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HARMONIC FOLIATIONS ON A COMPACT RIEMANNIAN MANIFOLD OF NON-NEGATIVE CONSTANT CURVATURE

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction. Let M be a compact oriented manifold and $\mathscr{F}$ a Riemannian and harmonic foliation with respect to a bundle-like metric. Kamber and Tondeur [3] proved the fundamental formula for a special variation of $\mathscr{F}$, and making use of it they showed in [4] that the index of a Riemannian and harmonic foliation on the sphere S^n $(n > 2)$ for which the standard metric is bundle-like is not smaller than $q + 1$, where q is the codimension of $\mathscr{F}$.

The purpose of this paper is to prove that any harmonic foliation on a compact Riemannian manifold of non-negative constant curvature for which the normal plane field is minimal (see § 1 for the definition) is totally geodesic. As a corollary we can state that any Riemannian and harmonic foliation on the sphere S^n $(n > 2)$ for which the standard metric is bundle-like is totally geodesic. Moreover, Escobales [1] has classified recently all totally geodesic foliations on the spheres for which the standard metrics are bundle-like. This means that harmonic foliations on the spheres for which the standard metrics are bundle-like have been completely classified.

On the other hand, a theorem of Ferus [2] gives an estimate for the codimension of a totally geodesic foliation of the sphere S^n. Thus we can apply these results to the foregoing theory of Kamber and Tondeur to sharpen their result.

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1. Preliminaries. We shall be in the C^∞-category. Let (M, g) be an n-dimensional Riemannian manifold, and $\mathscr{F}$ a foliation of codimension q on M. Then there arise two tensor fields associated with a foliated Riemannian manifold $(M, g, \mathscr{F})$ as follows. Denote by $V(M)$ the space of vector fields on M, and by ∇ the Riemannian connection on M. For any $X \in V(M)$ we decompose it as

$$X = X' + X'' ,$$

where X' (resp. X'') is tangent (resp. normal) to $\mathscr{F}$. Actually, choosing a suitable Riemannian metric on the tangent bundle $T(M)$ of M, we may decompose $T(M)$ as the direct product $\mathscr{F} \oplus \mathscr{F}^{\perp}$, where $\mathscr{F}^{\perp}$ is called a *normal plane field*. Then we define two tensors A and h of type (1, 2) on M by

$$(1.1)\qquad \begin{aligned} A(X,\ Y) &= -(\nabla_{Y''}X'')' \,, \\ h(X,\ Y) &= (\nabla_{Y'}X')'' \,, \quad X, Y \in V(M) \,. \end{aligned}$$

The restriction of h to each leaf of $\mathscr{F}$ is what is called the *second fundamental form* of the leaf. From now on we express them with respect to a locally defined orthonormal frame field, and derive some basic formulas among them and their derivatives. As for the range of indices we use the following convention unless otherwise stated:

$$\begin{aligned} &A, B, C, \cdots = 1, \cdots, n \,; \\ &i, j, k, \cdots = 1, \cdots, p \,; \\ &\alpha, \beta, \gamma, \cdots = p+1, \cdots, n \,, \end{aligned}$$

where $p = n - q$ denotes the dimension of $\mathscr{F}$. The summation $\sum$ is taken over all repeated indices. Let $\{e_1, \cdots, e_n\}$ be a local field of orthonormal frames on M such that $e_1, \cdots, e_p$ are always tangent to $\mathscr{F}$. Denote its dual forms by $\omega_1, \cdots, \omega_n$. The connection forms ω_{AB} with respect to ω_A are defined by the equations

$$(1.2)\qquad \begin{aligned} &\omega_{BA} + \omega_{AB} = 0 \,, \\ &d\omega_A + \sum \omega_{AB} \wedge \omega_B = 0 \,. \end{aligned}$$

The Riemannian connection ∇ on M is given by

$$(1.3)\qquad \nabla_{e_A} e_B = \sum \omega_{CB}(e_A) e_C \,.$$

It follows from (1.1) and (1.3) that

$$(1.4)\qquad \begin{aligned} h(e_i,\ e_j) &= \sum \omega_{\alpha i}(e_j) e_\alpha \,, \\ A(e_\alpha,\ e_\beta) &= \sum \omega_{\alpha j}(e_\beta) e_j \,. \end{aligned}$$

