HARMONIC FUNCTIONS AND GREEN'S INTEGRAL*

BY

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§1. *Introductory.*

In his monograph on the *Theory of Fourier's Series*† Bôcher has devoted a section to Poisson's integral,

$$F(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(\psi)\frac{1-r^2}{1-2r\cos(\psi-\varphi)+r^2}d\psi \qquad (r<1),$$

and has given for it the following simple and elegant interpretation:

"*If we imagine that at each point of the unit circle the value of $f(s)$ at that point has been marked, then the value of $F(r,\varphi)$ at any point P within the circle is equal to the average of these values as they would be read off by an observer at P who turns with uniform angular velocity and who is situated in a refracting medium which causes the rays of light reaching his eye to take the form of circular arcs orthogonal to the unit circle.*"

On the basis of this theorem, many of the theorems on harmonic functions for the circle become intuitive, and the course which formal proof must take becomes evident. It has therefore seemed worth while to undertake a generalization to other regions, including those of higher connectivity. In its broad outlines, the generalization is easy. In fact, Poisson's integral is a special case of Green's integral

$$u(\xi,\eta)=\frac{1}{2\pi}\int_C f(s)\frac{\partial}{\partial n}G[\xi,\eta;x(s),y(s)]\,ds$$

where $G(\xi,\eta;x,y)=\log(1/\rho)$ plus a continuous function, ρ being

$$\sqrt{(\xi-x)^2+(\eta-y)^2}.$$

If, (ξ,η) regarded as fixed, $H(\xi,\eta;x,y)$ is the negative of the function conjugate to G, so that $\partial G/\partial n=\partial H/\partial s$, the integral becomes

$$u(\xi,\eta)=\frac{1}{2\pi}\int_0^{2\pi}f(s)\,dH.$$

* Presented to the Society December 28, 1906 and November 26, 1910.

† Annals of Mathematics, ser. 2, vol. 7 (1906), p. 94.

Noticing, therefore, that the lines $H =$ const. are the lines of greatest steepness of Green's function, or the lines of flow, if Green's function is a velocity potential, and that, in the neighborhood of the point $P(\xi, \eta)$, H has the character of $\arctan (y-\eta)/(x-\xi)$, we arrive at the following interpretation of Green's integral:

"*The value of* $u(\xi, \eta)$ *is the average of the boundary values* $f(s)$ *as they would appear to an observer stationed at* $P(\xi, \eta)$ *in a medium refracting light so that the rays take the form of the lines of flow of Green's function.*"*

Although this interpretation is easily obtained, it is a different matter to show the existence of the derivatives involved, and to investigate the properties of Green's integral. The attack on these problems led to the author's studies which appeared in these Transactions of 1908.† It is the purpose of the present paper to extend the results there established, and to make applications to Green's integral and to the question of the unique determination of harmonic functions whose boundary values are discontinuous. Attention is also called to Theorems IV, V, and VI, which are believed to be new and useful.

§2. *Derivatives of harmonic functions on the boundary of a multiply connected region.*

As the extension by conformal mapping of theorems on harmonic functions for the circle seems feasible only in the case of simply connected regions, it appears desirable to extend the results of the above cited papers to multiply connected regions.‡ At the same time the extension will be made to derivatives of higher order. In the matter of multiple connectivity a paper of PLEMELJ§ will be found useful.

Let R be a finite closed connected region bounded by a system of curves C, consisting of a closed outer curve C_0 and k closed curves $C_1, C_2, C_3, \cdots C_k$ lying within C_0 and having no points in common with it or each other, and free

* It is of interest to note that this may also be considered a generalization of Gauss' theorem that the value of a harmonic function at the center of a circle is the mean of its values on the circumference. In fact, for this case, $H = \vartheta = s/r$. BÔCHER, *Note on Poisson's integral*, Bulletin of the American Mathematical Society, vol. 4 (1898), p. 424.

† *Potential functions on the boundary of their regions of definition*, vol. 9, (1908), pp. 39–50, and *Double distributions and the Dirichlet problem*, vol. 9, pp. 51–66. Hereafter referred to as P. F.

‡ To the literature of the subject previously cited should be added H. PETRINI, *Les dérivées premières et secondes du potentiel*, Acta Mathematica, vol. 31 (1908), p. 127ff. G. PUCCIANO, *Studio sui potenziali logarithmici di strato lineare semplice e doppio e delle loro derivate primo*, Rendiconti del Circolo Matematico di Palermo, vol. 23 (1907), pp. 374–393. These papers consider dependence of potentials of various distributions of attracting matter upon densities and moments of the distributions. The dependence upon boundary values of the harmonic function is another question, related, but not simply related, to the first.

§ *Uber lineare Randwertaufgaben der Potentialtheorie*, Monatshefte für Mathematik und Physik, vol. 15 (1904), pp. 337–411.

from double points. Let them have parameter equations $x = x(s)$, $y = y(s)$, where s is the length of arc, the curve C_i corresponding to the interval $l_{i-1} \leqq s < l_i$, where $l_{-1} = 0$, and $l_k = l$ is the sum of the lengths of all the curves of the system C. Continuity of a function $f(s)$ at one of the points l_i is to mean

$$f(l_i - 0) = f(l_{i-1} + 0) \qquad \text{and} \qquad f(l_{i+1} - 0) = f(l_i + 0).$$

The following hypotheses will be employed (P. F., p. 41).

($A^{(r)}$) *There exist three positive numbers, N, α and δ, independent of s and Δs such that for $|\Delta s| < \delta$, $|x^{(r)}(s + \Delta s) - x^{(r)}(s)| < N|\Delta s|^\alpha$, and $|y^{(r)}(s + \Delta s) - y^{(r)}(s)| < N|\Delta s|^\alpha$.*

($B^{(r)}$) *$f^{(r)}(s)$ is continuous, and the integral*

$$\int_0^\tau \left|\frac{f^{(r)}(s+t) - f^{(r)}(s-t)}{t}\right| dt$$

vanishes with τ uniformly with respect to s.

In both conditions, r is an integer greater than or equal to 1 and $x^{(r)}$ means the rth derivative of x. If they are fulfilled, we have the following theorem:

THEOREM I. *There exists a uniquely determined harmonic function on R, $u(x, y)$ which approaches the boundary values $f(s)$; all its derivatives with respect to x and y of order r are continuous in the closed region R.*

The proof which follows depends upon the representation of the harmonic function as the potential of double and simple distributions on C. The moments and densities of these distributions will first be found and a study will then be made of the properties of their potentials.

§3. *Representation of a harmonic function as the potential of distributions on C.*

The attempt to determine the moment $\varphi(t)$ of a double distribution so that its potential

$$u(\xi, \eta) = \int_0^l \varphi(t) \frac{\partial}{\partial n} \log \frac{1}{\rho} dt,$$

will assume the boundary values $\pi f(s)$, leads to the integral equation

$$f(s) = \varphi(s) + \lambda \int_0^l \varphi(t) K(s, t)\, dt, \tag{1}$$

from which $\varphi(s)$ is to be determined for $\lambda = 1$, and where $K(s, t)$ is the function

$$\frac{1}{\pi} \frac{\partial}{\partial t} \arctan \frac{y(s) - y(t)}{x(s) - x(t)}.$$

$K(s, t)$ is continuous for $s \neq t$, except at the points $s = l_i$, $t = l_{i\pm 1}$, and satisfies the same inequalities near the points of discontinuity as in the case of simple connectivity.* The changes of order in the integrals may be justified, and the Fredholm resolvent $L(s,t;\lambda)$ established just as before. This done, a difference arises, since in the present case $\lambda = 1$ is a pole † of $L(s,t;\lambda)$. All the poles of this function are known to be simple, and in the neighborhood of $\lambda = 1$ it will have a development

$$L(s,t;\lambda) = P(s,t)/(\lambda - 1) + Q(s,t) + R(s,t;\lambda - 1), \tag{2}$$

where $R(s,t;\lambda - 1)$ is a power series in $\lambda - 1$ beginning with the first power of this difference, converging uniformly in closed regions excluding the discontinuities of $K(s,t)$, and satisfying an inequality of the same character as $K(s,t)$ for small $\lambda - 1$ at the points of discontinuity. $P(s,t)$ and $Q(s,t)$ have the continuity properties stated for $K(s,t)$.

