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Harmonic functions and quadratic harmonic morphisms on Walker spaces

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Abstract: Let $(W, q, \mathcal{D})$ be a 4-dimensional Walker manifold. After providing a characterization and some examples for several special $(1,1)$-tensor fields on $(W, q, \mathcal{D})$, we prove that the proper almost complex structure J, introduced by Matsushita, is harmonic in the sense of García-Río et al. if and only if the almost Hermitian structure (J, q) is almost Kähler. We classify all harmonic functions locally defined on $(W, q, \mathcal{D})$. We deal with the harmonicity of quadratic maps defined on $\mathbb{R}^4$ (endowed with a Walker metric q) to the n-dimensional semi-Euclidean space of index r, and then between local charts of two 4-dimensional Walker manifolds. We obtain here the necessary and sufficient conditions under which these maps are harmonic, horizontally weakly conformal, or harmonic morphisms with respect to q.

Key words: 4-manifold, harmonic function, harmonic map, Walker manifold, almost complex structure

1. Introduction

Walker manifolds are of special interest in an increasing number of works in mathematical physics [8, 11–13, 26], particularly in general relativity [10].

A Walker manifold is a semi-Riemannian n-dimensional manifold (i.e. a manifold endowed with a non-degenerate symmetric $(0,2)$-tensor field of arbitrary signature [28]) with an r-dimensional lightlike distribution (see [17]), which is parallel w. r. t. the Levi-Civita connection. Constituting the background of several specific semi-Riemannian structures, these manifolds are involved in many physical contexts and they are useful for furnishing various interesting examples. Since the para-Kähler and hypersymplectic metrics are necessarily of Walker type, in the monograph [11], Walker metrics are considered in connection with almost para-Hermitian structures, for whose classification from the earlier period we quote [5] (see also [1, 16] and the references therein).

Dimension 4 is of special interest, first of all since this is the minimal dimension for the existence of a nontrivial almost para-Hermitian structure. Any 4-dimensional Walker manifold of nowhere zero scalar curvature has a natural almost para-Hermitian structure (see [11]). A second motivation for paying attention to 4-dimensional spaces comes from a physical point of view, related to classical mechanics.

At the beginning of the present paper we characterize the almost Hermitian, almost semi-Riemannian product, almost para-Hermitian, and almost anti-Hermitian structures on 4-dimensional Walker manifolds.

Our work is mainly devoted to harmonicity. The developments in harmonic map theory over the past

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decades have revealed that its methods and instruments were taken from a large spectrum of mathematical results that go beyond the analysis of manifolds and differential geometry. Harmonic maps were introduced in theoretical physics under the name of nonlinear sigma models, or chiral fields, with the purpose of describing pion-nucleon physics in a low energy approximation.

Of great interest in mathematics, harmonic maps, defined as critical points of the Dirichlet energy functional, were known before 1964, but research in this direction was undertaken by Eells and Sampson, who published their seminal work, [19], followed by a series of papers on the study of harmonic maps and morphisms (see The Atlas of Harmonic Morphisms, http://riemann.unica.it/ montaldo/homepage/atlas/). Moreover, harmonicity is studied in the context of sections in vector bundles, tensor fields, connections, and minimal distributions (see [7, 21, 29, 32, 33]).

In the first part of this paper we show that the proper almost complex structure J on a 4-dimensional Walker manifold $(W, q, \mathcal{D})$ (from [26]) is harmonic (in the sense of [21]) if and only if (J, q) is almost Kähler.

In the second part we characterize the harmonicity of smooth local functions defined on an open local chart of a 4-dimensional Walker manifold.

Another notion we deal with is that of quadratic maps, inspired from classical mechanics, where the kinetic energy is expressed as a quadratic form defined on the state space. Related to linear theories, such as, for instance, the problem of small oscillations, both kinetic and potential energies are taken as quadratic forms.

Aiming to obtain more examples of harmonic morphisms (which are a special class of harmonic maps, [3]), Baird, Baird and Wood, Ou, and Ou and Wood investigated in [2, 3, 30, 31] quadratic maps between Euclidean spaces or Euclidean spheres. Then Lu et al. went further to consider harmonic quadratic morphisms between semi-Euclidean spaces (see [24] and the references therein).