Thus the only components $h^A{}_{BC}$ (resp. $A^B{}_{CD}$) of h (resp. A) which may not vanish are

$$(1.5)\qquad h^\alpha{}_{ij} = \omega_{\alpha i}(e_j) \quad (\text{resp. } A^i{}_{\alpha\beta} = \omega_{\alpha i}(e_\beta)) \,.$$

Moreover the connection forms $\omega_{\alpha i}$ are given by

$$(1.6)\qquad \omega_{\alpha i} = \sum h^\alpha{}_{ij}\omega_j + \sum A^i{}_{\alpha\beta}\omega_\beta \,.$$

The foliation $\mathscr{F}$ is said to be *harmonic* or *minimal* (resp. *totally geodesic*) if $\sum h^\alpha{}_{ii} = 0$ (resp. $h^\alpha{}_{ij} = 0$).

After Kitahara [5] and Reinhart [9], we define the second fundamental form B of the normal plane field $\mathscr{F}^{\perp}$ by

$$B(X,Y)=\{A(X,Y)+A(Y,X)\}/2\,,\quad X,Y\in V(M)\,. \tag{1.7}$$

The normal plane field $\mathscr{F}^{\perp}$ is said to be *minimal* (resp. *totally geodesic*) if $\operatorname{Tr} B=\sum A^{j}{}_{\alpha\alpha}e_j=0$ (resp. $B=0$).

The curvature form $\Omega=(\Omega_{AB})$ of M is defined by

$$\Omega_{AB}=d\omega_{AB}+\sum\omega_{AC}\wedge\omega_{CB}\,. \tag{1.8}$$

We put

$$\Omega_{AB}=-\sum(R_{ABCD}/2)\omega_C\wedge\omega_D\,,\quad R_{ABCD}+R_{ABDC}=0\,. \tag{1.9}$$

Then the components R_{ABCD} of Ω satisfy

$$R_{ABCD}=-R_{BACD}=R_{CDAB}\,. \tag{1.10}$$

Since the distribution $\omega_\alpha=0$ is integrable by definition, we have

$$h^{\alpha}{}_{ij}=h^{\alpha}{}_{ji}\,. \tag{1.11}$$

The distribution $\omega_i=0$ is integrable if and only if

$$A^{i}{}_{\alpha\beta}=A^{i}{}_{\beta\alpha}\,. \tag{1.12}$$

On the contrary, the Riemannian metric g is bundle-like (see Molino [6] or Reinhart [8]) if and only if

$$A^{i}{}_{\alpha\beta}=-A^{i}{}_{\beta\alpha}\,. \tag{1.13}$$

Thus, the Riemannian metric is bundle-like if and only if $B=0$, and then the normal plane field $\mathscr{F}^{\perp}$ is minimal.

Now, for a tensor filed $T=(T^{A_1\cdots A_r}{}_{B_1\cdots B_s})$ on M, we define the covariant derivative $(T^{A_1\cdots A_r}{}_{B_1\cdots B_sC})$ by

$$\sum T^{A_1\cdots A_r}{}_{B_1\cdots B_sC}\omega_C=dT^{A_1\cdots A_r}{}_{B_1\cdots B_s}-\sum_{a=1}^{r}T^{A_1\cdots A_{a-1}CA_{a+1}\cdots A_r}{}_{B_1\cdots B_s}\omega_{CA_a}-\sum_{b=1}^{s}T^{A_1\cdots A_r}{}_{B_1\cdots B_{b-1}CB_{b+1}\cdots B}\omega_{CB_b}\,. \tag{1.14}$$

Then the exterior derivative of (1.6) gives

$$h^{\alpha}{}_{ijk}-h^{\alpha}{}_{ikj}=R_{\alpha ijk}\,, \tag{1.15}$$

$$h^{\alpha}{}_{ij\beta}-A^{i}{}_{\alpha\beta j}-\sum h^{\alpha}{}_{ik}h^{\beta}{}_{kj}-\sum A^{i}{}_{\alpha\gamma}A^{j}{}_{\gamma\beta}=R_{\alpha ij\beta}\,, \tag{1.16}$$

$$A^{i}{}_{\alpha\beta\gamma}-A^{i}{}_{\alpha\gamma\beta}+\sum h^{\alpha}{}_{ij}(A^{j}{}_{\beta\gamma}-A^{j}{}_{\gamma\beta})=-R_{\alpha i\gamma\beta}\,. \tag{1.17}$$