Setting the expression (2) in the following characteristic equations for the resolvent

$$\begin{aligned} K(s,t) &= L(s,t;\lambda) + \lambda \int_0^l L(s,r;\lambda) K(r,t)\,dr, \\ K(s,t) &= L(s,t;\lambda) + \lambda \int_0^l L(r,t;\lambda)\, K(s,r)\,dr, \end{aligned} \tag{3}$$

and equating the coefficients of like powers of $\lambda - 1$, we have

$$\begin{aligned} 0 &= P(s,t) + \int_0^l P(s,r)\, K(r,t)\,dr, \\ 0 &= P(s,t) + \int_0^l P(r,t)\, K(s,r)\,dr, \end{aligned} \tag{4}$$

and

$$\begin{aligned} K(s,t) &= Q(s,t) + \int_0^l Q(s,r)\, K(r,t)\,dr + \int_0^l P(s,r)\, K(r,t)\,dr, \\ K(s,t) &= Q(s,t) + \int_0^l Q(r,t)\, K(s,r)\,dr + \int_0^l P(r,t)\, K(s,r)\,dr. \end{aligned} \tag{5}$$

The second equation (4) shows that $P(s,t)$ is for any fixed t the moment of a double distribution whose potential is 0 in R. We conclude ‡ that this potential is 0 also outside of C_0 and constant inside of each C_i $(i > 0)$. Hence $P(s,t)$ must, for fixed t, be 0 on C_0 and constant on each C_i $(i > 0)$. Let

$$S_1(s),\ S_2(s),\ S_3(s),\ \cdots,\ S_k(s)$$

* P. F., p. 53. Also FREDHOLM, Acta Mathematica, vol. 27 (1903), p. 384. It should be added that if $r \geqq 2$ in the hypotheses $(A^{(r)})$ and $(B^{(r)})$, $K(s,t)$ is continuous throughout.

† See the article cited of PLEMELJ. The theorems there established may be applied here with the one caution that for $r = 1$ the possible discontinuities of $K(s,t)$ must be considered. The λ of his article is minus the λ of the present paper.

‡ The reasoning is well known. See, for instance, P. F., p. 58.

each equal 1 when s lies in the interval corresponding to the curve with the same index, and be 0 for all other values of s. Then $P(s, t)$ may be written

$$P(s,t) = S_1(s)\,T_1(t) + S_2(s)\,T_2(t) + S_3(s)\,T_3(t) + \cdots + S_k(s)\,T_k(t) \tag{6}$$

where $T_i(t)$ is the value of $P(s, t)$ when s corresponds to a point of C_i.

Using this value in the first equation (4), and letting s lie in the various intervals, we find the equations

$$0 = T_i(t) + \int_0^l T_i(r)\,K(r,t)\,dr \qquad (i = 1, 2, 3, \ldots, k), \tag{7}$$

which show that the $T_i(t)$ are the densities of simple distributions on C whose potentials have vanishing normal derivatives in the region outside of C. As

$$\int_0^l K(r,t)\,dt = 1,$$

we find upon integrating the equations (7),

$$\int_0^l T_i(t)\,dt = 0 \qquad (i = 1, 2, 3, \ldots, k),$$

so that the total masses are 0 and each potential vanishes at infinity. Hence each potential with density $T_i(t)$ is 0 on C_0 and constant on each $C_i\ (i > 0)$. We shall call them $V_i(\xi, \eta)$, and shall have use for them presently. They are linearly independent.*

If the first equation (4) be multiplied by $L(t, q; \lambda)\,dt$ and integrated, we find on comparing coefficients of $\lambda - 1$

$$P(s,t) = \int_0^l P(s,r)\,P(r,t)\,dt, \tag{8}$$

or

$$\sum_{i=1}^k S_i(s)\,T_i(t) = \sum_{i=1}^k T_i(t) \int_{l_{i-1}}^{l_i} \sum_{j=1}^k S_j(s)\,T_j(r)\,dr. \tag{9}$$

From this it follows, because of the linear independence† of the $T_i(t)$, that

$$\int_{l_{i-1}}^{l_i} T_i(t)\,dt \;\begin{array}{l} = 1\ (i = j) \\ = 0\ (i \neq j) \end{array}\Bigg\}, \qquad \int_0^{l_0} T_i(t)\,dt = -1 \quad (i = 1, 2, 3, \ldots, k), \tag{10}$$

the last equations following from the preceding ones because of the vanishing of the total mass.

* See PLEMELJ, loc. cit., p. 389.

† A fact proven in the theory of integral equations. See PLEMELJ, *Zur Theorie der Fredholmschen Functionalgleichung*, Monatshefte für Mathematik und Physik, vol. 15 (1904), pp. 110 and 113.

Proceeding to the solving of the integral equation

$$f(s) = \varphi(s) + \int_0^l \varphi(t) K(s,t)\,dt, \tag{11}$$

we multiply it by $P(r,s)\,ds$ and integrate, using the first equation (4). The result is

$$\int_0^l f(s) P(r,s)\,ds = 0, \tag{12}$$

a condition which must be satisfied if the equation (11) is to be solvable. If $f(s)$ does not satisfy this condition, the function

$$g(s) = f(s) - \int_0^l f(t) P(s,t)\,dt$$

does, as may be seen by employing the relation (8). The difference $f(s) - g(s)$ is constant on each curve C_i. Let us consider the equation

$$g(s) = \varphi(s) + \int_0^l \varphi(t) K(s,t)\,dt. \tag{13}$$

Multiply it by $Q(r,s)\,ds$ and integrate, using equations (5) and (4). The result is

$$\int_0^l g(s) Q(r,s)\,ds = \int_0^l \varphi(s) Q(r,s)\,ds + \int_0^l \varphi(t)[K(r,t) - Q(r,t) + P(r,t)]\,dt.$$

Equation (13) reduces this to

$$\varphi(s) = g(s) - \int_0^l g(t) Q(s,t)\,dt + \int_0^l \varphi(t) P(s,t)\,dt,$$

or

$$\varphi(s) = g(s) - \int_0^l g(t) Q(s,t)\,dt + \sum_{i=1}^{k} c_i S_i(s), \tag{14}$$

which must be the form of the solution, if it exists. As

$$S_i(s) + \int_0^l S_i(t) K(s,t)\,dt = 0,$$

the added summation may be omitted if, as in the present case, a particular solution suffices. That (14) actually does give a solution may be seen by multiplying it by $K(s,r)\,ds$ and integrating.

It therefore appears that it is not always possible to find a double distribution whose potential will take on the boundary values $\pi f(s)$. It is, on the other hand, possible to find one such that the boundary values of the potential will differ from $\pi f(s)$ by a function which is 0 on C_0 and constant on each of the curves C_i $(i > 0)$. Let $W(\xi, \eta)$ be such a potential. Then as the functions $V(\xi, \eta)$ are linearly independent, it will be possible to find constants $c_1, c_2, c_3, \cdots c_k$, such that the potential

$$c_1 V_1(\xi, \eta) + c_2 V_2(\xi, \eta) + c_3 V_3(\xi, \eta) + \cdots + c_k V_k(\xi, \eta)$$

of a simple distribution on C takes on, on the boundary, a value equal to the difference between $\pi f(s)$ and the boundary values of $W(\xi, \eta)$. Hence

$$u(\xi, \eta) = W(\xi, \eta) + c_1 V_1(\xi, \eta) + c_2 V_2(\xi, \eta) + c_3 V_3(\xi, \eta) + \cdots + c_k V_k(\xi, \eta)$$

is a harmonic function on R taking on the boundary values $\pi f(s)$.

§4. *Character of certain functions occurring in the solution of the integral equation.*

The functions

$$K_i(s, t) = \int_0^l K_{i-1}(s, r) K(r, t)\, dr \qquad [K_0(s, t) = K(s, t)]$$

have the same continuity properties as $K(s, t)$, except that if the iteration process is carried far enough, a bounded function is obtained (P. F., p. 54). The result of the next iteration is a function continuous throughout. The only proof that need be given here is for a point (s_0, s_0) of the line $s = t$. Let the increment $\Delta K(s, t)$ due to the increments Δs and Δt be written as a sum $J_1 + J_2$, where

$$J_1 = \int_{s_0-\eta}^{s_0+\eta} [K_{i-1}(s, r) K(r, t) - K_{i-1}(s_0, r) K(r, s_0)]\, dr,$$

and

$$J_2 = \int_0^{s_0-\eta} + \int_{s_0+\eta}^{l} [K_{i-1}(s, r) K(r, t) - K_{i-1}(s_0, r) K(r, s_0)]\, dr.$$

Then

$$|J_1| < B \int_{s_0-\eta}^{s_0+\eta} [|K(r, t)| + |K(r, s_0)|]\, dr,$$

where B is an upper bound for $|K_{i-1}(s, t)|$, and by the inequality to which $K(s, t)$ is subject,

$$|J_1| < 4BA \int_{s_0}^{s_0+\eta} (r - s_0)^{\alpha-1} dr = \frac{4BA\eta^\alpha}{\alpha},$$

and so can be made less than $\frac{1}{2}\epsilon$ by taking η small enough. Then because of the uniform continuity of the integrand in the remaining intervals, Δs and Δt can be made so small that $|J_2| < \frac{1}{2}\epsilon$. With these restrictions on Δs and Δt, $|\Delta K_i(s, t)| < \epsilon$.

