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# HARMONIC AND RELATIVELY AFFINE MAPPINGS

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The theory of harmonic mappings of a Riemannian space into another has been initiated by Eells and Sampson [2] and studied by Chern [1], Goldberg [1], [3], T. Ishihara [3], [5] and others.

In this paper, we study projective and affine mappings of a manifold with symmetric affine connection into another and harmonic and relatively affine mappings of a Riemannian space into another.

## 1. Differentiable mappings of a manifold with symmetric affine connection into another

Let $(M, \nabla)$ be a manifold of dimension $n$ with symmetric affine connection $\nabla$, and $(N, \bar{\nabla})$ a manifold of dimension $p$ with symmetric affine connection $\bar{\nabla}$, where $n, p \geq 2$. Let there be given a differentiable mapping $f: M \to N$ which we denote sometimes by $f: (M, \nabla) \to (N, \bar{\nabla})$. Manifolds, mappings and geometric objects which we discuss in this paper are assumed to be of differentiability class $C^\infty$. Take coordinate neighborhoods $\{U; x^h\}$ of $M$ and $\{\bar{U}, y^\alpha\}$ of $N$ in such a way that $f(U) \subset \bar{U}$, where $(x^h) = (x^1, x^2, \cdots, x^n)$ and $(y^\alpha) = (y^{\bar{1}}, y^{\bar{2}}, \cdots, y^{\bar{p}})$ are local coordinates of $M$ and $N$ respectively. The indices $h, i, j, k, l, m, r, s, t$ run over the range $\{1, 2, \cdots, n\}$, and the indices $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ the range $\{\bar{1}, \bar{2}, \cdots, \bar{p}\}$. The summation convention will be used with respect to these two systems of indices. Suppose that $f: (M, \nabla) \to (N, \bar{\nabla})$ is represented by equations

$$y^\alpha = y^\alpha(x^1, x^2, \cdots, x^n) \tag{1.1}$$

with respect to $\{U, x^h\}$ and $\{\bar{U}, y^\alpha\}$. We put

$$A_i^{\ \alpha} = \partial_i y^\alpha(x^1, x^2, \cdots, x^n) \ , \tag{1.2}$$

where $\partial_i = \partial/\partial x^i$. Then the differential $df$ of the mapping $f$ is represented by the matrix $(A_i^{\ \alpha})$ with respect to the local coordinates $(x^h)$ and $(y^\alpha)$ of $M$ and $N$.

When a function $\rho$, local or global, is given in $N$, throughout the paper we shall identify $\rho$ with the function $\rho \circ f$ induced in $M$. We denote by $\Gamma_{ji}^h$ the

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components of the affine connection $\nabla$ in $M$, and by $\Gamma^{\alpha}_{\gamma\beta}$ those of the affine connection $\bar{\nabla}$ in $N$.

In this and the next sections, $X$, $Y$ and $Z$ denote arbitrary vector fields in $M$ with local expressions $X = X^h\partial_h$, $Y = Y^h\partial_h$ and $Z = Z^h\partial_h$ respectively. Then $(A_i{}^\alpha X^i)\partial_\alpha$, where $\partial_\alpha = \partial/\partial y^\alpha$, is the local expression of the vector field $(df)X$ defined along $f(M)$. If we put in $U$

$$A_{ji}{}^\alpha = \nabla_j A_i{}^\alpha , \tag{1.3}$$

where

$$\nabla_j A_i{}^\alpha = \partial_j A_i{}^\alpha + \Gamma^\alpha_{\gamma\beta} A_j{}^\gamma A_i{}^\beta - \Gamma^h_{ji} A_h{}^\alpha , \tag{1.4}$$

then $(A_{ji}{}^\alpha X^j Y^i)\partial_\alpha$ is the local expression of a vector field $B$ defined along $f(M)$, and $A_{ji}{}^\alpha = A_{ij}{}^\alpha$.

Consider a curve $\gamma : I \rightarrow M$ in $M$, $I$ being an interval, and denote by $\bar{\gamma} = f \circ \gamma : I \rightarrow N$ the image of $\gamma$ by $f$. When $\gamma$ is locally represented by $x^h = x^h(t)$, $t$ being a parameter belonging to $I$, $\bar{\gamma}$ is so by $y^\alpha = y^\alpha(x^h(t))$. If $\gamma$ satisfies

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt}$$

with a certain function $\alpha(t)$ of $t$, then $\gamma$ is called a *path* of $(M, \nabla)$. It is easily seen that the above equations can be reduced to

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = 0$$

by a suitable change of the parameter $t$. In this case $\gamma$ is called a path with *affine parameter t*. A path in $N$ and the affine parameter on this path will be similarly defined.

