its connected components spectral, or at least have features predicted by the spectrum of the bigger set. This is not so as the following example (due to Steen Pedersen) shows. Details below!

If the set Ω is not assumed connected, the conclusion in Theorem 3.3.5 would be false. If a spectral set Ω is disconnected, then properties of the connected components are not immediately discerned from knowing that there is a spectrum for Ω.

EXAMPLE 7.1.1. An example showing this can be constructed by taking $\Omega = \Omega(p)$ to the following set obtained from a unit square, dividing it into two triangles along the main diagonal, followed by a translation of the triangle under the diagonal.

Details: Start with the following two open triangles making up a fixed unit-square, divided along the main diagonal. Now make a translation of the triangle under the diagonal by a non-zero integer amount p in the x-direction; i.e., by the vector $(p, 0)$, leaving the upper triangle alone .

Further, let $\Omega(p)$ be the union of the resulting two p-separated triangles. Hence the two connected components in $\Omega(p)$ will be two triangles; the second obtained from the first by a mirror image and a translation. Neither of these two disjoint open triangles is spectral; see [**Fug74**]. Nonetheless, as spectrum for $\Omega(p)$ we may take the unit-lattice $\mathbb{Z}^2$. To see this, one may use a simple translation argument in $L^2(\Omega(p))$, coupled with the fact that $e_\lambda(p) = 1$ for all $\lambda \in \mathbb{Z}^2$.

There are two interesting open connected sets in the plane that are known [**Fug74**] not to be spectral. They are the open disk and the triangle. Of those two non-spectral sets, the triangles may serve as building blocks for spectral sets. Not the disks!

By the reasoning from the $d = 1$ example, by analogy, one would expect that the existence of commuting selfadjoint extensions will force Ω to tile $\mathbb{R}^d$ by translations, at least if Ω is also connected. And in any case, one would expect that issues of spectrum and tile for bounded sets Ω in $\mathbb{R}^d$ are related.

Indeed, Fuglede showed that for special configuration of sets Ω, there are self-adjoint extensions, and that they are associated in a natural way with lattices L in $\mathbb{R}^d$. The spectrum of the representation U is a lattice L. By a *lattice* we mean a rank-d discrete additive subgroup of $\mathbb{R}^d$.

Suppose now that d commuting selfadjoint extensions exist for some bounded open domain Ω in $\mathbb{R}^d$; and suppose in addition the multiplicative condition is satisfied. When the spectral theorem of Stone–Naimark–Ambrose–Godement (the SNAG theorem) is applied to a particular choice of d associated commuting unitary one-parameter groups, Fuglede showed that Ω must then be a fundamental domain (also called a tile for translations) for the lattice L^* which is dual to L. If L is a lattice in $\mathbb{R}^d$, the *dual lattice* L^* is

$$L^* = \left\{ \lambda \in \mathbb{R}^d \mid \lambda \cdot s \in 2\pi\mathbb{Z} \text{ for all } s \in L \right\}.$$

But more importantly, Fuglede pointed out that the several-variable variant of the spectral theorem (the SNAG theorem), and some potential theory, shown that if in addition Ω is assumed connected, and has a "regular" boundary, then the existence of commuting selfadjoint extensions implies that $L^2(\Omega)$ has an orthonormal basis of complex exponentials

$$\{\exp(is \cdot x) \mid s \text{ in some discrete subset } S \subset \mathbb{R}^d\}.$$

In a later paper [**Jr82**], Jorgensen suggested that a pair of sets (Ω, S) be called a *spectral pair*, and that S called a spectrum of Ω. With this terminology, we can state Fuglede's conjecture as follows: A measurable subset Ω of $\mathbb{R}^d$ with finite positive Lebesgue measure has a spectrum, i.e., is the first part of a spectral pair, if and only if Ω tiles $\mathbb{R}^d$ with some set of translation vectors in $\mathbb{R}^d$.

Apparently Fuglede's work was all done around 1954, and was in many ways motivated by von Neumann's thinking about unbounded operators. He had apparently felt the need to first understand non-trivial examples that arise from tilings by translations with vectors that can have irregular configurations, and aren't related in any direct way to a lattice. Natural examples for Ω for $d = 2$ that come to mind are the open interior of the triangle or of the disk. But Fuglede proved in [**Fug74**] that these two planar sets do *not* have spectra, i.e., they do not have the basis property for any subset $S \subset \mathbb{R}^2$. Specifically, in either case, there is no $S \subset \mathbb{R}^2$ such that $\{e_s|_\Omega\ ;\ s \in S\}$ is an orthogonal basis for $L^2(\Omega)$. This of course is consistent with the spectrum-tile conjecture.

Very importantly, in his 1974 paper, Fuglede calculated a number of instructive examples that showed the significance of combinatorics and of finite cyclic groups in our understanding of spectrum and tilings; and he made precise what is now referred to as the Fuglede conjecture ([**JP94**, **JP00**, **PW01**, **Ped04**, **Jr82**]).

Fuglede's question about equivalence of the two properties (existence of orthogonal Fourier frequencies for a given measurable subset Ω in $\mathbb{R}^d$) and the existence of a subset which makes Ω tile $\mathbb{R}^d$ by translations, was for $d = 1, 2$, and perhaps 3. But of course the question is intriguing for any value of the dimension d. In recent years, a number of researchers, starting with Tao, have now produced examples in higher dimensions giving negative answers: By increasing the dimension, it is possible to construct geometric obstructions to tiling which do not have spectral theoretic counterparts; and vice versa; see e.g., [**FMM06**, **Tao04**, **LW95**, **LW96b**, **LW96c**, **LW96a**, **IKT03**, **Mat05**, **KL96**].

Decomposition of wavelet representations

Fundamental domains are important in direct integral decompositions for unitary representations where one often use fundamental domains as "parameters" in direct integral decompositions.

It is known (see [**DJS12**]) that there exists a direct integral decomposition for the general wavelet representation, and this in turn solves a question posed by Judith Packer [**LPT01**].

The framework in [**DJS12**] entails representations built from certain finite-to-one endomorphisms r in compact metric spaces X, and we study their dilations to automorphisms in induced solenoids. The wavelet representations are covariant systems formed from the dilated automorphisms. They depend on assigned measures μ on X. It is known that when the data (X, r, μ) are given the associated wavelet representation is typically reducible. By introducing wavelet filters associated to (X, r), one may build random walks in X, path-space measures, harmonic functions, and an associated Martin boundary.

Details. We begin with some preliminaries regarding measures on solenoids. Since our starting point is a given finite-to-one endomorphism r in a compact metric space X, it is then natural to look for a way of corresponding to this a unitary operator U in a Hilbert space $\mathscr{H}$, such that U together with (X, r) satisfy a covariance relation; see Theorem 7.1.3 below. The introduction of suitable measures on the associated solenoid (X_∞, r_∞), built from (X, r), then gets us a representation π of the algebra $L^\infty(X)$ such that U, together with r_∞, form a crossed-product in the sense of C^*-algebras. This is possible since r_∞ is an automorphism. We will refer to a crossed-product system $(\mathscr{H}, U, \pi)$ as a wavelet representation.

Indeed, the traditional wavelet representations fall within this wider framework of $(\mathscr{H}, U, \pi)$ covariant crossed products [**DJ07c**]. Specifically, in the special case when $X = \mathbb{T}$, and the endomorphism r is just the power mapping $r(z) = z^N$ (for a fixed integer $N > 1$), then it can be seen that a covariant crossed products indeed specializes to a unitary representation of a corresponding N-Baumslag-Solitar group; see e.g., [**DJ08a, Dut06**]. Even in the case of these classical Baumslag-Solitar groups, our understanding of the unitary representations and their decompositions is so far only partial.

DEFINITION 7.1.2. Let X be a compact metric space and $r : X \to X$ be a finite-to-one, onto, Borel measurable map. Let μ be a *strongly invariant* Borel probability measure on X, i.e.

$$\int f\,d\mu = \int \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y)\,d\mu(x), \tag{7.1.1}$$

for any bounded Borel function on X.

(1) A function m_0 on X is called a *quadrature mirror filter (QMF)* if

$$\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 = 1, \quad x \in X. \tag{7.1.2}$$

(2) Given a QMF m_0 we define

$$W(x) = \frac{|m_0(x)|^2}{\#r^{-1}(r(x))}, \quad x \in X. \tag{7.1.3}$$

Then the function W satisfies the following equation:

$$\sum_{r(y)=x} W(y) = 1, \quad x \in X. \tag{7.1.4}$$

(3) A function h on X is called R_W*-harmonic* if

$$\sum_{r(y)=x} W(y)h(y) = h(x), \quad x \in X. \tag{7.1.5}$$

In what follows we will assume that:

$$\text{the set of zeroes for } m_0 \text{ has } \mu\text{-measure zero.} \tag{7.1.6}$$

(Note that (7.1.4) can be interpreted as an assignment of transition probabilities: the probability of transition from x to $y \in r^{-1}(x)$ is equal to $W(y)$.)

THEOREM 7.1.3 ([**DJ07c**]). *There exists a Hilbert space $\mathscr{H}$, a unitary operator U on $\mathscr{H}$, a representation π of $L^\infty(X)$ on $\mathscr{H}$ and an element φ of $\mathscr{H}$ such that*

(1) *(Covariance) $U\pi(f)U^{-1} = \pi(f \circ r)$ for all $f \in L^\infty(X)$.*
(2) *(Scaling equation) $U\varphi = \pi(m_0)\varphi$*
(3) *(Orthogonality) $\langle \pi(f)\varphi, \varphi\rangle = \int f\,d\mu$ for all $f \in L^\infty(X)$.*
(4) *(Density) $\{U^{-n}\pi(f)\varphi \mid n \in \mathbb{N}, f \in L^\infty(X)\}$ is dense in $\mathscr{H}$.*

Moreover they are unique up to isomorphism.

DEFINITION 7.1.4. We call the system $(\mathscr{H}, U, \pi, \varphi)$ in Theorem 7.1.3 the *wavelet representation* associated to the function m_0.

We will be interested in the decomposition of the wavelet representation into irreducibles.