Referring now to a previous paper (P. F., pp. 55–57), we take n so large as to make $K_n(s,t)$ continuous throughout. The function $\Gamma(s,t;\lambda)$ there given is a power series in λ with continuous coefficients, and such that a dominant series with positive constant coefficients can be found, $G_0 + G_1\lambda + G_2\lambda^2 + \cdots$, which is convergent for every λ. In the expression for $L(s,t;\lambda)$,

$$L(s,t;\lambda) = -k(s,t;\lambda) \tag{15}$$
$$+\frac{1}{\delta(\lambda)}\left[\Gamma(s,t;\lambda) + \lambda\int_0^l \Gamma(r,t;\lambda)\,k(s,r;\lambda)\,dr\right];$$

the same statement may be made for the functions

$$N(s,t;\lambda) = \Gamma(s,t;\lambda) + \lambda\int_0^l \Gamma(r,t;\lambda)\,k(s,r;\lambda)\,dr,$$

since multiplying a uniformly continuous function by $K(s,r)\,dr$ and integrating gives a continuous function, and since

$$\int_0^l |K(s,r)|\,dr$$

is bounded. Indeed, if M be a bound for this integral, a dominant series for $N(s,t;\lambda)$ will be

$$G_0 + G_1\lambda + G_2\lambda^2 + \cdots + \lambda[M + M^2\lambda + \cdots + M^n\lambda^{n-1}][G_0 + G_1\lambda + \cdots],$$

which is the product of an always convergent series by a polynomial, and hence is always convergent. The same is true of the development of $N(s,t;\lambda)$ about any point of the plane, in particular of

$$N(s,t;\lambda) = N_p(s,t)(\lambda-1)^p + N_{p+1}(s,t)(\lambda-1)^{p+1} + \cdots.$$

As we have seen, the point $\lambda = 1$ is a simple pole of $L(s,t;\lambda)$, so that the development of $\delta(\lambda)$ about the point $\lambda = 1$ will begin with the $(p+1)$th power of $\lambda - 1$. It follows that $(\lambda-1)N(s,t;\lambda)/\delta(\lambda)$ is a power series in $\lambda - 1$ with continuous coefficients and has a dominant series with finite radius of convergence. A comparison of equations (2) and (15) therefore leads to the results:

$P(s,t)$ is continuous throughout.

$Q(s,t) = K(s,t) - K_1(s,t) + K_2(s,t) - \cdots \pm K_{n-1}(s,t)$ plus a function continuous throughout.

The first $n-1$ coefficients of $R(s,t;\lambda-1)$ are polynomials in $K(s,t)$, $K_1(s,t)$, $K_2(s,t)$, $\cdots K_{n-1}(s,t)$ plus continuous functions, and all further coefficients are continuous.

These statements hold for $r = 1$ in condition $(A^{(r)})$; if $r \geqq 2$, we may go farther, and assert the continuity of all the functions of s and t considered. For the study of the rth derivatives of harmonic functions, information will be needed concerning the $(r-1)$th derivative of $K(t, s)$ with respect to s. Since this derivative is a rational function of $x(t)$, $y(t)$ and of $x(s)$, $y(s)$ and their derivatives of order r or lower, the denominator being a power of $\rho^2 = [x(s) - x(t)]^2 + [y(s) - y(t)]^2$, $K_s^{(r-1)}(t, s)$ is uniformly algebraically continuous* in s in any closed region excluding the line $s = t$ and the points $s = l_i, t = l_{i\pm 1}$. For the neighborhood of $s = t$, a special study must be made (a similar study will yield similar results for the other points of discontinuity). What is needed (P. F., pp. 63–66) is an inequality

$$(16) \qquad |K_s^{(r-1)}(t, s)| < A\,|t - s|^{\alpha-1},$$

as well as one of the form

$$(17) \qquad |K_s^{(r-1)}(t, s) - K_s^{(r-1)}(t, 0)| < F s^{\frac{1}{2}\alpha} t^{\frac{1}{2}\alpha - 1},$$

the latter holding for $\eta > t > t^2 \geqq s \geqq 0$, with a similar one for the corresponding negative values of the arguments. A, α, F, and η are positive constants.

To establish these inequalities, consider first $q_s^{(r)}(s, t)$, where

$$q(s, t) = [y(s) - y(t)]/[x(s) - x(t)].$$

If this defining relation be cleared of fractions, and then differentiated r times with respect to s, the resulting equations may be solved for $q_s^{(r)}(s, t)$, and will give

$$(18) \quad q_s^{(r)}(s,t) = \frac{\begin{vmatrix} x(s)-x(t) & 0 & 0 & \cdots & 0 & y(s)-y(t) \\ x'(s) & x(s)-x(t) & 0 & \cdots & 0 & y'(s) \\ x''(s) & 2x'(s) & x(s)-x(t) & \cdots & 0 & y''(s) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ x^{(r-1)}(s) & {}^{r-1}C_1 x^{(r-2)}(s) & {}^{r-1}C_2 x^{(r-3)}(s) & \cdots & x(s)-x(t) & y^{(r-1)}(s) \\ x^{(r)}(s) & {}^{r}C_1 x^{(r-1)}(s) & {}^{r}C_2 x^{r-2}(s) & \cdots & {}^{r}C_1 x'(s) & y^{(r)}(s) \end{vmatrix}}{[x(s) - x(t)]^{r+1}},$$

${}^rC_1$, ${}^rC_2$, $\cdots$ being binomial coefficients. Let us imagine t fixed, and the axes so chosen that $x'(t) = 1$, $y'(t) = 0$. Then by the law of the mean, $[x(s) - x(t)]^{r+1} = (s - t)^{r+1}[1 + F(s, t)(s - t)]$, where $F(s, t)$ is a bounded function. Now add to the first row of the determinant above $(t - s)$ times the second row, $(t - s)^2/2!$ times the third, and so on to the last. The

* That is, there are three positive numbers, δ, A, and α, independent of s and t, such that for $|\Delta s| < \delta$, $|K_s^{(r-1)}(t, s + \Delta s) - K_s^{(r-1)}(t, s)| < A|\Delta s|^\alpha$.

law of the mean then shows that all the elements of the top row except the first and last contain $(t-s)^{r+1}$. The quotients of the first and last elements by $|s-t|^{r+\alpha}$ will be bounded, a result of condition $(A^{(r)})$. Since all the other elements of the determinant are bounded, it follows that $q_s^{(r)}(s, t)$ is a bounded function times $|s-t|^{\alpha-1}$. Finally, as

$$K_s^{(r-1)}(t, s) = \frac{1}{\pi}\frac{\partial^r}{\partial s^r}\arctan q(s, t) = \frac{1}{\pi}\frac{\partial^{r-1}}{\partial s^{r-1}}\frac{q'_s(s, t)}{1+[q(s, t)]^2},$$

and as all the derivatives of $q(s, t)$ with respect to s of order lower than the rth are bounded, the inequality (16) is established.