Now, using $y^\alpha = y^\alpha(x^h(t))$, (1.3) and (1.4), we find

$$\frac{d^2y^\alpha}{dt^2} + \Gamma^\alpha_{\gamma\beta}\frac{dy^\gamma}{dt}\frac{dy^\beta}{dt} = A_h{}^\alpha\left(\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}\right) + A_{ji}{}^\alpha\frac{dx^j}{dt}\frac{dx^i}{dt} . \tag{1.5}$$

We assume that an arbitrary path in $(M, \nabla)$ is mapped by $f$ into a path in $(N, \bar{\nabla})$. Such a mapping $f$ is said to be *projective*. Under this assumption, we have from (1.5)

$$\beta(t)\frac{dy^\alpha}{dt} = A_{ji}{}^\alpha\frac{dx^j}{dt}\frac{dx^i}{dt}$$

for any path $\gamma : x^h = x^h(t)$ in $(M, \nabla)$, $\beta(t)$ being a certain function of $t$. Thus, $\gamma$ being arbitrary, we find $\beta A_h{}^\alpha \xi^h = A_{ji}{}^\alpha \xi^j \xi^i$ for any direction $\xi = \xi^h\partial_h$ at any point of $M$, from which we conclude that

$$(1.6) \qquad A_{ji}{}^{\alpha} = p_j A_i{}^{\alpha} + p_i A_j{}^{\alpha}$$

for some local functions $p_i$ in $U$, which are the components of a 1-form in $M$. The converse being evident, we have

**Proposition 1.1.** *In order for a mapping $f: (M, \nabla) \to (N, \bar{\nabla})$ to be projective, it is necessary and sufficient that $A_{ji}{}^{\alpha}$ has the form* (1.6).

We next assume that an arbitrary path in $(M, \nabla)$ is mapped by $f$ into a path in $(N, \bar{\nabla})$ with the affine parameter preserved. Such a mapping $f$ is said to be *affine*. Under this assumption, we have from (1.5)

$$(1.5) \qquad A_{ji}{}^{\alpha} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0$$

for any path $\gamma: x^h = x^h(t)$ in $(M, \nabla)$. Thus, $\gamma$ being arbitrary, we have $A_{ji}{}^{\alpha}\xi^j\xi^i = 0$ for any direction $\xi = \xi^h \partial_h$ at any point of $M$, from which we conclude that $A_{ji}{}^{\alpha} = 0$. The converse being evident, we have

**Proposition 1.2.** *In order for a mapping $f: (M, \nabla) \to (N, \bar{\nabla})$ to be affine, it is necessary and sufficient that $A_{ji}{}^{\alpha} = 0$.*

## 2. Differentiable mapping of a Riemannian space into another

Let $(M, g)$ and $(N, \bar{g})$ be Riemannian spaces of dimensions $n$ and $p$ respectively. Let there be given a mapping $f: M \to N$ denoted sometimes by $f: (M, g) \to (N, g)$. We denote by $g_{ji}$ the components of the Riemannian metric $g$ in $M$, and by $g_{\gamma\beta}$ those of the Riemannian metric $\bar{g}$ in $N$. The Christoffel symbols formed with $g_{ji}$ and $g_{\gamma\beta}$ are denoted by $\left\{ {h \atop ji} \right\}$ and $\left\{ {\alpha \atop \gamma\beta} \right\}$ respectively. Thus, denoting by $\nabla$ the affine connection determined by $\left\{ {h \atop ji} \right\}$ and by $\bar{\nabla}$ that determined by $\left\{ {\alpha \atop \gamma\beta} \right\}$, we can regard $f$ as $f: (M, \nabla) \to (N, \bar{\nabla})$.

If we put

$$(2.1) \qquad g^*_{ji} = g_{\gamma\beta} A_j{}^{\gamma} A_i{}^{\beta} ,$$

then $g^*_{ji}$ are the components of the tensor $g^* = f^*\bar{g}$ induced in $M$ from $\bar{g}$ by $f$. For $g^* = \rho g$, $f: (M, g) \to (N, \bar{g})$ is said to be *conformal*, *homothetic* or *isometric* according as the function $\rho$ is positive, constant or equal to 1.

