HARMONIC CALCULUS ON P.C.F. SELF-SIMILAR SETS

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ABSTRACT. The main object of this paper is the Laplace operator on a class of fractals. First, we establish the concept of the renormalization of difference operators on post critically finite (p.c.f. for short) self-similar sets, which are large enough to include finitely ramified self-similar sets, and extend the results for Sierpinski gasket given in [10] to this class. Under each invariant operator for renormalization, the Laplace operator, Green function, Dirichlet form, and Neumann derivatives are explicitly constructed as the natural limits of those on finite pre-self-similar sets which approximate the p.c.f. self-similar sets. Also harmonic functions are shown to be finite dimensional, and they are characterized by the solution of an infinite system of finite difference equations.

0. INTRODUCTION

Mathematical analysis has recently begun on fractal sets. The pioneering works are the probabilistic approaches of Kusuoka [11] and Barlow and Perkins [2]. They have constructed and investigated Brownian motion on the Sierpinski gaskets. In their standpoint, the Laplace operator has been formulated as the infinitesimal generator of the diffusion process.

On the other hand, in [10], we have found the direct and natural definition of the Laplace operator on the Sierpinski gaskets as the limit of difference operators. In the present paper, we extend the results in [10] to a class of self-similar sets called p.c.f. self-similar sets which include the nested fractals defined by Lindstrøm [13]. Several examples of p.c.f. self-similar sets are given in the figures of §8. The reader can find an exposition of the original ideas of this work in §0 of [10].

In §1, we study some topological properties of general self-similar sets and define p.c.f. self-similar sets. Roughly speaking, p.c.f. self-similar sets are almost the same concept as "finitely ramified fractals" mainly used by physicists. We note that the Sierpinski carpet, where Barlow and Bass [1] have constructed a diffusion process, is not a p.c.f. self-similar set.

In §2, we introduce the concept of a quasi-harmonic structure on p.c.f. self-similar sets. It induces a sequence of the difference operators which correspond to the discrete Laplace operators. When a quasi-harmonic structure satisfies some condition for invariance, we call it a harmonic structure, and we will find

explicit and simple definitions of harmonic functions, the Green function, and the Laplace operator.

In §6, we treat the Dirichlet problem of Poisson's equation and Gauss-Green's formula on p.c.f. self-similar sets. Further, we will see that the Dirichlet problem of Poisson's equation is equivalent to some kind of infinite system of finite difference equations. This fact has been pointed out by Hata-Yamaguti [6] and Yamaguti-Kigami [15] in the simplest case.

In §7, we give an explicit and simple definition of Dirichlet forms associated with regular harmonic structures. And then, the Green function turns out to be the reproducing kernel of the Dirichlet form $(\mathscr{E}, \mathscr{F}_0)$.

As a whole, our approach establishes a kind of classical calculus on p.c.f. self-similar set, and it may be more convenient to the study of harmonic functions and the Laplace operator than the probabilistic approaches.

Finally we mention three related works. The first one and the second one are the probabilistic approaches on a class of self-similar sets.

Lindstrøm [13] has constructed the diffusion processes on nested fractals. The readers may refer to Example 8.5 and Remarks after Definition 1.10 and Definition 4.4.

Kusuoka [12] has given an explicit expression of Dirichlet forms on a class of self-similar sets by using products of random matrices. We also use random matrices A_s's obtained by quasi-harmonic structure. These random matrices are correspondent with those used by Kusuoka. We conjecture that our Dirichlet forms and the Dirichlet forms given by Kusuoka are the same in the correspondent cases.

Shima [14] and Fukushima-Shima [16] have studied the eigenvalue problem of the Laplace operator given by [10]. They apply "the decimation method" and determine the eigenvalues and eigenvectors completely. We conjecture that their method can be applied to our Laplace operator on p.c.f. self-similar sets.

In this paper, we adopt the "directory" structure in numbering the lemmas, propositions, and theorems. For example, Lemma 2.7.1 is the lemma for the proof of Lemma 2.7. Ordinarily, Lemma I.J.K is used only for the proof of Lemma or Proposition or Theorem I.J.

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1. Self-similar sets

In this section, we will define self-similar sets and study their fundamental properties.

First, we introduce the one-sided shift space and give some basic concepts and notations.

Definition 1.1. Let $S = \{s_1, s_2, \ldots, s_N\}$ be a finite set.

(1) The one-sided shift space $\Sigma(S)$ is defined by $\Sigma(S) = S^{\mathbb{N}}$.

(2) For $n \geq 0$, the collection of words consisting of n symbols $W_n(S)$ is defined by $W_n(S) = S^n$. In particular, $W_0(S) = \{\varnothing\}$ where $\varnothing$ denotes the empty word.

(3) $W_*(S) = \bigcup_{n \geq 0} W_n(S)$.

(4) $\Sigma_*(S) = W_*(S) \cup \Sigma(S)$.

We denote the ith symbol of $w \in \Sigma_*(S)$ by w_i or $(w)_i$ and write $w = w_1w_2w_3\cdots$.

It is easy to verify that $\Sigma(S)$ is compact with the metric $d_{\Sigma(S)}$ given by, for w and $v \in \Sigma(S)$,

$$d_{\Sigma(S)}(w, v) = \sum_{i=1}^{\infty} 2^{-i}(1 - \delta_{w_iv_i}),$$

where

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2. (1) For $w \in \Sigma_*(S)$,

$$|w| = \begin{cases} n & \text{if and only if } w \in W_n(S), \\ \infty & \text{if } w \in \Sigma(S). \end{cases}$$

(2) Let $w \in \Sigma_*(S)$, then for $n \geq 0$,

$$[w]_n = \begin{cases} w & \text{if } |w| \leq n, \\ w_1w_2\cdots w_n & \text{otherwise.} \end{cases}$$

In particular, $[w]_0 = \varnothing$ for all $w \in \Sigma_*(S)$.

(3) Let $w \in \Sigma_*(S)$, then

$$\sigma w = \begin{cases} \varnothing & \text{if } w = \varnothing, \\ w_2w_3\cdots w_n & \text{if } w \in W_n(S), \\ w_2w_3w_4\cdots & \text{if } w \in \Sigma(S). \end{cases}$$

(4) Let w and $v \in \Sigma_*(S)$, then $w \wedge v = [w]_k$, where $k = \max\{n | [w]_n = [v]_n\}$.

(5) Let $w \in W_*(S)$ and $v \in \Sigma_*(S)$, then $w \cdot v \in \Sigma_*(S)$ is defined by

$$(w \cdot v)_i = \begin{cases} w_i & \text{if } i \leq |w|, \\ v_{i-|w|} & \text{otherwise.} \end{cases}$$

We always identify $w \in W_*(S)$ with $\tilde{w}: \Sigma(S) \to \Sigma(S)$ defined by $\tilde{w}(v) = w \cdot v$. In particular, $\varnothing \in W_0(S)$ is identified with the identity map of $\Sigma(S)$.

The following definition of a self-similar structure is an abstraction of topological features from the concepts of the self-similar sets studied by Hutchinson [8] and Hata [5].

Definition 1.3. Let K be a compact metric space, S be a finite set, and, for each $s \in S$, let $F_s: K \to K$ be a continuous injection. Then $(K, S, \{F_s\}_{s\in S})$ is said to be a self-similar structure on K (or simply, K is self-similar) if there exists a continuous surjection $\pi: \Sigma(S) \to K$ satisfying

$$\pi \circ s = F_s \circ \pi \quad \text{for every } s \in S.$$

Further, for $w \in W_*(S)$, we define

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n},$$

where $n = |w|$, and $K_w = F_w(K)$. In particular, $F_\varnothing$ is the identity map of K.

The following result implies that π is uniquely determined for a given self-similar structure.

Proposition 1.4. *Let $(K, S, \{F_s\}_{s \in S})$ be a self-similar structure on K. Then, for all $w \in \Sigma(S)$, $\bigcap_{n \geq 0} K_{[w]_n}$ consists of a single point $\pi(w)$.*

Proof. For $w \in \Sigma(S)$, we have $\bigcap_{n \geq 0} [w]_n \cdot \Sigma(S) = \{w\}$, and the diameter of $[w]_n \cdot \Sigma(S) \to 0$ as $n \to \infty$. As π is continuous and $\pi([w]_n \cdot \Sigma(S)) = K_{[w]_n}$, we have $\{\pi(w)\} = \bigcap_{n \geq 0} K_{[w]_n}$.

Let $\mathscr{L} = (K, S, \{F_s\}_{s \in S})$ be a self-similar structure on K. It is easy to see that π is a homeomorphism if and only if $C_K(\mathscr{L}) = \bigcup_{s, t \in S, s \neq t} (K_s \cap K_t)$ is empty. Further, if $C_K(\mathscr{L}) = \varnothing$, then each F_s is a branch of the inverse of a well-defined map $\pi \circ \sigma \circ \pi^{-1}: K \to K$.

Definition 1.5. Let $\mathscr{L} = (K, S, \{F_s\}_{s \in S})$ be a self-similar structure on K. Then the critical set of $\mathscr{L}$ is defined by $C(\mathscr{L}) = \pi^{-1}(C_K(\mathscr{L}))$, and the post critical set of $\mathscr{L}$ is defined by $P(\mathscr{L}) = \bigcup_{n \geq 1} \sigma^n(C(\mathscr{L}))$.

Examples of self-similar structures are given in §8.

Hereafter, we discuss only one fixed self-similar structure, and so we use Σ, W_n, P, C, and so on instead of $\Sigma(S)$, $W_n(S)$, $P(\mathscr{L})$, $C(\mathscr{L})$, and so on.

Definition 1.6. Let $(K, S, \{F_s\}_{s \in S})$ be a self-similar structure on K. Then,

    (1) For $w \in W_*$, $B_w = F_w(\pi(P))$;
    (2) For $m \geq 0$, $P^{(m)} = \bigcup_{w \in W_m} wP$ and $V_m = \pi(P^{(m)})$;
    (3) $V_* = \bigcup_{m \geq 0} V_m$ and $V_m^\circ = V_m - V_0$ for $m = 0, 1, \ldots, *$.

Lemma 1.7. *Let K be a self-similar set. Then, for any w and $v \in W_n$ with $w \neq v$, $K_w \cap K_v = B_w \cap B_v$.*

Proof. As $F_{w \wedge v}$ is injective, we may assume $w_1 \neq v_1$ without loss of generality. Then, since $K_w \cap K_v \subset C_K$,
$$K_w \cap K_v \subset \pi(C \cap w\Sigma) \cap \pi(C \cap v\Sigma).$$
On the other hand, $C \subset \sigma^{-1}(P) = \bigcup_{s \in S} sP$. Hence, $C \cap w\Sigma \subset wP$ and $C \cap v\Sigma \subset vP$ and therefore $K_w \cap K_v \subset B_w \cap B_v$. Trivially, $K_w \cap K_v \supset B_w \cap B_v$, and thus we have proved the lemma.

Lemma 1.8. *If $\pi(w) \in V_0$, then $w \in P$. In other words, $\pi^{-1}(\pi(P)) = P$.*

Proof. As $V_0 = \pi(P)$, there exist $u \in P$ and a word $v \neq \varnothing$ such that $\pi(w) = \pi(u)$ and $v \cdot u \in C$. Then, $\pi(v \cdot u) = \pi(v \cdot w)$ and so, $v \cdot w \in C$. Thus we have $w \in P$.

Corollary 1.9. *For $m \geq 0$, if $\pi(w) \in V_m$, then $w \in P^{(m)}$. In other words, $\pi^{-1}(\pi(P^{(m)})) = P^{(m)}$.*

Proof. As $V_m = \bigcup_{v \in W_m} B_v$, $\pi(w) \in B_v$ for some $v \in W_m$. Using Lemma 1.7, $\pi(w) \in B_v \cap K_{[w]_m} = B_v \cap B_{[w]_m}$. This implies $\pi(w) \in B_{[w]_m} = F_{[w]_m}(\pi(P))$ and hence $\pi(\sigma^m w) \in \pi(P)$. Now using Lemma 1.8, we have $\sigma^m w \in P$. Thus we have
$$[w]_m \cdot \sigma^m w = w \in [w]_m P \subset P^{(m)}.$$

The last three results will underlie many arguments in this paper, and frequently, we may not mention using them.

Next, we introduce a class of canonical measures on self-similar sets.

Definition 1.10. Let $\eta = (\eta_s)_{s \in S}$ satisfy

$$(1.1) \qquad \sum_{s \in S} \eta_s = 1 \quad \text{and} \quad 0 < \eta_s < 1 \text{ for each } s \in S.$$

Then we denote by $\tilde{\mu}_\eta$ the unique Borel measure on Σ with, for all $w \in W_*$,

$$\tilde{\mu}_\eta(w\Sigma) = \eta_{w_1}\eta_{w_2}\cdots\eta_{w_n},$$

where $n = |w|$. Further, let K be a self-similar set. Then the η-self-similar measure on K, μ_η is given by $\pi_*(\tilde{\mu}_\eta)$, that is,

$$\mu_\eta(A) = \tilde{\mu}_\eta(\pi^{-1}(A))$$

for all Borel sets $A \subset K$.

The following theorem explains why μ_η is called η-self-similar.

Theorem 1.11. *Let K be a self-similar set, and let η satisfy* (1.1). *Then there exists a unique Borel measure μ on K such that $\mu(K) = 1$ and*

$$\mu(A) = \sum_{s \in S} \eta_s \mu(F_s^{-1}(A \cap K_s))$$

for all Borel sets $A \subset K$. The unique measure μ coincides with μ_η.
Proof. See Hutchinson [8] or Falconer [3].

Now we define a *post critically finite self-similar set*, which is the main object of our study.

Definition 1.12. Let K be a self-similar set. Then K is said to be post critically finite, or p.c.f. for short, if the post critical set P is finite.

If K is p.c.f., it follows immediately by the above definition that $C, P^{(m)}, B_w$, and V_m are all finite sets and V_* is countably infinite. Further, by Lemma 1.7, $K_w \cap K_v = B_w \cap B_v$ is finite. So a p.c.f. self-similar set may be called a finitely ramified fractal in physicists' terminology. Nested fractals defined by Lindstrøm [13] are p.c.f. self-similar sets. The set of all essential fixed points in his terminology equals π (a post critical set). Lindstrøm's nesting axiom holds for p.c.f. self-similar sets, but the axiom of symmetry is not satisfied in general.