With respect to the inequality (17), it should be noticed that if a function is subject to it, that function multiplied by a function which is uniformly algebraically continuous in s is subject to an inequality of the same kind. We need therefore only show that this inequality is a property of the quotients by $[x(s)-x(t)]^{r+1}$ of the elements of the first row of the above determinant after the addition of the specified multiples of the other rows. The first element alone will be considered here, as it is typical. It is

$$E(s, t) = x(s) - x(t) + x'(s)(t-s) + x''(s)(t-s)^2/2! + \cdots + x^{(r)}(s)(t-s)^r/r!$$

As for the divisor, if we choose our axes so that $x'(0) = 1$, we may replace it by $(t-s)^{r+1}$ since the multiplier necessary to correct this replacement, $\pm(s-t)^{r+1}/[x(s)-x(t)]^{r+1}$, is uniformly algebraically continuous in s. Indeed, its s-derivative is bounded for small s and t. If now $x(t)$, and $x(s)$ and its derivatives in the expression $E(s, t)$, be developed about the point 0, the result will be

$$\begin{aligned}\frac{E(s, t)}{(t-s)^{r+1}} - \frac{E(0, t)}{t^{r+1}} = \{&t^{r+1}[x^{(r)}(\vartheta_0 s)s^r + {}^rC_1 x^{(r)}(\vartheta_1 s)s^{r-1}(t-s) + \cdots \\ &+ {}^rC_{r-1}x^{(r)}(\vartheta_{r-1}s)s(t-s)^{r-1} + x^{(r)}(s)(t-s)^r - x^{(r)}(\vartheta' t)t^r] \\ &- (t-s)^{r+1}[x^{(r)}(0)t^r - x^{(r)}(\vartheta' t)t^r]\} \div r!\,t^{r+1}(t-s)^{r+1},\end{aligned}$$

in which it is important to notice that the two proper fractions ϑ' are the same. The numbers $\vartheta_0, \cdots, \vartheta_{r-1}$ lie between zero and one. If the condition $(A^{(r)})$ be consulted, and F be used to denote various bounded functions, it will be seen that this expression may be written

$$\begin{aligned}&\{t^{r+1}[x^{(r)}(0)(s+t-s)^r + F\cdot(s+t-s)^r s^\alpha - x^{(r)}(\vartheta' t)t^r] \\ &\quad - (t-s)^{r+1}t^r[x^{(r)}(0) - x^{(r)}(\vartheta' t)]\} \div r!\,t^{r+1}(t-s)^{r+1} \\ &= \{t^{2r+1}s^\alpha F + t^r[x^{(r)}(0) - x^{(r)}(\vartheta' t)][t^{r+1} - (t-s)^{r+1}]\} \div r!\,t^{r+1}(t-s)^{r+1} \\ &= F\{s^\alpha t^r/(t-s)^{r+1} + t^{\alpha-1}s[{}^{r+1}C_1 t^r - {}^{r+1}C_2 t^{r-1}s + \cdots + s^r]/(t-s)^{r+1}\},\end{aligned}$$

and in the interval in question, this is less than $F \cdot s^{\frac{1}{2}\alpha} t^{\frac{1}{2}\alpha - 1}$. The inequality (17) is thus established.

§5. *Character of the moment and of the densities of the potentials W and V_i.*

In this paragraph, $f(s)$ will be regarded as subject to the condition $(B^{(r)})$; $g(s)$ will then be subject to the same condition. With the properties of $Q(s, t)$ obtained in the last paragraph, it is easy to show that the solution (14) of the integral equation (15) is continuous. We proceed to a study of the derivatives of $\varphi(s)$, and to that end, consider the equation

$$g'(s) = \Phi(s) - \int_0^l \Phi(t) K(t, s)\, dt. \tag{19}$$

The necessary and sufficient condition for its solvability turns out to be

$$\int_0^l g'(s)\, ds = 0,$$

which is satisfied. A solution is

$$\Phi(s) = g'(s) + \int_0^l g'(t)\, \kappa(t, s)\, dt, \tag{20}$$

where $\kappa(s, t)$ is the function corresponding to $Q(s, t)$ in the development of $L(s, t; \lambda)$ about the point $\lambda = -1$. The function $\Phi(s)$ is continuous. Equation (19) may be integrated with respect to s, with the result

$$g(s) = \int_0^s \Phi(s)\, ds - \int_0^l \Phi(t) \int_0^s K(t, s)\, ds\, dt + c(s),$$

$c(s)$ being constant on each interval $l_i \leqq s < l_{i+1}$. As

$$\int_0^s K(t, s)\, ds = \frac{1}{\pi} \arctan \frac{y(s) - y(t)}{x(s) - x(t)} + c(s, t),$$

where $c(s, t)$ is constant in each rectangle $l_i \leqq s < l_{i+1}$, $l_j \leqq t < l_{j+1}$, we have, on integrating by parts,*

$$g(s) = \int_0^s \Phi(s)\, ds + \int_0^l \left[\int_0^t \Phi(t)\, dt \right] K(s, t)\, dt + c(s).$$

* The relations

$$\int_{l_{i-1}}^{l_i} \Phi(t)\, dt = 0$$

are needed here. They are a consequence of equation (20). It should be observed that from equations (3) it follows that

$$\int_{l_i}^{l_{i+1}} L(s, t; \lambda)\, dt$$

is independent of s on each s-interval, and hence this is true for $\kappa(s, t)$ also.

From this it follows that

$$\varphi(s) = \int_0^s \Phi(s)\,ds + c(s),$$

and hence $\varphi'(s) = \Phi(s)$, so that $\varphi'(s)$ is continuous, and satisfies the equation

$$g'(s) = \varphi'(s) - \int_0^l \varphi'(t) K(t, s)\,dt. \tag{21}$$

This equation may be differentiated $r - 1$ times with respect to s. The integral in the resulting equation will then satisfy condition $(B^{(r)})$,* and we may therefore make the statement:

THEOREM II. *The moment $\varphi(s)$ of the double distribution whose potential, $W(\xi, \eta)$, takes on the boundary values $\pi f(s)$ subject to condition $(B^{(r)})$, the boundary curve being subject to condition $(A^{(r)})$, satisfies condition $(B^{(r)})$.*

Turning to the densities of the simple distributions whose potentials are $V_i(\xi, \eta)$, that is, to the functions $T_i(s)$, we note that their continuity follows from that of $P(s, t)$. Moreover, as they satisfy equation (7), the reasoning applied to equation (21) enables us to state further:

THEOREM III. *The densities $T_i(s)$ of the simple distribution whose potentials are $V_i(\xi, \eta)$ are subject to the condition $(B^{(r-1)})$ if the boundary curve satisfies condition $(A^{(r)})$.*

§ 6. *Derivatives of harmonic functions on the boundary.*

Let $h_1, h_2, \ldots h_r$ denote r fixed directions in the plane, or, as variables, let $h_i = \xi \cos \alpha_i + \eta \sin \alpha_i$, α_i being constants. Consider first

$$W(\xi, \eta) = \int_0^l \varphi(t) \frac{\partial}{\partial n} \log\left(\frac{1}{\rho}\right) dt,$$

where $\rho^2 = [\xi - x(t)]^2 + [\eta - y(t)]^2$. If it be transformed by integration by parts and by use of the function $\Theta(\xi, \eta; t) = \arctan[\eta - y(t)]/\xi - x(t)]$, conjugate to $\log \rho$, its derivative with respect to h_1 will take the form

$$\frac{\partial W}{\partial h_1} = \int_0^l \varphi_{1,1}(t) \frac{\partial}{\partial t} \Theta(\xi, \eta; t)\,dt + \int_0^l \varphi_{1,2}(t) \frac{\partial}{\partial t} \log\left(\frac{1}{\rho}\right) dt \tag{22}$$

where

$$\varphi_{1,1}(t) = -\varphi'(t)\,[x'(t) \cos \alpha_1 + y'(t) \sin \alpha_1],$$

$$\varphi_{1,2}(t) = -\varphi'(t)\,[y'(t) \cos \alpha_1 - x'(t) \sin \alpha_1].$$

If integration by parts be applied to the formula (22) and the derivative taken with respect to h_2, and so on, there is obtained the formula

$$\frac{\partial^{r-1} W}{\partial h_1 \partial h_2 \cdots \partial h_{r-1}} = \text{const.} + \int_0^l \varphi'_{r-1,1}(t)\, \Theta(\xi, \eta; t)\,dt + \int_0^l \varphi'_{r-1,2}(t) \log\left(\frac{1}{\rho}\right) dt, \tag{23}$$

* For method of proof, see P. F., pp. 63–66.

in which, as a consequence of the condition $(A^{(r)})$ on $x(t)$, $y(t)$, and of the condition $(B^{(r)})$ on $\varphi(t)$ (see Theorem III), $\varphi_{r-1,1}(t)$ and $\varphi_{r-1,2}(t)$ satisfy conditions $(B^{(1)})$. Then, as

$$\frac{\partial}{\partial\nu}\log\left(\frac{1}{\rho}\right)=\frac{\partial\Theta}{\partial\sigma}, \qquad \frac{\partial}{\partial\sigma}\log\left(\frac{1}{\rho}\right)=\frac{\partial\Theta}{\partial\nu},$$

the derivative of W of order $r+1$ obtained from (23) by one more differentiation admits of exactly the same treatment as did W itself in the study *Potential Functions on the Boundary of their Regions of Definition*,* with the result:

The potential $W(\xi,\eta)$ has continuous derivatives of order r in the closed region R.