Differentiating (2.1) covariantly, we find

$$(2.2) \qquad \nabla_k g^*_{ji} = D_{kji} + D_{kij} ,$$

where we have put

$$D_{kji} = A_{kj}{}^{\gamma} A_i{}^{\beta} g_{\gamma\beta} . \tag{2.3}$$

Changing indices in (2.2), we obtain

$$\nabla_j g^*_{ki} = D_{jki} + D_{jik} , \tag{2.4}$$

$$\nabla_i g^*_{kj} = D_{ikj} + D_{ijk} . \tag{2.5}$$

Forming (2.2) + (2.4) − (2.5), we find

$$D_{kji} = \tfrac{1}{2}(\nabla_k g^*_{ji} + \nabla_j g^*_{ki} - \nabla_i g^*_{kj}) , \tag{2.6}$$

where we have used $D_{kji} = D_{jki}$ which is a direct consequence of $A_{kj}{}^{\alpha} = A_{jk}{}^{\alpha}$. When $\nabla g^* = 0$, that is, when $\nabla_k g^*_{ji} = 0$ is satisfied, $f: (M, g) \to (N, \bar{g})$ is said to be *relatively affine* (see [4]). Since we can see from (2.2) and (2.6) that $\nabla_k g^*_{ji} = 0$ and $D_{kji} = 0$ are equivalent, we now have

**Proposition 2.1.** *A mapping $f: (M, g) \to (N, \bar{g})$ is relatively affine if and only if $D_{kji} = 0$, i.e., if and only if $A_{kj}{}^{\gamma} A_i{}^{\beta} g_{\gamma\beta} = 0$.*

Thus any affine mapping is relatively affine.

The conditions $\nabla g^* = 0$ and $g^* = \rho^2 g$ imply $\rho^2 = \text{const}$. Thus we have

**Proposition 2.2.** *If a mapping $f: (M, g) \to (N, \bar{g})$ is relatively affine and at the same time conformal, then it is homothetic.*

It is easily seen that the rank of the mapping $f: (M, g) \to (N, \bar{g})$, i.e., the rank of $(A_i{}^{\alpha})$ is equal to the rank of $(g^*_{ji})$ at each point of $M$. If the mapping $f$ is relatively affine, then $\nabla g^* = 0$ which implies that $g^*$ is of constant rank $m$. Therefore, if $f$ is relatively affine, then $f$ is of constant rank $m$. Assume that $f$ is relatively affine and of constant rank $m < n$, and for any point $p$ of $M$ put $D_p = \{X \in T_p(M) \mid (df)_p X = 0\}$, which is a subspace of dimension $n - m$ in the tangent space $T_p(M)$ of $M$ at $p$. Therefore the correspondence $D: p \to D_p$ defines an $(n - m)$-dimensional distribution $D$ in $M$, which is called the *vertical distribution*. It is easily verified that a vector field $X$ belongs to the vertical distribution $D$ if and only if $A_i{}^{\alpha} X^i = 0$, or equivalently, if and only if $g^*_{ji} X^i = 0$. By considering such a vector field $X$ and differentiating $A_i{}^{\alpha} X^i = 0$ covariantly, we then obtain $A_{ji}{}^{\alpha} X^i + A_i{}^{\alpha} \nabla_j X^i = 0$. Thus transvecting $A_k{}^{\beta} g_{\beta\alpha}$ to this equation and using $D_{kji} = 0$, we have $g^*_{ki} \nabla_j X^i = 0$, i.e., $(df)(\nabla_Y X) = 0$. Consequently, we arrive at

**Proposition 2.3.** *Let a mapping $f: (M, g) \to (N, \bar{g})$ be relatively affine. If $M$ is connected, then $f$ is of constant rank $m$. When $0 < m < \dim M = n$, the vertical distribution $D$ is of dimension $n - m$ and parallel.*

As a corollary to Proposition 2.3, we have

**Proposition 2.4.** *Let $f: (M, g) \to (N, \bar{g})$ be relatively affine. If $(M, g)$ is a connected and irreducible Riemannian space, then $f$ is either of rank $n(= \dim M)$ or a constant mapping.*

We now put

$$A^{\alpha} = g^{ji} A_{ji}{}^{\alpha} , \tag{2.7}$$

where $(g^{ji}) = (g_{ji})^{-1}$. Then the vector field $T$ with components $A^{\alpha}$ defined along $f(M)$ is called the *tension field* of the mapping $f: (M, g) \to (N, \bar{g})$. It is well known that $f: (M, g) \to (N, \bar{g})$ is harmonic if and only if $T = 0$, i.e., if and only if $A^{\alpha} = 0$ (see [2]).

