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Author(s)	Masaoka, Hiroaki; Segawa, Shigeo
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## HARMONIC DIMENSION OF COVERING SURFACES AND MINIMAL FINE NEIGHBORHOOD

Dedicated to Professor Yukio Kusunoki on his seventieth birthday

HIROAKI MASAOKA and SHIGEO SEGAWA

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### Introduction

Consider an *end*  $\Omega$  in the sense of Heins [7]:  $\Omega$  is a relatively noncompact subregion of an open Riemann surface  $R$ , which is of null boundary and has a single ideal boundary component, and the relative boundary  $\partial\Omega (\neq \emptyset)$  of  $\Omega$  consists of finitely many analytic closed Jordan curves. We denote by  $\mathcal{P}(\Omega)$  the class of nonnegative harmonic functions on  $\Omega$  with vanishing boundary values on  $\partial\Omega$ . The minimum number of elements of  $\mathcal{P}(\Omega)$  generating  $\mathcal{P}(\Omega)$  provided that such a finite set exists, otherwise  $\infty$ , is referred to as the *harmonic dimension* of  $\Omega$ ,  $\dim \mathcal{P}(\Omega)$  in notation. In terms of Martin compactification, it is known that  $\dim \mathcal{P}(\Omega)$  coincides with the number of minimal boundary points in the Martin compactification of  $R$  (cf. e.g. [4]), and hence  $\dim \mathcal{P}(\Omega) = \dim \mathcal{P}(\Omega')$  for any pair  $(\Omega, \Omega')$  of ends of  $R$ .

Denote by  $D$  the punctured unit disc  $\{0 < |z| < 1\}$  and let  $W$  be a  $p$ -sheeted ( $1 < p < \infty$ ) unlimited covering surface of  $D$  such that the projection of branch points of  $W$  accumulates only at  $z = 0$ . Then  $W$  is naturally considered as a subregion of an open Riemann surface  $R$  which is a  $p$ -sheeted unlimited covering surface of  $\{0 < |z| \leq \infty\}$ . If  $R$  has a single ideal boundary component, it is seen that  $W$  is an end. We denote by  $\mathcal{E}_p$  the class of ends  $W$  of this kind. In this paper we are especially concerned with ends belonging to  $\mathcal{E}_p$ . First of all, it is noted that  $1 \leq \dim \mathcal{P}(W) \leq p$  for each  $W \in \mathcal{E}_p$  (cf. Heins [7]). Roughly speaking, if each sheet of  $W$  is closely connected with any of the other sheets, then  $\dim \mathcal{P}(W) = 1$  and if each sheet of  $W$  is faintly connected with the other sheets, then  $\dim \mathcal{P}(W) = p$ . This intuition is realized as follows. Consider two positive decreasing sequences  $\{a_n\}$  and  $\{b_n\}$  satisfying  $b_{n+1} < a_n < b_n < 1$  and  $\lim_{n \rightarrow \infty} a_n = 0$ . Set  $G = \{0 < |z| < 1\} - I$ , where  $I = \cup_{n=1}^{\infty} I_n$  and  $I_n = [a_n, b_n]$ . We take  $p$  copies  $G_1, \dots, G_p$  of  $G$  and join the upper edge of  $I_n$  on  $G_j$  with the lower edge of  $I_n$  on  $G_{j+1}$  ( $j \bmod p$ ) for every  $n$ . Then we obtain a  $p$ -sheeted covering surface  $W_1$  of  $D$  which belongs to  $\mathcal{E}_p$ . For this end  $W_1$  we have showed the following (cf. [9], [11] and [14]).

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**Theorem A.** (i) *If  $I$  is thin at  $z = 0$ , in the sense that  $z = 0$  is an irregular boundary point of  $G$  with respect to Dirichlet problem, then  $\dim \mathcal{P}(W_1) = p$ .*

(ii) *If  $I$  is thick (= not thin) at  $z = 0$ , then  $\dim \mathcal{P}(W_1) = 1$ .*

The proof of the above theorem in [9] and [11] essentially relies on symmetry of  $G$  relative to the real line and the fact that the cover transformation group of  $W_1$  acts transitively and cyclically on each fiber. Main purpose of this paper is to show the following theorem which characterizes harmonic dimension in terms of fine topology and gives generalization of Theorem A.

**Main Theorem.** *Let  $\mathcal{M}$  be the class of open connected subsets  $M$  of  $D$  such that  $M \cup \{0\}$  is a fine neighborhood of  $z = 0$ . For every  $W \in \mathcal{E}_p$ , it holds that*

$$\dim \mathcal{P}(W) = \max_{M \in \mathcal{M}} n_W(M),$$

where  $n_W(M)$  is the number of connected components of  $\pi^{-1}(M)$  and  $\pi$  is the projection of  $W$  onto  $D$ .

After Preliminaries (§1), the proof of Main Theorem will be given in §2. In §3 from Main Theorem we shall derive Proposition 3.1 and Theorems 3.1 and 3.2, which include Theorem A above. Applying Main Theorem, we shall also show that, for an arbitrary given integer  $q$  with  $1 \leq q \leq p$ , there exists a  $W \in \mathcal{E}_p$  such that  $\dim \mathcal{P}(W) = q$ .

Consider the cover transformation group  $\mathcal{G}_W$  of  $W \in \mathcal{E}_p$ . In this paper we say that  $W$  is *normal* if there always exists a  $\tau \in \mathcal{G}_W$  which carries a given point  $w$  into a prescribed point  $w'$  with same projection (cf. [6]). In §4 we show that the *harmonic dimension of  $W$  divides  $p$  if  $W$  is normal*. We also say that  $W$  is *cyclic* if  $W$  is normal and  $\mathcal{G}_W$  is cyclic. The end  $W_1$  in Theorem A is a typical example of cyclic covering surfaces. So it might be interesting whether the range of harmonic dimensions of cyclic covering surfaces in  $\mathcal{E}_p$  is  $\{1, p\}$ . Second purpose of §4 is to answer this question negatively. In fact, for each divisor  $q$  of  $p$ , we shall give an example  $W \in \mathcal{E}_p$  such that  $W$  is cyclic and  $\dim \mathcal{P}(W) = q$ .