As for the functions

$$V_i(\xi,\eta)=\int_0^l T_i(t)\log\left(\frac{1}{\rho}\right)dt,$$

the first derivatives may be written

$$\frac{\partial V_i}{\partial h_i}=\int_0^l \psi_{1,1}(t)\frac{\partial}{\partial t}\Theta(\xi,\eta;\,t)\,dt+\int_0^l \psi_{1,2}(t)\frac{\partial}{\partial t}\log\left(\frac{1}{\rho}\right)dt,$$

where

$$\psi_{1,1}(t)=T_i(t)[x'(t)\cos\alpha_1+y'(t)\sin\alpha_1],$$

$$\psi_{1,2}(t)=-T_i(t)[y'(t)\cos\alpha_1-x'(t)\sin\alpha_1].$$

These expressions are seen to have the form of those for $\varphi_{1,1}(t)$ and $\varphi_{1,2}(t)$, except that they involve $T_i(t)$ instead of its derivative. Hence, as $T_i(t)$ is subject to condition $(B^{(r-1)})$ by Theorem III, the functions $\psi_{r-1,1}(t)$ and $\psi_{r-1,2}(t)$, analogous to $\varphi_{r-1,1}(t)$ and $\varphi_{r-1,2}(t)$ above, satisfy the condition $(B^{(1)})$. We may therefore conclude that

The potentials $V_i(\xi,\eta)$ have continuous derivatives of order r in the closed region R.

The proof of Theorem I is thus completed.

§ 7. *Green's function and some of its properties.*

Green's function is defined by the equation

$$G(\xi,\eta;x,y)=\log\frac{1}{\rho(\xi,\eta;\,x,y)}-g(\xi,\eta;\,x,y) \tag{24}$$

where $g(\xi,\eta;x,y)$ is harmonic in (x,y) in R for fixed (ξ,η) and has the same boundary values as the logarithmic term in equation (24), and (ξ,η) are the coördinates of an interior point of R. $G(\xi,\eta;x,y)$ is therefore harmonic in (x,y) in any subregion of R excluding (ξ,η), and approaches 0 as (x,y)

* P. F., pp. 40–50, especially p. 43, third line, and p. 46, fourth line of § 3.

approaches any point of the boundary. It is known to be symmetric in its two pairs of arguments:

$$G(x, y; \xi, \eta) = G(\xi, \eta; x, y),$$

so that *for fixed* (x, y), $G(\xi, \eta; x, y)$ *is harmonic in* (ξ, η) *in any subregion of* R *excluding* (x, y).

Moreover, by Theorem I, *for fixed* (ξ, η) *the derivatives with respect to* x *and* y *of order* r *of* $G(\xi, \eta; x, y)$ *are continuous in any closed subregion of* R *excluding the point* (ξ, η). In the present section it will be assumed only that $r = 1$.

If (a, b) is a fixed interior point of R, there will be points, in case R is multiply connected, where $\partial G(a, b; x, y)/\partial x$ and $\partial G(a, b; x, y)/\partial y$ vanish simultaneously. It is important to know, however, that under the condition $(A^{(1)})$ these points do not occur on the boundary. Continuity of $x'(s)$ and $y'(s)$ alone is not sufficient to insure this, as may be seen by examining the conformal transformation $z = -\zeta \log \zeta$, which, although it has an infinite derivative for $\zeta = 0$, maps the upper half of the z-plane on a region whose boundary has a continuously turning tangent. The circle $\xi^2 + (\eta - 1)^2 = 1$ corresponds to a region of the z-plane with continuously turning tangent, and Green's function for this region has a vanishing normal derivative at the origin.

To prove that $\partial G(a, b; s)/\partial n > 0$* under the hypothesis $(A^{(1)})$, we choose our coördinate system so that $s = x = y = 0$ at the point at which we are to study the derivative, and take the x-axis in the direction of n. Then if $\zeta = \xi + i\eta = \rho e^{i\vartheta}$, there is a transformation $z = (\zeta - \zeta^{2m/(2m-1)})/\beta$, where m is a positive integer, and β a positive constant, which maps a circle $\rho = \alpha \cos \vartheta$ of the ζ-plane, of sufficiently small diameter, α, on a simply connected region S of the z-plane, S lying entirely within R except that its boundary is tangent to that of R at the origin. The existence of such a transformation is a consequence of the hypothesis $(A^{(1)})$. If then, $G(a, b; x, y)$ expressed in terms of (ξ, η) is $\Gamma(\xi, \eta)$, $\partial G/\partial n = \partial \Gamma/\partial \xi \cdot \partial \xi/\partial n$. If ds and $d\sigma$ are linear elements in the z-plane and ζ-plane respectively, $ds^2 = d\sigma^2 |1 - (2m/2m - 1) \cdot \zeta^{1/(2m-1)}|^2/\beta^2$, so that at the origin $\partial \xi/\partial n = \beta$, and $\partial G/\partial n$ is positive if $\partial \Gamma/\partial \xi$ is. But $\Gamma(\xi, \eta)$ is harmonic in the circle $\rho = \alpha \cos \vartheta$, and is continuous on the circumference. Its boundary values $\gamma(\sigma)$ are positive at every point except $\sigma = 0$, where $\gamma(\sigma) = 0$.

We are thus led to enquire whether the normal derivative of a non-constant harmonic function on the surface of a circle can vanish at the point where this function attains its minimum. It will not restrict the generality to assume a unit radius for the circle. Let ϵ be a positive constant less than π. Then $\gamma(\sigma)/(\sigma - \epsilon) > 0$ for $\epsilon < \sigma \leqq \pi$. This function, continuous in the interval given, does not approach its lower limit as $\sigma = \epsilon$, so that it attains this lower limit, say λ_+, which is positive. Let λ_- be the corresponding lower limit for

* $\dfrac{\partial}{\partial n} G(a, b; s) = \dfrac{\partial}{\partial n} G[a, b; x(s) - y'(s)n, y(s) + x'(s)n]\Big|_{n=0}$

$-\gamma(\sigma)/(\sigma+\epsilon)$, $-\pi \leqq \sigma < -\epsilon$, and let λ be positive and less than the smaller of λ_+ and λ_-. Then, if

$$\begin{aligned} \beta(\sigma) &= 0 && \text{for} \quad -\epsilon \leqq \sigma \leqq \epsilon, \\ \beta(\sigma) &= \lambda(\sigma-\epsilon) && \text{for} \quad \epsilon \leqq \sigma \leqq \pi, \\ \beta(\sigma) &= -\lambda(\sigma+\epsilon) && \text{for} \quad -\pi \leqq \sigma \leqq -\epsilon, \end{aligned}$$

it will follow that $\gamma(\sigma) \geqq \beta(\sigma)$, for $-\pi \leqq \sigma \leqq \pi$, and if $\gamma(\sigma) - \beta(\sigma) = \delta(\sigma)$, $\delta(\sigma) > 0$ for $\sigma \neq 0$, and $\delta(0) = 0$. If $B(\xi, \eta)$ and $\Delta(\xi, \eta)$ are the harmonic functions determined by the boundary values $\beta(\sigma)$ and $\delta(\sigma)$, $\Gamma(\xi, \eta) = B(\xi, \eta) + \Delta(\xi, \eta)$. A negative normal derivative for $\Delta(\xi, \eta)$ at $\sigma = 0$ is impossible, as it would imply a minimum of a harmonic function in the interior of its region. $B(\xi, \eta)$, having continuous boundary values, will be given by Poisson's integral, from which, upon integrating by parts and differentiating, we obtain

$$\frac{\partial B}{\partial \xi} = -\frac{\partial B}{\partial \rho}\bigg|_{\sigma=0} = \frac{1}{\pi}\int_{-\pi}^{+\pi} \frac{\beta'(\varphi)\sin\varphi\, d\varphi}{1+\rho^2-2\rho\cos\varphi} = \frac{2\lambda}{\pi}\int_{\epsilon}^{\pi} \frac{\sin\varphi\, d\varphi}{1+\rho^2-2\rho\cos\varphi}.$$

This derivative is continuous for $\rho = 1$ and has the value

$$\frac{\lambda}{\pi}\log\frac{2}{1-\cos\epsilon} > 0.$$

Thus $\partial G(a, b;\, s)/\partial n > 0$ at every boundary point; and because of the uniform continuity of the derivatives of $G(a, b;\, x, y)$, we have

THEOREM IV. *There is a positive constant* μ, *and a region containing a finite neighborhood of every point of* C, *such that*

$$\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2 > \mu$$

throughout this region. C is here subject to condition $(A^{(1)})$.