DEFINITION 7.1.5. We say that a subset F of X_∞ is a *fundamental domain* if, up to μ_∞-measure zero:

$$\bigcup_{n\in\mathbb{Z}} r_\infty^n(F) = X_\infty \quad \text{and} \quad r_\infty^n(F)\cap r_\infty^m(F)=\emptyset \text{ for } n\neq m.$$

DEFINITION 7.1.6. For $z=(z_0,z_1,\dots)$ in X_∞ define the following representation: consider the Hilbert space

$$\mathscr{H}_z := \left\{(\xi_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |\xi_n|^2 |\tilde{m}_n(z)|^2 < \infty\right\},$$

with inner product

$$\langle \xi,\eta\rangle_{\mathscr{H}_z} := \sum_{n\in\mathbb{Z}} \xi_n\overline{\eta_n}\,|\tilde{m}_n(z)|^2 .$$

Define the unitary operator

$$U_z(\xi_n)_{n\in\mathbb{Z}} = (m_0\circ\theta_0\circ r_\infty^n(z)\xi_{n+1})_{n\in\mathbb{Z}};$$

and the representation of π of $L^\infty(X)$:

$$\pi_z(f)(\xi_n)_{n\in\mathbb{Z}} = (f\circ\theta_0\circ r_\infty^n(z)\xi_n)_{n\in\mathbb{Z}}, \quad f\in L^\infty(X).$$

THEOREM 7.1.7 ([**DJS12**]). *In the hypotheses of Theorem* 7.1.3*, there exist a fundamental domain F. The wavelet representation associated to m_0 has the following direct integral decomposition:*

$$[\mathscr{H},U,\pi] = \int_F^\oplus [\mathscr{H}_z,U_z,\pi_z]\,d\mu_\infty(z),$$

where the component representations $[\mathscr{H}_z,U_z,\pi_z]$ in the decomposition are irreducible for a.e., z in F, relative to μ_∞.

PROOF. We refer the reader to the original paper [**DJS12**] for details. □

Common fundamental domains

Motivated by the study of Weyl-Heisenberg (or Gabor) frames, Deguang Han and Yang Wang proved that two lattices in $\mathbb{R}^n$ having the same finite co-volume have a common measurable fundamental domain (see [**HW01**]). A much more general result was proved in [**DHJP13**]: Consider two lattices in a group of polynomial growth, one acting on the left and the other acting on the right. Assuming that the two given lattices have the same co-volume, then they must have a common measurable fundamental domain.

The main question here is the necessary and sufficient conditions for the existence of a common tiling system. The answer is contained in Theorems 7.1.8 and 7.1.10 below, and the reader is referred to the original paper [**DHJP13**] for the proofs.

THEOREM 7.1.8. *Consider two commuting measure-preserving actions of some countable (possibly finite) discrete groups Γ and Λ on the same measure space $(M,\mathscr{B},m)$. Assume in addition that both actions have fundamental domains of finite positive measures, X for Γ and Y for Λ, and $m(X)\geq m(Y)$. Then the following affirmations are equivalent:*

(1) *The two actions have a common tiling system.*
(2) *For all sets $A \in \mathscr{B}$ which are invariant for both Γ and Λ, the following equality holds*

$$m(A \cap X) = \frac{m(X)}{m(Y)} \cdot m(A \cap Y). \tag{7.1.7}$$

We shall need the notion of *covolume*. It is defined for a fixed pair (G, Γ), where G is a locally compact group, and Γ is a discrete subgroup in G. To be specific, select a left-invariant Haar measure in G, denoted dx, i.e., the identity

$$\int_G f(yx)\, dx = \int_G f(x)\, dx \tag{7.1.8}$$

holds for all measurable functions f on G. Note that dx is unique up to a positive scalar multiple.

Let $M := G/\Gamma$ (called a homogeneous space) with G acting as a transitive transformation group on M. Let

$$\pi : G \longrightarrow M \tag{7.1.9}$$

be the canonical map, $\pi(x) = x\Gamma$, for $x \in G$, where $x\Gamma$ denotes the coset

$$x\Gamma = \{x\gamma \mid \gamma \in \Gamma\}. \tag{7.1.10}$$

One checks that there is then a unique G-invariant measure m on $M = G/\Gamma$, determined by the following duality formula: For $\varphi \in C_c(G)$ (= compact supported, continuous), and set

$$(\tau\varphi)(\pi(x)) = \sum_{\gamma \in \Gamma} \varphi(x\gamma). \tag{7.1.11}$$

The measure m on G/Γ is determined by:

$$\int_M (\tau\varphi)\, F\, dm = \int_G \varphi(x)\, (F \circ \pi)(x)\, dx \tag{7.1.12}$$

for all $\varphi \in C_c(G)$, and all measurable functions F on $M = G/\Gamma$.

More precisely, the two measures dx on G and m on M are determined such that the two operators

$$\begin{aligned} \varphi \longmapsto \tau\varphi : \quad & L^2(G) \longrightarrow L^2(m), \quad \text{and} \\ F \longmapsto F \circ \pi : & L^2(m) \longrightarrow L^2(G) \end{aligned}$$

are each other's adjoints.

DEFINITION 7.1.9. Given (G, Γ) as specified, i.e., G locally compact, and Γ a discrete subgroup in G. We say that Γ has finite co-volume, written

$$cov_G(\Gamma) := m(G/\Gamma) \tag{7.1.13}$$

where m denotes the G-invariant measure on $M = G/\Gamma$.

THEOREM 7.1.10. *Let G be a locally compact group of polynomial growth with Haar measure dx. Suppose Γ and Λ are two uniform lattices in G. Consider the action of Γ on G on the left and the action of Λ on G on the right. If $cov_G(\Gamma) = cov_G(\Lambda)$, then the two actions have a common tiling system.*

7.2. Symmetry for unitary representations of Lie groups

The material below is based primarily on ideas in the paper [**JT18c**] by Jorgensen et al.

The notion "*reflection-positivity*" came up first in a renormalization question in physics [**OS73**]: "How to realize observables in relativistic quantum field theory (RQFT)?" This is part of the bigger picture of quantum field theory (QFT) [**GJ79**]; and it is based on a certain analytic continuation (or reflection) of the *Wightman distributions* (from the Wightman axioms). However this initial approach to RQFT entails major issues both on the physics side (see the cited papers below); and on the mathematics side: the operator valued distributions take values in non-commuting unbounded operators, and the mathematical complications deriving from this fact alone are major. By contrast, in the *Euclidean models* we get commuting algebras of random variables. Of course, then there are other issues about making a translation (continuation) of Euclidean solutions back to the original problem from QFT. In more detail, the analytic continuation from RQFT to the Osterwalder-Schrader (OS) axioms induce Euclidean random fields; and Euclidean covariance. (See, e.g., [**OS73**, **OS75**, **GJ79**, **GJ87**, **Jor02**, **JP13**, **JJ17**, **JL17**].) For the unitary representations of the respective symmetry groups, we therefore change these groups as well: OS-reflection applied to the Poincaré group of relativistic fields yields the Euclidean group as its reflection. The starting point of the OS-approach to QFT is a certain positivity condition called "reflection positivity," see Definition 7.2.2.

Now, when it is carried out in concrete cases, the initial function spaces change; but, more importantly, the inner product which produces the respective Hilbert spaces of quantum states changes as well. What is especially intriguing is that, before reflection we may have a Hilbert space of functions, but after the time-reflection is turned on, then, in the new inner product, the corresponding completion, magically becomes a *Hilbert space of distributions.*

The motivating example here is derived from a certain version of the Segal–Bargmann transform (see Example 7.3.2). For more detail on the background and the applications, we refer to two previous joint papers [**JO98**] and [**JO00**], as well as [**Kle77**, **Kle78**, **KLS82**, **Jor86**, **Jor87**, **Nee94**, **Hal00**, **AJP07**, **JT17b**, **JJ16**, **JP15**, **JJM14**, **JR08**].

Our present purpose is to analyze in detail a number of geometric properties connected with the axioms of reflection positivity, as well as their probabilistic counterparts; especially the role of the Markov property.

In rough outline: It is possible to express Osterwalder-Schrader positivity (OS-p) purely in terms of a triple of projections in a fixed Hilbert space, and a reflection operator. For such three projections, there is a related property, often referred to as the Markov property. It is well known that the latter implies the former; i.e., when the reflection is given, then the Markov property implies OS-p, but not conversely.

In this section, we shall prove two theorems which flesh out much more precise relationships between the two. The word "Markov" traditionally makes reference to a random walk process where the Markov property in turn refers to past and future: Expectation of the future, conditioned by the past (details below). By contrast, our present initial definitions only make reference to three prescribed projection operators. Initially, there is not even mention of an underlying probability space. All this comes later. Now if we are in the context of a random walk process, then such a process may or may not have the Markov property; which is now instead defined relative to notions of past, present, and future, and the associated conditional expectations.

While our discussion of the Markov property is couched here in an axiomatic framework; and is motivated by our particular aims, we stress that Markov properties, Markov processes, and Markov fields form an active and very diverse area. While there are links from those directions to our present results, the connections are not always direct. For the readers benefit we have included the following citations [**Nel58b**, **Nel73a**, **Nel73b**, **Nel75**, **BDS16**, **KA17**, **LR17**] on Markov/random fields.

7.2.1. The geometry of reflections and positivity. Let $\mathscr{H}$ be a given Hilbert space, and let $U, \theta : \mathscr{H} \to \mathscr{H}$ be two unitary operators, such that:

$$\theta^2 = I_{\mathscr{H}},\ \theta^* = \theta, \text{ and} \tag{7.2.1}$$

$$\theta U \theta = U^*. \tag{7.2.2}$$

Note that (7.2.1) states that θ has spectrum equal to the two point set $\{\pm 1\}$. We think of (7.2.2) as a reflection symmetry for the given operator U. In this case, (7.2.2) states that U is unitarily equivalent to its adjoint U^*, and so U and its adjoint U^* have the same spectrum, but, except for trivial cases, U is not selfadjoint.

We further assume that there exists a closed subspace $\mathscr{H}_+ \subset \mathscr{H}$ s.t.

$$U\mathscr{H}_+ \subset \mathscr{H}_+, \text{ and} \tag{7.2.3}$$

$$\langle h_+, \theta h_+ \rangle \geq 0,\ \forall h_+ \in \mathscr{H}_+. \tag{7.2.4}$$

If E_+ is the projection onto $\mathscr{H}_+$, then (7.2.4) is equivalent to

$$E_+ \theta E_+ \geq 0 \tag{7.2.5}$$

with respect to the usual ordering of operators (see Definition 7.2.5).

When a triple of projections $\varepsilon = (E_0, E_\pm)$ is given, we then say that an associated reflection θ satisfies an *Osterwalder-Schrader positivity* (OS-p) condition if (7.2.5) is satisfied. Its spectral theoretic properties will be analyzed in detail in the main body of our paper.