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## 1. Preliminaries from potential theory

**1.1** We begin with recalling the definition of balayage. Consider an open Riemann surface  $F$  possessing a Green's function. Denote by  $\mathcal{S} = \mathcal{S}(F)$  the class of all nonnegative superharmonic functions on  $F$ . Let  $E$  be a subset of  $F$  and  $s$

belong to  $S$ . Then the *balayage*  $\widehat{R}_s^E = {}^F\widehat{R}_s^E$  of  $s$  relative to  $E$  on  $F$  is defined by

$$\widehat{R}_s^E(w) = \liminf_{x \rightarrow w} \inf\{u(x) : u \in S, u \geq s \text{ on } E\}$$

(cf. e.g. [2]). We here review fundamental properties of balayage (cf. [2], [3], [8], etc).

- Proposition 1.1.** (i) If  $E_1 \subset E_2$ , then  $\widehat{R}_s^{E_1} \leq \widehat{R}_s^{E_2}$ ;  
 (ii)  $\widehat{R}_s^{E_1 \cup E_2} \leq \widehat{R}_s^{E_1} + \widehat{R}_s^{E_2}$ ;  
 (iii) if  $u, v \in S$  and  $s, t > 0$ , then  $\widehat{R}_{su+tv}^E = s\widehat{R}_u^E + t\widehat{R}_v^E$ ;  
 (iv) if  $E_1$  and  $E_2$  are closed subsets of  $F$  and  $N$  is a common connected component of both  $F - E_1$  and  $F - E_2$ , then  $\widehat{R}_s^{E_1} = \widehat{R}_s^{E_2}$  on  $N$ .

The following lemma gives us a relation between balayage on  $F$  and that on a covering surface of  $F$  (cf. [11]).

**Lemma 1.1.** Let  $\tilde{F}$  be an unlimited covering surface of  $F$  with the canonical projection  $\pi$  from  $\tilde{F}$  onto  $F$ ,  $E$  a subset of  $F$  and  $s$  belong to  $S$ . Then it holds that

$$\tilde{F}\widehat{R}_{s \circ \pi}^{\pi^{-1}(E)} = {}^F\widehat{R}_s^E \circ \pi$$

on  $\tilde{F}$ .

**1.2** We next state definitions of thinness and fine neighborhood (cf. [3]).

**DEFINITION 1.1.** Let  $a$  be a point in  $\mathbf{C}$  and set  $F = \{|z - a| < r\}$  and  $g_a(z) = \log(r/|z - a|)$ . We say that a subset  $E$  of  $\mathbf{C}$  is *thin* at  $a$  if  ${}^F\widehat{R}_{g_a}^{E \cap F} \neq g_a$ .

It is easily seen that the above definition does not depend on a choice of  $r > 0$ . If  $E$  is closed and  $a$  belongs to  $E$  in the above, it is well-known that  $E$  is thin at  $a$  if and only if  $a$  is an irregular boundary point of  $F - E$  with respect to Dirichlet problem (cf. e.g. [3]). We also say that  $E$  is *thick* at  $a$  if  $E$  is not thin at  $a$ .

**DEFINITION 1.2.** A subset  $U$  in  $\mathbf{C}$  which contains  $a$  is said to be a *fine neighborhood* of  $a$  if  $\mathbf{C} - U$  is thin at  $a$ .

The notion of fine neighborhood is originally defined in the category of *fine topology*. However, since it is well-known that the above definition of fine neighborhood coincides with the original definition, we adopt the above for convenience.

As for thinness of circular projection of closed set in  $D$ , the following proposition is applied in §3 (cf. [8]).

**Proposition 1.2.** *Let  $E$  be a closed subset of  $D$  and put  $E' = \{|z| : z \in E\}$ . If  $E$  is thin at  $z = 0$ , then  $E'$  is thin at  $z = 0$ .*

In §3, we are also in need of the following proposition (cf. [5]).

**Proposition 1.3.** *Let  $U$  be a fine neighborhood of  $z = 0$ . Then there exists a polar subset  $Z$  of  $\{|z| = 1\}$  satisfying the following property ; for every ray  $l_\theta = \{\arg z = \theta\}$  with  $l_\theta \cap Z = \emptyset$ , there exists a positive number  $\rho$  such that  $l_\theta \cap \{|z| < \rho\} \subset U$ .*

**1.3** Let  $F$  be an open Riemann surface possessing a Green’s function. We denote by  $\Delta$  the Martin boundary of  $F$  and by  $k_\zeta$  the Martin kernel on  $F$  with pole at  $\zeta$ . The *minimal boundary* of  $F$ ,  $\Delta_1$  in notation, is defined as the set of all minimal points in  $\Delta$ . Similarly as thinness and fine neighborhood, *minimal thinness* and *minimal fine neighborhood* are defined as follows (cf. [3]).

**DEFINITION 1.3.** Let  $\zeta$  be a point in  $\Delta_1$  and  $E$  a subset of  $F$ . We say that  $E$  is *minimally thin* at  $\zeta$  if  ${}^F\widehat{R}_{k_\zeta}^E \neq k_\zeta$ .

**DEFINITION 1.4.** Let  $\zeta$  be a point in  $\Delta_1$  and  $U$  a subset of  $F$ . We say that  $U \cup \{\zeta\}$  is a *minimal fine neighborhood* of  $\zeta$  if  $F - U$  is minimally thin at  $\zeta$ .

The following proposition plays a fundamental role in the study of (minimal) thinness and (minimal) fine neighborhood (cf. [12]).