Moreover, from the definition of $(h^{A}{}_{BCD})$ and (1.6) it follows that we have

$$h^{l}{}_{ijk}=-\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{lk}\,, \tag{1.18}$$

$$h^{l}{}_{ij\alpha}=-\sum h^{\beta}{}_{ij}A^{l}{}_{\beta\alpha}\,, \tag{1.19}$$

$$h^{\alpha}{}_{i\beta j}=-\sum h^{\alpha}{}_{ik}h^{\beta}{}_{kj}\,, \tag{1.20}$$

$$h^\alpha{}_{i\beta\gamma} = -\sum h^\alpha{}_{ki} A^k{}_{\beta\gamma}\,, \tag{1.21}$$

$$h^\alpha{}_{\beta ij} = -\sum h^\alpha{}_{ik} h^\beta{}_{kj}\,, \tag{1.22}$$

$$h^\alpha{}_{\beta i\gamma} = -\sum h^\alpha{}_{ik} A^k{}_{\beta\gamma}\,. \tag{1.23}$$

The Ricci formulas on the second covariant derivatives of h are given by the exterior derivative of the definition of the components $h^A{}_{BCD}$. For later use we write down these equations:

$$h^\alpha{}_{ijkl} - h^\alpha{}_{ijlk} = \sum h^\beta{}_{ij} R_{\alpha\beta kl} + \sum h^\alpha{}_{mj} R_{imkl} + \sum h^\alpha{}_{im} R_{jmkl}\,, \tag{1.24}$$

$$h^\alpha{}_{ijk\beta} - h^\alpha{}_{ij\beta k} = \sum h^\gamma{}_{ij} R_{\alpha\gamma k\beta} + \sum h^\alpha{}_{lj} R_{ilk\beta} + \sum h^\alpha{}_{il} R_{jlk\beta}\,, \tag{1.25}$$

$$h^\alpha{}_{ij\beta\gamma} - h^\alpha{}_{ij\gamma\beta} = \sum h^\delta{}_{ij} R_{\alpha\delta\beta\gamma} + \sum h^\alpha{}_{kj} R_{ik\beta\gamma} + \sum h^\alpha{}_{ik} R_{jk\beta\gamma}\,, \tag{1.26}$$

$$A^i{}_{\alpha\beta jk} - A^i{}_{\alpha\beta kj} = \sum A^l{}_{\alpha\beta} R_{iljk} + \sum A^i{}_{\gamma\beta} R_{\alpha\gamma jk} + \sum A^i{}_{\alpha\gamma} R_{\beta\gamma jk}\,, \tag{1.27}$$

$$A^i{}_{\alpha\beta j\gamma} - A^i{}_{\alpha\beta\gamma j} = \sum A^k{}_{\alpha\beta} R_{ikj\gamma} + \sum A^i{}_{\delta\beta} R_{\alpha\delta j\gamma} + \sum A^i{}_{\alpha\delta} R_{\beta\delta j\gamma}\,, \tag{1.28}$$

$$A^i{}_{\alpha\beta\gamma\delta} - A^i{}_{\alpha\beta\delta\gamma} = \sum A^k{}_{\alpha\beta} R_{ik\gamma\delta} + \sum A^i{}_{\varepsilon\beta} R_{\alpha\varepsilon\gamma\delta} + \sum A^i{}_{\alpha\varepsilon} R_{\beta\varepsilon\gamma\delta}\,. \tag{1.29}$$

2. **Proof of Theorem.** Let $(M, g, \mathscr{F})$ be a foliated Riemannian manifold. We keep the notation in § 1. The global vector field $v = \sum v_A e_A$ on M is defined by

$$v_k = \sum h^\alpha{}_{ij} h^\alpha{}_{ijk}\,, \quad v_\alpha = 0\,.$$

The divergence δv of v is first calculated.