If K is p.c.f., the self-similar measure μ_η becomes simple as follows.

Lemma 1.13. *Let K be p.c.f. and let η satisfy* (1.1). *Then for all $w \in W_*$,*

$$\mu_\eta(K_w) = \eta_{w_1}\eta_{w_2}\cdots\eta_{w_n}.$$

where $n = |w|$.
Proof. It is obvious from the fact that $\pi^{-1}(K_w) = w\Sigma \cup$ a finite set.

Further discussion on p.c.f. self-similar sets is given in Appendix A. We show, in short, that a p.c.f. self-similar set is determined by

$$\mathscr{A} = (S, V_1, V_0, \{F_s|_{V_0}\}_{s \in S}),$$

which will be called the ancestor of p.c.f. self-similar set.

We will give some examples of p.c.f. self-similar sets in §8.

2. QUASI-HARMONIC STRUCTURE

In this section, we first give the concept of a quasi-harmonic structure which generates difference operators H_m on V_m and then introduce the notion of the

harmonic function and quasi-harmonic function as the kernel of the difference operators. Throughout the rest of this paper we fix a p.c.f. self-similar structure $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ and assume that K is connected.

Notations. (1) $l(V) = \{f | f: V \to \mathbb{R}\}$. We use $(f)_p$ or f_p to denote the value of $f \in l(V)$ at $p \in V$. For $p \in V$, $\mathbf{e}_p \in l(V)$ is defined by

$$\mathbf{e}_p^V(q) = \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

When no confusion may occur, we write $\mathbf{e}_p$ instead of $\mathbf{e}_p^V$.

In the following, U and V are finite sets.

(2) $L(U, V) = \{A | A: l(U) \to l(V) \text{ and } A \text{ is linear}\}$. In particular, $L(V) = L(V, V)$. We use $(A)_{pq}$ or A_{pq} to denote $(A\mathbf{e}_p)_q$ for $p \in V$ and $q \in U$. Note that $\sum_{q \in U} A_{pq} f_q = (Af)_p$. For $A \in L(U, V)$, $^tA \in L(V, U)$ is the transpose of A.

(3) If $V \cap U = \varnothing$, then we use $U + V$ to denote the disjoint union of U and V. Note that $l(U + V) = l(U) \oplus l(V)$, we write

$$f = \begin{pmatrix} (f)_U \\ (f)_V \end{pmatrix} \quad \text{for } f \in l(U + V),$$

where $(f)_U = f|_U$ and $(f)_V = f|_V$.

(4) For $D \in L(V)$ with $^tD = D$, we define an equivalence relation $\underset{D}{\sim}$ on V by

(i) $p \underset{D}{\sim} p$ for all $p \in V$.

(ii) For $p \neq q$, $p \underset{D}{\sim} q$ if and only if there exists $\{p_i\}_{i=1}^m \subset V$ with $p_1 = p$, $p_m = q$, and $D_{p_i p_{i+1}} \neq 0$ for $i = 1, 2, \ldots, m-1$.

An equivalence class is called a *D-irreducible part*. Also we say that D is irreducible if and only if D has one irreducible part V.

Now we define a quasi-harmonic structure.

Definition 2.1. A pair $(D, r) \in L(V_0) \times l(S)$ is called a quasi-harmonic structure, or q.h.s. for short, on K if and only if

(1) $r_s > 0$ for each $s \in S$,
(2) $^tD = D$,
(3) D is irreducible,
(4) $D_{pp} < 0$ and $\sum_{p \in V_0} D_{pq} = 0$ for each $p \in V_0$,
(5) $D_{pq} \geq 0$ if $p \neq q$.

Further, for a finite set V, we define $\mathscr{H}(V) \subset L(V)$ by $\mathscr{H}(V) = \{D | D$ satisfies (2) $\sim$ (5) where V_0 is replaced by $V\}$.

In this section we treat a fixed q.h.s. (D, r).

The bijection $F_w: V_0 \to B_w$ induces a natural isomorphism $(F_w)_*: l(V_0) \simeq l(B_w)$. By this identification, we think of D as an element of $L(B_w)$. Then we obtain a difference operator H_m on V_m form the q.h.s. (D, r) as follows.

Definition 2.2. A difference operator $H_m \in L(V_m)$ is defined by

$$H_m = \sum_{w \in W_m} r_w^{-1} \cdot {}^tR_w D R_w,$$

where $R_w \in L(V_m, B_w)$ is the restriction defined by $(R_w f) = f|_{B_w}$ and $r_w = r_{w_1} r_{w_2} \cdots r_{w_n}$. Furthermore, we write, for $m \geq 0$ and $p \in V_m$, $H_{m,p} f = (H_m f)_p$, and $H_p^* f = H_{i(p),p} f$, where $i(p) = \min\{n | p \in V_n\}$.

Remark. This definition of difference operators is a natural extension of those on Sierpinski gasket given in [10] to p.c.f. self-similar sets. In fact, if we let

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \text{and} \quad r = (1, 1, 1),$$

then

$$H_{m,p} f = \sum_{q \in V_{m,p}} (f(q) - f(p)),$$

where $V_{m,p} = \{q | (H_m)_{pq} \neq 0\}$. This coincides with the definition of $H_{m,p} f$ given in [10]. Further details are given in §8.

Lemma 2.3. *For* $m \geq 0$, $H_m \in \mathscr{H}(V_m)$.

Proof. (2), (4), and (5) of Definition 2.1 can be verified immediately from the definition of H_m. We may show that H_m is irreducible. Since D is irreducible on V_0, DR_w is irreducible on B_w. This implies that $p \underset{H_m}{\sim} q$ for p and $q \in B_w$ with $w \in W_m$. Now let $p \in B_w$ and $q \in B_v$ with $v, w \in W_m$. Then by the fact that K is connected, there exists $\{w_i\}_{i=1}^k \subset W_m$ with $w_1 = w$, $w_k = v$, and $K_{w_i} \cap K_{w_{i+1}} = B_{w_i} \cap B_{w_{i+1}} \neq \varnothing$ for $i = 1, 2, \ldots, k-1$. Hence, choosing $p_i \in B_{w_i} \cap B_{w_{i+1}}$, we can chain p and q by $p \underset{H_m}{\sim} p_1$, $p_i \underset{H_m}{\sim} p_{i+1}$, and $p_k \underset{H_m}{\sim} q$. This completes the proof of the lemma.

The above lemma shows that if $H_{m,p} f = 0$, then $f(p)$ is a kind of average of the $f(q)$'s where $(H_m)_{pq} \neq 0$. This observation motivates the following definition of a harmonic function.

Definition 2.4. A continuous function f on K is said to be harmonic if and only if $(H_m f)_{V_m^\circ} = 0$ for all $m \geq 1$.

It is unfortunate that the difference equations defining a harmonic function are overdetermined in general, and we may scarcely expect that there exist nontrivial harmonic functions without some further assumptions on the q.h.s. (D, r). This problem is treated in §4. For a while, we investigate functions satisfying some necessary conditions for being harmonic.

Definition 2.5. A continuous function f on K is said to be quasi-harmonic, or q.h. for short, if and only if $H_p^* f = 0$ for all $p \in V_*^\circ$.

Theorem 2.6. *Suppose that*

(1) $$\#(B_s \cap V_0) \leq 1 \quad \text{for each } s \in S,$$

where $\#(A)$ *is the number of elements in* A. *Then for each* $\rho \in l(V_0)$, *there exists a unique quasi-harmonic function* f *with* $f|_{V_0} = \rho$.

Note that (1) is a condition not on q.h.s. but on the self-similar structure. The proof of Theorem 2.6 is given in Appendix B. We observe some heuristic arguments below and will show a part of them later on.

First we decompose H_m into

$$(2.1) \qquad H_m f = \begin{pmatrix} T_m & {}^t J_m \\ J_m & X_m \end{pmatrix} \begin{pmatrix} (f)_{V_0} \\ (f)_{V_m^\circ} \end{pmatrix},$$

where $T_m \in L(V_0)$, $J_m \in L(V_0, V_m^\circ)$, and $X_m \in L(V_m^\circ)$. In particular, we write $T = T_1$, $J = J_1$, and $X = X_1$.

Now let f be q.h., write $f|_{V_0} = f_0$ and $f|_{V_1^\circ} = f_1$, and consider a procedure to express f_1 by f_0. Obviously, for $p \in V_1^\circ$, $i(p) = 1$, and $H_p^* f = (Jf_0 + Xf_1)_p$. Hence, if X is invertible, we can obtain that $f_1 = -X^{-1}Jf_0$. So we have

$$(f)_{V_1} = \begin{pmatrix} I \\ -X^{-1}J \end{pmatrix} (f)_{V_0},$$

where I is the identity in $L(V_0)$. Furthermore, noting that $V_0 = B_\varnothing$, we have, for each $s \in S$,

$$(2.2) \qquad (f)_{B_s} = A_s (f)_{B_\varnothing},$$

where $A_s = R_s \begin{pmatrix} I \\ -X^{-1}J \end{pmatrix}$.

Next, letting $S_w = F_w(V_1^\circ)$, then the above procedure will turn out to be effective in getting $f|_{S_w}$ from $f|_{B_w}$. Here we introduce notation and observe some facts about B_w and S_w before stating the procedure.

(1) We denote $f|_{B_w}$ by $(f)_w^b$ or f_w^b and also denote $f|_{S_w}$ by $(f)_w^s$ or f_w^s. When no confusion can arise, we use f^b and f^s instead of f_w^b and f_w^s.

(2) Using the bijections $F_w : V_0 \to B_w$ and $F_w|_{V_1^\circ} : V_1^\circ \to S_w$, we always identity

$$L(B_w) \simeq L(V_0), \qquad L(B_w, S_w) \simeq L(V_0, V_1^\circ),$$
$$L(S_w, B_w) \simeq L(V_1^\circ, V_0), \qquad L(S_w) \simeq L(V_1^\circ).$$

(3) $D_w^* \in L(B_w + S_w, B_w)$ and $H_w^* \in L(B_w + S_w, S_w)$ is defined by

$$\begin{pmatrix} D_w^* \\ H_w^* \end{pmatrix} = r_w^{-1} \begin{pmatrix} T & {}^t J \\ J & X \end{pmatrix} \begin{pmatrix} f_w^b \\ f_w^s \end{pmatrix}.$$

Then, we can easily see that, for $p \in S_w$,

$$i(p) = |w| + 1 \quad \text{and} \quad H_p^* f = (H_w^* f)_p.$$

Now applying the above procedure, we obtain that $f^s = -X^{-1}Jf^b$ for a quasi-harmonic function f. Further (2.2) becomes

$$(2.3) \qquad f_{ws}^b = A_s f_w^b,$$

for all $w \in W_*$ and $s \in S$. Hence we can see that

$$(2.4) \qquad f_w^b = A_w (f)_{V_0},$$

where $A_w = A_{w_n} A_{w_{n-1}} \cdots A_{w_1}$.

From the preceding discussions, we next show some results about X_m.

Lemma 2.7. *For each $m \geq 1$,*

(1) X_m *is invertible and $X_m^{-1} \leq 0$, where $A \leq 0$ means $A_{pq} \leq 0$ for all p and q.*

(2) *Let $g_{pq}^m = -(X_m^{-1})_{pq}$ for p and $q \in V_m^\circ$. Then $g_{pp}^m \geq g_{pq}^m$ for each $q \in V_m^\circ$.*

We will prove Lemma 2.7 by using the following fact.

Lemma 2.7.1. *Let* V *be a finite set and let* $X \in L(V)$ *. Suppose that*

(I) $^tX = X$ *and* X *is irreducible,*

(II) $X_{pp} < 0$ *and* $\sum_{q \in V} X_{pq} \leq 0$ *for any* $p \in V$ *,*

(III) $X_{pq} \geq 0$ *if* $p \neq q$ *,*

(IV) $\sum_{q \in V} X_{pq} < 0$ *for some* $p \in V$ *.*

Then we have

(1) X *is invertible and* $X^{-1} \leq 0$ *.*

(2) *Let* $g_{pq} = (-X^{-1})_{pq}$ *for* p *and* $q \in V$ *. Then,* $g_{pp} \geq g_{pq}$ *.*

Proof. Let

$$x_{pq} = \begin{cases} X_{pq} & \text{if } p \neq q, \\ -X_{pq} & \text{if } p = q, \end{cases}$$

and let $y_p = \sum_{q \neq p} x_{pq}/x_{pp}$. Then (II) implies

$$0 < y_p \leq 1 \quad \text{for all } p \in V,$$

and (IV) becomes

(IV)$'$

$$0 < y_p < 1 \quad \text{for some } p \in V.$$

To prove (1), it suffices to show that

(2.5) $$\text{if } Xf \geq 0, \quad \text{then } f \leq 0,$$

because (2.5) implies that $Xf = 0$ if and only if $f = 0$ and $X^{-1}f \leq 0$ for any $f \geq 0$. Now assuming that $Xf \geq 0$ and $\max_{q \in V} f_q = M > 0$. Then choosing $p \in V$ with $f_p = M$, we have

$$y_p f_p \geq \sum_{q \neq p} (x_{pq}/x_{pp}) f_q \geq f_p.$$

Note that from $f_p = M > 0$ and $f_p \geq f_q$, we can see that

$$y_p = 1 \quad \text{and} \quad f_p = f_q \text{ if } x_{pq} \neq 0.$$

Since X is irreducible, the above discussion implies inductively that $y_q = 1$ for all $q \in V$. This contradicts (IV)$'$. Thus we have shown (2.5). To prove (2), fix $p \in V$, let $d_q = g_{pq}$ for each $q \in V$ and assume that $M = \max_{q \in V} d_q > d_p$. Then choosing $r \in V$ with $d_r = M$, as $g_{pq} = (-X^{-1})_{pq}$, we have

$$\sum_{q \neq r} x_{rq} d_q - x_{rr} d_r = 0,$$

and hence

$$d_r \leq \sum_{q \neq r} (x_{rq}/x_{rr}) d_q \leq y_r d_r.$$

Since $d_r = M \geq d_q \geq 0$ for $q \neq r$, we can conclude that

$$y_r = 1, \qquad d_q = M \text{ if } x_{rq} \neq 0, \quad \text{and} \quad x_{rp} = 0.$$

By the fact that X is irreducible, the above discussion implies inductively that $x_{qp} = 0$ for all $q \neq p$. This contradicts to the fact that X is irreducible. Therefore we can conclude Lemma 2.7.1.