**Proposition 1.4.** *Let  $\zeta$  be a point in  $\Delta_1$  and  $E$  a closed subset of  $F$ . If  $E$  is minimally thin at  $\zeta$ , then  ${}^F\widehat{R}_{k_\zeta}^E$  is a potential and there exists a unique connected component  $U$  of  $F - E$  such that  ${}^F\widehat{R}_{k_\zeta}^E < k_\zeta$  on  $U$ . Moreover, under the condition of Definition 1.1, if  $E$  is a closed subset of  $F$  and thin at  $a$ , then  ${}^F\widehat{R}_{g_a}^E$  is a potential and there exists a unique connected component  $U$  of  $F - E$  such that  ${}^F\widehat{R}_{g_a}^E < g_a$  on  $U$ .*

We close Preliminaries by stating the following, which is easily verified from the above proposition (cf. [12]).

**Proposition 1.5.** *Let  $\zeta$  be a point in  $\Delta_1$  and  $U$  an open subset of  $F$  such that  $U \cup \{\zeta\}$  is a minimal fine neighborhood of  $\zeta$ . Then there exists a unique connected component  $V$  of  $U$  such that  $V \cup \{\zeta\}$  is a minimal fine neighborhood of  $\zeta$ . Moreover, let  $a$  be a point in  $\mathbf{C}$  and  $U$  an open subset of  $\mathbf{C}$  such that  $U \cup \{a\}$  is a fine neighborhood of  $a$ . Then there exists a connected component  $V$  of  $U$  such that*

$V \cup \{a\}$  is a fine neighborhood of  $a$ .

**2. Proof of Main Theorem**

**2.1** Throughout this section, let  $W$  belong to  $\mathcal{E}_p$ , where  $\mathcal{E}_p$  is defined in Introduction. Denote by  $\partial W$  the relative boundary of  $W$  and by  $\pi = \pi_W$  the projection of  $\overline{W} = W \cup \partial W$  onto  $\{0 < |z| \leq 1\}$ . Note that  $\pi(\partial W) = \{|z| = 1\}$ . We consider the Martin compactification  $W^*$  of  $W$ . Then  $W^*$  takes a form  $W^* = W \cup \partial W \cup \Delta^W$ , where  $\Delta^W$  is the *Martin ideal boundary* of a bordered surface  $\overline{W}$ . We also denote by  $\Delta_1^W$  the set of minimal points in  $\Delta^W$ . It is well-known that  $\dim \mathcal{P}(W)$  coincides with the number of points in  $\Delta_1^W$  (cf. e.g. [4]). We again note that

$$(2.1) \quad 1 \leq \dim \mathcal{P}(W) \leq p$$

for every  $W \in \mathcal{E}_p$  (cf. [7]). For simplicity of notation, here and hereafter denote by  $\widehat{R}_f^E$  the balayage  ${}^W \widehat{R}_f^E$  on  $W$  and set  $g(z) = \log(1/|z|)$ . We first maintain the following

**Lemma 2.1.** *Let  $\zeta$  belong to  $\Delta_1^W$  and  $N$  be an open subset of  $W$ . Suppose that  $N \cup \{\zeta\}$  is a minimal fine neighborhood of  $\zeta$ . Then  $\pi(N) \cup \{0\}$  is a fine neighborhood of  $z = 0$ .*

*Proof.* By (2.1), we can put  $\Delta_1^W = \{\zeta_1, \dots, \zeta_n\}$  ( $n \leq p$ ), where  $\zeta_1 = \zeta$ . Let  $k_i$  be the Martin kernel with pole at  $\zeta_i$  ( $i = 1, \dots, n$ ). In view of Proposition 1.5, we may assume that  $N$  is connected. By Proposition 1.4,

$$(2.2) \quad \widehat{R}_{k_1}^{W-N} < k_1$$

on  $N$ . It is easily seen that  $g \circ \pi \geq ck_i$  with  $c > 0$  ( $i = 1, \dots, n$ ). Therefore the Martin representation theorem (cf. [4], [8]) implies that there exist positive numbers  $c_i$  satisfying

$$(2.3) \quad g \circ \pi = \sum_{i=1}^n c_i k_i.$$

From this with Lemma 1.1 and Proposition 1.1 it follows that

$$D \widehat{R}_g^{D-\pi(N)} \circ \pi = \widehat{R}_{g \circ \pi}^{W-\pi^{-1}(\pi(N))} \leq \widehat{R}_{g \circ \pi}^{W-N} = \sum_{i=1}^n c_i \widehat{R}_{k_i}^{W-N} \leq \sum_{i=1}^n c_i k_i.$$

Hence, in view of (2.2) and (2.3), we have

$${}^D\widehat{R}_g^{D-\pi(N)} \circ \pi < \sum_{i=1}^n c_i k_i = g \circ \pi$$

on  $N$ , i.e.  ${}^D\widehat{R}_g^{D-\pi(N)} < g$  on  $\pi(N)$ . Since  ${}^D\widehat{R}_g^E = {}^{D_0}\widehat{R}_g^E$  in general, where  $D_0 = \{|z| < 1\}$ , this means that  $\pi(N) \cup \{0\}$  is a fine neighborhood of  $z = 0$ .  $\square$

**2.2** As stated in Main Theorem, let  $\mathcal{M}$  be the class of open connected subsets  $M$  of  $D$  such that  $M \cup \{0\}$  is a fine neighborhood of  $z = 0$ , or equivalently,  $D - M$  is thin at  $z = 0$ . We next claim

**Lemma 2.2.** *Let  $M$  belong to  $\mathcal{M}$  and  $N$  be an arbitrary connected component of  $\pi^{-1}(M)$ , where  $\pi = \pi_W$ . Then there exists a point  $\zeta \in \Delta_1^W$  such that  $N \cup \{\zeta\}$  is a minimal fine neighborhood of  $\zeta$ .*