REMARK 7.2.1. In our discussion of (7.2.2)-(7.2.3), we state things in the simple case of just a single unitary operator U, but our conclusions will apply *mutatis mutandis* also to the case when U is instead a strongly continuous unitary representation of a suitable non-commutative Lie group G. In the Lie group case, there is a distinguished one-parameter subgroup of G corresponding to a choice of time-direction. Hence the corresponding restriction will be a unitary one-parameter group, and the forward direction will be the positive half-line $\mathbb{R}_+$, viewed as a sub-semigroup. If G is a Lie group, we shall also be concerned with sub-semigroups. Condition (7.2.3) will refer to invariance of $\mathscr{H}_+$ under this sub-semigroup. In all these cases, we shall simply refer to U with regards to (7.2.2)-(7.2.3), even if it is not a single unitary operator. (For details, see, e.g., [**JKL89**, **JNO16**, **JO98**, **JO00**, **Jor87**, **Jor02**].) In case of a single unitary operator U, of course by iteration we will automatically have a representation of the group $\mathbb{Z}$ of integers, and in this case the sub-semigroup will be understood to be $\mathbb{N}_0$.

Note on terminology. Given a fixed Hilbert space $\mathscr{H}$, we shall make use of the following identification between projections P in $\mathscr{H}$, on the one hand, and the corresponding closed subspaces $P\mathscr{H} \subset \mathscr{H}$ on the other. By projection P, we mean an operator P in $\mathscr{H}$ satisfying $P^2 = P = P^*$. Conversely, if $\mathscr{L} \subset \mathscr{H}$ is a fixed closed subspace, then by general theory, we know that there is then a unique projection, say Q, such that $Q\mathscr{H} = \mathscr{L} = \{h \in \mathscr{H}\ ; Qh = h\}$.

In some of our discussions below, there will be more than one Hilbert space, say $\mathscr{H}$ and $\mathscr{K}$; and they may arise inside calculations. In those cases, it will be convenient to mark the inner products and norms with subscripts, $\langle\cdot,\cdot\rangle_{\mathscr{K}}$, $\|\cdot\|_{\mathscr{K}}$ etc.

In the discussion of reflection positivity, there will typically be three projections E_0, $E_\pm$ at the outset, and the corresponding closed subspaces will be denoted, $\mathscr{H}_0 := E_0\mathscr{H}$, $\mathscr{H}_\pm := E_\pm\mathscr{H}$.

We shall denote such a system of projections $(E_\pm, E_0)$ by ε. If a reflection θ (see (7.2.1)) maps $\mathscr{H}_+$ to $\mathscr{H}_-$ (plus minus parity), we say that $\theta \in \mathscr{R}(\varepsilon)$. If also (7.2.2) and (7.2.3) hold, we shall say that $\theta \in \mathscr{R}(\varepsilon, U)$.

A central theme in our considerations will be the interplay between the following two structures; both referring to a Hilbert space $\mathscr{H}$, and a unitary operator U in $\mathscr{H}$, or a unitary representation of a group G. Given $(\mathscr{H}, U)$, we shall then consider the following two structures:

(1) *Reflections* θ. By this we mean selfadjoint unitary operators, see (7.2.1), such that (7.2.2) holds.
(2) *Triple of projections* $\varepsilon = (E_0, E_\pm)$, or equivalently triple of closed subspaces $(\mathscr{H}_0, \mathscr{H}_\pm)$, the correspondences being $\mathscr{H}_\pm = E_\pm\mathscr{H}$, and $\mathscr{H}_0 = E_0\mathscr{H}$.

The motivation for both (1) and (2) is stochastic processes, with $\mathscr{H}_+$ referring to "future", $\mathscr{H}_-$ to "past", and $\mathscr{H}_0$ to "present".

Given $\varepsilon = (E_0, E_\pm)$, we shall set

(7.2.6)
$$Ref\,(\varepsilon) = \Big\{\theta\ ; \theta \text{ satisfies (7.2.1), (7.2.2) and (7.2.3), and } E_-\theta E_+ = \theta E_+\Big\}.$$

Given θ, a reflection, we shall set

$$Subsp\,(\theta) = \{\varepsilon = (E_0, E_\pm) \text{ such that } E_-\theta E_+ = \theta E_+\}. \tag{7.2.7}$$

One checks immediately that:

$$E_-\theta E_+ = \theta E_+ \Longleftrightarrow \theta\mathscr{H}_+ \subseteq \mathscr{H}_-. \tag{7.2.8}$$

Hence in addition to (7.2.8) we must pay attention to the interplay between θ and E_0, or equivalently, between θ and the closed subspace $\mathscr{H}_0 = E_0\mathscr{H}$.

Two conditions for (θ, E_0) are especially relevant:

(1) $\theta E_0 = E_0$. If $\theta = 2P - I_{\mathscr{H}}$, this is equivalent to $PE_0 = E_0$, or stated differently, $E_0 \leq P$ where "$\leq$" refers to ordering of projections, see Definition 7.2.3 below.
(2) weaker: $\theta E_0 = E_0\theta$.

Definitions and lemmas

In our study of reflections, and reflection positivity, we shall need a number of fundamental concepts from the theory of operators in Hilbert space. While they are in the literature, they are not collected in a single reference. For readers not in operator theory, we include below those basic facts in the form they will be needed inside the paper. A new feature is the notion of *signed quadratic forms* and *subspaces which are positive* with respect to such a given signed quadratic form; see Lemma 7.2.8.

DEFINITION 7.2.2. When U, θ, and E_+ satisfy these conditions, i.e., (7.2.1)-(7.2.5), we then say that *Osterwalder-Schrader reflection positivity* holds, abbreviated OS-p.

Below we discuss the standard ordering of projections. What will be important is that this ordering may be stated in terms of anyone of six equivalent properties. Each one will be relevant for the applications to follow; to geometry, to spectral theory, and to analysis of conditional expectations.

DEFINITION 7.2.3 (Order on projections).

(1) A *projection* in a Hilbert space $\mathscr{H}$ is an operator P satisfying $P = P^2 = P^*$.
(2) If E and P are two projections, we say that $E \leq P$ iff (Def.) one of the following equivalent conditions holds:
   (a) $E\mathscr{H} \subseteq P\mathscr{H}$;
   (b) $\|Eh\| \leq \|Ph\|$, $\forall h \in \mathscr{H}$;
   (c) $\langle h, Eh\rangle \leq \langle h, Ph\rangle$, $\forall h \in \mathscr{H}$;
   (d) $PE = E$;
   (e) $EP = E$;
   (f) for vectors $h \in \mathscr{H}$, the following implication holds: $Eh = h \Longrightarrow Ph = h$.

PROOF. This is standard operator theory, and can be found in books. See e.g. [**JT17b**]. □

We shall need this ordering in an analysis of system (7.2.1)-(7.2.5). From the conditions $\theta^* = \theta$, $\theta^2 = I_{\mathscr{H}}$ (reflection) we conclude that $\theta = 2P - I_{\mathscr{H}}$ where P is the projection onto $\{h \in \mathscr{H} \mid \theta h = h\}$.

LEMMA 7.2.4. *Let θ be a reflection, and let P be the projection such that $\theta = 2P - I_{\mathscr{H}}$, and let E_0 be a projection; then TFAE:*

(1) $\theta E_0 = E_0$*;*
(2) $E_0 \leq P$*, i.e.,* $E_0 h = h \Longrightarrow \theta h = h$*.*

PROOF. We have the following equivalences:

$$\theta E_0 = E_0 \Longleftrightarrow (2P - I_{\mathscr{H}})\, E_0 = E_0 \Longleftrightarrow PE_0 = E_0,$$

and the result now follows from the equivalent statements in Definition 7.2.3. □

DEFINITION 7.2.5. Fix a Hilbert space $\mathscr{H}$, and let A and B be two selfadjoint operators in $\mathscr{H}$. We say that $A \leq B$ iff (Def.) $\langle h, Ah\rangle \leq \langle h, Bh\rangle$, for $\forall h \in \mathscr{H}$.

Note that in case A and B are projections, this order relation agrees with that in Definition 7.2.3. Also $A \geq 0$, i.e., $\langle h, Ah\rangle \geq 0$, $\forall h \in \mathscr{H}$, states that the spectrum of A is a closed subset of $[0, \infty)$.

DEFINITION 7.2.6. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{L}_\pm$ be two subspaces. Equip $\mathscr{L}_+ \times \mathscr{L}_-$ with the following *signed quadratic form*,

$$\langle x, y\rangle_{sig} := \langle k_+, l_+\rangle_{\mathscr{H}} - \langle k_-, l_-\rangle_{\mathscr{H}}\,, \tag{7.2.9}$$

for all $x = (k_+, k_-)$, $y = (l_+, l_-)$ in $\mathscr{L}_+ \times \mathscr{L}_-$.

A subspace $\mathscr{P} \subset \mathscr{L}_+ \times \mathscr{L}_-$ is said to be *positive* iff (Def.) for all $x = (k_+, k_-) \in \mathscr{P}$, we have

$$\langle x, x\rangle_{sig} = \|k_+\|^2_{\mathscr{H}} - \|k_-\|^2_{\mathscr{H}} \geq 0. \tag{7.2.10}$$

REMARK 7.2.7. In the subsequent discussion we shall make use of a number of key ideas from the theory of operators in indefinite inner product spaces, often referred to as Krein spaces. In the specialized literature, however, Krein-spaces constitute a special family of indefinite inner product spaces. The particular properties needed in the present paper will be made clear in the context when needed. We refer the reader to the papers [**AD98**, **CDLdS89**, **Die61**, **DR90**, **GKn62**, **Jor79**, **KnvS66**, **Phi61**] for additional details.

LEMMA 7.2.8. *Let $\mathscr{H}$, $\mathscr{L}_\pm$, and $\langle\cdot,\cdot\rangle_{sig}$ be as in Definition* 7.2.6. *Then a subspace $\mathscr{P} \subset \mathscr{L}_+ \times \mathscr{L}_-$ is positive if and only if there is a contractive linear operator $\mathscr{L}_+ \xrightarrow{C} \mathscr{L}_-$ (w.r.t. the original norm from $\mathscr{H}$) such that $\mathscr{P}$ is the graph of C, and so $\mathscr{P} = \{(k_+, Ck_+)\ ;\ k_+ \in \mathscr{L}_+\}$,*

$$\langle x, x\rangle_{sig} = \|k_+\|^2_{\mathscr{H}} - \|Ck_+\|^2_{\mathscr{H}}. \tag{7.2.11}$$

PROOF. It is clear that the graph of a contraction is a positive subspace in $\mathscr{L}_+ \times \mathscr{L}_-$.