Proof of Lemma 2.7. X_m can be decomposed into irreducible parts, that is,

$$V_m^\circ = V_m^{(1)} + V_m^{(2)} + \cdots + V_m^{(i_m)},$$

and

$$X_m = X_m^{(1)} \oplus X_m^{(2)} \oplus \cdots \oplus X_m^{(i_m)},$$

where $V_m^{(i)}$ is an X_m-irreducible part and $X_m^{(i)} \in L(V_m^{(i)})$. Then it suffices to show that each $X_m^{(i)}$ satisfies the assumptions of Lemma 2.7.1. Using Lemma 2.3 and (2.1), we can easily verify (I), (II), and (III). Since H_m is irreducible, for each i, we can choose $p_* \in V_m^{(i)}$ and $q_* \in V_0$ with $(H_m)_{p_* q_*} > 0$.

Noting that $\sum_{q \in V_m} (H_m)_{p_* q} = 0$, we have $\sum_{q \in V_m^\circ} (X_m)_{p_* q} \leq -(H_m)_{p_* q_*} < 0$. Therefore we can conclude that (IV) holds for each $X_m^{(i)}$.

The last lemma will play an important role in §5. We next give a result on A_s.

Lemma 2.8. *For each* $s \in S$, A_s *is a stochastic matrix, that is,* $A_s \geq 0$ *and* $A_s \mathbf{e} = \mathbf{e}$, *where* $\mathbf{e} = \sum_{p \in V_0} \mathbf{e}_p$.

Proof. As $X^{-1} \leq 0$ and $J \geq 0$, we have $A_s \geq 0$. Next for the q.h. function $f \equiv 1$, $(f)_s^b = (f)_{V_0} = \mathbf{e}$. Hence (2.2) implies $A_s = \mathbf{e}$.

As an immediate consequence of Lemma 2.8, we have the "maximal principle" for quasi-harmonic functions as follows.

Theorem 2.9. *Let* f *be quasi-harmonic. Then, for any* x *in* K, *we have*

$$\min_{p \in V_0} f(p) \leq f(x) \leq \max_{p \in V_0} f(p).$$

At the end of this section, we introduce the notion of piecewise quasi-harmonic functions.

Definition 2.10. A continuous function f on K is said to be m-quasi-harmonic, or m-q.h. for short, if and only if $f \circ F_w$ is quasi-harmonic for all $w \in W_m$.

3. QUASI-HARMONIC EXPANSION

This section is devoted to the expansion of a function by a system of piecewise quasi-harmonic functions. This is called a quasi-harmonic expansion, or q.h. expansion for short. We always fix a p.c.f. self-similar structure $(K, S, \{F_s\}_{s \in S})$ and a q.h. structure (D, r) on K. Further we will need sufficiently many piecewise quasi-harmonic functions for the q.h. expansion. So, through this section, we assume that

Assumption (2). There exists a unique q.h. function f with $f|_{V_0} = \rho$ for any $\rho \in l(V_0)$.

We have shown in Theorem 2.6 that (1) implies (2). Now assuming (2), we have

Theorem 3.1. *For each* $\rho \in l(V_m)$, *there exists a unique* m-q.h. f *with* $f|_{V_m} = \rho$. *Especially, for each* $p \in V_m$, *we denote the unique* m-q.h. f *with* $f|_{V_m} = \mathbf{e}_p$ *by* ψ_p^m. *Then for any* $\rho \in l(V_m)$, *the unique* m-q.h. f *with* $f|_{V_m} = \rho$ *is given by* $f = \sum_{p \in V_m} \rho_p \psi_p^m$.

We will use $\{\psi_p\}_{p \in V_*}$ as the basis of the q.h. expansion, where $\psi_p = \psi_p^{i(p)}$. Using Theorem 2.9, we can obtain another important property of m-q.h. functions, which is called the maximal principle.

Theorem 3.2. *Let f be m-q.h. Then, for each $w \in W_*$ with $|w| \geq m$ and for any $x \in K_w$,*

$$\min_{q \in B_w} f(q) \leq f(x) \leq \max_{q \in B_w} f(q).$$

Definition 3.3. Let $f \in l(V_*)$. Then $\alpha \in l(V_*)$ is called a quasi-harmonic expansion of f if, for all $q \in V_*$,

$$\sum_{p \in V_m} \alpha_p \psi_p(q) \to f(q) \quad \text{as } m \to \infty.$$

When no confusion may occur, we denote $\psi_p|_{V_*}$ by ψ_p. Also, we denote $H_w^*(f|_{B_w+S_w})$ by $H_w^* f$.

In the following, we show that every $f \in l(V_*)$ has a unique q.h. expansion.

Lemma 3.4. *Let $f \in l(V_*)$. If $\alpha \in l(V_*)$ is a q.h. expansion of f, then*

$$(3.1) \qquad \begin{cases} \alpha_p = f(p) & \text{for each } p \in V_0, \\ \alpha_w^s = r_w X^{-1} H_w^* f & \text{for each } w \in W_*. \end{cases}$$

Lemma 3.4.1. *For $w \in W_m$ and $p \in V_*$,*

$$H_w^* \psi_p = \begin{cases} r_w^{-1} X \mathbf{e}_p & \text{if } p \in S_w, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. First if $p \in S_w$, then

$$H_w^* \psi_p = r_w^{-1}(J(\psi_p)^b + X(\psi_p)^s) = r_w^{-1} X \mathbf{e}_p.$$

Next, if p does not belong to S_w and $i(p) \geq |w| + 1$, then $\psi_p|_{B_w+S_w} = 0$. Therefore $H_w^* \psi_p = 0$. Finally, if $i(p) < m$, then ψ_p is $i(p)$-q.h. and therefore $H_w^* \psi_p = 0$.

Proof of Lemma 3.4. Let $f_m = \sum_{p \in V_m} \alpha_p \psi_p$, then for all $w \in W_*$,

$$\lim_{m \to \infty} H_w^* f_m = H_w^* f,$$

and for each $p \in V_0$, $f_m(p) = \alpha_p = f(p)$.

On the other hand, using Lemma 3.4.1, we have, for $m \geq |w| + 1$,

$$H_w^* f_m = \sum_{p \in S_w} \alpha_p r_w^{-1} X \mathbf{e}_p = r_w^{-1} X \sum_{p \in S_w} \alpha_p \mathbf{e}_p = r_w^{-1} X \alpha_w^s.$$

Letting $m \to \infty$, we obtain $\alpha_w^s = r_w X^{-1} H_w^* f$ as required.

The above lemma says that each $f \in l(V_*)$ has at most one q.h. expansion given by (3.1). So we define $\alpha(f) \in l(V_*)$ by

$$\begin{cases} (\alpha(f))_p = f(p) & \text{for } p \in V_0, \\ (\alpha(f))_w^s = r_w X^{-1} X_w^* f & \text{for } w \in W_*. \end{cases}$$

We denote $(\alpha(f))_w^s$ by $\alpha_w(f)$. We also define, for all $m \geq 0$, $P_m f = \sum_{p \in V_m} (\alpha(f))_p \psi_p$.

Lemma 3.5. *Let* $f \in l(V_*)$, *then* $P_m f = \sum_{p \in V_m} f(p) \psi_p^m$.

Proof. Since $P_m f$ is m-q.h., by Theorem 3.1, we may show by induction on m that

$(3.2)_m$ $f|_{V_m} = (P_m f)|_{V_m}.$

$(3.2)_0$ is obvious by definition. Now suppose $(3.2)_m$. Note that

$$(3.3) \qquad\qquad P_{m+1} f = P_m f + \sum_{p \in V_{m+1} - V_m} (\alpha(f))_p \psi_p.$$

Hence, if $p \in V_m$, then $(P_{m+1} f)_p = (P_m f)_p = f(p)$. Next for each $w \in W_m$,

$$(P_m f)_w^s = -X^{-1} J (P_m f)_w^b = -X^{-1} J f_w^b.$$

This together with (3.3) shows that

$$(P_{m+1} f)_w^s = (P_m f)_w^s + \alpha_w(f) = -X^{-1} J f_w^b + X^{-1}(J f_w^b + X f_w^s) = f_w^s.$$

This implies that $(P_{m+1} f)_p = f(p)$ for any $p \in V_{m+1} - V_m$.

Combining Lemma 3.4 and Lemma 3.5, we have

Theorem 3.6. *Each* $f \in l(V_*)$ *has the unique quasi-harmonic expansion* $\alpha(f)$.

In the rest of this section, we consider q.h. expansion for continuous functions. The set of all continuous functions on K is denoted by $C(K)$, which is equipped with the supremum norm defined by $|f|_K = \sup_{x \in K} |f(x)|$.

Since V_* is dense in K , the restriction map $i: C(K) \to l(V_*)$ given by $i(f) = f|_{V_*}$ becomes an inclusion map. Hereafter, $C(K)$ is regarded as a subset of $l(V_*)$ in this manner. We also regard $l(V_m)$ as a subset of $l(V_*)$ by the natural map i_m defined by, for $\rho \in V_m$, $i_m(\rho) = \sum_{p \in V_m} \rho_p \psi_p^m$. Then, $P_m: l(V_*) \to l(V_m)$ defined above equals to the projection map for the inclusion i_m .

Noting that $l(V_m) \subset C(K)$, we have

Theorem 3.7. *Let* $f \in l(V_*)$. *Then* $f \in C(K)$ *if and only if* $P_m f$ *is uniformly convergent on* K *as* $m \to \infty$. *And if* $f \in C(K)$, *then* $P_m f$ *converges to* f *uniformly on* K .

Proof. If $P_m f$ is uniformly convergent on K as $m \to \infty$, then the limit, say, $\tilde{f}$ is continuous on K . Therefore from

$$\tilde{f}|_{V_m} = (P_m f)|_{V_m} = f|_{V_m} ,$$

we have $\tilde{f} = f$ and $f \in C(K)$.

Next if $f \in C(K)$, let

$$\varepsilon_m = \max_{w \in W_m} \sup_{x, y \in K_w} |f(x) - f(y)|.$$

Then since f is uniformly continuous on K , $\lim_{m \to \infty} \varepsilon_m = 0$. On the other hand, using Theorem 3.2, we can obtain, for any x ,

$$|P_m f(x) - f(x)| \le |P_m f(x) - f(p)| + |f(p) - f(x)| \le 2\varepsilon_m ,$$

where p is chosen so that $x \in K_w$ and $p \in B_w$ for some $w \in W_m$. Hence $|P_m f - f|_K \le 2\varepsilon_m$ and so $P_m f$ converges to f uniformly on K .

4. HARMONIC STRUCTURE

In this section, we introduce the concept of harmonic structure, which has all the properties required in the further study of calculus on p.c.f. self-similar sets. For example, under harmonic structure, each quasi-harmonic function will turn out to be harmonic.

Definition 4.1. Let (D, r) be a quasi-harmonic structure. Then $\mathscr{S}_r(D) \in L(V_0)$ is defined by

$$\mathscr{S}_r(d) = T - {}^t J X^{-1} J.$$

Harmonic structure will be defined as an eigenvector of the nonlinear transformation $\mathscr{S}_r$. When no confusion can arise, we drop the r of $\mathscr{S}_r$ and write $\mathscr{S}$.

Lemma 4.2. For each $w \in W_*$ and any $f \in l(B_w + S_w)$,

$$r_w^{-1} \mathscr{S}(D) f_w^b = D_w^* f - {}^t J X^{-1} H_w^* f.$$

Proof. This follows immediately by the definition of D_w^* and H_w^* in §2.

The following result will not be applied in this paper. It implies, however, the notable fact that $\mathscr{S}_r : \mathscr{H}(V_0) \to \mathscr{H}(V_0)$, which may be a starting point of some further study.

Proposition 4.3. For each quasi-harmonic structure (D, r), $(\mathscr{S}_r(D), r)$ is also a quasi-harmonic structure.

For the proof of Proposition 4.3, we decompose V_1° into X-irreducible parts $U_1, U_2, \ldots, U_m$ and we define

$$\partial U_i = \{p | p \in V_0 \text{ and } J_{pq} \neq 0 \text{ for some } q \in U_i\}.$$

Lemma 4.3.1. For $p \in V_0$, let $\psi_p \in l(V_1)$ be the quasi-harmonic function with $\psi_p|_{V_0} = \mathbf{e}_p$. Then, for $q \in V_1^\circ$, $\psi_p(q) > 0$ if, and only if $q \in U_i$ and $p \in \partial U_i$ for some $i = 1, 2, \ldots, m$.

Proof. First if $q \in U_i$ and $p \in (\partial U_i)^c$, then $\psi_p|_{\partial U_i} \equiv 0$ and so $\psi_p(q) = 0$. Hence if $\psi_p(q) > 0$ then $q \in U_i$ and $p \in \partial U_i$ for some $i = 1, 2, \ldots, m$. Conversely, if $q \in U_i$ and $\psi_p(q) = 0$, then $\psi_p(q) = \min_{r \in V_1} \psi_p(r)$. As $H_{1,q} \psi_p = 0$, $\psi_p(r) = 0$ for all r with $(H_1)_{qr} \neq 0$. Inductively, we can show that $\psi_p|_{U_i \cup \partial U_i} \equiv 0$. Therefore $p \in (\partial U_i)^c$.

Using Lemma 4.2, we have $\mathscr{S}(D) \mathbf{e}_p = D_w^* \psi_p$, and hence, for all p and $r \in V_0$,

$$(4.1) \qquad \mathscr{S}(D)_{rp} = h_{rp} + \sum_{q \in V_1^\circ} h_{rq} \psi_p(q),$$

where $h_{ab} = (H_1)_{ab}$.

Lemma 4.3.2. For p and $r \in V_0$, if p and $r \in \partial U_i$ for some $i = 1, 2, \ldots, m$, then $\mathscr{S}(D)_{rp} > 0$.

Proof. There exists $q \in U_i$ with $h_{rq} > 0$. And by Lemma 4.3.1, $\psi_p(q) > 0$. Hence by (4.1), $\mathscr{S}(D)_{rp} > 0$.