Here we extend their work by considering other semi-Riemannian (precisely Walker) metrics instead of semi-Euclidean ones.

For this purpose, we study quadratic maps from a 4-dimensional Walker space $(\mathbb{R}^4, q)$ to the n-dimensional semi-Euclidean space of index r. Our main results provide necessary and sufficient conditions under which the introduced quadratic maps are harmonic, horizontally weakly conformal, or harmonic morphisms. Moreover, the same properties are characterized for the quadratic maps between local charts of two 4-dimensional Walker spaces.

2. Preliminary results

We recall here the notion of Walker manifolds, to which the monograph [11] is devoted, and whose importance is related to the generalizations of Riemann extensions, the Osserman problem, almost para-Hermitian structures, and so on (see [11–13, 22, 26, 34]).

Definition 2.1 *[35] A Walker manifold $(W, q, \mathcal{D})$ is an n-dimensional semi-Riemannian manifold (W, q), with an r-dimensional lightlike distribution $\mathcal{D}$ (that is, q restricted to $\mathcal{D}$ is zero), which is parallel w. r. t. the Levi-Civita connection of q.*

A reason to focus on dimension 4 comes from classical mechanics, when a coordinate space (M, g) is given, for example, by a 2-dimensional Riemannian manifold, with g defined by the kinetic energy. The total space of the tangent bundle TM of the coordinate space M is the 4-dimensional state space, whose elements can be expressed as quadruples (x, y, z, t). For instance, the body launched obliquely in space is described by its position $p = (x, y) \in M$ and its speed $v = (\dot{x}, \dot{y})$, with the state of the particle $(p, v) = (x, y, \dot{x}, \dot{y}) \in TM$.

Let W be a 4-dimensional Walker manifold, with the lightlike distribution $\mathcal{D}$ of maximum dimension ($r=2$). Then the metric q is neither Riemannian nor Lorentz, but neutral (i.e. the positive and negative indices of inertia are equal; hence, in this case, the signature is $(2,2)$). From [35], around any point of W there exist adapted local coordinates systems (x,y,z,t) w. r. t. which the matrix of q has the following expression:

$$Q = \begin{pmatrix} O & I \\ I & S \end{pmatrix}, \tag{2.1}$$

where O and I denote respectively the 2-dimensional null and identity matrices, and $S = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a symmetric matrix, having the entries a,b,c as smooth functions of (x,y,z,t).

For the almost standard geometric objects of a 4-dimensional Walker manifold, see [23].

In the adapted local coordinates (x,y,z,t) on the Walker space (W,q), the corresponding local frame of vector fields $\{\partial_x, \partial_y, \partial_z, \partial_t\}$ will also be denoted by $\{\partial_{x^1}, \partial_{x^2}, \ \partial_{x^3}, \partial_{x^4}\}$.

The inverse of the matrix Q (from (2.1)) in the adapted local coordinates is

$$(q^{\alpha\beta})_{\alpha,\beta=\overline{1,4}} = \begin{pmatrix} -S & I \\ I & O \end{pmatrix}.$$

The Christoffel symbols $\Gamma_{ij}^k, \ i,j,k=\overline{1,4}$ of the Levi-Civita connection of a Walker metric (2.1), obtained in [14], can be expressed by the following matrices:

$$\begin{gathered}
\Gamma^1 = \frac{1}{2}\begin{pmatrix} O & \begin{pmatrix} a_x & c_x \\ a_y & c_y \end{pmatrix} \\ \begin{pmatrix} a_x & a_y \\ c_x & c_y \end{pmatrix} & \begin{pmatrix} aa_x + ca_y + a_z & a_t + ac_x + cc_y \\ a_t + ac_x + cc_y & ab_x + cb_y - b_z + 2c_t \end{pmatrix} \end{pmatrix}, \\
\Gamma^2 = \frac{1}{2}\begin{pmatrix} O & \begin{pmatrix} c_x & b_x \\ c_y & b_y \end{pmatrix} \\ \begin{pmatrix} c_x & c_y \\ b_x & b_y \end{pmatrix} & \begin{pmatrix} ca_x + ba_y - a_t + 2c_z & b_z + cc_x + bc_y \\ b_z + cc_x + bc_y & cb_x + bb_y + b_t \end{pmatrix} \end{pmatrix}, \\
\Gamma^3 = -\frac{1}{2}\begin{pmatrix} O & O \\ O & \begin{pmatrix} a_x & c_x \\ c_x & b_x \end{pmatrix} \end{pmatrix}, \qquad \Gamma^4 = -\frac{1}{2}\begin{pmatrix} O & O \\ O & \begin{pmatrix} a_y & c_y \\ c_y & b_y \end{pmatrix} \end{pmatrix},
\end{gathered} \tag{2.2}$$

where $(\Gamma_{ij}^k)_{i,j,k=\overline{1,4}}$ are the entries of Γ^k.