Conversely, suppose $\mathscr{P}$ is a given positive subspace; then

$$\|k_+\|^2_{\mathscr{H}} - \|k_-\|^2_{\mathscr{H}} \geq 0,\ \forall\,(k_+, k_-) \in \mathscr{P}. \tag{7.2.12}$$

Using (7.2.12), we see that if (k_+, k_-) and $\left(k_+, k'_-\right)$ are both in $\mathscr{P}$, then $k_- = k'_-$; and so $k_+ \xmapsto{C} Ck_+ = k_-$ defines a unique contractive operator $\mathscr{L}_+ \xrightarrow{C} \mathscr{L}_-$. As a result, we get that $\mathscr{P}$ is then the graph of this contraction C. □

We shall need the following additional details regarding the bijective correspondence discussed in Lemma 7.2.8. They are included below:

DEFINITION 7.2.9. Let $\mathscr{L}_\pm$ be a pair of Hilbert spaces. If $\mathscr{M} \subset \mathscr{L}_+$ is a linear subspace, and C is a linear operator with domain $dom\,(C) = \mathscr{M}$; then we say that C is contractive iff (Def.)

$$\|Cm\|_{\mathscr{L}_-} \leq \|m\|_{\mathscr{L}_+},\ \forall m \in \mathscr{M}. \tag{7.2.13}$$

LEMMA 7.2.10. *Let $\mathscr{H}$, $\mathscr{L}_\pm$, and $\langle\cdot,\cdot\rangle_{sig}$ be as above. Then there is a bijective correspondence between*

(1) *$\mathscr{P} \subseteq \mathscr{L}_+ \times \mathscr{L}_-$ s.t. $\langle x, x\rangle_{sig} \geq 0$, $\forall x \in \mathscr{P}$; and*
(2) *$C : \mathscr{L}_+ \to \mathscr{L}_-$ contractive with $dom\,(C) \subseteq \mathscr{L}_+$.*

The bijective correspondence (1)↔(2) is given by

$$\mathscr{P} = G\,(C) \tag{7.2.14}$$

where the graph G is defined as follows:

$$G\,(C) := \textit{graph of } C = \left\{\begin{pmatrix} k_+ \\ Ck_+ \end{pmatrix}\ ;\ k_+ \in dom\,(C)\right\},\ \textit{and}$$

$$\left\langle \begin{pmatrix} k_+ \\ Ck_+ \end{pmatrix}, \begin{pmatrix} k'_+ \\ Ck'_+ \end{pmatrix} \right\rangle_{sig} = \left\langle k_+, k'_+\right\rangle_{\mathscr{L}_+} - \left\langle Ck_+, Ck'_+\right\rangle_{\mathscr{L}_-}. \tag{7.2.15}$$

We say that $\mathscr{P}$ is <u>*Krein-positive*</u>.

PROOF. (sketch) The direction (2)→(1) is clear, and (1)→(2) follows from the following implication: Given $\mathscr{P}$ as in (1), i.e., Krein-positive, then the implication below holds:

$$\begin{pmatrix} 0 \\ k_- \end{pmatrix} \in \mathscr{P} \Longrightarrow k_- = 0. \tag{7.2.16}$$

Indeed, if $\left(\begin{smallmatrix} 0 \\ k_- \end{smallmatrix}\right) \in \mathscr{P}$, then

$$\left\langle \begin{pmatrix} 0 \\ k_- \end{pmatrix}, \begin{pmatrix} 0 \\ k_- \end{pmatrix} \right\rangle_{sig} = - \|k_-\|^2_{\mathscr{L}_-} \geq 0$$

so $k_- = 0$.

Note that (7.2.16) shows that if $\left(\begin{smallmatrix} k_+ \\ k_- \end{smallmatrix}\right) \in \mathscr{P}$, then $Ck_+ := k_-$ is a well defined contraction. Indeed, suppose $\left(\begin{smallmatrix} k_+ \\ k_- \end{smallmatrix}\right), \left(\begin{smallmatrix} k_+ \\ k'_- \end{smallmatrix}\right) \in \mathscr{P}$, then $\left(\begin{smallmatrix} 0 \\ k_- - k'_- \end{smallmatrix}\right) \in \mathscr{P}$; and so $k_- - k'_- = 0$, and $k_- = k'_-$; i.e., the desired conclusion.

Existence of C follows: When $\mathscr{P}$ is given as in (1). We saw that when $\mathscr{P}$ is specified as in (1), and C is the corresponding contraction $\mathscr{L}_+ \to \mathscr{L}_-$ with $dom\,(C) \subseteq \mathscr{L}_+$, then $\mathscr{P} = G\,(C)$, so if C and C' both satisfy $G\,(C) = G\,(C') = \mathscr{P}$, then their graphs agree as subspaces of $\mathscr{L}_+ \times \mathscr{L}_-$. The uniqueness follows since linear operators $\mathscr{L}_+ \to \mathscr{L}_-$ are uniquely determined from their respective graphs as subspaces of $\mathscr{L}_+ \times \mathscr{L}_-$. □

COROLLARY 7.2.11. *Let $\mathscr{P}$ and $\mathscr{P}'$ be two Krein-positive subspaces of $\mathscr{L}_+ \times \mathscr{L}_-$, and let C, C' be the corresponding contractions, i.e., $\mathscr{P} = G\,(C)$ and $\mathscr{P}' = G\,(C')$, then TFAE:*

(1) *containment of subspaces*

$$\mathscr{P} \subseteq \mathscr{P}'; \text{ and} \tag{7.2.17}$$

(2) *$dom\,(C) \subset dom\,(C')$ with $Cm = C'm$, $\forall m \in dom\,(C)$.*

PROOF. Immediate from Lemma 7.2.10. Also see [**AD98**] for Krein spaces and extensions of contractions. □

COROLLARY 7.2.12. *Fix $\mathscr{L}_\pm$, a pair of Hilbert spaces as above, and consider Krein-positive subspaces $\mathscr{P}$ of $\mathscr{L}_+ \times \mathscr{L}_-$, equipped with the partial order (7.2.17) above. Then every Krein-positive subspace $\mathscr{P}$ is contained in a maximal Krein-positive subspace $\mathscr{P}_M$; i.e., $\mathscr{P} \subseteq \mathscr{P}_M$, and the implication below holds*

$$\mathscr{P}_M \subseteq Q\,(\textit{a Krein-positive subspace}) \Longrightarrow \mathscr{P}_M = Q.$$

(1) *Note that given $\mathscr{P}$, the choice of $\mathscr{P}_M$ is not unique.*
(2) *Note that by $\mathscr{P} = G\,(C)$ from Lemma 7.2.8, the maximal choices $\mathscr{P}_M$ correspond to maximal contractive extensions via: $\mathscr{P} = G\,(C)$ and $\mathscr{P}_M = G\,(C_M)$.*

PROOF. Zorn's lemma. □

Reflections with given spaces $\mathscr{H}_+$ and $\mathscr{H}_-$

The material in the previous subsection will serve to give a characterization of families of reflections; they will be computed from positive subspaces relative to certain signed quadratic forms; see especially Corollary 7.2.16. Signed quadratic forms in an infinite dimensional setting were first studied systematically by M. G. Krein et al [**GKn62**,**KnvS66**], and R. S. Phillips et al [**Phi61**,**Die61**,**CDLdS89**,**DR90**][1].

LEMMA 7.2.13. *Let $\mathscr{H}$, $\mathscr{H}_+$, $\mathscr{H}_0$, and θ be as in Lemma 7.2.4. Let P be the projection onto $\{h \in \mathscr{H}\ ; \theta h = h\}$. Then*

$$\mathscr{H} = P\mathscr{H} \oplus (1-P)\,\mathscr{H}. \tag{7.2.18}$$

The decomposition is orthogonal and therefore unique,

$$h = u + v, \quad Pu = u, \quad Pv = 0; \tag{7.2.19}$$

i.e., the ± 1 eigenspaces for θ.

Fix a closed subspace $\mathscr{H}_+$. The OS-positivity $\langle h_+, \theta h_+\rangle \geq 0$, $\forall h_+ \in \mathscr{H}_+$, holds if and only if $\mathscr{H}_+$ is contained in the graph of a contractive operator

$$C : P\mathscr{H} \longrightarrow P^{\perp}\mathscr{H}, \tag{7.2.20}$$

i.e., $\mathscr{H}_+ \subseteq \{u + Cu\ ; u \in P\mathscr{H}\}$.

PROOF. Decompose vectors $h_+ \in \mathscr{H}_+$ as in (7.2.18)-(7.2.19), and assume OS-positivity, then

$$\langle h_+, \theta h_+\rangle = \|u\|^2 - \|v\|^2 \geq 0; \quad h_+ = u \oplus v \text{ as in } (7.2.19)\,. \tag{7.2.21}$$

But then the assignment $C : u \longmapsto v$ will define a contractive operator C as stated in the lemma. Indeed, suppose $h_+ = u \oplus v$ is as in (7.2.21). Since $\|u\|^2 - \|v\|^2 \geq 0$; if $u = 0$, it follows that $v = 0$; and so $Cu := v$ is well defined as a contractive operator (see Lemma 7.2.8).

When a contraction $C : P\mathscr{H} \to P^{\perp}\mathscr{H}$ is given, then the corresponding closed subspace $\mathscr{H}_+$ is $\mathscr{H}_+ = \{u + Cu\ ; u \in P\mathscr{H}\}$; and the reflection $\theta = \theta_C$ is determined by $\theta\,(u + Cu) := u - Cu$, and $\langle h_+, \theta h_+\rangle = \|u\|^2 - \|Cu\|^2 \geq 0$ follows.

Since the converse implication is clear, the lemma is proved. ☐

COROLLARY 7.2.14. *Given $\mathscr{H}$, $\mathscr{H}_+$, and $\mathscr{H}_0$, as stated in Lemma 7.2.13. Then there is a bijection between the admissible reflections θ, on the one hand, and partially defined contractions defined as in (7.2.20), on the other $C : \mathscr{H}_+(\theta) \longrightarrow \mathscr{H}_-(\theta)$ where*

$$\mathscr{H}_+(\theta) = \{h \in \mathscr{H}\ ; \theta h = h\}\,,$$
$$\mathscr{H}_-(\theta) = \{k \in \mathscr{H}\ ; \theta k = -k\}\,.$$

COROLLARY 7.2.15. *Let θ be a reflection, and let $P = proj\,\{x \in \mathscr{H}\ ; \theta x = x\}$ so that $\theta = 2P - I_{\mathscr{H}}$. Let C be the corresponding contraction.*

Given a projection E_0 such that $E_0 \leq P$, then TFAE:

(1) $E_0 \leq E_+$; *and*
(2) $E_0 \leq \ker(C)$.