Proof of Lemma 4.3. We may show that $\mathscr{S}_r(D) \in \mathscr{H}(V_0)$. Noting that $-{}^t J X^{-1} J \geq 0$ and $\mathscr{S}(D) \mathbf{e} = 0$, where $\mathbf{e} = \sum_{p \in V_0} \mathbf{e}_p$, we can easily verify that (2), (4),

and (5) of Definition 2.1 hold. Therefore we may show that $\mathscr{F}(D)$ is irreducible. Then, for p and $q \in V_0$, as H_1 is irreducible, there is a sequence $\{q_i\}_{i=1}^n$ such that $q_1 = p$, $q_n = q$, and $h_{q_i q_{i+1}} > 0$ for $i = 1, 2, \ldots, n-1$. Let $\{q_i\}_{i=1}^n \cap V_0 = \{p_i\}_{i=1}^k$ where $p_i = q_{i_k}$ with $1 = i_1 < i_2 < \cdots < i_k = n$.

Now if $i_{j+1} = i_j + 1$, then $h_{p_j p_{j+1}} > 0$ and hence by (4.1) we have $p_j \underset{\mathscr{F}(D)}{\sim} p_{j+1}$. If $i_{j+1} > i_j + 1$, then p_j and $p_{j+1} \in \partial U_i$ for some i and by Lemma 4.3.2, we have $p_j \underset{\mathscr{F}(D)}{\sim} p_{j+1}$. Therefore we have $p \underset{\mathscr{F}(\widetilde{D})}{\sim} q$. This completes the proof.

Definition 4.4. A quasi-harmonic structure (D, r) is said to be a harmonic structure if there exists $\lambda > 0$ such that

$$\mathscr{F}_r(d) = \lambda^{-1} D.$$

This λ is called the characteristic exponent of the harmonic structure (D, r) and is denoted by $\lambda = \lambda(D, r)$. Further, a harmonic structure (D, r) is said to be regular if $r_s < \lambda(D, r)$ for each $s \in S$.

Remark. One may ask whether there is any harmonic structure on a given p.c.f. self-similar set. In general, we are not ready to answer this question. In the case of nested fractal, Lindstrøm has treated essentially the same problem in [13]. In our words, he has shown that there exists a regular symmetric harmonic structure on every nested fractal. K. Hattori et al. [7] has also treated a similar problem by another approach. From their discussion, we can deduce that there exists a p.c.f. self-similar set where $\mathscr{F}_r$ has no eigenvector with positive eigenvalue for some r.

Examples of harmonic structures are given in §8. We now give some remarkable facts derived from the definition of harmonic structure.

Lemma 4.5. *Let* (D, r) *be a harmonic structure with* $\lambda = \lambda(D, r)$. *If* f *is* m-*q.h., then, for each* $p \in V_m$ *and* $k \geq 0$, $H_{m+k, p} f = \lambda^{-k} H_{m, p} f$.

Proof. Using induction on k, we need to show only the case when $k = 1$. For $w \in W_*$ with $p \in B_w$, by Lemma 4.2, we have

$$r_w^{-1}(D f_w^b)_p = \lambda(D_w^* f - {}^t J X^{-1} H_w^* f)_p.$$

As f is m-q.h., we obtain $r_w^{-1}(D f_w^b)_p = \lambda(D_w^* f)_p$. Here, summing the above equality for all $w \in W_*$ with $p \in B_w$, we have $H_{m, p} f = \lambda H_{m+1, p} f$ as required.

Lemma 4.6. *If* (D, r) *is a harmonic structure, then every quasi-harmonic function is harmonic.*

Proof. If f is q.h., then Lemma 4.5 implies that for all $p \in V_*$ and all $m \geq i(p)$,

$$H_{m, p} f = \lambda^{-(m-i(p))} H_p^* f = 0.$$

Theorem 4.7. *Let* (D, r) *be a harmonic structure. Then for any* $\rho \in l(V_0)$, *there exists a unique harmonic function* f *with* $f|_{V_0} = \rho$.

Proof. If the assumption (1) in Theorem 2.6 is satisfied, then combining Theorem 2.6 and Lemma 4.6, we can deduce the above result. In the general case, we let $\mathscr{L}_m = (K, S_m, \{F_w\}_{w \in S_m})$, where $\mathscr{L} = (K, S, \{F_s\}_{s \in S})$ is the original self-similar structure and $S_m = W_m(s)$. Also, for the original harmonic structure (D, r), we let $r^{(m)} = (r_w)_{w \in S_m}$.

Then, $\mathscr{L}_m$ is a self-similar structure and $(D, r^{(m)})$ is a harmonic structure on $\mathscr{L}_m$. Now we can easily see that $\mathscr{L}_m$ satisfies (1) for sufficiently large m. And so, by the above discussion, there exists a unique $\mathscr{L}_m$-harmonic function f with $f|_{V_0} = \rho$ for each $\rho \in l(V_0)$, where $\mathscr{L}_m$-harmonic means harmonic with respect to $\mathscr{L}_m$ and $(D, r^{(m)})$. We may prove that $\mathscr{L}_m$-harmonic function is $\mathscr{L}$-harmonic. Noting that

$$V_0(\mathscr{L}_m) = V_0(\mathscr{L}), \qquad V_n(\mathscr{L}_m) = V_{nm}(\mathscr{L}), \quad \text{and} \quad H_n(\mathscr{L}_m) = H_{nm}(\mathscr{L}),$$

Lemma 4.6 implies that if f is $\mathscr{L}_m$-harmonic, for all $n \geq 0$,

$$(4.2) \qquad (H_{nm}(\mathscr{L})f)_{V_{nm}^\circ} = 0.$$

On the other hand, using Lemma 4.2, we can show that if

$$(H_k(\mathscr{L})f)_{V_k^\circ} = 0 \quad \text{for some } k \geq 0,$$

then

$$(H_j(\mathscr{L})f)_{V_j^\circ} = 0 \quad \text{for all } j \leq k.$$

This together with (4.2) shows that f is $\mathscr{L}$-harmonic.

By the last theorem, we can use the theory of quasi-harmonic expansion in §3 under a harmonic structure. Hereafter, if (D, r) is a harmonic structure, we drop "quasi-" of quasi-harmonic function, m-quasi-harmonic function, quasi-harmonic expansion and so forth.

The following two results will play an important role in §5 and §6.

Lemma 4.8. *Let* (D, r) *be a harmonic structure with* $\lambda = \lambda(D, r)$. *Then, for all* $p \in V_m$ *and* $q \in V_n$,

$$\lambda^m H_{m,p} \psi_q^n = \lambda^n H_{n,q} \psi_p^m.$$

Proof. First if $m = n$, then $H_{m,p} \psi_q^n = (H_m e_q)_p = (H_m)_{pq}$. Since ${}^t H_m = H_m$, we have $H_{m,p} \psi_q^n = H_{n,q} \psi_p^m$.

Next, if $m \neq n$, suppose $m > n$, then ψ_q^n is n-harmonic and hence Lemma 4.5 implies that $H_{m,p} \psi_q^n = \lambda^{-(m-n)} H_{n,p} \psi_q^n$. Applying the result when $m = n$ and noting that $\psi_p^m|_{V_n} = \psi_p^n|_{V_n}$,

$$H_{m,p} \psi_q^n = \lambda^{-(m-n)} H_{n,q} \psi_p^n = \lambda^{-(m-n)} H_{n,q} \psi_p^m.$$

This completes the proof of Lemma 4.8.

Lemma 4.9. *If* (D, r) *is a harmonic structure with* $\lambda = \lambda(D, r)$, *then for any* $m \geq 1$, *we have*

$$T_m - {}^t J_m X_m^{-1} J_m = \lambda^{-m} D.$$

Proof. By Theorem 4.7, for each $\rho \in l(V_0)$, there exists a harmonic function f with $f|_{V_0} = \rho$. Making use of Lemma 4.5, we have $(H_m f)_{V_0} = \lambda^{-m} D \rho$. Noting that $f|_{V_m^\circ} = -X_m^{-1} J_m \rho$, we obtain

$$(H_m f)_{V_0} = (T_m - {}^t J_m X_m^{-1} J_m) \rho.$$

Consequently, we can get the required equality.

As is shown in §8, a harmonic structure is not always regular. We can see, however, the following result.

Theorem 4.10. *If* (D, r) *is a harmonic structure, then* $r_s < \lambda(D, r)$ *for some* $s \in S$.

Theorem 4.10 will be used for the proof of existence of "admissible" measures in the case of nonregular harmonic structure in §5.

Lemma 4.10.1. *Let* $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ *be a p.c.f. self-similar structure. If* $\omega \in \Sigma$ *is periodic, that is,* $\sigma^n(\omega) = \omega$ *for some* n, *then* $\pi^{-1}(\pi(\omega)) = \{\omega\}$.

Proof. For $w \in W_*$, we define $w^k \in W_*$ for $k = 1, 2, \ldots$ inductively by $w^1 = w$ and $w^{k+1} = w \cdot w^k$. And also, we define $\dot{w} \in \Sigma$ by $\dot{w} = w \cdot w \cdot w \ldots$. Then if ω is periodic, there exists $w \in W_*$ with $\omega = \dot{w}$. Suppose $\tilde{\omega} \in \pi^{-1}(\pi(\omega))$ with $\tilde{\omega} \neq \omega$, then for each $k \geq 1$,

$$\pi(w^k \cdot \tilde{\omega}) = \pi(w^k \cdot \omega) = \pi(\omega).$$

Hence $\pi^{-1}(\pi(\omega))$ contains infinitely many elements. This contradicts to the fact that $\mathcal{L}$ is p.c.f.

Lemma 4.10.2. *Let* (D, r) *be a harmonic structure on* $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$. *If* $\mathcal{L}$ *satisfies the assumption* (1) *of Theorem* 2.6 *then, for any* $p \in V_0$, $(-\mathcal{F}(D))_{pp}$ $< (-T)_{pp}$.

Proof. Recalling (4.1), we have $(-\mathcal{F}(D))_{pp} = (-T)_{pp} - \sum_{q \in V_1^\circ} \psi_{pq} h_{qp}$. Using a similar discussion as in the proof of Lemma 4.3, if $h_{pq} > 0$, then $\psi_{pq} > 0$. Further, if (1) holds, then $\{q | q \in V_1^\circ \text{ and } h_{pq} > 0\} \neq \varnothing$ and so $\sum_{q \in V_1^\circ} \psi_{pq} h_{qp} > 0$. This completes the proof of Lemma 4.10.2.

Lemma 4.10.3. *Let* (D, r) *be a harmonic structure. If* $\dot{w} \in P$ *for* $w \in W_*$, *then* $r_w < \lambda(D, r)^{|w|}$.

Proof. Recalling $H_1 = \sum_{s \in S} r_s^{-1} \cdot {}^t R_s D R_s$, we have

$$(4.3) \qquad\qquad T_{pp} = \sum_{(q, s) \in Q_p} r_s^{-1} D_{qq},$$

where $Q_p = \{(q, s) | q \in V_0 \text{ and } F_s(q) = p\}$. Let $w = w_1 w_2 \cdots w_m$ with $w_i \in S$ and let $p_i = \pi(\sigma^i(\dot{w}))$ for $i = 1, 2, \ldots, m$. Then by Lemma 4.10.1, we obtain, for $i = 1, 2, \ldots, m$, $Q_{p_i} = \{(p_{i+1}, w_i)\}$, where $p_{m+1} = p_1$. Hence by (4.3),

$$T_{p_i p_i} = r_{w_i}^{-1} D_{p_{i+1} p_{i+1}}.$$

Now suppose the assumption (1) of Theorem 2.6 holds, then using Lemma 4.10.2, we have $T_{p_i p_i} < \lambda r_{w_i}^{-1} T_{p_{i+1} p_{i+1}}$, and so $T_{p_1 p_1} < \lambda^{|w|} r_w^{-1} T_{p_1 p_1}$. Consequently $r_w < \lambda^{|w|}$.

In the general case, we change the self-similar structure $\mathcal{L}$ by $\mathcal{L}_m$ as in the proof of Theorem 4.7. We choose m so that $\mathcal{L}_m$ can satisfy (1). Then, by Lemma 4.9, $\lambda(D, r^{(m)}) = \lambda^m$. Therefore the above arguments will imply that $r_w^m < \lambda^{m|w|}$. Hence we have $r_w < \lambda^{|w|}$.

Proof of Theorem 4.10. By the definition of P, if $\mathcal{L}$ is p.c.f., then P contains a periodic element $\dot{w}$. By Lemma 4.10.3, $r_w < \lambda^{|w|}$. And so, $r_s < \lambda$ for at least one $s \in S$.

5. GREEN'S FUNCTION

In this section, we introduce the Green function g associated with a harmonic structure. And, for some appropriate measure μ, we define the Green

operator G_μ as an integral operator whose kernel is g. $-G_\mu$ will turn out to be the inverse of the Laplace operator in §6. We always fix a harmonic structure (D, r) with $\lambda = \lambda(D, r)$.

Definition 5.1. (1) For $w \in W_*$ and $x \in K$, we define $\psi_w(x) \in l(S_w)$ by $(\psi_w(x))_p = \psi_p(x)$ for each $p \in S_w$.

(2) For $m \geq 1$, Ψ_m and $g_m : K \times K \to \mathbb{R}$ are defined by

$$\Psi_m(x, y) = - \sum_{w \in W_{m-1}} r_w {}^t \psi_w(x) X^{-1} \psi_w(y),$$

and

$$g_m(x, y) = \sum_{k=1}^{m} \lambda^{-k} \Psi_k(x, y).$$

By the above definition, we can easily see the following facts.

Lemma 5.2. (1) $\Psi_m \geq 0$,

(2) $\operatorname{supp} \Psi_m \subset \bigcup_{w \in W_{m-1}} K_w \times K_w$,

(3) $\Psi_m|_{V_{m-1} \times V_{m-1}} \equiv 0$,

(4) *for some* $C > 0$, $|\Psi_m|_{K \times K} \leq C(R/\lambda)^m$, *for all* $m \geq 0$, *where* $R = \max_{s \in S} r_s$.

Definition 5.3. For $(x, y) \in K \times K$, we define

$$g(x, y) = \begin{cases} \lim_{m \to \infty} g_m(x, y) & \text{if the limit exists and is finite,} \\ \infty & \text{if } g_m(x, y) \text{ diverges to } \infty \text{ as } m \to \infty. \end{cases}$$

By Lemma 5.2–(1), we can see that $g_m(x, y)$ is monotonically increasing as $m \uparrow \infty$. Hence, the preceding definition is well defined. Making use of Lemma 5.2, we can also verify the following results.