*Proof.* Let  $\Delta_1^W = \{\zeta_1, \dots, \zeta_n\}$  and  $k_i$  be the Martin kernel with pole at  $\zeta_i$  ( $i = 1, \dots, n$ ). In the same way as in the proof of the preceding lemma (see (2.3)), there exist positive numbers  $c_i$  such that

$$(2.4) \quad g \circ \pi = \sum_{i=1}^n c_i k_i.$$

By definition and Proposition 1.4, we have  ${}^D\widehat{R}_g^{D-M} < g$  on  $M$ . Therefore Proposition 1.1, (2.4) and Lemma 1.1 imply that

$$\sum_{i=1}^n c_i \widehat{R}_{k_i}^{W-\pi^{-1}(M)} = \widehat{R}_{g \circ \pi}^{W-\pi^{-1}(M)} = {}^D\widehat{R}_g^{D-M} \circ \pi < g \circ \pi = \sum_{i=1}^n c_i k_i$$

on  $\pi^{-1}(M)$ . Hence there exists a  $\nu \in \{1, \dots, n\}$  such that  $\widehat{R}_{k_\nu}^{W-\pi^{-1}(M)} \neq k_\nu$  on  $N$ . On the other hand, in view of (iv) of Proposition 1.1, it is easily seen that  $\widehat{R}_{k_\nu}^{W-\pi^{-1}(M)} = \widehat{R}_{k_\nu}^{W-N}$  on  $N$ . Therefore we conclude that  $\widehat{R}_{k_\nu}^{W-N} \neq k_\nu$  on  $N$ , i.e.  $N \cup \{\zeta_\nu\}$  is a minimal fine neighborhood of  $\zeta_\nu$ .  $\square$

**2.3** In addition to Lemmas 2.1 and 2.2, we need the following lemma (cf. [10]) for the proof of Main Theorem.

**Lemma 2.3.** *Let  $E$  be a subset of  $W$ . If  $E$  is minimally thin at every  $\zeta \in \Delta_1^W$ , then  $\pi(E)$  is thin at  $z = 0$ .*

Before starting with the proof of Main Theorem, recall the definition of  $n_W(M)$ . For each  $M \in \mathcal{M}$ , let  $n_W(M)$  be the number of connected components of  $\pi^{-1}(M) = \pi_W^{-1}(M)$ .

**Proof of Main Theorem.** Set  $\dim \mathcal{P}(W) = n$  and  $\Delta_1^W = \{\zeta_1, \dots, \zeta_n\}$ . We first show that there exists an  $M \in \mathcal{M}$  such that  $n \leq n_W(M)$ , which implies that  $\dim \mathcal{P}(W) \leq \max_{M \in \mathcal{M}} n_W(M)$ . Let  $\{N_1, \dots, N_n\}$  be mutually disjoint open connected subsets of  $W$  such that each  $N_i \cup \{\zeta_i\}$  is a minimal fine neighborhood of  $\zeta_i$ . Set  $S = \bigcap_{i=1}^n \pi(N_i)$  and  $E = \bigcap_{i=1}^n N_i^c$ . Since each  $\pi(N_i) \cup \{0\}$  is a fine neighborhood of  $z = 0$  by means of Lemma 2.1,  $S \cup \{0\}$  is also a fine neighborhood of  $z = 0$ . Since each  $N_i^c$  is minimally thin at  $\zeta_i$  by definition,  $E$  is minimally thin at each  $\zeta_i$ . Hence  $\pi(E)$  is thin at  $z = 0$  by means of Lemma 2.3. Therefore  $(S - \pi(E)) \cup \{0\}$  is a fine neighborhood of  $z = 0$ . Then, by Proposition 1.5, there exists a connected component  $M$  of  $S - \pi(E)$  belonging to  $\mathcal{M}$ . Let  $w$  be a point in  $N_i \cap \pi^{-1}(M)$  and  $C$  be an arbitrary curve in  $W$  such that  $w$  is the initial point of  $C$  and  $\pi(C)$  is contained in  $M$ . Then it is seen that the end point of  $C$  belongs to  $N_i$ . By this reasoning, for each  $i = 1, \dots, n$ , we can take a connected component  $O_i$  of  $\pi^{-1}(M)$  such that  $O_i \subset N_i$ . It is evident that  $O_i$ 's are mutually disjoint. Consequently we conclude that  $n_W(M) \geq n$ .

We next show that  $n \geq n_W(M)$  for every  $M \in \mathcal{M}$ , which implies that  $\dim \mathcal{P}(W) \geq \max_{M \in \mathcal{M}} n_W(M)$ . Set  $m = n_W(M)$  and let  $\{N_1, \dots, N_m\}$  be the totality of connected components of  $\pi^{-1}(M)$ . For each  $N_i$ , by virtue of Lemma 2.2, there exists an  $\eta_i \in \Delta_1^W$  such that  $N_i \cup \{\eta_i\}$  is a minimal fine neighborhood of  $\eta_i$ . Since  $N_i \cap N_j = \emptyset$  if  $i \neq j$ , we see that  $\eta_i \neq \eta_j$  if  $i \neq j$ . Therefore we obtain that  $\{\eta_1, \dots, \eta_m\} \subset \Delta_1^W$  or  $m \leq \dim \mathcal{P}(W)$ .

The proof is herewith complete. □

By virtue of Main Theorem and (2.1), the following corollaries are instantly verified.