Consider the divergence of the vector field with local expression $(g^{hi} A_i{}^{\gamma} A^{\beta} g_{\gamma\beta}) \partial_h$ in $M$. We then obtain

$$\nabla_l (g^{li} A_i{}^{\gamma} A^{\beta} g_{\gamma\beta}) = A^{\gamma} A^{\beta} g_{\gamma\beta} + A_i{}^{\gamma} (\nabla_l A^{\beta}) g^{li} g_{\gamma\beta} ,$$

where we have put

$$\nabla_i A^{\alpha} = \partial_i A^{\alpha} + \left\{ {\alpha \atop \gamma\beta} \right\} A_i{}^{\gamma} A^{\beta} . \tag{2.8}$$

Thus we have

**Proposition 2.5.** *A mapping $f: (M, g) \to (N, \bar{g})$ is harmonic if $M$ is compact and $\nabla T = 0$ which means $\nabla_i A^{\alpha} = 0$.*

## 3. Laplacian of $\|df\|^2$

We shall compute Laplacian of $\|df\|^2$ for later use. We now put in $U$

$$\nabla_k A_{ji}{}^{\alpha} = \partial_k A_{ji}{}^{\alpha} + \left\{ {\alpha \atop \gamma\beta} \right\} A_k{}^{\gamma} A_{ji}{}^{\beta} - \left\{ {m \atop kj} \right\} A_{mi}{}^{\alpha} - \left\{ {m \atop ki} \right\} A_{jm}{}^{\alpha} . \tag{3.1}$$

Then $(\nabla_k A_{ji}{}^{\alpha} X^k Y^j Z^i) \partial_{\alpha}$ is the local expression of a vector field defined along $f(M)$. Taking account of (1.3), (1.4) and (3.1), we obtain the following formula of Ricci-type:

$$\nabla_k \nabla_j A_i{}^{\alpha} - \nabla_j \nabla_k A_i{}^{\alpha} = R_{\delta\gamma\beta}{}^{\alpha} A_k{}^{\delta} A_j{}^{\gamma} A_i{}^{\beta} - R_{kji}{}^{h} A_h{}^{\alpha} , \tag{3.2}$$

where $R_{\delta\gamma\beta}{}^{\alpha}$ and $R_{kji}{}^{h}$ are the components of the curvature tensors of $\bar{g}$ and $g$ respectively. We are now going to compute Laplacian of $\|df\|^2$. We then have

$$\begin{aligned} \tfrac{1}{2} \Delta \|df\|^2 &= \tfrac{1}{2} g^{lk} \nabla_l \nabla_k (A_j{}^{\beta} A_i{}^{\alpha} g^{ji} g_{\beta\alpha}) \\ &= g^{lk} (\nabla_l \nabla_k A_j{}^{\beta}) A_i{}^{\alpha} g^{ji} g_{\beta\alpha} + \|B\|^2 , \end{aligned} \tag{3.3}$$

where

$$\|B\|^2 = A_{lk}{}^{\beta} A_{ji}{}^{\alpha} g^{lj} g^{ki} g_{\beta\alpha} . \tag{3.4}$$

Thus using (3.2) and putting $R_{\delta\gamma\beta\alpha} = R_{\delta\gamma\beta}{}^{\lambda} g_{\lambda\alpha}$, from (3.3) we obtain

$$\frac{1}{2}\Delta\|df\|^2 = (\nabla_j A^\beta)A_i{}^\alpha g^{ji} g_{\beta\alpha} + \|B\|^2 + R_{\delta\gamma\beta\alpha}A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} + R_i{}^h g^*_{hj} g^{ij} , \tag{3.5}$$

where $R_j{}^h = R_{jk}g^{hk}$ are the mixed components of the Ricci tensor of $(M, g)$ and $\nabla_j A^\alpha$ are defined by (2.8). Thus taking account of (3.5) we have

**Lemma 3.1.** *For a harmonic mapping $f: (M, g) \to (N, \bar{g})$, we have*

$$\frac{1}{2}\Delta\|df\|^2 = \|B\|^2 + R_{\delta\gamma\beta\alpha}A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} + R_j{}^h g^*_{hi} g^{ji} . \tag{3.6}$$