As it was pointed out in [12], an orthonormal basis for a Walker metric (2.1), involving canonical coordinates, is:

$$\begin{gathered}
e_1 = \tfrac{1-a}{2}\partial_x + \partial_z, \ e_2 = -c\partial_x + \tfrac{1-b}{2}\partial_y + \partial_t \\
e_3 = -\tfrac{1+a}{2}\partial_x + \partial_z, \ e_4 = -c\partial_x - \tfrac{1+b}{2}\partial_y + \partial_t.
\end{gathered} \tag{2.3}$$

3. Special $(1,1)$-tensor fields on Walker 4-manifolds

Recall some special vector bundle endomorphisms (given by certain G-structures), defined by $(1,1)$-tensor fields on manifolds.

Definition 3.1 *Let (M,g) be a semi-Riemannian n-dimensional manifold endowed with a (1,1)-tensor field F (i.e. an endomorphism of TM), such that*

$$F^2 = \varepsilon \,\mathrm{Id}, \text{and } g(F\cdot, F\cdot) = \varepsilon' g(\cdot,\cdot),$$

where Id is the identity endomorphism of TM and $\varepsilon, \varepsilon' \in \{-1,1\}$. Then (F,g) is called an almost Hermitian structure with indefinite metric [4, 9], almost para-Hermitian structure (note that g has to be neutral; see [5, 6, 16] and the references therein), almost semi-Riemannian product structure [27], or almost anti-Hermitian structure (i.e. almost complex structure of Norden type) [15, 25], provided the pair $(\varepsilon, \varepsilon')$ is respectively $(-1,1)$, $(1,-1)$, $(1,1)$, or $(-1,-1)$.

By a straightforward calculation we obtain:

Lemma 3.2 *Let $(W,q,\mathcal{D})$ be a 4-dimensional Walker manifold, endowed with a tensor field J of type $(1,1)$, expressed in adapted local coordinates by*

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix},$$

where J_i, $i=\overline{1,4}$ are matrices of order 2. Then J is an almost Hermitian structure with indefinite metric q if and only if

$$a)\ J_1J_2 + J_2J_4 = J_3J_1 + J_4J_3 = O,$$
$$b)\ J_1^2 + J_2J_3 = J_3J_2 + J_4^2 = -I, \qquad c)\ J_4 = -(J_1^t + J_3^t S),$$

where both J_3 and $J_2 + SJ_4$ are skew-symmetric.

Example 3.3 *Let $(W,q,\mathcal{D})$ be a 4-dimensional Walker manifold. One of the structures characterized in Lemma 3.2 is the proper almost Hermitian structure from [26], which has the form*

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.1}$$

w. r. t. the orthonormal basis $\{e_i\}_{i=\overline{1,4}}$ given by (2.3). It induces a positive $\frac{\pi}{2}$-rotation on the degenerate distribution $\mathcal{D}$ spanned by ∂_x, ∂_y. The proper almost complex structure given by (3.1) is completely determined by the metric, as follows:

$$\begin{gathered} J\partial_x = \partial_y,\ J\partial_y = -\partial_x, \\ J\partial_z = -c\partial_x + \tfrac{a-b}{2}\partial_y + \partial_t,\ J\partial_t = \tfrac{a-b}{2}\partial_x + c\partial_y - \partial_z. \end{gathered} \tag{3.2}$$

In a similar way as in Lemma 3.2, we obtain:

Lemma 3.4 *Let $(W, q, \mathcal{D})$ be a 4-dimensional Walker manifold, endowed with a tensor field F of type $(1,1)$, expressed in adapted local coordinates by*