**Corollary 2.1.** *If there exists an  $M \in \mathcal{M}$  such that  $n_W(M) = p$ , then  $\dim \mathcal{P}(W) = p$ .*

**Corollary 2.2.** *If  $n_W(M) = 1$  for every  $M \in \mathcal{M}$ , then  $\dim \mathcal{P}(W) = 1$ .*

### 3. Applications of Main Theorem

**3.1** In this section, we are concerned with application of Main Theorem. Let  $\{I_n\}_{n=1}^\infty$  and  $\{J_n\}_{n=1}^\nu$  ( $0 \leq \nu \leq \infty$ ) be sequences of closed segments in  $D$  accumulating only at  $z = 0$  such that  $I_m \cap I_n = J_m \cap J_n = \emptyset$  ( $m \neq n$ ) and  $I_m \cap J_n = \emptyset$ . We also assume that  $I_n \subset \{\arg z = \theta_n\}$  for a sequence  $\{\theta_n\}$ . Set

$$I = \bigcup_{n=1}^\infty I_n, \quad J = \bigcup_{n=1}^\nu J_n \quad \text{and} \quad S = S(I, J) = D - I - J.$$



We consider a subclass  $\mathcal{F}_p(I, J)$  of  $\mathcal{E}_p$  which consists of ends  $W \in \mathcal{E}_p$  satisfying the following three conditions, where we denote by  $\pi_W$  the projection of  $W$  onto  $D$ :

- (i)  $\pi_W^{-1}(S)$  consists of  $p$  connected components  $S_1, \dots, S_p$ , where each  $S_i$  is a copy of  $S$ ,
- (ii) every branch point of  $W$  lies over a point in the set of end points of  $\{I_n\}$  and  $\{J_n\}$ ,
- (iii) for every end point  $z$  of  $\{I_n\}$ , there exists a branch point of  $W$  of order  $p - 1$  (i.e. of multiplicity  $p$ ) which lies over  $z$ .

We first maintain

**Proposition 3.1.** *Let  $W$  be in  $\mathcal{F}_p(I, J)$ . If  $I \cup J$  is thin at  $z = 0$ , then  $\dim \mathcal{P}(W) = p$ .*

The above result was originally proved in [11]. We here give an alternative and very short proof of the above by applying Corollary 2.1.

*Proof.* By assumption,  $S$  belongs to  $\mathcal{M}$ . It follows from the above condition (i) that  $n_W(S) = p$ . Therefore, by virtue of Corollary 2.1, we have  $\dim \mathcal{P}(W) = p$ . □

**3.2** Here and hereafter, we set  $E' = \{|z| : z \in E\}$  for a subset  $E$  of  $D$ .

**Theorem 3.1.** *Let  $W$  be in  $\mathcal{F}_p(I, J)$ . Suppose that  $\{I'_n\}$  are mutually disjoint. If  $I'$  is thick at  $z = 0$  and  $J'$  is thin at  $z = 0$ , then  $\dim \mathcal{P}(W) = 1$ .*

*Proof.* It suffices to show that  $n_W(M) = 1$  for every  $M \in \mathcal{M}$ . For this purpose, set  $E = D - M$ . Since  $E$  is thin at  $z = 0$ , it follows from Proposition 1.2 that  $E'$  is thin at  $z = 0$ . Denoting by  $K$  the totality of end points of  $\{I_n\}$ , we see that  $E' \cup J' \cup K'$  is thin at  $z = 0$ , and therefore  $I' - (E' \cup J' \cup K')$  is thick at  $z = 0$ . Hence there exist an  $r$  with  $0 < r < 1$  and a positive integer  $m$  such that  $r$  belongs to  $I'_m - (E' \cup J' \cup K')$ . Setting  $C_1 = \{|z| = r\}$ , we deduce that  $C_1 \subset M$ ,  $C_1 \cap J = \emptyset$  and  $C_1 \cap I$  consists of a single point  $a$  which contained in  $I_m - K$ . We shall show that  $\pi_W^{-1}(C_1)$  consists of a single closed Jordan curve. Consider a closed Jordan curve  $C_2$  passing through  $a$  such that  $C_2 - \{a\}$  is contained in  $S$ ,  $C_2$  surrounds only one end point of  $I_m$  and the inside of  $C_2$  is contained in  $S \cup I_m$ . We give  $C_i$  ( $i = 1, 2$ ) anti-clockwise orientation and parametrization  $z = z_i(t)$  ( $0 \leq t \leq 1$ ) with  $z_i(0) = z_i(1) = a$ . Take an arbitrary point  $b \in \pi_W^{-1}(a)$  and denote by  $\Gamma_i : w = w_i(t)$  ( $0 \leq t \leq 1$ ) the lift of  $C_i$  by  $\pi_W^{-1}$  satisfying  $w_i(0) = b$ . Then, since  $C_1 - \{a\}$  and  $C_2 - \{a\}$  are contained in  $S$ , we see that  $w_1(1) = w_2(1)$ . On the other hand, it follows from the conditions (ii) and (iii) in no. 3.1 that  $\pi_W^{-1}(C_2)$  consists of a single closed Jordan curve. Consequently, we obtain that

$\pi_W^{-1}(C_1)$  also consists of a single closed Jordan curve. Hence, in view of  $C_1 \subset M$ , it is not difficult to see that  $\pi_W^{-1}(M)$  is connected, which completes the proof.  $\square$

It is easily seen that Proposition 3.1 and Theorem 3.1 yield Theorem A in Introduction.

**3.3** Applying the same argument as in the above proof, we can prove the following, which gives another sufficient condition in order that  $\dim \mathcal{P}(W) = 1$ .

**Theorem 3.2.** *Let  $W$  be in  $\mathcal{F}_p(I, J)$  and  $l_\theta = \{\arg z = \theta\}$ . Suppose that  $I \subset l_\theta$  and there exists a positive number  $\varepsilon$  such that  $J \subset D - \{|\arg z - \theta| < \varepsilon\}$ . If both of  $I$  and  $l_\theta - I$  are thick at  $z = 0$ , then  $\dim \mathcal{P}(W) = 1$ .*