Proposition 5.4. g_m *converges to* g, *as* $m \to \infty$, *uniformly on each compact subset of* $K \times K - \{(x, x) | x \in K\}$. *Furthermore, if* (D, r) *is regular, then* g_m *converges to* g, *as* $m \to \infty$, *uniformly on* $K \times K$.

We next state the assumptions on measures under which the Green operator can be defined.

Definition 5.5. A measure μ is said to be admissible with respect to (D, r) if

(1) μ is a regular Borel measure on K and $\mu(K) = 1$,

(2) $\mu(0) > 0$ for each open set $0 \subset K$,

(3) $\mu(V_0) = 0$,

(4) $\lim_{m \to \infty} \int_K g_D^m \, d\mu < \infty$, where g_D^m is defined by $g_D^m(x) = g_m(x, x)$.

Remark. Using some usual discussion on measures and integration, we can see that (4) is equivalent to

(4') $g_D \in L^1(K, \mu)$,

where g_D is defined by $g_D(x) = g(x, x)$.

The assumptions (1)–(3) in the above definition are not so restrictive, for example, they hold for the self-similar measures μ_η introduced in §1. If (D, r) is regular, then (4) becomes trivial because g_D^m converges to g_D uniformly on $K \times K$. In this case, every measure with (1)–(3) is admissible with respect to (D, r). On the other hand, if (D, r) is nonregular, then (4) is not trivial. In this case, however, we can see that

Proposition 5.6. *There exists a self-similar measure* μ_η *which is admissible with respect to* (D, r).

Lemma 5.6.1. *Let* μ_η *be a self-similar measure.*

$$\text{If } \sum_{s \in S} r_s \eta_s < \lambda, \quad \text{then } \lim_{m \to \infty} \int_K g_D^m \, d\mu < \infty.$$

Proof. Note that $C = \sup_{x \in K_w}(-{}^t\psi_w(x)X^{-1}\psi_w(x))$ is independent of $w \in W_*$. By Lemma 1.13, we can see that

$$\int_K \Psi_m(x, x)\mu_\eta(dx) \leq C \sum_{w \in W_{m-1}} r_w \mu_\eta(K_w) \leq C \left(\sum_{s \in S} r_s \eta_s\right)^{m-1}.$$

Hence,

$$\int_K g_D^m \, d\mu \leq C\lambda^{-1} \sum_{k=1}^m \left(\lambda^{-1} \sum_{s \in S} r_s \eta_s\right)^{k-1}.$$

This completes the proof of Lemma 5.6.1.

Proof of Proposition 5.6. By Theorem 4.10, $r_s < \lambda$ for some $s \in S$. Hence we can choose $\eta = (\eta_s)_{s \in S}$ so that $\sum_{s \in S} r_s \eta_s < \lambda$. So, by Lemma 5.6.1, μ_η satisfies (4) and therefore μ is admissible.

Now we are concerned with the Green operator G_μ associated with (D, r). In the rest of this section, μ is always an admissible measure with respect to (D, r).

Definition 5.7. For $f \in C(K)$ and $x \in K$, we define

$$(G_\mu^m f)(x) = \int_K g_m(x, y)f(y)\mu(dy),$$

for all $m \geq 0$, and

$$(G_\mu f)(x) = \int_K g(x, y)f(y)\mu(dy).$$

In the course of later discussion, we can show that $G_\mu^m f$ converges to $G_\mu f$ uniformly on K and therefore $G_\mu f$ is continuous on K. As a result we will obtain the following main theorem of this section.

Theorem 5.8. (1) *For any* $f \in C(K)$, $G_\mu f \in C(K)$ *and*

$$|G_\mu f|_K \leq \left(\int_K g_D \, d\mu\right) |f|_K.$$

(2) *For any* $m \geq 0$ *and any* $p \in V_m$,

$$\lambda^m H_{m,p}(G_\mu f) = \begin{cases} -\int_K \psi_p^m f \, d\mu & \text{if } p \in V_m^\circ, \\ -\int_K (\psi_p^m - \psi_p) f \, d\mu & \text{if } p \in V_0. \end{cases}$$

In the following we prove the above theorem step by step.

Lemma 5.8.1. *For any* $w \in W_m$,

$$g(F_w(x), F_w(y)) - g_m(F_w(x), F_w(y)) = \lambda^{-m} r_w g(x, y).$$

Proof. Recalling the definition of ψ_w and F_w, we can easily see that

$$\psi_u \circ F_w = \begin{cases} \psi_v & \text{if } u = w \cdot v, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for $w \in W_m$,

$$g(F_w(x), F_w(y)) - g_m(F_w(x), F_w(y))$$

$$= -\sum_{i=1}^{\infty} \lambda^{-(m+i)} \sum_{u \in W_{m+i-1}} r_u {}^t\psi_u(F_w(x)) X^{-1} \psi_u(F_w(y))$$

$$= \lambda^{-m} r_w \left(-\sum_{i=1}^{\infty} \lambda^{-i} \sum_{v \in W_{i-1}} r_v {}^t\psi_v(x) X^{-1} \psi_v(y) \right)$$

$$= \lambda^{-m} r_w g(x, y).$$

Lemma 5.8.2. *For any* $m \geq 0$ *and any* $p \in V_m$,

$$\lambda^m H_{m,p} g^x = \begin{cases} -\psi_p^m(x) & \text{for } p \in V_m^\circ, \\ -\psi_p^m(x) + \psi_p(x) & \text{for } p \in V_0, \end{cases}$$

where g^x *is defined by* $g^x(y) = g(x, y)$.

Proof. Using Lemma 4.8, we can verify that

$$\lambda^m H_{m,p} \psi_w = \lambda^{|w|+1} H_w^* \psi_p^m.$$

Hence

$$\lambda^m H_{m,p} g^x = -\sum_{i=1}^{\infty} -\lambda^{-i} \sum_{w \in W_{i-1}} r_w {}^t\psi_w(x) X^{-1} (\lambda^m H_{m,p} \psi_w)$$

$$= -\sum_{w \in W_*} {}^t\psi_w(x) \alpha_w(\psi_p^m)$$

$$= -\psi_p^m(x) + (P_0 \psi_p^m)(x).$$

Next, we define $G_m \in L(V_m^\circ)$ by $(G_m)_{pq} = g(p, q)$ $(= g_m(p, q))$ for p and $q \in V_m^\circ$. Then we have

Lemma 5.8.3. $G_m = \lambda^{-m}(-X_m)^{-1}$.

Proof. Lemma 5.8.2 implies that, for p and $q \in V_m^\circ$, $(\lambda^m X_m g^p)_q = -\psi_p^m(q)$. Hence we have $\lambda^m X_m G_m = -I$, where I is the identity. This implies the required equality.

Lemma 5.8.4. *For all* x *and* $y \in K$, $g(x, y) \leq g(y, y)$.

Proof. We let $x \neq y$ because this lemma is trivial if $x = y$. Choosing $w \in \Sigma$ so that $\pi(w) = y$, then

$$g_m(y, y) = \sum_{p \in B_{[w]_m}} g_m(p, p)(\psi_p^m(y))^2.$$

Hence

$$\min_{p \in B_{[w]_m}} g_m(p, p) \leq g_m(y, y).$$

And so, we can choose $p_m \in B_{[w]_m}$ so that

$$\limsup_{m \to \infty} g_m(p_m, p_m) \leq g(y, y).$$

On the other hand, we can choose $q_m \in V_m$ so that

$$q_m \to x \quad \text{as } m \to \infty,$$

and

$$\lim_{m \to \infty} g_m(q_m, p_m) = g(x, y).$$

Now combining Lemma 2.7 and Lemma 5.8.3, we have, for all $m \geq 1$,

$$g_m(q_m, p_m) \leq g_m(p_m, p_m).$$

Letting $m \to \infty$, we obtain $g(x, y) \leq g(y, y)$ as required.

Lemma 5.8.5. *For all x and $y \in K$,*

$$g(x, y) - g_m(x, y) \leq g(y, y) - g_m(y, y).$$

Proof. If $g(x, y) - g_m(x, y) > 0$, then by Lemma 5.2(2), there exists $w \in W_m$ such that $x = F_w(\overline{x})$ and $y = F_w(\overline{y})$ for some $\overline{x}$ and $\overline{y} \in K$. Hence by Lemma 5.8.1,

$$g(x, y) - g_m(x, y) = \lambda^{-m} r_w g(\overline{x}, \overline{y}),$$

and

$$g(y, y) - g_m(y, y) = \lambda^{-m} r_w g(\overline{y}, \overline{y}).$$

Therefore by Lemma 5.8.4, we can obtain the required inequality.

Proof of Theorem 5.8. (1) For any $f \in C(K)$, by Lemma 5.8.4 and Lemma 5.8.5, we have, for all $m \geq 0$,

$$|g(x, y)f(y) - g_m(x, y)f(y)| \leq (g_D(y) - g_D^m(y))|f|_K,$$

where $g_0 \equiv 0$. Hence, for all $x \in K$, $(G_\mu f)(x)$ is a finite value and

$$(5.1) \qquad |(G_\mu f)(x) - (G_\mu^m f)(x)| \leq \left(\int_K (g_D - g_D^m) \, d\mu \right) |f|_K.$$

Therefore $G_\mu^m f$ converges to $G_\mu f$ uniformly on K, and so $G_\mu f \in C(K)$. Also letting $m = 0$ in (5.1), we have $|G_\mu f|_K \leq (\int_K g_D \, d\mu)|f|_K$.

(2) We can easily see that

$$\lambda^m H_{m,p}(G_\mu f) = \int_K (\lambda^m H_{m,p} g^y) f(y) \mu(dy).$$

Hence by Lemma 5.8.2, we have the required results.

6. THE LAPLACE OPERATOR

The object of this section is to study the Laplace operator Δ_μ associated with a harmonic structure (D, r) and an admissible measure μ. In §6 and §7, (D, r) is always a harmonic structure with $\lambda = \lambda(D, r)$ and μ is an admissible measure with respect to (D, r).

The Laplace operator is defined as a limit of difference operators Δ_μ^m as follows.

Definition 6.1. (1) $\Delta_\mu^m: l(V_m) \to l(V_m^\circ)$ is defined by, for each $p \in V_m^\circ$, $(\Delta_\mu^m f)_p = \lambda^m \mu_{m,p}^{-1} H_{m,p} f$, where $\mu_{m,p} = \int_K \psi_p^m d\mu$. When no confusion may occur, we also denote $\Delta_\mu^m \circ P_m: C(K) \to l(V_m^\circ)$ by Δ_μ^m.

(2) Let f be a continuous function on K. If there exists $\varphi \in C(K)$ such that

$$\max_{p \in V_m^\circ} |(\Delta_\mu^m f)_p - \varphi(p)| \to 0 \quad \text{as } m \to \infty,$$

then we let $\Delta_\mu f = \varphi$. The domain of Δ_μ is denoted by $\mathscr{D}_\mu$.

By Theorem 6.9 and Corollary 6.10, we will see that the above definition of the Laplace operator justifies the terminologies such as "harmonic function" or "Green function" in the usual sense.

We next introduce the Neumann derivatives at a point of V_0.

Definition 6.2. For $f \in C(K)$ and $p \in V_0$, we let

$$(df)_p = \lim_{m \to \infty} -\lambda^m H_{m,p} f,$$

if the limit of the right-hand side exists and is finite.

As μ is fixed throughout this section, we often use Δ, and $\mathscr{D}$ instead of Δ_μ and $\mathscr{D}_\mu$.

Lemma 6.3. *Let* $u \in C(K)$ *and* $v \in \mathscr{D}$. *Then, as* $m \to \infty$,

$$\lambda^m \sum_{p \in V_m^\circ} u(p) H_{m,p} v \to \int_K u \Delta v \, d\mu.$$

Proof. We let

$$f_m(x) = \lambda^m \sum_{p \in V_m^\circ} (u(p) \mu_{m,p}^{-1} H_{m,p} v) \psi_p^m(x),$$

and

$$f(x) = u(x) \Delta v(x).$$

Then we may show that, as $m \to \infty$,

$$(6.1) \qquad \int_K f_m \, d\mu \to \int_K f \, d\mu.$$

By the definition of Δ, similar discussions as in the proof of Theorem 3.7 imply that, as $m \to \infty$, f_m converges to f uniformly on each compact set in $K - V_0$. Also, the f_m's are equibounded. Therefore, by the Lebesgue convergence theorem, we can prove (6.1).

Applying the last lemma, we can verify that the Neumann derivatives exist for any $f \in \mathscr{D}$.

Lemma 6.4. *Let* $f \in \mathscr{D}$ *and* $p \in V_0$. *Then*

$$(df)_p = -(Df)_p + \int_K \psi_p \Delta f \, d\mu.$$

Lemma 6.4.1. *For* $p \in V_0$,

$$(Df)_p = \lambda^m H_{m,p} f + \lambda^m \sum_{q \in V_m^\circ} \psi_p(q) H_{m,q} f.$$

Proof. By Lemma 4.9,

$$T_m - {}^t J_m X_m^{-1} J_m = \lambda^{-m} D.$$

Hence,

$$(6.2) \qquad Df = \lambda^m (H_m f)_{V_0} - \lambda^m \cdot {}^t J_m X_m^{-1} (H_m f)_{V_m^\circ}.$$

On the other hand, $(H_m \psi_p)_{V_m^\circ} = 0$ implies $-X_m^{-1} J_m \mathbf{e}_p = (\psi_p)_{V_m^\circ}$. And so, we have

$$(6.3) \qquad (-X_m^{-1} J_m)_{qp} = \psi_p(q).$$

Combining (6.2) and (6.3), we can complete the proof.

Proof of Lemma 6.4. Using Lemma 6.3, we have, as $m \to \infty$,

$$\lambda^m \sum_{q \in V_m^\circ} \psi_p(q) H_{m,q} f \to \int_K \psi_p \Delta f \, d\mu.$$

Hence by Lemma 6.4.1, we obtain

$$\lim_{m \to \infty} -\lambda^m H_{m,p} f = -(Df)_p + \int_K \psi_p \Delta f \, d\mu$$

as required.