Let $e_{(1)}, \cdots, e_{(n)}$ be $n$ orthonormal vectors at each point of $(M, g)$ such that

$$g_{ji} = e_{(1)j}e_{(1)i} + \cdots + e_{(n)j}e_{(n)i} , \tag{3.7}$$

$$g^*_{ji} = \lambda_1 e_{(1)j}e_{(1)i} + \cdots + \lambda_n e_{(n)j}e_{(n)i} , \tag{3.8}$$

where $e_{(s)}{}^h$ are the components of $e_{(r)}$, and $e_{(r)i} = e_{(r)}{}^h g_{hi}$. Then we find

$$\lambda_1, \cdots, \lambda_n \geq 0 . \tag{3.9}$$

If we now put $\bar{e}_{(s)} = (df)e_{(s)}$, then $\bar{e}_{(s)}$ has components of the form $e_{(s)}{}^\alpha = A_i{}^\alpha e_{(s)}{}^i$. Therefore we get

$$R_{\delta\gamma\beta\alpha}A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} = \sum_{r\neq s} R_{\delta\gamma\beta\alpha}e_{(r)}{}^\delta e_{(s)}{}^\gamma e_{(r)}{}^\beta e_{(s)}{}^\alpha$$

and hence

$$R_{\delta\gamma\beta\alpha}A_l{}^j A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} = -\sum_{r\neq s} \bar{\sigma}(\bar{e}_{(r)}, \bar{e}_{(s)})\lambda_r\lambda_s , \tag{3.10}$$

where $\bar{\sigma}(\bar{X}, \bar{Y})$ denotes the sectional curvature of $(N, \bar{g})$, $\bar{X}$ and $\bar{Y}$ being any two linear independent vectors at any point of $(N, \bar{g})$.

On the other hand, we can easily find

$$\sum_{r\neq s} \lambda_r\lambda_s = -\sum_s (\lambda_s - \tilde{\lambda})^2 + n(n-1)\tilde{\lambda}^2 , \tag{3.11}$$

where we have put

$$\tilde{\lambda} = \frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \geq 0 . \tag{3.12}$$

$n\tilde{\lambda}$ is sometimes denoted by

$$\text{Trace } g^* = n\tilde{\lambda} = g^*_{ji}g^{ji} \geq 0 . \tag{3.13}$$

We here consider the following condition:

(C) *There is a constant c such that* $c \geq \bar{\sigma}(\bar{X}, \bar{Y})$ *for any two linearly independent vectors* $\bar{X}$ *and* $\bar{Y}$ *at any point of* $(N, \bar{g})$.

Then using (3.10) and (3.11) we obtain

$$R_{\delta\gamma\beta\alpha}A_l{}^{\delta}A_i{}^{\gamma}A_k{}^{\beta}A_j{}^{\alpha}g^{lk}g^{ji} \geq c \sum_s (\lambda_s - \tilde{\lambda})^2 - n(n-1)c\tilde{\lambda}^2 , \tag{3.14}$$

when condition (C) is satisfied.

Next, using (3.7) and (3.8), we have

$$R_j{}^h g^*_{hi} g^{ji} = \lambda_1(R_{ji}e_{(1)}{}^j e_{(1)}{}^i) + \cdots + \lambda_n(R_{ji}e_{(n)}{}^j e_{(n)}{}^i) , \tag{3.15}$$

where $R_{ji} = R_j{}^h g_{hi}$ are the components of the Ricci tensor of $(M, g)$. Assume $M$ to be compact and put

$$\frac{r}{n} = \min R_{ji}A^j A^i , \tag{3.16}$$

where $A = A^h \partial_h$ runs over the unit sphere bundle over $(M, g)$. Then by (3.15) and (3.16) we find

$$R_j{}^h g^*_{hi} g^{ji} \geq r\tilde{\lambda} , \tag{3.17}$$

and use of (3.14), (3.17) and Lemma 3.1 thus gives

**Lemma 3.2.** *For a harmonic mapping* $f\colon (M, g) \to (N, \bar{g})$ *we have*

$$\tfrac{1}{2}\Delta \|df\|^2 \geq \|B\|^2 + c \sum_s (\lambda_s - \tilde{\lambda})^2 + n(n-1)c\tilde{\lambda}^2 + r\tilde{\lambda} , \tag{3.18}$$

*when M is compact and condition* (C) *is satisfied.*

## 4. Theorems

First we shall give some remarks. If $\|B\|^2 = 0$, then we have $B = 0$ which means that $f\colon (M, g) \to (N, \bar{g})$ is affine. If $\lambda_1 = \cdots = \lambda_n = \tilde{\lambda}$, then $g^* = \tilde{\lambda} g$, which means that $f\colon (M, g) \to (N, \bar{g})$ is conformal when $\tilde{\lambda} \neq 0$ everywhere and that $f$ is a constant mapping when $\tilde{\lambda} = 0$ everywhere and $M$ is connected. Thus, if $\|B\|^2 = 0$ and $\lambda_1 = \cdots = \lambda_n$, and $M$ is connected, then $f$ is a homothetic or constant mapping, because of Proposition 2.2. Consequently from Lemma 3.2 we have