*Proof.* Without loss of generality, we may assume that  $\theta = 0$ . As well as Proof of the preceding theorem, we have only to show that  $n_W(M) = 1$  for every  $M \in \mathcal{M}$ . By means of Proposition 1.3, there exist two rays  $l_\alpha$  and  $l_\beta$  with  $-\varepsilon < \alpha < 0 < \beta < \varepsilon$  and a positive number  $\rho$  such that  $\{re^{i\alpha} : 0 < r < \rho\}$  and  $\{re^{i\beta} : 0 < r < \rho\}$  are contained in  $M$ . Since  $(D - M)'$  is thin at  $z = 0$  by Proposition 1.2 and  $I$  is thick at  $z = 0$ , there exists a positive number  $s$  with  $s < \rho$  such that  $C_s = \{se^{i\gamma} : \alpha \leq \gamma \leq \beta\} \subset M$  and  $s \in I - K$ , where  $K$  is the totality of end points of  $\{I_n\}$ . By similar reasoning, there exists a positive number  $t$  with  $t < s$  such that  $C_t = \{te^{i\gamma} : \alpha \leq \gamma \leq \beta\} \subset M$  and  $t \in l_\theta - I$ . Joining  $C_t$ ,  $\{re^{i\alpha} : t \leq r \leq s\}$ ,  $C_s$  and  $\{re^{i\beta} : t \leq r \leq s\}$  in order, we obtain a closed curve  $C$  in  $M$ . Since  $C - \{s\} \subset S$  and  $s \in I - K$ , applying the same argument as in the preceding proof, we can conclude that  $\pi_W^{-1}(C)$  is connected. Hence, by the fact  $C \subset M$ , we see that  $n_W(M) = 1$  and  $\dim \mathcal{P}(W) = 1$ . This completes the proof.  $\square$

We here remark that, in Theorem 3.2, the assumption that  $J \subset D - \{|\arg z - \theta| < \varepsilon\}$  for  $\varepsilon > 0$  can not be removed. Let  $I$  be the same as in Theorem 3.2 and set  $I_n = \{te^{i\theta} : a_n \leq t \leq b_n\}$ . Take a decreasing sequence  $\{\theta_n\}$  with  $\lim_{n \rightarrow \infty} \theta_n = \theta$ . Put  $J_n = \{te^{i\theta_n} : a_n \leq t \leq b_n\}$  and  $J = \cup_{n=1}^\infty J_n$ . For this pair  $(I, J)$ , choose a  $W \in \mathcal{F}_2(I, J)$  such that the projection of the totality of branch points of  $W$  coincides with the totality of end points of  $\{I_n\}$  and  $\{J_n\}$ . Denote by  $L_{2n}$  (resp.  $L_{2n-1}$ ) the closed segment  $[a_n e^{i\theta}, a_n e^{i\theta_n}]$  (resp.  $[b_n e^{i\theta}, b_n e^{i\theta_n}]$ ). Making the convergence  $\theta_n \rightarrow \theta$  be sufficiently rapid, we can suppose that  $L = \cup_{n=1}^\infty L_n$  is thin at  $z = 0$ . Then it holds that  $\dim \mathcal{P}(W) = 2$ . In fact,  $M = D - L$  belongs to  $\mathcal{M}$  and  $\pi_W^{-1}(M)$  consists of two components. Hence, by Corollary 2.1, we have that  $\dim \mathcal{P}(W) = 2$ .

We also remark that, in Theorem 3.2, the assumption that  $l_\theta - I$  is thick at  $z = 0$

can not be removed (cf. [11] and see also no. 4.4).

**3.4** Although it seems to be well-known that, for an arbitrarily given integer  $q$  with  $1 \leq q \leq p$ , there exists an end  $W \in \mathcal{E}_p$  such that  $\dim \mathcal{P}(W) = q$ , we can find no references to show this explicitly. For the sake of completeness, applying Main Theorem, we shall give an example  $W_q \in \mathcal{E}_p$  with  $\dim \mathcal{P}(W_q) = q$  for every  $q \in \{1, \dots, p\}$ .

If  $q$  is 1 or  $p$ , then there exists a  $W \in \mathcal{E}_p$  with  $\dim \mathcal{P}(W) = q$  by means of Theorem A in Introduction. Assume that  $1 < q < p$ . For convenience' sake, we use the notation  $\mathcal{E}_n$  for  $n = 1$ :  $\mathcal{E}_1 = \{D\}$ , where  $D$  is considered to be an end of  $\{0 < |z| \leq \infty\}$ . By Theorem A and the fact that  $\dim \mathcal{P}(D) = 1$ , there exists an end  $V_1 \in \mathcal{E}_{q-1}$  (resp.  $V_2 \in \mathcal{E}_{p-q+1}$ ) such that  $\dim \mathcal{P}(V_1) = q - 1$  (resp.  $\dim \mathcal{P}(V_2) = 1$ ). Denote by  $\pi_1$  (resp.  $\pi_2$ ) the projection of  $V_1$  (resp.  $V_2$ ) onto  $D$ . Consider a sequence  $\{K_n\}_{n=1}^\infty$  of mutually disjoint closed segments in  $D$  such that  $K$  is thin at  $z = 0$  and there exist no branch points of  $V_1$  (resp.  $V_2$ ) on  $\pi_1^{-1}(K)$  (resp.  $\pi_2^{-1}(K)$ ), where  $K = \bigcup_{n=1}^\infty K_n$ . For every  $n$ , let  $\kappa_{1n}$  (resp.  $\kappa_{2n}$ ) be a closed segment in  $V_1$  (resp.  $V_2$ ) with  $\pi_1(\kappa_{1n}) = K_n$  (resp.  $\pi_2(\kappa_{2n}) = K_n$ ). Joining  $V_1$  with  $V_2$  crosswise along slits  $\kappa_{1n}$  and  $\kappa_{2n}$  for every  $n$ , we obtain an end  $W_q$  which belongs to  $\mathcal{E}_p$ . To prove that  $\dim \mathcal{P}(W_q) = q$ , we denote by  $\pi$  the projection of  $W_q$  onto  $D$ . By virtue of Main Theorem, we can find an  $M \in \mathcal{M}$  with  $n_{V_1}(M) = q - 1$ . Let  $M'$  be the connected component of  $M - K$  such that  $z = 0$  is a boundary point of  $M'$ . Then  $M'$  also belongs to  $\mathcal{M}$ , because  $K$  is thin at  $z = 0$ . Since  $n_{V_1}(M') = q - 1$  and  $n_{V_2}(M') \geq 1$ , it is easily seen that  $n_{W_q}(M') \geq q$ , and hence  $\dim \mathcal{P}(W_q) \geq q$  by Main Theorem. On the other hand, by Main Theorem, we have that  $n_{V_2}(M) = 1$  for every  $M \in \mathcal{M}$ . From this and the fact that  $n_{V_1}(M) \leq q - 1$  for every  $M \in \mathcal{M}$ , it follows that  $n_{W_q}(M) \leq q$  for every  $M \in \mathcal{M}$ , and hence  $\dim \mathcal{P}(W_q) \leq q$  by means of Main Theorem. The proof of  $\dim \mathcal{P}(W_q) = q$  is herewith complete.