Next we establish Gauss-Green's formula, that is,

Theorem 6.5. *Let u and $v \in \mathscr{D}$. Then*
(1) $\int_K (u \Delta v - v \Delta u) \, d\mu = \sum_{p \in V_0} (u(p)(dv)_p - v(p)(du)_p)$.
(2) $\int_k \Delta u \, d\mu = \sum_{p \in V_0} (du)_p$.

To prove Theorem 6.5, we need some results on symmetric forms $\mathscr{E}_m$ on $l(V_m)$.

Definition 6.6. $\mathscr{E}_m$ is a symmetric form on $l(V_m)$ defined by

$$\mathscr{E}_m(u, v) = -\lambda^m \, {}^t u H_m v.$$

For ease of notation, we write, for u and $v \in l(V_*)$,

$$\mathscr{E}_m(u, v) = \mathscr{E}_m(P_m u, P_m v).$$

By this manner, we frequently regard $\mathscr{E}_m$ as a symmetric form on $l(V_*)$.

By the above definition, we can immediately see that

Lemma 6.7.

$$\mathscr{E}_m(u, v) = -\lambda^m \sum_{p \in V_m} u(p) H_{m,p} v$$

$$= \lambda^m \sum_{p \in V_m} \sum_{q \in V_m} h_{pq}^m (u(p) - u(q))(v(p) - v(q)),$$

where $h_{pq}^m = (H_m)_{pq}$.

Lemma 6.8. *Let $u \in C(K)$ and $v \in \mathscr{D}$. Then*

$$\lim_{m \to \infty} \mathscr{E}_m(u, v) = \sum_{p \in V_0} u(p)(dv)_p - \int_K u \Delta v \, d\mu.$$

Proof. By Lemma 6.7,

$$\mathscr{E}_m(u, v) = \sum_{p \in V_0} u(p)(-\lambda^m H_{m,p} v) - \lambda^m \sum_{p \in V_m^\circ} u(p) H_{m,p} v.$$

Hence Lemma 6.3 and Lemma 6.4 imply the required equality.

Proof of Lemma 6.5. Obviously $\mathscr{E}_m(u, v) = \mathscr{E}_m(v, u)$. Therefore by Lemma 6.8, we can prove (1). And letting $u \equiv 1$ in (1), we can prove (2).

Next, we are concerned with the Dirichlet problem of Poisson's equation as follows.

Theorem 6.9. *For given $\varphi \in C(K)$ and given $\rho \in l(V_0)$, there exists a unique $f \in \mathscr{D}$ such that*

$$(6.4) \qquad \begin{cases} \Delta f = \varphi, \\ f|_{V_0} = \rho, \end{cases}$$

and this f is given by $f = \sum_{p \in V_0} \rho_p \psi_p - G_\mu \varphi$.

Corollary 6.10. *f is harmonic if and only if $f \in \mathscr{D}$ and $\Delta f = 0$.*

For the proof of Theorem 6.8, we need further results on the symmetric forms $\mathscr{E}_m$ given in the following.

Definition 6.11. (1) For $w \in W_*$, $(\cdot, \cdot)_w$ is a symmetric form on $l(V_*)$ defined by

$$(u, v)_w = -r_w^{-1} \cdot {}^t\alpha_w(u) X \alpha_w(v).$$

(2) $\Phi_m(\cdot, \cdot)$ is a symmetric form on $l(V_*)$ defined by

$$\begin{cases} \Phi_m(u, v) = \sum_{w \in W_{m-1}} (u, v)_w & \text{for } m \geq 1, \\ \Phi_0(u, v) = \mathscr{E}_0(u, v). \end{cases}$$

Lemma 6.12. *$(f, f)_w \geq 0$ and the equality holds if and only if $\alpha_w(f) = 0$.*

Proof. Using Lemma 2.7, we can see that $-X$ is positive definite. This implies Lemma 6.12.

The following is a remarkable fact about the symmetric forms $\mathscr{E}_m$. It is a key result for studying the Dirichlet form $\mathscr{E}$, the limit of $\mathscr{E}_m$ in §7.

Lemma 6.13. *For $m \geq 0$,*

$$\mathscr{E}_m(u, v) = \sum_{k=0}^m \lambda^k \Phi_k(u, v).$$

Lemma 6.13.1. *For each $w \in W_*$, let $\langle u, v \rangle_w = -r_w^{-1} \, {}^t u^b D v^b$. Then*

$$\mathscr{E}_m(u, v) = \lambda^m \sum_{w \in W_m} \langle u, v \rangle_w.$$

Proof. Obvious by the definition of H_m.

Lemma 6.13.2. *For each $w \in W_*$,*

$$\sum_{s \in S} \langle u, v \rangle_{ws} - \lambda^{-1} \langle u, v \rangle_w = (u, v)_w.$$

Proof. First recalling

$$\alpha_w(f) = r_w^{-1} X H_w^* f = r_w^{-1} X^{-1} (J \quad X) \begin{pmatrix} f_w^b \\ f_w^s \end{pmatrix},$$

then we have

(6.5) $$\qquad (u, v)_w = -r_w^{-1} ({}^t u_w^b \; {}^t u_w^s) \begin{pmatrix} {}^t J X^{-1} J & {}^t J \\ J & X \end{pmatrix} \begin{pmatrix} v_w^b \\ v_w^s \end{pmatrix}.$$

We also obtain

(6.6) $$\qquad \langle u, v \rangle_w = -r_w^{-1} ({}^t u_w^b \; {}^t u_w^s) \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_w^b \\ v_w^s \end{pmatrix},$$

and

(6.7) $$\qquad \sum_{s \in S} \langle u, v \rangle_{ws} = -r_w^{-1} ({}^t u_w^b \; {}^t u_w^s) \begin{pmatrix} T & {}^t J \\ J & X \end{pmatrix} \begin{pmatrix} v_w^b \\ v_w^s \end{pmatrix}.$$

Now since (D, r) is a harmonic structure, we have $T - {}^t J X^{-1} J = \lambda^{-1} D$. This together with (6.5), (6.6), and (6.7) completes the proof of Lemma 6.13.2.

Proof of Lemma 6.13. Using Lemma 6.13.2, we see that

$$\lambda^m \sum_{w \in W_{m-1}} \sum_{s \in S} \langle u, v \rangle_{ws} - \lambda^{m-1} \sum_{w \in W_{m-1}} \langle u, v \rangle_w = \lambda^m \Phi_m(u, v).$$

Applying Lemma 6.13.1, this implies

$$\mathscr{E}_m(u, v) - \mathscr{E}_{m-1}(u, v) = \lambda^m \Phi_m(u, v).$$

Hence we can obtain the required result.

Corollary 6.14. *For each* $m \geq 0$, $\mathscr{E}_m(u, u) \leq \mathscr{E}_{m+1}(u, u)$. *And* $\mathscr{E}_m(u, u) = \lim_{k \to \infty} \mathscr{E}_k(u, u)$ *if and only if* u *is* m-*harmonic.*

Proof. By Lemma 6.12, $\mathscr{E}_{m+1}(u, u) - \mathscr{E}_m(u, u) = \lambda^{m+1} \Phi_{m+1}(u, u) \geq 0$. And we can see that $\mathscr{E}_m(u, u) = \lim_{k \to \infty} \mathscr{E}_k(u, u)$ if and only if $\Phi_n(u, u) = 0$ for all $n \geq m$. Hence by Lemma 6.12, $\alpha_w(f) = 0$ for $w \in W_* - W_{m-1}$. This is equivalent to the statement that f is m-harmonic.

Proof of Theorem 6.9. For the uniqueness, let $\varphi \equiv 0$ and $\rho \equiv 0$ and show that $f \equiv 0$. If $f|_{V_0} \equiv 0$ and $\Delta f \equiv 0$, then Lemma 6.8 implies that

$$\lim_{m \to \infty} \mathscr{E}_m(f, f) = 0.$$

Using Corollary 6.14, we can see that f is harmonic. Therefore by $f|_{V_0} \equiv 0$, we obtain $f \equiv 0$. Now, we may prove that if $f = -G_\mu \varphi$, then $\Delta f = \varphi$. By Theorem 5.8,

$$\mu_{m,p}^{-1} \lambda^m H_{m,p} f = \mu_{m,p}^{-1} \int_K \psi_p^m \varphi \, d\mu.$$

Noting that φ is uniformly continuous on K and

$$\max_{p \in V_m^\circ} (\text{diameter of supp } \psi_p^m) \to 0 \quad \text{as } m \to \infty,$$

we can verify that

$$\max_{p \in V_m^\circ} |\lambda^m \mu_{m,p}^{-1} H_{m,p} f - \varphi(p)| \to 0 \quad \text{as } m \to \infty.$$

Therefore $\Delta f = \varphi$.

Finally we remark that the Dirichlet problem for Poisson's equation (6.4) is equivalent to the following infinite system of finite difference equations on V_*,

$$\begin{cases} f|_{V_0} = \rho, \\ \lambda^{i(p)} H_p^* f = \int_K \psi_p \varphi \, d\mu \quad \text{for each } p \in V_*. \end{cases}$$

Such an equivalence was obtained in [15] and [10] for simple cases.

7. DIRICHLET FORM

In this section, we construct a Dirichlet form $\mathscr{E}$ associated with a regular harmonic structure as a limit of symmetric forms $\mathscr{E}_m$ on $l(V_m)$. In the present paper, we will not introduce the detailed study on Dirichlet forms. The reader may refer to Fukushima [4] for the definition and the results on Dirichlet forms.

Throughout this section, we fix a regular harmonic structure (D, r) with $\lambda = \lambda(D, r)$ and an admissible measure μ with respect to (D, r). We remark that under a regular harmonic structure, Green's function g is continuous on $K \times K$ and every self-similar measure is admissible.

By Corollary 6.14, we can see that $\lim_{m \to \infty} \mathscr{E}_m(f, f)$ exists for each $f \in l(V_*)$ if we let the value of limit be ∞. We define a symmetric form $\mathscr{E}$ as follows.

Definition 7.1. A subspace $\mathscr{F} \subset l(V_*)$ is defined by

$$\mathscr{F} = \left\{ f \,\middle|\, f \in l(V_*) \text{ and } \lim_{m \to \infty} \mathscr{E}_m(f, f) < \infty \right\}.$$

And a symmetric form $\mathscr{E}$ on $\mathscr{F}$ is defined by $\mathscr{E}(u, v) = \lim_{m \to \infty} \mathscr{E}_m(u, v)$.

Obviously, $\mathscr{E}$ is nonnegative definite and $\mathscr{E}_m(u, v) = \mathscr{E}(P_m u, P_m v)$. Now, Corollary 6.14 shows that

Proposition 7.2. $l(V_m) \subset \mathscr{F}$ for all $m \geq 0$. And $f \in l(V_m)$ if and only if $\mathscr{E}_m(f, f) = \mathscr{E}(f, f)$.

Also by Lemma 6.8, we can obtain

Proposition 7.3. Let μ be admissible. Then $\mathscr{D}_\mu \subset \mathscr{F}$ and

$$\mathscr{E}(u, v) = \sum_{p \in V_0} u(p)(dv)_p - \int_K u \Delta_\mu v \, d\mu,$$

for $u \in C(K) \cap \mathscr{F}$ and $v \in \mathscr{D}_\mu$.

Here we state the main results of this section.

Theorem 7.4. Let μ be admissible. Then,
  (1) $\mathscr{F} \subset C(K)$,
  (2) $(\mathscr{F}, \mathscr{E})$ is a regular local Dirichlet space on $L^2(K, \mu)$,
  (3) $\mathscr{E}(f, g^x) = f(x) - (P_0 f)(x)$ for all $f \in \mathscr{F}$ and all $x \in K$,
  (4) $(\mathscr{F}, \mathscr{E})$ is the minimal closed extension of $(\mathscr{D}_\mu, \mathscr{E})$,
  (5) $\mathscr{D}_\mu$ is dense in $C(K)$.

In the following, we prove the above theorem step by step.
For ease of notation, we let

$$[f]_m = \sqrt{\mathscr{E}_m(f, f)} \quad \text{and} \quad [f] = \sqrt{\mathscr{E}(f, f)}.$$

First we observe some facts immediately obtained by the preceding definitions.

Lemma 7.4.1. *Let* u *and* $v \in \mathscr{F}$. *Then,*

(1) $[P_m u] = [u]_m$,

(2) $[u]_m \uparrow [u]$ *as* $m \uparrow \infty$,

(3) $[P_m u - P_n u]^2 = [u]_m^2 - [u]_n^2$ *for* $m > n$,

(4) $|\mathscr{E}(u, v)| \leq [u][v]$.

Lemma 7.4.2. *Let* $f \in l(V_*)$ *and* $x \in K$. *Then*

$$\mathscr{E}_m(f, g^x) = (P_m f)(x) - (P_0 f)(x).$$

Proof. By the definition of g and α_w, we can see that

$$\alpha_w(g^x) = -\lambda^{-(|w|+1)} r_w X^{-1} \psi_w(x).$$

Hence $(f, g^x)_w = \lambda^{-(|w|+1)} \cdot {}^t\alpha_w(f)\psi_w(x)$. Therefore by Lemma 6.13,

$$\mathscr{E}_m(f, g^x) = \sum_{k=1}^{m} \lambda^k \Phi_k(f, g^x) = \sum_{w \in W_m} {}^t\alpha_w(f)\psi_w(x).$$

This implies the required result.

It follows immediately from the last lemma that

Lemma 7.4.3. *For any* $x \in K$, $g^x \subset \mathscr{F}$, *and* $\mathscr{E}(g^x, g^x) = g(x, x)$.

Lemma 7.4.4. *If* $f \in \mathscr{F}$, *then* $f \in C(K)$ *and for all* $x \in K$,

$$\mathscr{E}(f, g^x) = f(x) - (P_0 f)(x).$$

Proof. By Lemma 3.7, we may show that $P_m f$ converges uniformly on K. Using Lemma 7.4.1(4), we have

$$|\mathscr{E}(P_m f - P_n f, g^x)| \leq [P_m f - P_n f][g^x],$$

for all $x \in K$ and $m > n$. Applying Lemma 7.4.1(3), we obtain

$$|P_m f(x) - P_n f(x)| \leq ([f]_m^2 - [f]_n^2)^{1/2} g_D(x).$$

Hence

$$|P_m f - P_m f|_K \leq ([f]_m^2 - [f]_n^2)^{1/2} |g_D|_K.$$

Again using Lemma 7.4.1(2), this implies that $P_m f$ converges uniformly on K.