For the further study of Green's function and Green's integral, a theorem due to OSGOOD * on the convergence of an infinite sequence of harmonic functions, and one on the boundedness of the derivatives of a harmonic function, will be of use. The first may be stated as follows:

OSGOOD'S THEOREM. *If* $u_1, u_2, u_3, \cdots u_i, \cdots$ *is an infinite sequence of functions, harmonic in a region* T, *converging at every point (or at a set of points everywhere dense) in* T *to a function* U, *and if there is a constant* L *such that* $|u_i| < L$ *for every* i, *then* U *is harmonic in* T. *Moreover the convergence is uniform in every closed subregion containing only interior points of* T.

The second theorem is as follows:

THEOREM V. *Let* S *denote a finite number of segments of the boundary* C *of* R,

* Annals of Mathematics, ser. 2, vol. 3 (1901–02), p. 26.

and T a closed subregion of R, having no points in common with the boundary of R except interior points of S. Then if u is a harmonic function on R which vanishes on S, and which is bounded on the rest of C, the first derivatives of u are bounded in T. C is here subject to condition $(A^{(1)})$.* The theorem admits the dependence of u on a parameter, of which, however, the bound referred to must be independent.

To prove this, let us first suppose R simply connected. If $H\ (a, b; x, y)$ is the negative of the conjugate of Green's function $G(a, b;\ x,\ y)$ for R, the transformation $w = re^{u} = e^{-G+iH}$ maps R conformly on the surface of the unit circle, and $u(x, y)$ becomes a function of (r, t),

$$U(r, t) = \frac{1}{2\pi}\int_{\Sigma}\frac{U(1, \vartheta)(1 - r^2)\,d\vartheta}{1 - 2r\cos(t - \vartheta) + r^2},$$

where the intervals Σ correspond to the points of C not in S. The points of the closed map of T will all have finite distances from the points of Σ, so that the derivatives of $U(r, t)$ will be bounded in the map of T. Hence, as the derivatives of first order of the mapping function are continuous in the closed region R, the derivatives of $u(x, y)$ will be bounded in T. If R is not simply connected, two simply connected regions can be found which will completely cover it, and by means of them the theorem can be generalized to hold for the case of multiple connectivity without difficulty.

From the formula for $U\ (r, t)$ may be inferred the additional result:

Theorem VI. *If, under the conditions of the last theorem, $U\ (x, y)$ contains a parameter, and approaches* 0 *uniformly on the rest of C as the parameter approaches a limit, the first derivatives of $U\ (x, y)$ approach* 0 *uniformly in T.*

As an application, let us establish the fact that $\partial G\ (\xi, \eta; s)/\partial n$ is harmonic. Let (ξ, η) be confined to the surface of a circle K' lying entirely within R. Let K'' be a larger concentric circle also entirely within R, the difference of their radii being denoted by δ. Then if Δ denote the greatest diameter of R, $G = \log(\Delta/\rho) - (g + \log\Delta) \geqq 0$. But $g + \log\Delta$ has boundary values that are never negative, and hence $0 \leqq g + \log\Delta \leqq \log(\Delta/\rho)$. Therefore $0 \leqq G \leqq \log(\Delta/\rho) + (g + \log\Delta) \leqq 2\log(\Delta/\rho)$, and with $(x,\ y)$ confined to $R - K''$, $\rho \geqq \delta$, so that $0 \leqq G \leqq 2\log(\Delta/\delta)$. Now let T represent a closed subregion of $R-K''$, containing, except for a segment S of C, only interior points of $R - K''$. Then G fulfills the conditions of Theorem V for the regions $R - K''$ and T, so that its first derivatives, and hence also its first difference quotients with respect to x and y, are bounded in $R - K''$. But as the derivatives of G exist on the boundary, a sequence of difference quotients can be selected which satisfy the requirements of Osgood's theorem, and we may conclude

* An evident generalization, allowing the values of u on S to differ from 0 is obtained by adding to u any harmonic function on R with bounded derivatives of first order.

The function $\partial G(\xi, \eta; s)/\partial n$ *is harmonic in* (ξ, η) *in the open region consisting of the interior points of* R.

§ 8. *Green's integral.*

A second application of Osgood's theorem establishes

THEOREM VII. *If* $f(s)$ *is bounded and summable in* LEBESGUE'S *sense, or if* $|f(s)|$ *is summable, Green's integral*

$$u(\xi, \eta) = \frac{1}{2\pi}\int_0^l f(s)\frac{\partial}{\partial n}G(\xi, \eta; s)\,ds$$

is harmonic in R.

In order to study the behavior of this integral as (ξ, η) approaches a boundary point, we shall need the following

FUNDAMENTAL LEMMA.* *Given three points,* π_-, π, π_+ *on one of the curves* C_i, *and a positive constant* ϵ, *it is possible to find a circle with center at* π *and with finite radius, such that when* (ξ, η) *lies within this circle,* $\partial G(\xi, \eta; s)/\partial n < \epsilon$ *for all values of* s *other than those belonging to the arc* $\pi_- \pi \pi_+$.

To prove this, describe about π two concentric circles cutting the arc $\pi_- \pi \pi_+$ each in only two points. Call the subregion of R included in the inner one K, and that outside the outer one T. Then with (x, y) lying in T and (ξ, η) in K, the derivatives of G of first order with respect to x and y are bounded, say by the constant B, by Theorem V. Given an arbitrary constant, γ, we take upon the circular part of the boundary of T a finite number of points, such that every point of this arc is less than a distance $\gamma / 2B$ along the arc from one of these selected points. If then the value of G at these selected points can be made less than $\frac{1}{2}\gamma$, because of the boundedness of the derivatives the value on the whole circular arc will be less than γ. That G can be made less than $\frac{1}{2}\gamma$ at a finite number of points on the arc by restricting (ξ, η) to a circle about π follows from the fact that for any fixed (x, y) G approaches 0 as (ξ, η) approaches a boundary point. The conditions of Theorem VI are thus fulfilled, and it follows that the derivatives of G with respect to x and y approach 0 uniformly in T, and in particular, so does $\partial G(\xi, \eta; s)/\partial n$. The lemma is thus demonstrated.

By means of the lemma and the fact that

$$\int_0^l \frac{\partial}{\partial n}G(\xi, \eta; s)\,ds = 2\pi,$$

a number of important properties of Green's integral may be proved,† among them

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* Cf. BÔCHER, loc. cit., p. 94, theorem IV.

† Cf. BÔCHER, loc. cit., p. 98.

THEOREM VIII. *Green's integral*

$$u(\xi, \eta) = \frac{1}{2\pi}\int_0^l f(s)\frac{\partial}{\partial n}G(\xi, \eta; s)\, ds$$

approaches the boundary values $f(s)$ at every point where this function is continuous.

With this the interpretation given to Green's integral in the first paragraph is justified, the existence of all the symbols and the validity of the processes involved having been established.

A method of reducing the study of Green's integral in the neighborhood of a boundary point, even in the case of a multiply connected region, to that of Poisson's integral will be found useful. Let S represent a segment of the boundary of R containing in its interior the point at which the behavior of the integral is to be studied. Join the end-points of S by a curve running through R and bounding a simply connected region R_0, the bounding curve forming with S a closed curve satisfying condition $(A^{(1)})$. To this curve assign the boundary values $f_0(s)$, equal to $f(s)$ on S and 0 elsewhere. If the further quantities connected with the simply connected region be marked by the subscript 0, the equation holds:

$$(25)\quad u(\xi, \eta) - u_0(\xi, \eta) = \frac{1}{2\pi}\int_S f(s)\left[\frac{\partial G}{\partial n} - \frac{\partial G_0}{\partial n}\right]ds + \frac{1}{2\pi}\int_{C-S} f(s)\frac{\partial G}{\partial n}\,ds,$$

or, as (ξ, η) approaches the boundary point,

$$(26)\qquad \lim\,[u(\xi, \eta) - u_0(\xi, \eta)] = \lim\frac{1}{2\pi}\int_S f(s)\left[\frac{\partial G}{\partial n} - \frac{\partial G_0}{\partial n}\right]ds.$$

A sufficient condition for the interchange of the integration and limit signs is that the integrand be summable and bounded.* But this is true of the harmonic function $G - G_0$ and its first derivatives in R_0, and it is therefore only necessary to postulate these properties for $f(s)$. Then as $G - G_0$ approaches 0 uniformly on the boundary of R_0, its derivatives also approach zero uniformly in R_0, by Theorem V. Hence $\lim\,[u(\xi, \eta) - u_0(\xi, \eta)] = 0$, and the behavior of u as (ξ, η) approaches a point of the boundary is the same as that of u_0.† But if the region R_0 be mapped on the surface of the unit circle by means of $G_0(a, b; x, y)$ and its conjugate, Green's integral $u_0(\xi, \eta)$ for R_0 becomes

* LEBESGUE, *Leçons sur l'integration*, p. 114. Various other conditions may be employed, such as the convergence of the integral

$$\int |f(s)|\,ds.$$

† As here sketched, this method gives information only about the behavior of $u(\xi, \eta)$, but not of its derivatives. If it is feasible to give to $f_0(s)$ the values of $u(\xi, \eta)$ on that part of the boundary of R_0 which is interior to R, $u_0(\xi, \eta)$ will be identical with $u(\xi, \eta)$ and conclusions can then be drawn concerning the derivatives of $u(\xi, \eta)$. In what follows, however the method is used only in connection with $u(\xi, \eta)$ itself.