**4. Normal and cyclic covering surfaces**

**4.1** We start with fixing terminology and notation. Let  $F$  be an open Riemann surface and  $\tilde{F}$  an *unlimited* covering surface of  $F$ . Denote by  $\pi = \pi_{\tilde{F}}$  the canonical projection of  $\tilde{F}$  onto  $F$ . A conformal mapping  $\tau$  of  $\tilde{F}$  onto itself is said to be a *cover transformation* of  $\tilde{F}$  if  $\pi(\tau(w)) = \pi(w)$  for every  $w \in \tilde{F}$ . Denote by  $\mathcal{G}_{\tilde{F}}$  the group of cover transformations of  $\tilde{F}$ . We proceed to the statement of definitions of normal and cyclic covering surfaces. Although normality is usually defined only for unramified covering surfaces(cf. [1]), we adopt the following definition, in which normality is defined even for ramified covering surfaces (cf. [6]).

**DEFINITION 4.1.** We say that  $\tilde{F}$  is *normal* if  $\mathcal{G}_{\tilde{F}}$  is transitive on  $\pi^{-1}(z)$  for every  $z \in F$ , that is, for every pair  $(w_1, w_2)$  of  $\pi^{-1}(z)$ , there exists a  $\tau \in \mathcal{G}_{\tilde{F}}$  such

that  $\tau(w_1) = w_2$ .

**DEFINITION 4.2.** We say that  $\tilde{F}$  is *cyclic* if  $\tilde{F}$  is normal and  $\mathcal{G}_{\tilde{F}}$  is cyclic.

If  $\tilde{F}$  is  $p$ -sheeted ( $1 < p < \infty$ ), it is not difficult to see that the order of  $\mathcal{G}_{\tilde{F}}$  is a divisor of  $p$  and that  $\tilde{F}$  is normal if and only if  $\mathcal{G}_{\tilde{F}}$  is exactly of order  $p$  (cf. e.g. [1]).

**4.2** Assuming that  $W \in \mathcal{E}_p$  is normal, we are interested in relation between the harmonic dimension of  $W$  and the number  $p$  of sheets of  $W$ . Our first assertion of this section is the following

**Theorem 4.1.** *If  $W \in \mathcal{E}_p$  is normal, then the harmonic dimension of  $W$  divides  $p$ .*

*Proof.* We first put  $\dim \mathcal{P}(W) = m$ . By means of Main Theorem, there exists an  $M \in \mathcal{M}$  such that  $n_W(M) = m$ . We denote by  $\{N_1, \dots, N_m\}$  the totality of connected components of  $\pi^{-1}(M)$  and set  $H = \{\tau \in \mathcal{G}_W : \tau(N_1) = N_1\}$ . It is evident that  $H$  is a subgroup of  $\mathcal{G}_W$ . Let  $r$  be the order of  $H$  and put  $q = p/r$ . Consider the decomposition of  $\mathcal{G}_W$  by left cosets of  $H$ :

$$(4.1) \quad \mathcal{G}_W = \tau_1 H + \dots + \tau_q H,$$

where  $\tau_1 H = H$ .

Observe that every  $\tau \in \mathcal{G}_W$  induces a permutation on  $\{N_1, \dots, N_m\}$ . Therefore we can consider a mapping  $\varphi : \tau_i \mapsto N_\nu$  from  $\{\tau_1, \dots, \tau_q\}$  to  $\{N_1, \dots, N_m\}$  such that  $\tau_i(N_1) = N_\nu$ . Suppose that  $\varphi(\tau_i) = \varphi(\tau_j)$ . Then we have that  $\tau_j^{-1}\tau_i(N_1) = N_1$  or  $\tau_j^{-1}\tau_i \in H$ , and hence  $\tau_i = \tau_j$  by (4.1). This implies that  $\varphi$  is injective. On the other hand, for each  $N_\nu$ , there exists a  $\tau \in \mathcal{G}_W$  satisfying  $\tau(N_1) = N_\nu$  by assumption. Putting  $\tau = \tau_i\sigma$  for a  $\sigma \in H$ , we see that  $\tau_i(N_1) = \tau_i\sigma(N_1) = \tau(N_1) = N_\nu$ . This implies that  $\varphi$  is also surjective. Consequently we obtain that  $\varphi$  is bijective, and hence  $m = q$ . This completes the proof.  $\square$

**4.3** Put  $\mathcal{H}_p = \{\dim \mathcal{P}(W) : W \in \mathcal{E}_p, W \text{ is cyclic}\}$  and denote by  $\mathcal{I}_p$  the totality of divisors of  $p$ . Our second assertion of this section is that  $\mathcal{I}_p \subset \mathcal{H}_p$ . In fact, for an arbitrarily given  $q \in \mathcal{I}_p$ , we shall construct a  $W \in \mathcal{E}_p$  such that  $W$  is cyclic and  $\dim \mathcal{P}(W) = q$ . On the other hand, by Theorem 4.1, we see that  $\mathcal{H}_p \subset \mathcal{I}_p$ . Consequently we obtain