We let, for u and $v \in \mathscr{F}$,

$$\mathscr{E}^*(u, v) = \mathscr{E}(u, v) + \int_K uv \, d\mu.$$

Also, $\mathscr{F}_0$ is defined by

$$\mathscr{F}_0 = \{f | f \in \mathscr{F} \text{ and } P_0 f \equiv 0\}.$$

Lemma 7.4.5. $\mathscr{F}_0$ *is a Hilbert space with the inner product* $\mathscr{E}^*$.

Proof. By Lemma 7.4.4, if $f \in \mathscr{F}_0$, then

(7.1) $$|f|_K \leq [f]|g_D|_K.$$

Hence there exists $C > 0$ such that for all $f \in \mathscr{F}_0$,

$$[f] \leq (\mathscr{E}^*(f, f))^{1/2} \leq C[f].$$

Therefore, we may show that $\mathscr{F}_0$ is a Hilbert space by the inner product $\mathscr{E}$. Let $\{f_m\}_{m \geq 1}$ be a Cauchy sequence in $(\mathscr{F}_0, \mathscr{E})$, that is, there exists $\{\varepsilon_m\}_{m \geq 1}$ such that $\lim_{m \to \infty} \varepsilon_m = 0$ and $[f_m - f_n] < \varepsilon_n$ for all $m \geq n$. Then, for each $k \geq 0$, $\{P_k f_m\}_{m \geq 1}$ is a Cauchy sequence in $l(V_k)$ and hence there exists $f \in l(V_*)$ satisfying that, for each $k \geq 0$, $P_k f_m \to P_k f$ as $m \to \infty$ in $l(V_k)$. Now, for $m \geq n$,

$$[P_k f_m - P_k f_n] = [f_m - f_n]_k \leq [f_m - f_n] < \varepsilon_n.$$

Hence, letting $m \to \infty$, we obtain $[P_k f - P_k f_n] = [f - f_n]_k \leq \varepsilon_n$. Again letting $k \to \infty$, we can see that $f \in \mathscr{F}_0$ and $[f - f_n] \leq \varepsilon_n$. Hence f_m converges to f in $(\mathscr{F}_0, \mathscr{E})$.

Lemma 7.4.6. $\mathscr{F}$ *is a Hilbert space with the inner product* $\mathscr{E}^*$.

Proof. Note that $\mathscr{F} = \mathscr{F}_0 \oplus l(V_0)$ and use Lemma 7.4.5.

Proof of Theorem 7.4. We have shown (1) and (3) in the course of the above arguments.

(2) By Lemma 7.4.6, we see that $\mathscr{E}$ is a closed form on $L^2(K, \mu)$.

(i) *Regularity.* Recalling Proposition 7.2, we can verify that $\bigcup_{m \geq 0} l(V_m)$ is a core.

(ii) *Markov property.* For $u \in \mathscr{F}$, we define $\overline{u} \in l(V_*)$ by

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } 0 < u(x) < 1, \\ 1 & \text{if } 1 \leq u(x), \\ 0 & \text{if } u(x) \leq 0. \end{cases}$$

Then, for each p and $q \in V_*$,

$$(\overline{u}(p) - \overline{u}(q))^2 \leq (u(p) - u(q))^2.$$

Using Lemma 6.7, we have $\mathscr{E}_m(\overline{u}, \overline{u}) \leq \mathscr{E}_m(u, u)$. Therefore $\overline{u} \in \mathscr{F}$ and $\mathscr{E}(\overline{u}, \overline{u}) \leq \mathscr{E}(u, u)$. Thus, $\mathscr{E}$ is stable under the unit contraction.

(iii) *Local property.* Let f_1 and $f_2 \in \mathscr{F}$ with $\text{supp} f_1 \cap \text{supp} f_2 = \varnothing$. Then, for sufficiently large m, there exists $U_i \subset W_m$ such that

$$\text{supp} f_i \subset \bigcup_{w \in U_i} K_w \quad \text{and} \quad \left(\bigcup_{w \in U_1} K_w \right) \cap \left(\bigcup_{w \in U_2} K_w \right) = \varnothing.$$

Hence using Lemma 6.7, we can see that $\mathscr{E}(f_1, f_2) = 0$.

Thus we have shown that $(\mathscr{F}, \mathscr{E})$ is a regular local Dirichlet space on $L^2(K, \mu)$.

(4) Since $\mathscr{F} = \mathscr{F}_0 \oplus l(V_0)$ and $l(V_0) \subset \mathscr{D}$, we may prove that $\mathscr{D}_0 = \mathscr{F}_0 \cap \mathscr{D}$ is a dense subset of the Hilbert space $(\mathscr{F}_0, \mathscr{E})$. For $p \in V_*$, we let $\varphi_p^m = \mu_{m,p}^{-1} \psi_p^m$.

Then, using Proposition 7.3, Lemma 7.4.3, and Lemma 7.4.4,

$$[G_\mu \varphi_p^m - g^p]^2 = [G_\mu \varphi_p^m]^2 - 2\mathscr{E}(G_\mu \varphi_p^m, g^p) + g(p, p)$$

$$= \int_{K \times K} g(x, y)\varphi_p^m(x)\varphi_p^m(y)\mu(dy)\mu(dx)$$

$$- 2 \int_K g(p, y)\varphi_p^m(y)\mu(dy) + g(p, p).$$

By a similar discussion as in the proof of Theorem 6.9, we can see that

$$[G_\mu \varphi_p^m - g^p] \to 0 \quad \text{as } m \to \infty.$$

On the other hand, by Lemma 5.8.2, $-\sum_{q \in V_m} h_{pq}^m g^q = \psi_p^m$. Hence,

$$\left[G_\mu \left(-\sum_{q \in V_m} h_{pq}^m \varphi_q^m \right) - \psi_p^m \right] \to 0 \quad \text{as } m \to \infty.$$

Therefore, $\mathrm{cl}(\mathscr{D}_0)$ in $(\mathscr{F}_0, \mathscr{E})$ contains every ψ_p^m. Note that $\bigcup_{m \geq 1} l(V_m)$ is dense in $(\mathscr{F}_0, \mathscr{E})$, we can conclude that $\mathscr{D}_0$ is dense in $(\mathscr{F}_0, \mathscr{E})$.

(5) Note that $\bigcup_{m \geq 0} l(V_m)$ is dense in $C(K)$. Using (7.1), we can show that $\mathscr{D}$ is dense in $C(K)$.

EXAMPLES

This section is devoted to five examples of p.c.f. self-similar sets and harmonic structures on them.

First in item (I), we describe S, C, q for each $q \in \pi^{-1}(C)$, P and p for each $p \in \pi^{-1}(P)$. According to the discussion in Appendix A, item (I) gives complete information about π and determines the equivalence class of the self-similar structure.

Secondly in item (II), we give some harmonic structures.

Finally in item (III), we describe the concrete self-similar structure, where each self-similar set is immersed in $\mathbb{C}$ and each F_s is the contracting similitude of $\mathbb{C}$.

In each example, we use $\dot{s}$ for $s \in S$ which is defined by $(\dot{s})_i = s$ for all $i \geq 1$.

Example 8.1. Interval.

(II) $S = \{1, 2\}$.

$C = \{1\dot{2}, 2\dot{1}\}$, $q = \pi(1\dot{2}) = \pi(2\dot{1})$,

$P = \{\dot{1}, \dot{2}\}$, $p_1 = \pi(\dot{1})$, $p_2 = \pi(\dot{2})$.

(II) $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$,

$r = (\alpha, 1 - \alpha)$ for $0 < \alpha < 1$, $\lambda = 1$.

When $\alpha = \frac{1}{2}$, $(\mathscr{F}, \mathscr{E}^*)$ equals the H^1-Sobolev spaces.

$$\text{(III)} \ p_1 = 0, \ p_2 = 1, \ q = \tfrac{1}{2},$$
$$F_1(z) = \tfrac{1}{2}z, \ F_2(z) = \tfrac{1}{2}z - \tfrac{1}{2}.$$

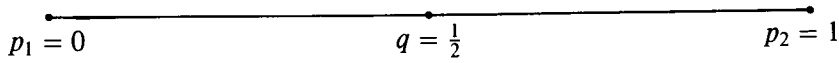$p_1 = 0$ $q = \tfrac{1}{2}$ $p_2 = 1$

FIGURE 1

Example 8.2. Sierpinski gasket.

(I) $S = \{1, 2, 3\}$.
$C = \{2\dot{3}, 3\dot{2}, 1\dot{3}, 3\dot{1}, 1\dot{2}, 2\dot{1}\}, \ q_1 = \pi(1\dot{2}) = \pi(2\dot{1}),$
$q_2 = \pi(1\dot{3}) = \pi(3\dot{1}), \ q_3 = \pi(1\dot{2}) = \pi(2\dot{1}).$
$P = \{\dot{1}, \dot{2}, \dot{3}\}, \ p_s = \pi(\dot{s})$ for $s = 1, 2, 3$.

(II) $D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$,

$r = (1, 1, 1), \ \lambda = \tfrac{5}{3}$.

(III) $p_1, p_2,$ and p_3 are the vertices of regular triangle.
$q_1 = \tfrac{1}{2}(p_2 p_3), \ q_2 = \tfrac{1}{2}(p_1 p_3), \ q_3 = \tfrac{1}{2}(p_1 p_2),$
where $\tfrac{1}{2}(p_i p_j)$ is the midpoint of p_i and p_j.
$F_s(z) = \tfrac{1}{2}(z - p_s) + p_s$.
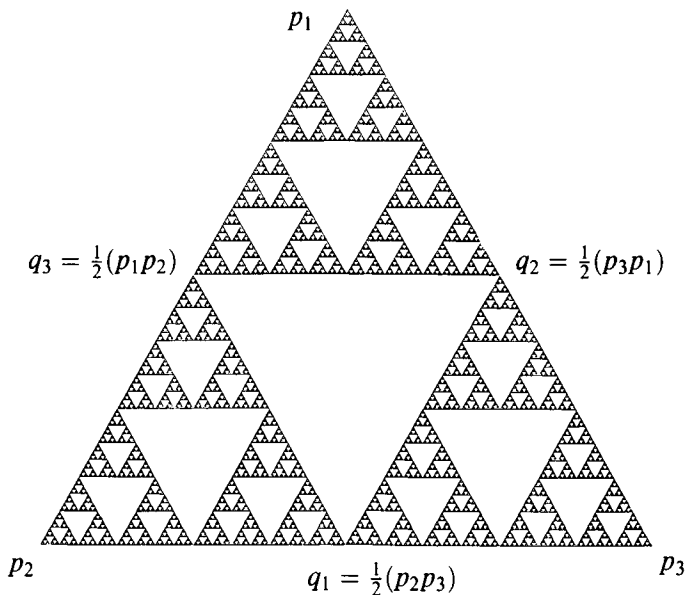

FIGURE 2

Example 8.3.

(I) $S = \{1, 2, 3\}$.

$C = \{1\dot{2}, 2\dot{1}, 3\dot{1}\}$, $q = \pi(1\dot{2}) = \pi(2\dot{1}) = \pi(3\dot{1})$.

$P = \{\dot{1}, \dot{2}\}$; $p_1 = \pi(\dot{1})$, $p_2 = \pi(\dot{2})$.

(II) $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$,

$r = (\alpha, 1 - \alpha, \beta)$ for $0 < \alpha < 1$ and $\beta > 0$. $\lambda = 1$.

If $\beta \geq 1$, then this harmonic structure is not regular.

(III) $p_1 = 0$, $p_2 = 1$, $q = \frac{1}{2}$.

$F_1(z) = \frac{1}{2}z$, $F_2(x) = \frac{1}{2}z - \frac{1}{2}$, $F_3(z) = \frac{\sqrt{-1}}{4}z + \frac{1}{2}$.

$$\frac{1}{2} + \frac{1}{4}\sqrt{-1}$$
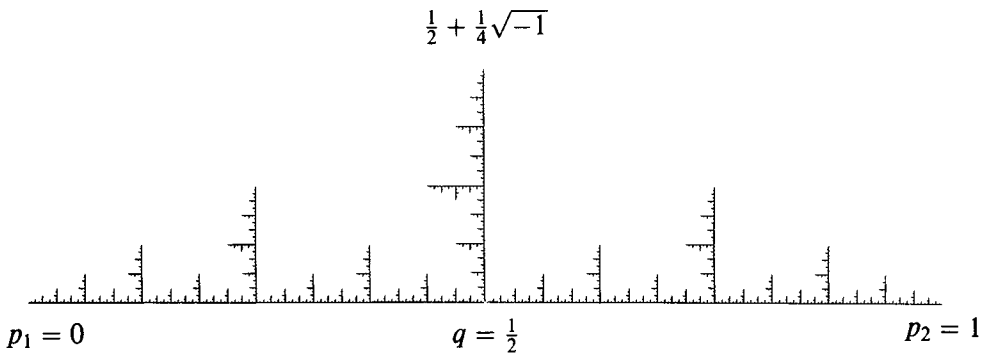
$p_1 = 0$ $q = \frac{1}{2}$ $p_2 = 1$

FIGURE 3

Example 8.4. Hata's tree-like set.

This tree-like self-similar set was found by Hata [5].

(I) $S = \{1, 2\}$.

$C = \{11\dot{2}, 2\dot{1}\}$, $q = \pi(11\dot{2}) = \pi(2\dot{1})$.

$P = \{\dot{1}, \dot{2}, 1\dot{2}\}$; $p_s = \pi(\dot{s})$ for $s \in S$, $p_3 = \pi(1\dot{2})$.

This self-similar structure does not satisfy (1).

(II) $D = \begin{pmatrix} -(1 + \alpha^{-1}) & 1 & \alpha^{-1} \\ 1 & -1 & 0 \\ \alpha^{-1} & 0 & -\alpha^{-1} \end{pmatrix}$,

$r = (\alpha, 1 - \alpha^2)$ for $0 < \alpha < 1$, $\lambda = 1$.

(III) $p_1 = 0$, $p_2 = 1$, $p_3 = \beta$, $q = |\beta|^2$.

$F_1(z) = \beta\overline{z}$, $F_2(z) = (1 - |\beta|^2)\overline{z} + |\beta|^2$,

where $|\beta| < 1$, $|1 - \beta| < 1$ and $\text{Im}\,\beta \neq 0$.