PROOF. We shall identify closed subspaces in $\mathscr{H}$ with the corresponding projections; see Definition 7.2.3. By Corollary 7.2.14, $\theta = \theta_C$ has the form

$$\theta\,(u + Cu) = u - Cu,\ u \in P\mathscr{H},$$

where $C : P\mathscr{H} \to P^{\perp}\mathscr{H}$, is a uniquely determined contraction.

[1] We thank Professor D. Alpay for calling our attention to [**Die61**,**CDLdS89**,**DR90**].

Let $x_0 \in E_0$; then $x_0 \in \mathscr{H}_+$ if and only if $\exists\,(!)\, u \in P\mathscr{H}$ such that $x_0 = u + Cu$. So

$$0 = \underbrace{(u - x_0)}_{\in P\mathscr{H}} + \underbrace{Cu}_{\in P^\perp\mathscr{H}},$$

and both terms are zero; i.e., $u = x_0$, and $Cu = Cx_0 = 0$. The equivalence (1) $\Longleftrightarrow$ (2) now follows. □

COROLLARY 7.2.16. *Let θ be a reflection in a Hilbert space $\mathscr{H}$, and let $P :=$ proj $\{x \in \mathscr{H} \,;\, \theta x = x\}$. Let $C : P\mathscr{H} \longrightarrow P^\perp\mathscr{H}$ be the corresponding contraction. Assume the subspaces $\mathscr{H}_\pm$ satisfy $\mathscr{H}_+ = \{x + Cx \,;\, x \in P\mathscr{H}\}$, and $\mathscr{H}_- = \theta(\mathscr{H}_+) = \{x - Cx \,;\, x \in P\mathscr{H}\}$. We now have:*

$$\mathscr{H}_+ \cap \mathscr{H}_- = \ker(C) = \mathscr{H}_+ \cap P \tag{7.2.22}$$

where we identify subspaces with the corresponding projections.

PROOF. The implication "$\supset$" is immediate from Corollary 7.2.15. Now, let $h \in \mathscr{H}_+ \cap \mathscr{H}_-$. Hence, there are vectors $x, y \in P\mathscr{H}$ such that $h = x + Cx = y - Cy$. Hence,

$$\underbrace{y - x}_{\in P\mathscr{H}} = \underbrace{Cx + Cy}_{\in P^\perp\mathscr{H}}; \tag{7.2.23}$$

so both sides of (7.2.23) must be zero. We get $y = x$, and $Cx = 0$; so $h = x \in \ker(C)$ which is the desired conclusion (7.2.22). □

REMARK 7.2.17. In Corollary 7.2.16, we assumed $\mathscr{H}_+ = \{x + Cx \,;\, x \in P\mathscr{H}\}$; but this is not necessarily satisfied in the general formulation (see (7.2.3)-(7.2.4)).

For example, let $\mathscr{H} = \mathbb{C}^3$ with the standard orthonormal basis $\{e_j\}_{j=1}^3$. Set

$$\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ and } \quad \mathscr{H}_+ = span\left\{e_1 + \frac{1}{2}e_3\right\}.$$

So $\mathscr{H}_+$ is 1-dimensional. The contraction C is given by

$$C : span\{e_1\} \longrightarrow span\{e_3\}$$
$$Ce_1 = \frac{1}{2}e_3;$$

yields $\mathscr{H}_+ = span\{e_1 + Ce_1\}$. Then we have $\theta = 2P - I$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \quad E_+\theta E_+ \geq 0, \text{ where}$$

E_+ denotes the projection onto $\mathscr{H}_+$. It is clear that

$$\mathscr{H}_+ \subsetneq \{x + Cx \,;\, x \in P\mathscr{H}\}, \text{ proper containment,}$$

since $\dim P = 2$.

Now, extend the contraction to $C : P\mathscr{H} \longrightarrow P^\perp\mathscr{H}$ via

$$Ce_2 = 0;$$

then $\ker(C) = span\{e_2\}$. Thus, we get $\mathscr{H}_\pm = span\left\{e_1 \pm \frac{1}{2}e_3\right\}$, but

$$0 = \mathscr{H}_+ \cap \mathscr{H}_- = \mathscr{H}_+ \cap P \neq \ker(C) = span\{e_2\}.$$

REMARK 7.2.18. In the general configuration the two projections $E_\pm$ from Corollary 7.2.16 can be more complicated. If it is only assumed that the system $(E_\pm, \theta)$ satisfies the OS-p condition in (7.2.5), $\mathscr{H}_\pm := E_\pm \mathscr{H}$, then the best that can be said about $\mathscr{H}_+ \cap \mathscr{H}_-$ is the following:

Let $Q := E_+ \wedge E_- =$ the projection onto $\mathscr{H}_+ \cap \mathscr{H}_-$; then the following limit holds (in the strong operator topology):

$$Q = \lim_{n\to\infty} (E_+ E_-)^n . \tag{7.2.24}$$

This conclusion follows from a general fact in operator theory, see e.g. [**Aro50**, sect.12], and also [**JT17b**]. Moreover, the limit in (7.2.24) is known to be monotone (decreasing.)

7.2.2. New Hilbert space from reflection positivity (renormalization). Given a Hilbert space $\mathscr{H}$ and three closed subspaces (equivalently, systems of projections, ε). In this very general setting, it is possible to give answers to the following questions: What are the conditions on a given system ε which admits reflections θ? Suppose reflections exist, then fix ε: What then is the variety of all compatible reflections θ? Characterize the maximal reflections.

Given ε, and an admissible reflection (ε, θ), what are the unitary operators U in $\mathscr{H}$ which define reflection symmetries with respect to (ε, θ)? Given (ε, θ), what is the relationship between operator theory in $\mathscr{H}_+$, and that of the induced Hilbert space $\mathscr{K}$? Explore dichotomies at the two levels.

Let $\mathscr{H}$, $\mathscr{H}_+$, θ, and U be as above. In particular, we assume that $E_+\theta E_+ \geq 0$. Set

$$\mathscr{N} = \ker(E_+\theta E_+) = \{h_+ \in \mathscr{H}_+ \,;\, \langle h_+, \theta h_+\rangle = 0\}, \text{ and} \tag{7.2.25}$$

$$\mathscr{K} = (\mathscr{H}_+/\mathscr{N})^{\sim}, \tag{7.2.26}$$

where "$\sim$" in (7.2.26) means Hilbert completion with respect to the sesquilinear form: $\mathscr{H}_+ \times \mathscr{H}_+ \to \mathbb{C}$, given by

$$\langle h_+, h_+\rangle_{\mathscr{K}} := \langle h_+, \theta h_+\rangle, \tag{7.2.27}$$

a renormalized inner product; see (7.2.4)-(7.2.5).

Set $q(h_+) = \text{class}(h_+) = h_+ + \mathscr{N}$, consider q as a contractive operator,

(7.2.28)

$$\begin{tikzcd} \mathscr{H}_+ \arrow[r] \arrow[rr, bend left, "q"] & \mathscr{H}_+/\mathscr{N} \arrow[r] & (\mathscr{H}_+/\mathscr{N})^{\sim} \end{tikzcd} = \text{Hilbert completion} = \mathscr{K}.$$

REMARK. Constructing physical Hilbert spaces entail completions, often a completion of a suitable space of functions. What can happen is that the completion may fail to be a Hilbert space of *functions*, but rather a suitable Hilbert space of *distributions*. Recall that a completion, say $\mathscr{H}$, is defined axiomatically, and the "real" secret is revealed only when the elements in $\mathscr{H}$ are identified; see Example 7.3.2 below.

Factorizations of $E_+\theta E_+$

Given the basic framework of OS reflection positivity, the operator $E_+\theta E_+$ plays a crucial role since OS positivity is defined directly from this operator. We show that the operator q from (7.2.28) offers a canonical factorization of $E_+\theta E_+ = q^*q$. But we further show that this factorization is universal; see Corollary 7.2.22.

THEOREM 7.2.19. *Let $\mathscr{H}$, θ, E_+ be as above, $\mathscr{H}_+ := E_+\mathscr{H}$. Then TFAE:*

(1) *$E_+\theta E_+ \geq 0$, OS-positivity; and*
(2) *there is a Hilbert space $\mathscr{L}$, and a bounded operator $B : \mathscr{H}_+ \to \mathscr{L}$ such that*

$$E_+\theta E_+ = B^*B; \tag{7.2.29}$$

see Figure 7.2.1.

REMARK 7.2.20. We show below that $\mathscr{H}_+ \xrightarrow{q} \mathscr{K}$ is a *universal* solution to the factorization problem (7.2.29) (see Corollary 7.2.22).

PROOF OF THEOREM 7.2.19. The implication (1)$\Longrightarrow$(2) is contained in Lemma 7.2.21 below. Indeed, if (1) holds, then we may take $\mathscr{L} = \mathscr{K}$, and $B = q : \mathscr{H}_+ \to \mathscr{K}$; see (7.2.28).

Conversely; suppose (2) holds (see Figure 7.2.1), then it is immediate that $E_+\theta E_+ = B^*B \geq 0$, by general theory; see Definition 7.2.5 above. $\square$

$$\begin{tikzcd}
\mathscr{H}_+ \arrow[r, bend left, "B"] \arrow[d, bend right, "E_+\theta E_+"'] & \mathscr{L} \arrow[dl, bend left, "B^*"] \\
\mathscr{H}_+ &
\end{tikzcd}$$

FIGURE 7.2.1. A factorization of $E_+\theta E_+$.

LEMMA 7.2.21. *Let $\mathscr{H}$, θ, E_+ be as above. We assume further that $E_+\theta E_+ \geq 0$, i.e., OS-positivity holds. Set $\mathscr{H}_+ = E_+\mathscr{H}$. Let $\mathscr{K}$ be the induced Hilbert space*

$$\mathscr{K} = \left(\mathscr{H}_+ / \{h_+ \,;\, \langle h_+, \theta h_+\rangle = 0\}\right)^{\sim} \tag{7.2.30}$$

*as in (*7.2.28*), and let $q : \mathscr{H}_+ \to \mathscr{K}$ be the canonical contraction. Then the adjoint operator $q^* : \mathscr{K} \to \mathscr{H}_+$ is given by*

$$q^*\left(q\left(h_+\right)\right) = E_+\theta h_+, \ \forall h_+ \in \mathscr{H}_+. \tag{7.2.31}$$

*In particular, the formula (*7.2.31*) defines q^* unambiguously.*

PROOF. (i) We first show that the formula (7.2.31) defines an operator: We must show that if

$$\langle h_+, \theta h_+\rangle = 0, \tag{7.2.32}$$

then $E_+\theta h_+ = 0$. But by Schwarz, for all $l_+ \in \mathscr{H}_+$, we have

$$\left|\langle l_+, \theta h_+\rangle\right|^2 \leq \langle l_+, \theta l_+\rangle \langle h_+, \theta h_+\rangle \underset{\text{by (7.2.32)}}{=} 0$$

and so $E_+\theta h_+ = 0$ as required in (7.2.31).