**Theorem 4.2.** *There exists a cyclic covering surface  $W \in \mathcal{E}_p$  with  $\dim \mathcal{P}(W) = q$  if and only if  $q$  is a divisor of  $p$ .*

Suppose  $q \in \mathcal{I}_p$  and put  $p = qr$ . Consider a sequence  $\{a_n\}_{n=1}^\infty$  with  $2^{-n} < a_n < 2^{-n+1}$ . Making each  $a_n$  draw near  $2^{-n+1}$ , we can assume that  $\cup_{n=1}^\infty [a_n, 2^{-n+1}]$  is thin at  $z = 0$  (cf. e.g. [16]) and

$$(4.2) \quad \sum_{n=1}^\infty \log \frac{a_n}{2^{-n}} = \infty.$$

For each  $j = 1, \dots, q$  and each  $n = 1, 2, \dots$ , set

$$I_n^j = \{te^{i\frac{2j\pi}{q}} : 2^{-n} \leq t \leq a_n\} \quad \text{and} \quad G = D - \cup_{j=1}^q I^j,$$

where  $I^j = \cup_{n=1}^\infty I_n^j$ . For convenience' sake, we denote by  $I_n^{j+}$  (resp.  $I_n^{j-}$ ) the left (resp. right) side edge of the slit  $I_n^j$  in  $D$  with respect to the ray  $\{\arg z = 2j\pi/q\}$ . Take  $p$  copies  $G_1, \dots, G_p$  of  $G$ . Joining  $I_n^{j+}$  on  $G_i$  with  $I_n^{j-}$  on  $G_{i+1}$  ( $i \bmod p$ ) for every  $j = 1, \dots, q$  and every  $n = 1, 2, \dots$ , we obtain a covering surface  $W_{pq}$  of  $D$  belonging to  $\mathcal{E}_p$ . It is clear that  $W_{pq}$  is cyclic. Our goal is to show that  $\dim \mathcal{P}(W_{pq}) = q$ . The proof is accomplished in no. 4.4.

**4.4** We first show that  $\dim \mathcal{P}(W_{pq}) \leq q$ . For this purpose, we need the following (cf. [13] and [15])

**Proposition 4.1.** *Let  $\Omega$  be an end in the sense of Heins and  $\{A_n\}_{n=1}^\infty$  a sequence of mutually disjoint subset of  $\Omega$  such that each  $A_n$  consists of at most  $\mu$  mutually disjoint annuli  $\{A_{nm}\}_{m=1}^{\mu_n}$  ( $\mu_n \leq \mu$ ) and  $A_{n+1}$  separates  $A_n$  from the ideal boundary of  $\Omega$  for every  $n$ . Suppose that  $\sum_{n=1}^\infty \text{mod } A_n = \infty$ , where  $\text{mod } A_{nm}$  are moduli of  $A_{nm}$  and  $(\text{mod } A_n)^{-1} = (\text{mod } A_{n1})^{-1} + \dots + (\text{mod } A_{n\mu_n})^{-1}$ . Then  $\dim \mathcal{P}(\Omega)$  is at most  $\mu$ .*

Set  $A_n = \pi^{-1}(\{2^{-n} < |z| < a_n\})$  ( $n = 1, 2, \dots$ ), where  $\pi$  is the projection of  $W_{pq}$  onto  $D$ . By construction of  $W_{pq}$ , it is not difficult to see that each  $A_n$  consists of  $q$  disjoint annuli  $A_{n1}, \dots, A_{nq}$  which are conformally equivalent to  $\{1 < |z| < \sqrt[r]{2^n a_n}\}$ , where  $r = p/q$ . This yields that

$$(\text{mod } A_n)^{-1} = (\text{mod } A_{n1})^{-1} + \dots + (\text{mod } A_{nq})^{-1} = qr(\log(2^n a_n))^{-1}.$$

Hence, by (4.2), we have that

$$\sum_{n=1}^\infty \text{mod } A_n = \frac{1}{p} \sum_{n=1}^\infty \log(2^n a_n) = \infty.$$

It is evident that  $A_{n+1}$  separates  $A_n$  from the ideal boundary of  $W_{pq}$ . Consequently, by means of Proposition 4.1, we see that  $\dim \mathcal{P}(W_{pq}) \leq q$ .

We next show that  $\dim \mathcal{P}(W_{pq}) \geq q$ . Set

$$J^j = D \cap \left\{ \arg z = \frac{2j\pi}{q} \right\} - I^j, \quad E = \bigcup_{j=1}^q \overline{J^j} \quad \text{and} \quad S = D - E,$$

where  $\overline{J^j}$  is the closure of  $J^j$ . Since each  $\overline{J^j}$  is thin at  $z = 0$  by assumption, it follows that  $E$  is thin at  $z = 0$ , and hence  $S$  belongs to  $\mathcal{M}$  defined in no. 2.2. It is not difficult to see that  $\pi^{-1}(S)$  consists of  $q$  components as well as  $A_n$ , i.e.  $n_{W_{pq}}(S) = q$ . Therefore, by virtue of Main Theorem, we conclude that  $\dim \mathcal{P}(W_{pq}) \geq q$ . The proof of  $\dim \mathcal{P}(W_{pq}) = q$  is herewith complete.

REMARK. Set  $I = I^1$  and  $J = \bigcup_{j=2}^q I^j$  for  $\{I^j\}_{j=1}^q$  in no. 3.1. Then the above  $W_{pq}$  for  $q > 1$  shows that, in Theorem 3.2, the assumption that  $l_\theta - I$  is thick at  $z = 0$  can not be removed.

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H. Masaoka  
Department of Mathematics  
Faculty of Science  
Kyoto Sangyo University  
Kamigamo–Motoyama  
Kitaku, Kyoto 603, Japan

S. Segawa  
Department of Mathematics  
Daido Institute of Technology  
Daido, Minami  
Nagoya 457, Japan