LEMMA 2.1.

$$\begin{aligned}\delta v = {} & \sum v_i A^i{}_{\alpha\alpha} + \sum h^\alpha{}_{ijk} h^\alpha{}_{ijk} + \sum h^\alpha{}_{ij} R_{\alpha ijkk} + \sum h^\alpha{}_{ij} R_{\alpha kikj} \\ & + \sum h^\alpha{}_{ij} h^\beta{}_{kk} h^\alpha{}_{ij\beta} + \sum h^\alpha{}_{ij} h^\alpha{}_{kkij} + \sum (h^\beta{}_{ik} R_{\alpha\beta jk} + h^\alpha{}_{lk} R_{iljk} + h^\alpha{}_{il} R_{kljk}) h^\alpha{}_{ij} \\ & + \sum h^\alpha{}_{ij} h^\alpha{}_{lk} h^\beta{}_{ij} h^\beta{}_{lk} + 2 \sum h^\alpha{}_{ij} h^\beta{}_{ik} h^\alpha{}_{jl} h^\beta{}_{lk}\,.\end{aligned}$$

PROOF. From the definition of (v_{AB}), we have

$$\sum v_{\alpha A}\omega_A = dv_\alpha - \sum v_A \omega_{A\alpha} = -\sum v_i \omega_{i\alpha}\,,$$

which implies

$$\sum v_{\alpha\alpha} = \sum v_i A^i{}_{\alpha\alpha}\,. \tag{2.1}$$

Moreover we have

$$\begin{aligned}\sum v_{kA}\omega_A &= dv_k - \sum v_A \omega_{Ak} = d(\sum h^\alpha{}_{ij} h^\alpha{}_{ijk}) - \sum v_i \omega_{ik} \\ &= \sum h^\alpha{}_{ijk}(h^\alpha{}_{ijA}\omega_A - h^\beta{}_{ij}\omega_{\alpha\beta} + h^\alpha{}_{lj}\omega_{li} + h^\alpha{}_{il}\omega_{lj}) \\ &\quad + \sum h^\alpha{}_{ij}(h^\alpha{}_{ijkA}\omega_A - h^l{}_{ijk}\omega_{\alpha l} + h^\alpha{}_{ljk}\omega_{li} + h^\alpha{}_{ilk}\omega_{lj} \\ &\quad + h^\alpha{}_{ijl}\omega_{lk} - h^\beta{}_{ijk}\omega_{\alpha\beta} + h^\alpha{}_{\beta jk}\omega_{\beta i} + h^\alpha{}_{i\beta k}\omega_{\beta j} + h^\alpha{}_{ij\beta}\omega_{\beta k}) \\ &\quad - \sum h^\alpha{}_{jl} h^\alpha{}_{jli}\omega_{ik} \\ &= \sum h^\alpha{}_{ijk} h^\alpha{}_{ijA}\omega_A + \sum h^\alpha{}_{ij}(h^\alpha{}_{ijkA}\omega_A - h^l{}_{ijk}\omega_{\alpha l} + 2h^\alpha{}_{i\beta k}\omega_{\beta j} + h^\alpha{}_{ij\beta}\omega_{\beta k})\,,\end{aligned}$$

which together with (1.18) and (1.22) gives

$$\sum v_{ii} = \sum h^\alpha{}_{ijk}h^\alpha{}_{ijk} + \sum h^\alpha{}_{ij}h^\alpha{}_{ijkk} + \sum h^\alpha{}_{ij}h^\beta{}_{ij}h^\alpha{}_{kl}h^\beta{}_{kl} + 2\sum h^\alpha{}_{ij}h^\alpha{}_{jk}h^\beta{}_{kl}h^\beta{}_{li} + \sum h^\alpha{}_{ij}h^\beta{}_{kk}h^\alpha{}_{ij\beta} . \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} h^\alpha{}_{ijkk} &= R_{\alpha ijkk} + h^\alpha{}_{ikjk} && \text{(by (1.15))} \\ &= R_{\alpha ijkk} + \sum h^\beta{}_{ik}R_{\alpha\beta jk} + \sum h^\alpha{}_{lk}R_{iljk} + \sum h^\alpha{}_{il}R_{kljk} + h^\alpha{}_{ikkj} && \text{(by (1.24))} \\ &= R_{\alpha ijkk} + \sum h^\beta{}_{ik}R_{\alpha\beta jk} + \sum h^\alpha{}_{lk}R_{iljk} + \sum h^\alpha{}_{il}R_{kljk} + R_{\alpha kikj} + h^\alpha{}_{kkij} . && \text{(by (1.15))} \end{aligned}$$

This, (2.1) and (2.2) complete the proof.