(ii) Since q^* is contractive, it is determined uniquely by its values on a dense subspace of vectors in $\mathscr{K}$; in this case $\{q\left(h_+\right) \,;\, h_+ \in \mathscr{H}_+\}$.

$$\begin{tikzcd} \mathscr{H}_+ \arrow[rr, "B=b\,q"] \arrow[dr, "q"'] & & \mathscr{L} \\ & \mathscr{K} \arrow[ur, "b"'] & \end{tikzcd}$$

FIGURE 7.2.2. Universality of q.

(iii) It remains to verify that

(7.2.33)
$$\langle q^*(q(h_+)), l_+\rangle_{\mathscr{H}} = \langle E_+\theta h_+, l_+\rangle_{\mathscr{H}} = \langle h_+, \theta l_+\rangle_{\mathscr{H}} \; (= \langle q(h_+), q(l_+)\rangle_{\mathscr{K}}),$$

$\forall h_+, l_+ \in \mathscr{H}_+$. Details:

$$\begin{aligned} \mathrm{LHS}_{(7.2.33)} &= \langle E_+\theta h_+, l_+\rangle = \langle \theta h_+, E_+ l_+\rangle \\ &= \langle \theta h_+, l_+\rangle = \langle h_+, \theta l_+\rangle = \mathrm{RHS}_{(7.2.33)} \end{aligned}$$

where we used the assumptions (7.2.1) and (7.2.5). In the computation, we omitted the subscript $\mathscr{H}$ in the inner products. □

COROLLARY 7.2.22. *The solution $q : \mathscr{H}_+ \to \mathscr{K}$ to the factorization problem $E_+\theta E_+ = q^*q$ (see (7.2.29)), in the OS-p case, is universal in the sense that if $\mathscr{H}_+ \xrightarrow{B} \mathscr{L}$ is any solution to (7.2.29) in Theorem 7.2.19, then there is a unique isomorphism $\mathscr{K} \xrightarrow{b} \mathscr{L}$ such that $b\,q = B$, see Figure 7.2.2; and $b^*b = I_{\mathscr{K}}$, so b is isometric.*

PROOF. Let $\mathscr{H}_+ \xrightarrow{B} \mathscr{L}$ be a solution to (7.2.29) in Theorem 7.2.19; we then define the isomorphism b (so as to complete the diagram in Figure 7.2.2) as follow:

For $h_+ \in \mathscr{H}_+$, set

$$b(q(h_+)) := B(h_+). \tag{7.2.34}$$

Now this defines an operator $b : \mathscr{K} \to \mathscr{L}$, since if $q(h_+) = 0$, then $0 = q^*q(h_+) = E_+\theta E_+ = B^*B(h_+)$, so $0 = \langle h_+, B^*Bh_+\rangle = \|Bh_+\|^2$, and so $Bh_+ = 0$ as required.

Now it is immediate from (7.2.34), that this operator $b : \mathscr{K} \to \mathscr{L}$ has the desired properties, in particular that the universality holds; see Figure 7.2.2. □

LEMMA 7.2.23. *Let $\mathscr{H}$ be a Hilbert space, and θ a reflection in $\mathscr{H}$ (see (7.2.1)). Let $P := proj\,\{x \in \mathscr{H}\,;\, \theta x = x\}$, so $\theta = 2P - I_{\mathscr{H}}$. Let $\mathscr{K}$ be the new Hilbert space in (7.2.28). Let $C : P\mathscr{H} \longrightarrow P^{\perp}\mathscr{H}$ be the contraction, such that*

$$\mathscr{H}_+ = \{x + Cx\,;\, x \in P\mathscr{H}\}, \tag{7.2.35}$$

and $\theta(x + Cx) = x - Cx$; then for $h_+ = x + Cx$, we have

$$\langle h_+, \theta h_+\rangle_{\mathscr{H}} = \|h_+\|^2_{\mathscr{K}} = \left\|(I_{\mathscr{H}} - C^*C)^{\frac{1}{2}}\,x\right\|^2_{\mathscr{H}}. \tag{7.2.36}$$

PROOF. By $\mathscr{K}$ we refer here to the completion (7.2.28); see also Figure 7.2.3. For the LHS in (7.2.36), we have

$$\begin{aligned}\langle h_+, \theta h_+ \rangle &= \langle x + Cx, x - Cx \rangle \\ &= \|x\|^2 - \|Cx\|^2 \\ &= \|x\|^2 - \langle x, C^*Cx \rangle \\ &= \langle x, (I - C^*C)\, x \rangle \\ &= \left\| (I - C^*C)^{\frac{1}{2}}\, x \right\|^2 = \mathrm{RHS}_{(7.2.36)},\end{aligned}$$

where we have dropped the subscript $\mathscr{H}$ in the computation. □

LEMMA 7.2.24. *Let the setting be as above, see (*7.2.1*)-(*7.2.3*). Then* $\widetilde{U} : \mathscr{K} \to \mathscr{K}$*, given by*

$$\widetilde{U}\,(\mathit{class}\, h_+) = \mathit{class}\,(Uh_+)\,,\ h_+ \in \mathscr{H}_+ \tag{7.2.37}$$

where class h_+ *refers to the quotient in (*7.2.25*), is* selfadjoint *and* contractive *(see Figure* 7.2.3*).*

PROOF. (See [**Kle77**, **Jor86**, **Jor87**, **JO98**, **Jor02**].) Despite the fact that proof details in one form or the other are in the literature, we feel that the spectral theoretic features of the argument have not been stressed; at least not in a form which we shall need below.

Denote the "new" inner product in $\mathscr{K}$ by $\langle\cdot,\cdot\rangle_{\mathscr{K}}$, and the initial inner product in $\mathscr{H}$ by $\langle\cdot,\cdot\rangle$.

$\widetilde{U}$ *is symmetric*: Let $x, y \in \mathscr{H}_+$, then

$$\langle x, \widetilde{U}y\rangle_{\mathscr{K}} = \langle x, \theta Uy\rangle = \langle x, U^*\theta y\rangle = \langle Ux, \theta y\rangle = \langle \widetilde{U}x, y\rangle_{\mathscr{K}}$$

which is the desired conclusion.

$\widetilde{U}$ *is contractive*: Let $x \in \mathscr{H}_+$, then

$$\begin{aligned}\left\|\widetilde{U}x\right\|_{\mathscr{K}}^2 &= \langle Ux, \theta Ux\rangle = \langle Ux, U^*\theta x\rangle \\ &= \left\langle U^2x, \theta x\right\rangle = \left\langle \widetilde{U}^2x, x\right\rangle_{\mathscr{K}} \\ &\le \left\|\widetilde{U}^2x\right\|_{\mathscr{K}} \cdot \|x\|_{\mathscr{K}} \qquad \text{(by Schwarz in } \mathscr{K}) \\ &\le \left\|\widetilde{U}^4x\right\|_{\mathscr{K}}^{\frac{1}{2}} \cdot \|x\|_{\mathscr{K}}^{1+\frac{1}{2}} \qquad \text{(by the first step)} \\ &\le \left\|\widetilde{U}^{2^{n+1}}x\right\|_{\mathscr{K}}^{\frac{1}{2^n}} \cdot \|x\|_{\mathscr{K}}^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}. \qquad \text{(by iteration)}\end{aligned}$$

By the spectral-radius formula, $\lim_{n\to\infty} \left\|\widetilde{U}^{2^n}x\right\|_{\mathscr{K}}^{\frac{1}{2^n}} = 1$; and we get $\left\|\widetilde{U}x\right\|_{\mathscr{K}}^2 \le \|x\|_{\mathscr{K}}^2$, which is the desired contractivity. □

$$
\begin{tikzcd}
\mathscr{H} \arrow[rr, "U", "\theta U\theta = U^*"'] & & \mathscr{H} & \text{unitary } U \\
\bigcup & & \bigcup & \\
\mathscr{H}_+ \arrow[d, "q"'] & U \arrow[d] & \mathscr{H}_+ \arrow[d, "q"] & \begin{array}{c}\text{invariant under } U\\ \langle h_+, \theta h_+\rangle \geq 0\end{array} \\
\mathscr{K} = (\mathscr{H}_+/\mathscr{N})^{\sim} \arrow[rr, bend right, "\widetilde{U}"'] & \widetilde{U} & \mathscr{K} = (\mathscr{H}_+/\mathscr{N})^{\sim} & \begin{array}{c}\text{induced operator}\\ \theta\text{-normalized}\\ \text{inner product}\\[1ex] \widetilde{U} \text{ is contractive}\\ \text{and selfadjoint}\end{array}
\end{tikzcd}
$$

FIGURE 7.2.3. Reflection positivity. A unitary operator U transforms into a selfadjoint contraction $\widetilde{U}$.

REMARK 7.2.25. In the proof of Lemma 7.2.24, we have made an identification:

$$\mathscr{H}_+ \ni x \longleftrightarrow q(x) \in \mathscr{K},$$

see (7.2.28). So the precise vectors are as follows: $\widetilde{U}q(x) = q(Ux)$, $(x \in \mathscr{H}_+)$; see Figure 7.2.3. The proof is in two steps:

Step 1. We verify the two conclusions for $\widetilde{U}$ (symmetry and contractivity) but only initially for the dense space of vectors in $\mathscr{K}$: $\{q(x)\,;\, x \in \mathscr{H}_+\}$.

Step 2. Having the two properties verified on a dense subspace in $\mathscr{K}$, it follows that the same conclusions will hold also on $\mathscr{K} :=$ completion of $\{q(x)\,;\, x \in \mathscr{H}_+\}$. The reason is that the two properties are preserved by passing to limits; now limit in the $\mathscr{K}$-norm.