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where F_i, $i = \overline{1,4}$ are matrices of order 2. Then (F, q) is an

i) almost semi-Riemannian product structure if and only if F satisfies the following relations:

$$\begin{aligned} a)\ & F_1F_2 + F_2F_4 = F_3F_1 + F_4F_3 = O, \\ b)& F_1^2 + F_2F_3 = F_3F_2 + F_4^2 = I, \\ c)& F_4 = F_1^t + F_3^t S, \end{aligned} \tag{3.3}$$

where both F_3 and $F_2 + SF_4$ are symmetric.

ii) almost para-Hermitian structure if and only if (3.3, a, b) and

$$F_4 = -(F_1^t + F_3^t S)$$

are satisfied, where both F_3 and $F_2 + SF_4$ are skew-symmetric.

iii) almost complex structure of Norden type if and only if F satisfies (3.3 a, c) and

$$F_1^2 + F_2F_3 = F_3F_2 + F_4^2 = -I,$$

where both F_3 and $F_2 + SF_4$ are symmetric.

We recall that an almost paracomplex structure is an almost product structure whose eigenvalues ± 1 have the same multiplicity (see [5, 6]).

Corresponding to the almost complex structure (3.1) considered in [26], we construct now, in the context of almost paracomplex structures, the following:

Example 3.5 *On a 4-dimensional Walker manifold, the structure defined w. r. t. the orthonormal basis $\{e_i\}_{i=\overline{1,4}}$ from (2.3) by*

$$F = \begin{pmatrix} a & 0 & 1-a & 0 \\ 0 & b & 0 & 1-b \\ 1+a & 0 & -a & 0 \\ 0 & 1+b & 0 & -b \end{pmatrix} \tag{3.4}$$

turns out to be an almost paracomplex structure, as it satisfies the relations (3.3 a, b) in Lemma 3.4. The dimensions of the eigendistributions corresponding to the eigenvalues $+1$ and -1 are equal. We note that ∂_x and ∂_y are two of its eigenvectors that span the degenerate distribution $\mathcal{D}$ and therefore $\mathcal{D}$ is an eigendistribution of F. From now on we call F the canonical almost paracomplex structure on a Walker manifold. Moreover, F is an almost semi-Riemannian product structure if and only if $a = b = 0$.

Lemma 3.6 *The orthonormal basis $\{e_i\}_{i=\overline{1,4}}$ given by (2.3) satisfies:*

$$\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i} e_i) = \frac{a_x}{2}\partial_x + c_x\partial_y + \frac{b_y}{2}\partial_y,$$

where

$$\varepsilon_i = \begin{cases} 1 \text{ if } i = 1,2 \\ -1 \text{ if } i = 3,4. \end{cases} \tag{3.5}$$

Proof From relations (2.3) and (2.2) we obtain

$$\nabla_{e_1} e_1 = \Big(\frac{1+a}{2}\frac{a_x}{2} + c\frac{a_y}{2}\Big)\partial_x + \Big(\frac{1-a}{2}c_x + \frac{c}{2}a_x + \frac{b}{2}a_y - a_t + 2c_z\Big)\partial_y - \frac{a_x}{2}\partial_z - \frac{a_y}{2}\partial_t,$$

$$\nabla_{e_2} e_2 = \Big(\frac{a}{2}b_x + \frac{c}{2}b_y - \frac{b_x}{2} + c_t\Big)\partial_x + \frac{1+b}{2}\frac{b_y}{2}\partial_y - \frac{b_x}{2}\partial_z - \frac{b_y}{2}\partial_t.$$

Based on a straightforward calculation, by using relations (2.3), (2.2), and

$$e_3 = e_1 - \partial_x, \ \ e_4 = e_2 - \partial_y, \tag{3.6}$$

we complete the proof. □

By adapting to the context of Walker manifolds the notion introduced by García-Río et al. in [21], we give the following definition, which we use later on:

Definition 3.7 *A $(1,1)$-tensor field F on a Walker manifold $(W,q,\mathcal{D})$ is called harmonic if its induced map $F:(TW,q^C)\to(TW,q^C)$ between semi-Riemannian manifolds is harmonic, where q^C is the complete lift of q.*

For the harmonicity of $(1,1)$-tensor fields on semi-Riemannian manifolds, w. r. t. general natural metrics, see [7].