Poisson's integral. Among the results obtained by application of this method is the following:

THEOREM IX. *If, at a point of the boundary, say* $s = 0$, *the function* $f(s)$ *has a finite break, the limit of Green's integral,* $u(\xi, \eta)$, *as* (ξ, η) *approaches* $x(0), y(0)$ *along a line making with the negative and positive directions of the tangent to the boundary at* $x(0), y(0)$ *the supplementary angles* α *and* β *respectively, is* $[\beta f(0-) + \alpha f(0+)]/2\pi$.

§ 9. *The unique determination of harmonic functions.*

If $f(s)$ is continuous, it is proven by the theorem stating the non-existence of a maximum of a harmonic function in the interior of R, that there is at most one harmonic function on R approaching the boundary values $f(s)$ in the strict sense of continuity of a two-dimensional approach. If the demand for a strictly continuous approach be relinquished (and the approach to the value of $f(s)$ along a normal, for instance, be substituted), the uniqueness breaks down. Thus the harmonic function $2xy/(x^2 + y^2)^2$, for the upper half of the (x, y)-plane, approaches 0 at every point of the boundary except at the origin, and here also, if the approach be along the normal. If the boundary values themselves are discontinuous, we can no longer properly speak of a harmonic function being determined by them. The problem of finding conditions to replace strict continuity, which, in conjunction with the boundary values determine a harmonic function, has been studied by PRYM,* SCHWARZ,† JULES RIEMANN,‡ FATOU,§ PLANCHEREL,|| and others.

The first part of the treatment which is to follow was suggested by the work of FATOU; the ideas underlying the latter part were published in outline during the preparation of this paper by PLANCHEREL (loc. cit.). They are of sufficient interest to warrant some further development in this paragraph. The method of procedure will be this. Having determined some properties of Green's integral having to do with its behavior on the boundary, we are assured that

* *Zur Integration der Differentialgleichung*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

Journal für reine und angewandte Mathematik, vol. 73 (1871), p. 340.

† *Zur Integration der partiellen Differentialgleichung*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

Gesammelte Mathematische Abhandlungen, vol. 2, p. 175, Journal für reine und angewandte Mathematik, vol. 74 (1872), p. 218.

‡ *Sur le problème de Dirichlet*, Annales Scientifique de l'école normale supérieure (1888), p. 331.

§ *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), p. 339.

|| Bulletin des Sciences Mathématiques, ser. 2, vol. 34 (1910), p. 111.

at least one harmonic function exists which has these properties. We then proceed to ask whether a different harmonic function can exist with these same properties. Should the answer be negative, we shall have found conditions sufficient to determine uniquely a harmonic function.

In the first place, Fatou shows in the article cited, for the case that R is the unit circle, that if $f(s)$ is periodic, bounded and summable,

$$u(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\frac{\partial}{\partial n} G(\xi, \eta; s)\, ds$$

approaches $f(s)$ as (ξ, η) approaches $[x(s), y(s)]$ along a radius, except at points of a set of measure zero, and it is evident that $u(\xi, \eta)$ is bounded. Moreover, no other bounded harmonic function has the same boundary values in the same sense. This result admits an extension, both in the matter of the curves of approach, and of the region R. To establish the generalization, we prove first the following:

THEOREM X. *Let the boundary of R be subject to condition $(A^{(2)})$. If, then, a bounded harmonic function $u(\xi, \eta)$ on R approaches a limit as (ξ, η) approaches a boundary point π along any single curve meeting the boundary orthogonally with finite curvature, it will approach the same limit along the normal and along every curve meeting the boundary at π orthogonally and with finite curvature.*

As the boundary has finite curvature, a circle may be inscribed in R touching the boundary at π and having no other point in common with the boundary. In this circular region, u, being bounded, and approaching continuous boundary values $g(s)$ everywhere save possibly at π, will be given by Poisson's integral, by Fatou's theorem; that is,

$$u(\rho, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{g(s)(1-\rho^2)\, ds}{1 - 2\rho\cos(\varphi - s) + \rho^2}. \tag{27}$$

If the axes be taken so that $\varphi = 0$ corresponds to the x-axis and π to the point $(1, 0)$, the curve meeting the boundary orthogonally with finite curvature may be written $y = F(x)(1-x)^2$, where $F(x)$ is a bounded function, or also in the form

$$\rho \sin\varphi = F(\rho)(1-\rho)^2. \tag{28}$$

We have, then, to prove

$$\lim_{\rho=1}\int_{-\pi}^{+\pi} g(s)\left\{\frac{1-\rho^2}{1 - 2\rho\cos s + \rho^2} - \frac{1-\rho^2}{1-2\rho\cos(\varphi - s) + \rho^2}\right\} ds = 0, \tag{29}$$

φ being defined as a function of ρ by equation (28). To carry out the proof, write the integral as the sum of two, $J_1 + J_2$, with the same integrand, the interval for J_1 being from $-\delta$ to $+\delta$, and J_2 having the rest of the interval $-\pi$ to $+\pi$ for its domain. Then, given $\epsilon > 0$, δ and $1 - \rho$ may be taken so small as to make $|J_1| < \frac{1}{2}\epsilon$, as will be shown; while for fixed δ, J_2 approaches 0 with $1 - \rho$, so that $1 - \rho$ can be taken so small that $|J_2| < \frac{1}{2}\epsilon$. Equation (29) will thus be established, and with it Theorem X.

To make evident the property attributed to J_1, we write it in the form

$$J_1 = \int_{-\delta}^{+\delta} g(s) \frac{(1-\rho^2)4\rho \sin \frac{1}{2}\varphi \sin(\frac{1}{2}\varphi - s)}{[1 - 2\rho\cos s + \rho^2][1 - 2\rho\cos(\varphi - s) + \rho^2]} ds .$$

By equation (28), $2\rho \sin \frac{1}{2}\varphi = F(\rho)(1-\rho)^2/\cos\frac{1}{2}\varphi$, and φ is small in the neighborhood of π, so that there is a finite constant B such that

$$|2g(s)F(\rho)/\cos\tfrac{1}{2}\varphi| < B,$$

and

$$|J_1| \leqq \frac{B(1-\rho)^2|\sin(\frac{1}{2}\varphi - \vartheta\delta)|}{1 - 2\rho\cos(\vartheta\epsilon) + \rho^2} \cdot \int_{-\delta}^{+\delta} \frac{(1-\rho^2)}{1 - 2\rho\cos(s - \varphi) + \rho^2} ds .$$

The integral is a harmonic function with boundary values 2π and 0, and is hence less than or equal to 2π; moreover, $1 - 2\rho\cos\vartheta\delta + \rho^2 > (1-\rho)^2$, so that $|J_1| < 2\pi B |\sin(\frac{1}{2}\varphi - \vartheta\delta)|$. But as φ approaches 0 with $1 - \rho$, J_1 evidently vanishes with φ and δ.