LEMMA 7.2.26. *Let $\mathscr{H}$, $\mathscr{H}_+$, and θ be as above. Set*

$$
\begin{aligned}
\mathscr{A}_+ :=& \big\{U \in \mathscr{H} \to \mathscr{H},\ \textit{bounded operators},\\
&\quad U\mathscr{H}_+ \subset \mathscr{H}_+ \left(E_+UE_+ = UE_+\right),\ \textit{and } \theta U = U^*\theta\big\},
\end{aligned}
$$

then $U, V \in \mathscr{A}_+ \Longrightarrow UV \in \mathscr{A}_+$, and $(UV)^{\sim} = \widetilde{U}\widetilde{V}$, where $\widetilde{U}$ is determined by

$$\widetilde{U}(q(h_+)) = q(Uh_+),\ \forall h_+ \in \mathscr{H}_+.$$

PROOF. Immediate from Lemma 7.2.24. □

LEMMA 7.2.27. *Let $\mathscr{H}$ be a fixed Hilbert space with subspaces $\mathscr{H}_\pm$ and $\mathscr{H}_0$. Let $E_\pm$ and E_0 denote the respective projections. Let θ be a reflection, i.e., $\theta^2 = I_{\mathscr{H}}$, $\theta^* = \theta$. Assume*

$$
\begin{aligned}
E_-\theta E_+ &= \theta E_+;\\
E_+\theta E_- &= \theta E_-;\ \textit{and}\\
\theta E_0 &= E_0.
\end{aligned}
\tag{7.2.38}
$$

(1) *Suppose $\theta : \mathscr{H}_+ \to \mathscr{H}_-$ is onto. Then we have the following equivalence*

$$E_+\theta E_+ \geq 0 \Longleftrightarrow E_-\theta E_- \geq 0. \tag{7.2.39}$$

(2) *Suppose* (1) *holds, then we get two completions*

$$\mathscr{K}_{\pm} := \left(\mathscr{H}_{\pm} / \left\{h_{\pm} \,;\, \langle h_{\pm}, \theta h_{\pm}\rangle = 0\right\}\right)^{\sim}, \tag{7.2.40}$$

see (7.2.28) *above. Then* θ *induces two isometries* $\widetilde{\theta} : \mathscr{K}_{+} \to \mathscr{K}_{-}$, $\widetilde{\theta} : \mathscr{K}_{-} \to \mathscr{K}_{+}$,

(3) *In general, the isometries from* (2) *are not onto. Indeed,* $\widetilde{\theta} : \mathscr{K}_{+} \to \mathscr{K}_{-}$ *is onto if and only if* $\mathscr{H}_{-} \ominus \theta\mathscr{H}_{+} = 0$; *and* $\widetilde{\theta} : \mathscr{K}_{-} \to \mathscr{K}_{+}$ *is onto if and only if* $\mathscr{H}_{+} \ominus \theta\mathscr{H}_{-} = 0$.

Proof. The key step in the proof of the lemma is (7.2.39). Indeed we have the following:

$$\begin{gathered}
E_{+}\theta E_{+} \geq 0; \\
\Updownarrow \\
\langle h_{+}, \theta h_{+}\rangle \geq 0, \ \forall h_{+} \in \mathscr{H}_{+}; \\
\Updownarrow \\
\left\langle \theta h_{+}, \theta^{2} h_{+}\right\rangle \geq 0, \ \forall h_{+} \in \mathscr{H}_{+}; \\
\Updownarrow \\
\langle h_{-}, \theta h_{-}\rangle \geq 0, \ \forall h_{-} = \theta\left(h_{+}\right) \in \mathscr{H}_{-},
\end{gathered}$$

where we used assumption (7.2.38) above.

Moreover, for all $h_{+} \in \mathscr{H}_{+}$, we have:

$$\begin{aligned}
\left\|\operatorname{class}\left(\theta h_{+}\right)\right\|_{\mathscr{K}_{-}}^{2} &= \langle \theta h_{+}, \theta\theta h_{+}\rangle \\
&= \langle h_{+}, \theta h_{+}\rangle = \left\|\operatorname{class}\left(h_{+}\right)\right\|_{\mathscr{H}_{+}}^{2}.
\end{aligned}$$

The remaining part of the proof is left to the reader. □

We now turn to a closer examination of the unitary reflection operator U from (7.2.1)-(7.2.3). Given θ as in (7.2.1), i.e., $\theta = \theta^{*}$, $\theta^{2} = I_{\mathscr{H}}$; we assume that $\mathscr{H}_{\pm}$ are two closed subspaces in $\mathscr{H}$ such that $\theta\mathscr{H}_{+} \subset \mathscr{H}_{-}$; or, equivalently, $E_{-}\theta E_{+} = \theta E_{+}$, where $E_{\pm}$ denote the respective projection for the corresponding subspaces $\mathscr{H}_{\pm}$; i.e.,

$$\mathscr{H}_{\pm} = \left\{h_{\pm} \in \mathscr{H} \,;\, E_{\pm}h_{\pm} = h_{\pm}\right\}. \tag{7.2.41}$$

Finally, we shall assume that the OS-positivity condition $E_{+}\theta E_{+} \geq 0$ holds; and so we are in a position to apply Lemma 7.2.13 and Corollary 7.2.14 above.

A given unitary operator U in $\mathscr{H}$ is said to be a *reflection-symmetry* iff (Def.)

$$\theta U \theta = U^{*}; \text{ and} \tag{7.2.42}$$

$$U\mathscr{H}_{+} \subseteq \mathscr{H}_{+} \text{ (equivalently, } E_{+}UE_{+} = UE_{+}.) \tag{7.2.43}$$

Theorem 7.2.28. *Let* $\mathscr{H}$, $\mathscr{H}_{\pm}$, θ, *and* U *be as above, i.e., we are assuming OS-positivity; and further that* U *satisfies* (7.2.42)-(7.2.43). *Let* P *be the projection onto* $\{h \in \mathscr{H} \,;\, \theta h = h\}$, *i.e., we have* $\theta = 2P - I_{\mathscr{H}}$.

(1) *Then*

$$PUE_{+} = PU^{*}\theta E_{+}. \tag{7.2.44}$$

(2) *If $C : P\mathscr{H} \longrightarrow P^{\perp}\mathscr{H}$ denotes the contraction from Lemma 7.2.13 and Corollary 7.2.14, then there is a unique operator $U_P : P\mathscr{H} \longrightarrow P\mathscr{H}$ such that $U_P = PUP$; and, if $h_+ = x + Cx$, $x \in P\mathscr{H}$, then*

$$\left\|\widetilde{U}q\left(h_+\right)\right\|_{\mathscr{K}}^2 = \left\|U_P x\right\|_{\mathscr{H}}^2 - \left\|CU_P x\right\|_{\mathscr{H}}^2. \tag{7.2.45}$$

(3) *In particular, since $\widetilde{U}$ is contractive by Lemma 7.2.24, we have*

$$\left\|U_P x\right\|_{\mathscr{H}}^2 - \left\|CU_P x\right\|_{\mathscr{H}}^2 \leq \left\|x\right\|_{\mathscr{H}}^2 - \left\|Cx\right\|_{\mathscr{H}}^2, \ \forall x \in P\mathscr{H}.$$

PROOF. Note that (1) is immediate from (7.2.2) and Corollary 7.2.14.

The first half is immediate from definition of the contraction C from Lemma 7.2.13. For $h_+ = x + Cx$, $x \in P\mathscr{H}$, we have

$$\left\langle h_+, \theta h_+\right\rangle_{\mathscr{H}} = \left\|q\left(h_+\right)\right\|_{\mathscr{K}}^2 = \left\|x\right\|_{\mathscr{H}}^2 - \left\|Cx\right\|_{\mathscr{H}}^2,$$

and

$$\left\|\widetilde{U}\left(q\left(h_+\right)\right)\right\|_{\mathscr{K}}^2 = \left\|q\left(Uh_+\right)\right\|_{\mathscr{K}}^2 = \left\|U_P x\right\|_{\mathscr{H}}^2 - \left\|CU_P x\right\|_{\mathscr{H}}^2;$$

and eq. (7.2.45) in (2) follows.

Now (3) is immediate from (1)-(2) combined with the fact that $\widetilde{U}$ is contractive in $\mathscr{K}$; see Lemma 7.2.24. □

COROLLARY 7.2.29. *Let $\mathscr{H}$, $\mathscr{H}_\pm$, $\mathscr{H}_0$, $E_\pm$, E_0, θ, be as in the statement of Lemma 7.2.27. Let $\mathscr{K}_\pm$ be the corresponding induced Hilbert spaces, see (7.2.40). Now set*

$$\mathscr{H}_\pm^{ex} = \textit{closed span of } \left\{h_0 + h_\pm \,;\, h_0 \in \mathscr{H}_0,\ h_\pm \in \mathscr{H}_\pm\right\}, \tag{7.2.46}$$

and let $E_\pm^{ex}$ denote the corresponding projections, i.e., $E_\pm^{ex} := E_0 \vee E_\pm$. Then the following analogies of (7.2.38) hold:

$$E_-^{ex}\theta E_+^{ex} = \theta E_+^{ex}; \textit{ and} \tag{7.2.47}$$

$$E_+^{ex}\theta E_-^{ex} = \theta E_-^{ex}. \tag{7.2.48}$$

Moreover, we have the implication

$$E_+\theta E_+ \geq 0 \Longrightarrow E_+^{ex}\theta E_+^{ex} \geq 0, \tag{7.2.49}$$

if and only if

$$\left|\left\langle h_+, h_0\right\rangle\right|^2 \leq \left\langle h_+, \theta h_+\right\rangle \left\|h_0\right\|^2, \ \forall h_+ \in \mathscr{H}_+, \forall h_0 \in \mathscr{H}_0. \tag{7.2.50}$$

PROOF. By Lemma 7.2.27, it is easy to prove one of the two formula (7.2.47)-(7.2.48).

In detail, we must show that if $h_0 \in \mathscr{H}_0$, $h_+ \in \mathscr{H}_+$, then $\theta\left(h_0 + h_+\right) \in \mathscr{H}_-^{ex}$; see (7.2.46). But this is clear since

$$\theta\left(h_0 + h_+\right) = \theta h_0 + \theta h_+ = h_0 + \theta h_+, \tag{7.2.51}$$

and $\theta h_+ \in \mathscr{H}_-$ by (7.2.38). We also used $\theta h_0 = h_0$ which is (ii) in Lemma 7.2.4.