The characterization of the above notion, given in [21], can be expressed on a Walker manifold as follows:

Theorem 3.8 *An endomorphism field F on a Walker manifold $(W,q,\mathcal{D})$ is harmonic if it is divergence-free, that is, if its codifferential w. r. t. the Levi-Civita connection of q vanishes (i.e. $\delta F = 0$).*

We apply Theorem 3.8 to obtain the following:

Proposition 3.9 *On a 4-dimensional Walker manifold $(W,q,\mathcal{D})$, the proper almost Hermitian structure J given by (3.1) is harmonic if and only if the structure (J,q) is almost Kähler.*

Proof The codifferential of J w. r. t. the Walker metric q (independent on the basis chosen) can be computed by using (2.3) and (3.5), as follows:

$$\begin{aligned} \delta J &= \operatorname{trace}(\nabla_{.}J)\cdot = \sum_{i=1}^{4}\varepsilon_i(\nabla_{e_i}J)e_i = \sum_{i=1}^{4}\varepsilon_i[\nabla_{e_i}(Je_i) - J\nabla_{e_i}e_i] \\ &= \nabla_{e_1}e_2 - J\nabla_{e_1}e_1 - \nabla_{e_2}e_1 - J\nabla_{e_2}e_2 - \nabla_{e_3}e_4 + J\nabla_{e_3}e_3 + \nabla_{e_4}e_3 - J\nabla_{e_4}e_4 \\ &= [e_1,e_2] + [e_4,e_3] - J\Big(\frac{a_x}{2}\partial_x + c_x\partial_y + \frac{b_y}{2}\partial_y\Big) = \frac{1}{2}(a_y+b_y)\partial_x - \frac{1}{2}(a_x+b_x)\partial_y. \end{aligned}$$

The above calculus was done by applying (3.2), Lemma 3.6, (3.6), and (2.2) as well. It follows that $\delta J = 0$ if and only if

$$a_x + b_x = 0, \quad a_y + b_y = 0. \tag{3.7}$$

Since (3.7) is equivalent to the fact that J is almost Kähler [26, Theorem 2] (cf. [12]), the proposition is proved. $\square$

4. Harmonic functions on Walker 4-manifolds

A harmonic local C^2-function on a Walker 4-manifold (W, q) has to satisfy the Laplace equation w. r. t. the Walker metric q:

$$\Delta f = \text{trace}(\nabla.df)\cdot = \sum_{i=1}^{4} \varepsilon_i [e_i(e_i f) - (\nabla_{e_i} e_i) f] = 0, \tag{4.1}$$

where $\varepsilon_i, i = \overline{1,4}$ are given by (3.5) and $\{e_i\}_{i=\overline{1,4}}$ is an arbitrary orthonormal basis with respect to q.

In the sequel, on a Walker manifold we denote by h_x, h_y, h_z, h_t the partial derivatives of any smooth function $h(x, y, z, t)$ with respect to x, y, z, t, and similarly for the second-order partial derivatives.

Theorem 4.1 *Let f be a (local) C^2-function defined on an open set $\mathcal{U}$ of a Walker 4-manifold $(W, q, \mathcal{D})$ and for any point $p \in \mathcal{U}$, let $(\mathcal{V}, x, y, z, t)$ be an arbitrary adapted local chart around p. Then the following assertions are equivalent:*

(i) f is harmonic.

(ii) On $\mathcal{V} \bigcap \mathcal{U}$ there exists a C^2-function $H(x, y, z, t)$ such that f satisfies the following PDE system:

$$\begin{cases} af_x + cf_y - 2f_z = H_y, \\ cf_x + bf_y - 2f_t = -H_x. \end{cases} \tag{4.2}$$

(iii) There exists a C^2-function $H(x, y, z, t)$ on $\mathcal{V} \bigcap \mathcal{U}$ such that f satisfies the system of implicit equations:

$$P_1(E_1, E_2, E_3, t) = 0, \;\; P_2(\widetilde{E}_1, \widetilde{E}_2, z, \widetilde{E}_3) = 0,$$

where P_1 (resp. P_2) is an arbitrary C^2-function depending on E_i (resp. $\widetilde{E}_i$), $i = \overline{1,3}$, which are the first integrals of the characteristic system

$$\frac{dx}{a} = \frac{dy}{c} = \frac{dz}{-2} = \frac{df}{H_y} \quad (resp.\ \frac{dx}{c} = \frac{dy}{b} = \frac{dt}{-2} = \frac{df}{-H_x}).$$