**Theorem 4.1.** *Let* $f\colon (M, g) \to (N, \bar{g})$ *be a harmonic mapping of a Riemannian space* $(M, g)$ *of dimension n into another Riemannian space* $(N, \bar{g})$, *and assume M to be compact and connected. Then*

(i) $f\colon (M, g) \to (N, \bar{g})$ *is a constant or homothetic mapping of rank n everywhere, if* $(M, g)$ *has positive definite Ricci tensor and there is a constant* $c > 0$ *such that* $c \geq \bar{\sigma}$, $\bar{\sigma}$ *being the sectional curvature of* $(N, \bar{g})$, *and the following condition is satified*:

$$(A_1) \qquad \text{Trace } g^* \leq \frac{r}{(n-1)c} ,$$

*where r is defined by* (3.16);

(ii) $f\colon (M, g) \to (N, \bar{g})$ *is a constant mapping, if the following condition is satisfied*:

$$(A_2) \qquad \bar{\sigma} \leq 0 \text{ and } (M, g) \text{ has positive definite Ricci tensor.}$$

In case (i) of Theorem 4.1, if $\dim M = n = \dim N$, then $f$ is a regular and homothetic mapping of $(M, g)$ onto a connected component of $(N, \bar{g})$; if $\dim M = n < \dim N$, then $f\colon (M, g^*) \to (N, \bar{g})$ is an isometric immersion, which is totally geodesic, and $g^* = \rho^2 g$ with constant $\rho^2 > 0$. Thus, in case (i) of Theorem 4.1 if $(N, \bar{g})$ is a sphere $(S^p, \bar{g}_0)$ of constant curvature, then $(M, g)$ is necessarily a sphere $(S^n, g_0)$ of constant curvature.

We now assume that $r = 0$ and $\bar{\sigma} \leq 0$. Using (3.10) and (3.17), from Lemma 3.1 we have

$$\tfrac{1}{2}\Delta \|df\|^2 \geq \|B\|^2 + R_j{}^h g^*_{hi} g^{ji} \geq \|B\|^2 .$$

Thus, if $M$ is compact, then $R_j{}^h g^*_{hi} g^{ji} = 0$, which and (3.15) imply

$$(4.1) \qquad \lambda_1 (R_{ji} e_{(1)}{}^j e_{(1)}{}^i) + \cdots + \lambda_n (R_{ji} e_{(n)}{}^j e_{(n)}{}^i) = 0 .$$

Hence it follows from (4.1) that

$$(4.2) \qquad \lambda_s (R_{ji} e_{(s)}{}^j e_{(s)}{}^i) = 0 , \qquad (s = 1, 2, \cdots, n),$$

since $\lambda_s (R_{ji} e_{(s)}{}^j e_{(s)}{}^i) \geq 0$. (4.2) means that the Ricci tensor of $(M, g)$ is of rank $\leq n - m$ when $f$ is of rank $m$ everywhere. Consequently taking account of Proposition 2.3 we obtain

**Theorem 4.2.** *Let $f\colon (M, g) \to (N, \bar{g})$ be a harmonic mapping of a Riemannian space $(M, g)$ into another Riemannian space $(N, \bar{g})$, and assume $M$ to be compact and connected. Then either $f$ is an affine mapping of constant rank $m \geq 0$ and the Ricci tensor of $(M, g)$ is of rank $\leq n - m$, or $f$ is a constant mapping, if the following condition is satisfied*:

($A_3$) $\bar{\sigma} \leq 0$, *and $(M, g)$ has positive semi-definite Ricci tensor and $r = 0$, where $r$ is defined by* (3.16). *In this case,* Trace $g^*$ *is necessarily constant.*

In Theorem 4.2, if $(M, g)$ is connected and irreducible, then $f$ is a constant mapping because of Proposition 2.4; if $f$ is of rank $n$ everywhere and $(N, \bar{g})$ is a flat torus, then $(M, g)$ is also a flat torus, and the isometric immersion $f\colon (M, g^*) \to (N, \bar{g})$ is totally geodesic when $\dim M < \dim N$.

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