The second conclusion follows from this, since if $\left\langle h_+, \theta h_+\right\rangle \geq 0$, $\forall h_+ \in \mathscr{H}_+$; then

$$\begin{aligned}\left\langle h_+ + h_0, \theta\left(h_+ + h_0\right)\right\rangle &\underset{\text{by (7.2.51)}}{=} \left\langle h_+ + h_0, \theta h_+ + h_0\right\rangle \\ &= \left\langle h_+, \theta h_+\right\rangle + \left\langle h_+, h_0\right\rangle + \left\langle h_0, \theta h_+\right\rangle + \left\|h_0\right\|^2.\end{aligned}$$

Now use $\left\langle h_0, \theta h_+\right\rangle = \left\langle \theta h_0, h_+\right\rangle = \left\langle h_0, h_+\right\rangle$, and the result follows. □

REMARK 7.2.30. In the statement of Corollary 7.2.29, we impose the technical assumption (7.2.50). The following example shows that this restricting condition (7.2.50) does not always hold; i.e., that Corollary 7.2.29 cannot be strengthened.

EXAMPLE 7.2.31 (Also see Remark 7.2.17). Let $\mathscr{H} = \mathbb{C}^3$ with standard orthonormal basis $\{e_j\}_{j=1}^3$. Consider the reflection

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and set

$$\begin{aligned} \mathscr{H}_+ &= span\left\{e_1 + \frac{1}{2}e_3\right\}, \\ \mathscr{H}_- &= span\left\{e_1 - \frac{1}{2}e_3\right\}, \\ \mathscr{H}_0 &= span\left\{e_1\right\}. \end{aligned}$$

For $h_+ := e_1 + \frac{1}{2}e_3$, and $h_0 := e_1$, we get $\left|\langle h_+, h_0\rangle\right|^2 = 1$, but

$$\langle h_+, \theta h_+\rangle \left\|h_0\right\|^2 = \left\langle e_1 + \frac{1}{2}e_3, e_1 - \frac{1}{2}e_3 \right\rangle \left\|e_1\right\|^2 = \frac{3}{4}.$$

Hence condition (7.2.50) does not hold.

Note that $h_+ - h_0 \in \mathscr{H}_+^{ex}$, and

$$\langle h_+ - h_0, \theta\left(h_+ - h_0\right)\rangle = \left\langle \frac{1}{2}e_3, -\frac{1}{2}e_3 \right\rangle = -\frac{1}{4} < 0;$$

i.e., the positivity condition $E_+^{ex}\theta E_+^{ex} \geq 0$ in (7.2.49) is not satisfied.

COROLLARY 7.2.32. *Let $\mathscr{H}$, θ, and $\mathscr{H}_0$, $\mathscr{H}_\pm$ be as in Corollary* 7.2.29, *assume* (7.2.50), *and let $\mathscr{K}_\pm^{ex}$ be the corresponding induced Hilbert spaces; see* (7.2.46) *applied to $\mathscr{H}_\pm^{ex}$. Then the two quotient mappings $\mathscr{H}_0 \to \mathscr{K}_\pm^{ex}$ are isometric.*

7.3. Symmetry in physics and reflection positive constructions via unitary representations of Lie groups

In this section we introduce certain unitary representations which are given to act on the fixed Hilbert space. (See e.g., [**JT18c**] and the papers cited there.) We consider a given Hilbert space $\mathscr{H}$ which carries a reflection symmetry (in the sense of Osterwalder-Schrader) as defined in Section 7.2.1. If the unitary representation under consideration, say U, is a representation of a group G, then reflection-symmetry will refer to a suitable semigroup S in G, so a sub-semigroup. The setting is of interest even in the three cases when G is $\mathbb{Z}$, $\mathbb{R}$, or some Lie group from quantum physics. In the cases $G = \mathbb{Z}$, or $\mathbb{R}$, the semigroups are obvious, and, in each case, they define a causality. (The case $G = \mathbb{Z}$ is simply the study of a single unitary operator.) Nonetheless, the choice of semigroup in the case when G is a Lie group is more subtle. However, many of the important spectral theoretic properties may be developed initially in the cases $G = \mathbb{Z}$, or $\mathbb{R}$, where the essential structures are more transparent.

LEMMA 7.3.1. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a unitary one-parameter group in $\mathscr{H}$, such that $\theta U_t \theta = U_{-t}$, $t \in \mathbb{R}$, and $U_t\mathscr{H}_+ \subset \mathscr{H}_+$, $t \in \mathbb{R}_+$; then*

$$S_t = \widetilde{U_t} : \mathscr{K} \longrightarrow \mathscr{K},$$

is a selfadjoint contraction semigroup, *$t \in \mathbb{R}_+$, i.e., there is a selfadjoint generator L in $\mathscr{K}$ (see Figure* 7.3.1*),*

$$\langle k, Lk \rangle_{\mathscr{K}} \geq 0,\ \forall k \in dom\,(L)\,, \tag{7.3.1}$$

where

$$S_t (= \widetilde{U}_t) = e^{-tL},\ t \in \mathbb{R}_+,\ \textit{and} \tag{7.3.2}$$

$$S_{t_1} S_{t_2} = S_{t_1+t_2},\ t_1, t_2 \in \mathbb{R}_+. \tag{7.3.3}$$

PROOF. See [**GJ79**, **GJ87**, **Jor87**, **JO00**].

□

$$\begin{array}{ccc} A & \mathscr{H} \xrightarrow{U_t = e^{-tA}} \mathscr{H} & A^* = -A \\ \big\downarrow & & \\ L & \mathscr{K} \xrightarrow[S_t = e^{-tL}]{[S_t]_{t\in\mathbb{R}_+}} \mathscr{K} & L^* = L,\ L \geq 0 \end{array}$$

FIGURE 7.3.1. Transformation of skew-adjoint A into selfadjoint semibounded L.

7.3.1. An example. We include details below (Example 7.3.2) to stress the distinction between an abstract Hilbert-norm completion on the one hand, and a concretely realized Hilbert space on the other.

EXAMPLE 7.3.2 ([**JO98**, **JO00**]). Let $0 < s < 1$ be given, and let $\mathscr{H} = \mathscr{H}_s$ be the Hilbert space whose norm $\|f\|_s$ is given by

$$\|f\|_s^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f\,(x)}\, |x-y|^{s-1} f\,(y)\, dxdy. \tag{7.3.4}$$

Let $a \in \mathbb{R}_+$ be given, and set

$$(U\,(a)\,f)\,(x) = a^{s+1} f\left(a^2 x\right). \tag{7.3.5}$$

It is clear that then $a \mapsto U\,(a)$ is a unitary representation of the multiplicative group $\mathbb{R}_+$ acting on the Hilbert space $\mathscr{H}_s$. It can be checked that $\|f\|_s$ in (7.3.4) is finite for all $f \in C_c\,(\mathbb{R})$ (= the space of compactly supported functions on the line). Now let $\mathscr{H}_+$ be the closure of $C_c\,(-1,1)$ in $\mathscr{H}_s$ relative to the norm $\|\cdot\|_s$ of (7.3.4). It is then immediate that $U\,(a)$, for $a > 1$, leaves $\mathscr{H}_+$ invariant, i.e., it restricts to a semigroup of isometries $\{U\,(a)\,; a > 1\}$ acting on $\mathscr{K}_s$. Setting

$$(\theta f)\,(x) = |x|^{-s-1} f\left(\frac{1}{x}\right), \quad x \in \mathbb{R} \setminus \{0\}\,, \tag{7.3.6}$$

we check that θ is then a period-2 unitary in $\mathscr{H}_s$, and that

$$\theta U\,(a)\,\theta = U\,(a)^* = U\left(a^{-1}\right) \tag{7.3.7}$$

and

$$\langle f, \theta f \rangle_{\mathscr{H}_s} \geq 0, \quad \forall f \in \mathscr{H}_+, \tag{7.3.8}$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}_s}$ is the inner product

$$\langle f_1, f_2 \rangle_{\mathscr{H}_s} := \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f_1(x)} \, |x - y|^{s-1} \, f_2(y) \, dx dy. \tag{7.3.9}$$

In fact, if $f \in C_c(-1, 1)$, the expression in (7.3.8) works out as the following reproducing kernel integral:

$$\int_{-1}^{1} \int_{-1}^{1} \overline{f(x)} \, (1 - xy)^{s-1} \, f(y) \, dx dy, \tag{7.3.10}$$

and we refer to [**Jor86**, **JO98**, **JO00**, **Jor02**] for more details on this example. Also see [**NO15**].

Hence up to a constant, the norm $\|\cdot\|_s$ of (7.3.9) may be rewritten as

$$\int_{\mathbb{R}} |\xi|^{-s} \left| \hat{f}(\xi) \right|^2 d\xi, \tag{7.3.11}$$

and the inner product $\langle \cdot, \cdot \rangle_s$ as

$$\int_{\mathbb{R}} |\xi|^{-s} \, \overline{\hat{f}_1(\xi)} \hat{f}_2(\xi) \, d\xi, \tag{7.3.12}$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx \tag{7.3.13}$$

is the usual Fourier transform suitably extended to $\mathscr{H}_s$, using Stein's singular integrals. Intuitively, $\mathscr{H}_s$ consists of functions on $\mathbb{R}$ which arise as $\left(\frac{d}{dx}\right)^s f_s$ for some f_s in $L^2(\mathbb{R})$. This also introduces a degree of "non-locality" into the theory, and the functions in $\mathscr{H}_s$ cannot be viewed as locally integrable, although $\mathscr{H}_s$ for each s, $0 < s < 1$, contains $C_c(\mathbb{R})$ as a dense subspace. In fact, formula (7.3.11), for the norm in $\mathscr{H}_s$, makes precise in which sense elements of $\mathscr{H}_s$ are "fractional" derivatives of locally integrable functions on $\mathbb{R}$, and that there are elements of $\mathscr{H}_s$ (and of $\mathscr{K}_s$) which are not locally integrable.

A main conclusion in [**Jor02**] for this example is that, when $\mathscr{H}_+$ and $\mathscr{K}$ are as in (7.3.10), then the natural contractive operator q from (7.2.26)-(7.2.28) is automatically 1-1, i.e., its kernel is 0.

Remark 7.3.3. Note that, in general, the spectral type changes in passing from U to $\widetilde{U}$ in Lemma 7.2.24; see also Figure 7.2.3. For example, U from (7.3.5) above has absolutely continuous spectrum, while $\widetilde{U}$ has purely discrete (atomic) spectrum: When $a > 1$, one checks that the spectrum of $\widetilde{U}(a)$ is the set $\left\{a^{-2n} \, ; \, n \in \mathbb{N}\right\}$.

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