This gives the extension of Fatou's theorem with respect to the approach curves. In order to obtain for R the result which he gives for the circle, we employ the method of § 8, mapping upon the circle a sub-region R_0 of R, the boundary of R also being subject to condition $(A^{(2)})$. This condition will insure the continuity of the second derivatives of the mapping functions, and curves meeting the boundary orthogonally with finite curvature will be transformed into curves having the same property. We thus have

THEOREM XI. *If $f(s)$ is bounded and summable, Green's integral,*

$$u(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} f(s) \frac{\partial}{\partial n} G(\xi, \eta; s)\, ds,$$

is bounded and in general approaches $f(s)$ along curves meeting the boundary orthogonally with finite curvature.*

Let $U(\xi, \eta)$ be a harmonic function with the properties stated for Green's integral in the theorem just given. We shall show that it is identical with the function defined by Green's integral, and that it is therefore uniquely determined by the stated properties. To this end, let $G(a, b; x, y)$ be Green's function for R, and $H(a, b; x, y)$ the negative of its conjugate. Then for small enough c, $G(a, b; x, y) = c$ gives a set of curves neighboring on the boundary of R which go over into this boundary as c approaches 0. More-

* "In general" here, and in what follows is to mean "except possibly at points forming a set of measure zero."

over, since $U(\xi, \eta)$ is continuous on $G = c$, we may write

$$U(a, b) = \frac{1}{2\pi}\int_0^{2\pi} U(c, H)\,dH = \frac{1}{2\pi}\int_{G=c} U(c, s')\,\frac{\partial G}{\partial n'}\,ds'.$$

As, by hypothesis, $U(\xi, \eta)$ approaches in general the boundary values $f(s)$ along curves meeting the boundary orthogonally with finite curvature, and hence, in particular, by Theorem X, along the curves $H =$ const., we may pass to the limit $c = 0$, obtaining

$$U(a, b) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\,dH = \frac{1}{2\pi}\int_0^{l} f(s)\,\frac{\partial}{\partial n}\,G(a, b; s)\,ds,$$

as was to be proved. The result may be stated as follows:

THEOREM XII. *If $f(s)$ is bounded and summable, there is one and only one bounded harmonic function on R which approaches in general the boundary values $f(s)$ along curves meeting the boundary orthogonally with finite curvature. This function is given by Green's integral.* The boundary of R is here subject to condition $(A^{(2)})$.

Should the restriction on $f(s)$ that it be bounded prove inconvenient, the conditions given by Plancherel furnish another set of determining conditions for a harmonic function. The boundary will be subject to condition $(A^{(1)})$, and we shall consider the curve set $G(a, b; x, y) = c$, as before. Let s' denote the length of arc of the curve $G = c$ measured from a fixed point, and $u(s')$ the value of Green's integral on this curve. We may then state

THEOREM XIII.* *If $f(s)$ is continuous except at a finite number of points, and if $|f(s)|$ has a convergent integral, then*

$$\lim_{c=0}\int_{G=c} |u(s')|\,ds' = \int_{G=0} |f(s)|\,ds.$$

To establish this, we divide the interval $0 \leqq s \leqq l$ into two sets of segments, σ, containing all the points of discontinuity of $f(s)$ as interior points, and the complementary set, Σ. With a proper first restriction on c we may find a positive constant, $\mu < 1$, such that

$$\mu < \frac{\partial}{\partial n}\,G(a, b; s'), \tag{30}$$

the direction n' having the same relation to the curve $G = c$ as n to the boundary $G = 0$ of R, and the inequality holding also in the limit $c = 0$. Then, for any

* The theorem apparently admits an extension to a countable number of discontinuities. The approach of c to 0 may either be continuous, or through discrete values. See Plancherel, loc. cit., for interesting examples bearing on the problem.

given positive ϵ, σ may be taken so small that

(31) $$\int_\sigma |f(s)|\,ds < \tfrac{1}{4}\mu\epsilon < \tfrac{1}{4}\epsilon.$$

To obtain a similar inequality for $u(s')$, that is for the values of Green's integral on $G = c$, we establish a correspondence between the values of s and s' by pairing the values of these variables which belong to the same values of H, so that

$$dH = \frac{\partial G}{\partial n}\,ds = \frac{\partial G}{\partial n'}\,ds'.$$

Let σ' and $\sum'$ be the intervals for s' corresponding to σ and $\sum$ for s. The desired inequality is, then,

(32) $$\int_{\sigma'} |u(s')|\,ds' < \tfrac{1}{2}\epsilon.$$

It may be established by a consideration of the dominant function

$$v(\xi, \eta) = \frac{1}{2\pi}\int_0^l |f(s)|\frac{\partial}{\partial n}G(\xi, \eta;\, s)\,ds,$$

which, by Osgood's theorem, may be shown to be harmonic, and which, by the fundamental lemma approaches $|f(s)|$ wherever this function is continuous. As $v(\xi, \eta)$ is continuous at all points of $G = c$, it may be represented by Green's integral along this curve, so that

$$v(a, b) = \frac{1}{2\pi}\int_0^{2\pi} v(s')\,dH = \frac{1}{2\pi}\int_0^l |f(s)|\frac{\partial G}{\partial n}\,ds,$$

or

(33) $$\int_{G=c} v(s')\frac{\partial G}{\partial n'}\,ds' = \int_{G=0} |f(s)|\frac{\partial G}{\partial n}\,ds.$$

As, however, $v(s')$ approaches $|f(s)|$ uniformly at the points of $\sum$, c may be taken so small that

$$\left|\int_{\Sigma'} v(s')\frac{\partial G}{\partial n'}\,ds' - \int_{\Sigma} |f(s)|\frac{\partial G}{\partial n}\,ds\right| < \tfrac{1}{4}\mu\epsilon.$$

Hence by (33),

$$\int_{\sigma'} v(s')\frac{\partial G}{\partial n'}\,ds' < \int_\sigma |f(s)|\frac{\partial G}{\partial n}\,ds + \tfrac{1}{4}\mu\epsilon$$

or, by (31)

$$\int_{\sigma'} v(s')\frac{\partial G}{\partial n}\,ds' < \tfrac{1}{2}\mu\epsilon,$$

so that because of the inequality (30),

$$\int_{\sigma'} v(s')\,ds' < \tfrac{1}{2}\mu\epsilon.$$

Then, as $|u(s')| \leqq v(s')$, the inequality (32) follows.

Finally, as $u(s')$ and $\partial G / \partial n'$ approach their limits uniformly along Σ, we have

$$\lim_{c=0} \int_{\Sigma'} \left\{ \left| u(s') \right| \left(\frac{\partial G}{\partial n} \div \frac{\partial G}{\partial n'} \right) - \left| f(s) \right| \right\} ds = 0,$$

or

$$\lim_{c=0} \int_{\Sigma'} \left| u(s') \right| ds' = \int_{\Sigma} \left| f(s) \right| ds,$$

so that by taking c small enough, we secure the inequality

$$\left| \int_{\Sigma'} \left| u(s') \right| ds' - \int_{\Sigma} \left| f(s) \right| ds \right| < \tfrac{1}{3}\epsilon. \tag{34}$$

Adding the inequalities (31), (32), and (34), we obtain

$$\left| \int_{G=c} \left| u(s') \right| ds' - \int_0^l \left| f(s) \right| ds \right| < \epsilon,$$

which proves Theorem XIII.

Considering now any harmonic function $U(\xi, \eta)$ approaching the function $f(s)$ of the last theorem at all points where $f(s)$ is continuous, and satisfying the limiting equation there established for $u(\xi, \eta)$, we may write

$$U(a, b) = \frac{1}{2\pi} \int_{G=c} U(c, H)\, dH = \frac{1}{2\pi} \int_{G=c} U(s') \frac{\partial G}{\partial n'} ds',$$

so that

$$U(a, b) - \frac{1}{2\pi} \int_0^l f(s) \frac{\partial}{\partial n} G(a, b; s)\, ds$$
$$= \frac{1}{2\pi} \left\{ \int_{G=c} U(s') \frac{\partial G}{\partial n'} ds' - \int_0^l f(s) \frac{\partial G}{\partial n} ds \right\}.$$

By a division of the interval of integration, and a process of reasoning nearly identical with that used to establish Theorem XII, we arrive at the conclusion that the limit as c approaches 0 of the right hand member of this equation is zero. Hence the left hand member, which does not depend on c, must vanish, and $U(\xi, \eta)$ is given by Green's integral. From this follows

THEOREM XIV. *If $f(s)$ is continuous except at a finite number of points, and if $|f(s)|$ has a convergent integral, then there is one and only one harmonic function $u(\xi, \eta)$ on R which approaches $f(s)$ where this function is continuous and which has the property*

$$\lim_{c=0} \int_{G=c} \left| u(s') \right| ds' = \int_0^l \left| f(s) \right| ds.$$

This harmonic function is given by Green's integral. The boundary of R is here subject to condition $(A^{(1)})$.

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