LEMMA 2.2. *If the foliation is harmonic, then we have*

$$\sum h^\alpha{}_{iiA} = 0 , \tag{2.3}$$

$$\sum h^\alpha{}_{iijk} = -2\sum h^\alpha{}_{il}h^\beta{}_{lj}h^\beta{}_{ik} . \tag{2.4}$$

PROOF. From the definition of $(h^A{}_{BCD})$ we have

$$\sum h^\alpha{}_{iiA}\omega_A = \sum dh^\alpha{}_{ii} + \sum h^A{}_{ii}\omega_{\alpha A} + \sum h^\alpha{}_{Ai}\omega_{iA} + \sum h^\alpha{}_{iA}\omega_{iA} = 0 ,$$

which proves (2.3). Similarly, we have

$$\begin{aligned} \sum h^\alpha{}_{iijA}\omega_A &= \sum dh^\alpha{}_{iij} + \sum h^A{}_{iij}\omega_{\alpha A} + \sum h^\alpha{}_{Aij}\omega_{iA} + \sum h^\alpha{}_{iAj}\omega_{iA} + \sum h^\alpha{}_{iiA}\omega_{jA} \\ &= \sum h^k{}_{iij}\omega_{\alpha k} + 2\sum h^\alpha{}_{\beta ij}\omega_{i\beta} . && \text{(by (2.3))} \end{aligned}$$

Hence we have from (1.18) and (1.20)

$$h^\alpha{}_{iijk} = \sum h^l{}_{iij}h^\alpha{}_{lk} - 2\sum h^\alpha{}_{\beta ij}h^\beta{}_{ik} = -2\sum h^\alpha{}_{il}h^\beta{}_{lj}h^\beta{}_{ik} . \qquad \text{q.e.d.}$$

Now we can prove the following:

THEOREM 2.3. *Let (M, g) be a compact Riemannian manifold of constant sectional curvature c $(\geqq 0)$. Let $\mathscr{F}$ be a harmonic foliation such that the normal plane field $\mathscr{F}^\perp$ is minimal. Then the foliation $\mathscr{F}$ is totally geodesic.*

PROOF. We may assume that M is orientable, because otherwise we may consider its double covering space instead. Then for the vector field v defined above we have

$$\int_M \delta v * 1 = 0 ,$$

where $*1$ denotes the volume element of M. Since M is of constant curvature c, we have

$$R_{ABCD} = c(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{DB}) ,$$

and so $R_{ABCDE}=0$. By assumption we have $\sum A^{i}{}_{\alpha\alpha}=0$. Then Lemma 2.1 and (2.4) imply

$$\text{(2.5)}\qquad \int_M [\sum h^{\alpha}{}_{ijk}h^{\alpha}{}_{ijk} + cp\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{ij} + \sum h^{\alpha}{}_{ij}h^{\beta}{}_{ij}h^{\alpha}{}_{kl}h^{\beta}{}_{kl} + 2\sum \mathrm{Tr}\,(H^{\alpha}H^{\alpha}H^{\beta}H^{\beta} - H^{\alpha}H^{\beta}H^{\alpha}H^{\beta})] * 1 = 0\,,$$

where H^{α} denotes the $p\times p$ matrix $(h^{\alpha}{}_{ij})$. Since the matrix $H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha}$ is skew-symmetric, we find

$$\begin{aligned} 0 &\geqq \sum \mathrm{Tr}\,[(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})] \\ &= 2\sum \mathrm{Tr}\,(H^{\alpha}H^{\beta}H^{\alpha}H^{\beta} - H^{\alpha}H^{\alpha}H^{\beta}H^{\beta})\,. \end{aligned}$$

Therefore each term in (2.5) is non-negative. In particular, we have $\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{kl}=0$, and so $h^{\alpha}{}_{ij}=0$. q.e.d.

COROLLARY 1. *Let (M, g) be a compact Riemannian manifold of constant curvature c ($\geqq 0$). Let $\mathscr{F}$ be a harmonic foliation such that the Riemannian metric is bundle-like. Then the foliation $\mathscr{F}$ is totally geodesic.*

In the case of $c=0$ in Corollary 1, it follows from (1.16) and the fact that $\mathscr{F}$ is totally geodesic that A vanishes identically (cf. Ranjan [7]). Thus we have:

COROLLARY 2. *Let (M, g) be a compact flat Riemannian manifold. Let $\mathscr{F}$ be a harmonic foliation such that $\mathscr{F}^{\perp}$ is minimal. Then $\mathscr{F}^{\perp}$ is integrable and tatally geodesic.*

REMARK. Theorem 2.3 does not hold if we replace the assumption "of constant curvature c ($\geqq 0$)" by "with positive Ricci curvature" (cf. Takagi and Yorozu [10], Theorem 3.4).

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