In Figure 4, $\beta = 0.4 - 0.3\sqrt{-1}$.

$$p_1 = 0 \qquad q \qquad\qquad\qquad p_2 = 1$$

$$\beta = 0.4 - 0.3\sqrt{-1}$$
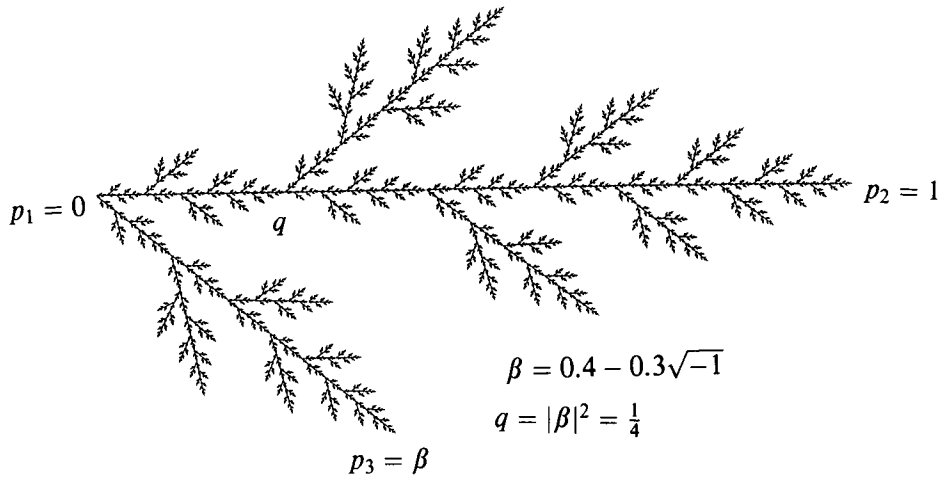$$q = |\beta|^2 = \tfrac{1}{4}$$

$$p_3 = \beta$$

FIGURE 4

Example 8.5. Nested fractal.

$S = \{1, 2, \ldots, N\}$. Each F_s is a ν-similitude of $\mathbb{R}^n$, that is, $|F_s(x) - F_s(y)| = \nu|x - y|$ for all $x, y \in \mathbb{R}^n$.

V_0 is the subset of the fixed points of F_s's. An element of V_0 is called an essential fixed point by Lindstrøm.

Let P be the matrix of invariant transition probabilities of random walk on V_0 given by Lindstrøm. Then $(P - I, (\nu, \ldots, \nu))$ is a regular harmonic structure on the nested fractal, where I is the identity matrix. We conjecture that our Dirichlet forms and the Dirichlet forms associated with the diffusion processes given by Lindstrøm [13] are the same.

APPENDIX A

We begin with some obvious facts.

Lemma A.1. *Let* $(K, S, \{F_s\}_{s\in S})$ *be a p.c.f. self-similar structure. Then,*

(1) $V_1 = \bigcup_{s\in S} F_s(V_0)$.
(2) *If* $V_0 \neq \varnothing$, *then for all* $s \in S$, $F_s(V_0) - V_0 \neq \varnothing$.

Definition A.2. Let S, V, and U be finite sets with $V \supset U$ and let $G_s: U \to V$ be an injection for each $s \in S$. Then $\mathscr{A} = (S, V, U, \{G_s\}_{s\in S})$ is called an ancestor if

(1) $V = \bigcup_{s\in S} G_s(U)$.
(2) *If* $U \neq \varnothing$, *then for all* $s \in S$, $G_s(U) - U \neq \varnothing$.

In particular, for a p.c.f. self-similar structure $\mathscr{L} = (K, S, \{F_s\}_{s\in S})$, $\mathscr{A}(\mathscr{L}) = (S, V_1, V_0, \{F_s|_{V_0}\}_{s\in S})$ is called the ancestor of $\mathscr{L}$.

By the following procedure, we can construct a p.c.f. self-similar structure from a given ancestor.

Definition A.3. Let $\mathscr{A} = (S, V, U, \{G_s\}_{s\in S})$ be an ancestor. Then, for each $x \in V$, $\mathscr{A}_x \subset \Sigma(S)$ is the collection of all w satisfying that there exists

$\{x_i\}_{i\geq 1} \subset U$ such that $G_{w_1}(x_1) = x$ and $G_{w_i}(x_i) = x_{i-1}$ for all $i \geq 2$. Further a relation $\underset{\mathscr{A}}{\sim}$ on $\Sigma(S)$ is defined by

$$w \underset{\mathscr{A}}{\sim} v \text{ if and only if } w = v \text{ or there exist } x \in V, \ w_*, \ v_* \in \mathscr{A}_x$$

and $u \in W_*(S)$ satisfying $w = u \cdot w_*$ and $v = u \cdot v_*$.

Lemma A.4. *Let $\mathscr{A} = (S, V, U, \{G_s\}_{s\in S})$ be an ancestor. Then,*

(1) $\underset{\mathscr{A}}{\sim}$ *is an equivalence relation on $\Sigma(S)$.*

(2) *If $w \underset{\mathscr{A}}{\sim} v$, then $s \cdot w \underset{\mathscr{A}}{\sim} s \cdot v$ for all $s \in S$.*

(3) *$K_{\mathscr{A}} = \Sigma(S)/\underset{\mathscr{A}}{\sim}$ equipped with the quotient topology from $\Sigma(S)$ is metrizable.*

(4) *For each $s \in S$, $F_s^{\mathscr{A}} = \pi_{\mathscr{A}} \circ s \circ \pi_{\mathscr{A}}^{-1}$ is a well-defined continuous injection, where $\pi_{\mathscr{A}} : \Sigma(S) \to K_{\mathscr{A}}$ is the natural projection.*

(4) of the last lemma follows from results in Kameyama [9]. The other parts follow immediately from Definition A.3.

Theorem A.5. *Let $\mathscr{A} = (S, V, U, \{G_s\}_{s\in S})$ be an ancestor. Then $\mathscr{L}(\mathscr{A}) = (K_{\mathscr{A}}, S, \{F_s^{\mathscr{A}}\}_{s\in S})$ is a p.c.f. self-similar structure.*

It is natural to expect that $\mathscr{L}(\mathscr{A}(\mathscr{L})) \simeq \mathscr{L}$ for every p.c.f. self-similar structure $\mathscr{L}$. Before stating the results, we may give the exact meaning of "$\simeq$".

Definition A.6. Let $\mathscr{L} = (K, S, \{F_s\}_{s\in S})$ and $\mathscr{K} = (L, T, \{G_t\}_{t\in T})$ be two self-similar structures. Then, $\mathscr{L} \leq \mathscr{K}$ if and only if there exist a bijection $\tau : S \to T$ and a continuous surjection $h : K \to L$ such that $G_{\tau(s)} \circ h = h \circ F_s$ for any $s \in S$. $\mathscr{L} \simeq \mathscr{K}$ if and only if $\mathscr{L} \geq \mathscr{K}$ and $\mathscr{L} \leq \mathscr{K}$.

Definition A.7. Let $\mathscr{A} = (S, V, U, \{G_s\}_{s\in S})$ and $\mathscr{B} = (T, W, X, \{H_t\}_{t\in T})$ be two ancestors. Then $\mathscr{A} \leq \mathscr{B}$ if and only if there exist a bijection $\tau : S \to T$ and an injection $h : V \to W$ such that $h(U) \subset X$ and $H_{\tau(s)} \circ h = h \circ G_s$ for each $s \in S$.

Finally, we can give the main result of Appendix A.

Theorem A.8. *Let $\mathscr{A}$ and $\mathscr{B}$ be ancestors and let $\mathscr{L}$ and $\mathscr{K}$ be p.c.f. self-similar structures. Then,*

(1) *If $\mathscr{L} \leq \mathscr{K}$, then $\mathscr{A}(\mathscr{L}) \leq \mathscr{A}(\mathscr{K})$.*

(2) *If $\mathscr{A} \leq \mathscr{B}$, then $\mathscr{L}(\mathscr{A}) \leq \mathscr{L}(\mathscr{B})$.*

(3) *$\mathscr{L}(\mathscr{A}(\mathscr{L})) \simeq \mathscr{L}$.*

(4) *$\mathscr{A}(\mathscr{L}(\mathscr{A})) \leq \mathscr{A}$.*

The proof is straightforward and left to the reader.

APPENDIX B

This appendix is devoted to the proof of Theorem 2.6.

Theorem 2.6. *Suppose that*

(1) $$\#(B_s \cap V_0) \leq 1 \quad \text{for each } s \in S.$$

Then, for each $\rho \in l(V_0)$, there exist a unique quasi-harmonic function f with $f|_{V_0} = \rho$.

In the following, we let $V_0 = \{1, 2, \ldots, M\}$ and $l(V_0) = \mathbb{R}^M$ for ease of notation.

Before starting the discussion, we summarize the results of the heuristic arguments in §2.

Lemma B.1. *Let $A_s = R_s(_{-X^{-1}J}^{\;\;I})$ for each $s \in S$. Then A_s is a stochastic matrix and for every quasi-harmonic function f and any $w \in W_*$,*

$$(f)_w^b = A_w(f)_{V_0},$$

where $A_w = A_{w_m} A_{w_{m-1}} \cdots A_{w_1}$ if $w = w_1 w_2 \cdots w_m$.

Now, for $f \in l(V_0)$, we define $v(f) = \max_{1 \le i, j \le M} |f_i - f_j|$.

Lemma B.2. *For any $f \in l(V_0)$ with $v(f) \ne 0$, $v(A_s f) < v(f)$ for each s.*

Proof. As $v(f) \ne 0$, we have

$$L_1 = \max_{j \in V_0} f_j > L_2 = \min_{j \in V_0} f_j.$$

We extend f to a function on V_1 by $f|_{V_1^\circ} = -X^{-1}Jf$. Then, $A_s f = R_s f$ and, for all $s \in S$,

$$L_2 \le \min_{j \in B_s} f_j \le \max_{j \in B_s} f_j \le L_1.$$

Now, if $v(f) = v(A_s)$, then there exist p and $q \in B_s$ such that $f(p) = L_1$ and $f(q) = L_2$. By the assumption (1), p or $q \in V_1^\circ$ and so, without loss of generality, we may assume $p \in V_1^\circ$. As D is irreducible, hence we can choose $\{q_1, q_2, \ldots, q_{n+1}\} \subset B_s$ so that

(i) $q_1 = p$, $q_{n+1} = q$,
(ii) $\{q_1, q_1, \ldots, q_n\} \subset V_1^\circ$,
(iii) $D_{F_s^{-1}(q_i)F_s^{-1}(q_{i+1})} > 0$ for $i = 1, 2, \ldots, n$.

Then, by the definition of H_1,

$$(H_1)_{q_i q_{i+1}} \ge D_{F_s^{-1}(q_i)F_s^{-1}(q_{i+1})} > 0.$$

On the other hand, if $f(r) = L_1$ for $r \in V_1^\circ$, then $f(r') = L_1$ for all r' with $(H_1)_{rr'} \ne 0$ because

$$H_{1,r}f = \sum_{r'} (H_1)_{rr'} (f(r') - f(r)) = 0.$$

Therefore, applying the above discussion inductively, we can show that $f(q_i) = L_1$ for $i = 1, 2, \ldots, n + 1$. This contradicts the fact that $f(q_{n+1}) = f(q) = L_2 < L_1$. Hence we have $v(A_s f) < v(f)$.

Lemma B.3. *There exists a constant C with $0 < C < 1$ such that, for all $f \in l(V_0)$ and $w \in W_*$, $v(A_w f) \le C^{|w|} v(f)$.*

Proof. We define $V = \{f | f \in l(V_0)$ and $\sum_{i=1}^M f_i = 0\}$ and $Q: l(V_0) \to V$ by, for each i, $(Qf)_i = f_i - M^{-1} \sum_{i=1}^M f_i$. Then, it follows that $v(AQf) = v(Af)$ and $v(f) = v(Qf)$. Hence, we have

$$\max_{v(f) \ne 0} v(A_s f)/v(f) = \max_{f \in V, v(f)=1} v(A_s f)/v(f).$$

Now, as $\{f | f \in V, v(f) = 1\}$ is compact, using Lemma B.2, we have $C_s = \max_{v(f) \neq 0} v(A_s f)/v(f) < 1$. Therefore, let $C = \max_{s \in S} C_s$, then $0 < C < 1$ and $v(A_w f) \leq C^{|w|} v(f)$ for all $f \in l(V_0)$ and $w \in W_*$.

Next, for given $\rho \in l(V_0)$, we define $f \in l(V_*)$ by, for each $w \in W_*$, $f_w^b = A_w \rho$.

Lemma B.4. *Let* $\{p_n\}_{n=1}^{\infty} \subset V_*$ *and* $p_n \to p$ *as* $n \to \infty$ *for some* $p \in K$. *Then* $\{f(p_i)\}_{i=1}^{\infty}$ *is convergent as* $n \to \infty$.

Proof. Let $K_{m,p} = \bigcup_{w \in W_m, p \in K_w} K_w$, then $K_{m,p} \supset K_{m+1,p}$ and $K_{m,p}$ is a neighborhood of p. Hence, for given $\varepsilon > 0$, we can choose m and n_0 so that $2C^m v(\rho) < \varepsilon$ and, if $n > n_0$, $p_n \in K_{m,p}$.

Recalling that A_s is a stochastic matrix, we have

$$\min_{q \in K_{m,p} \cap V_m} f(q) \leq f(p_n) \leq \max_{q \in K_{m,p} \cap V_m} f(q).$$

And so, using Lemma B.3, we can see that, if $k_1, k_2 > n_0$, then

$$|f(p_{k_1}) - f(p_{k_2})| \leq 2 \max_{w \in W_m, p \in K_w} v(A_w f) < \varepsilon.$$

Hence $\{f(p_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

Using the above Lemma, we can extend $f \in l(V_*)$ to a continuous function f on K. In fact, if $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset V_*$ and both sequences converges to the same point $p \in K$, let

$$\tilde{p}_n = \begin{cases} p_m & \text{if } n = 2m + 1, \\ q_m & \text{if } n = 2m. \end{cases}$$

Then using Lemma B.4, $\{f(\tilde{p}_n)\}$ is convergent and so we can show that

$$\lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} f(q_n).$$

Thus, this f is the harmonic function with $f|_{V_0} = \rho$. Uniqueness is obvious because $f|_{V_*}$ is uniquely determined as is observed in Lemma B.4.

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