Proof After a straightforward computation we obtain

$$\begin{aligned} e_1(e_1 f) &= -\tfrac{a_z}{2} f_x + f_{zz} + \tfrac{1-a}{2}\Big(-\tfrac{a_x}{2} f_x + \tfrac{1-a}{2} f_{xx} + 2f_{xz}\Big), \\ e_2(e_2 f) &= c^2 f_{xx} + \Big(\tfrac{1-b}{2}\Big)^2 f_{yy} + f_{tt} - 2cf_{xt} - c(1-b)f_{xy} + (1-b)f_{yt} + \\ &+ \Big(cc_x - c_y\tfrac{1-b}{2} - c_t\Big) f_x + \Big(c\tfrac{b_x}{2} - \tfrac{1-b}{2}\tfrac{b_y}{2} - \tfrac{b_t}{2}\Big) f_y, \end{aligned}$$

and similarly for $e_3(e_3 f)$ and $e_4(e_4 f)$, where we take into account the relations (3.6).

From Lemma 3.6 and (4.1) we obtain:

$$\begin{aligned} \Delta f &= -af_{xx} - bf_{yy} - 2cf_{xy} + 2f_{xz} + 2f_{yt} - (a_x + c_y)f_x - (b_y + c_x)f_y \\ &= -\partial_x(af_x + cf_y - 2f_z) - \partial_y(bf_y + cf_x - 2f_t). \end{aligned} \tag{4.3}$$

Then $\Delta f = 0$ if and only if there exists a smooth function $H(x, y, z, t)$, whose partial derivatives w. r. t. x and y, denoted by H_x and H_y, satisfy the relations (4.2).

From the theory of quasilinear systems of partial derivatives, we complete the proof. □

Corollary 4.2 *Let f be a C^2-function defined on an adapted local chart $(\mathcal{V}, x, y, z, t)$ of a Walker 4-manifold.*

a) Then f is harmonic provided it depends on z and t only.

b) When the metric coefficients are constant on $\mathcal{V}$, and f is independent on x, then f is harmonic if and only if it satisfies the implicit equation

$$P(y + \frac{b}{2}t, z, yT - bf) = 0,$$

where P (resp. T) is an arbitrary C^2-function depending on three variables (resp. on z and t only).

Proof The assertion a) follows directly from (4.3). Concerning b), we have

$$\Delta f = 0 \Leftrightarrow \partial_y(bf_y - 2f_t) = 0 \Leftrightarrow bf_y - 2f_t = T(z, t),$$

which gives

$$\frac{dy}{b} = \frac{dt}{-2} = \frac{df}{T},$$

which by integration completes the proof. □

5. Harmonic morphisms on Walker 4-spaces

Extended from the Riemannian case to the semi-Riemannian one, we recall the following notions for which we refer to the monograph [3]:

Definition 5.1 *Let $\varphi : (M, g) \to (N, h)$ be a C^2-map between semi-Riemannian manifolds. Then:*

i) The energy density of φ is defined by

$$e_\varphi = \frac{1}{2}|d\varphi|^2,$$

where $|\cdot|^2$ is the Hilbert–Schmidt square norm on $T^*M \otimes \varphi^{-1}TN$ induced by g and h, that is,

$$|d\varphi|^2 = \mathrm{trace}_g(\varphi^* h) = g^{ij} h(d\varphi(X_i), d\varphi(X_j)),$$

where $\{X_i\}$ is an arbitrary local frame on M.

ii) The energy (integral) of φ over any compact domain D of M is the real number

$$E(\varphi; D) = \int_D e(\varphi) v_g,$$

where v_g is the volume measure associated to g.

iii) φ *is called harmonic if it is an extremal of the energy functionals* $E(\cdot, D)$ *for all compact domains* D *in* M*.*

iv) The tension field $\tau(\varphi)$ *is given by:*

$$\tau(\varphi) = \mathrm{div}(d\varphi) = \mathrm{trace}_g(\nabla^{\varphi^{-1}TN} d\varphi),$$

where $\nabla^{\varphi^{-1}TN} d\varphi$ *is called the second fundamental form of* φ*, defined by:*

$$\nabla^{\varphi^{-1}TN} d\varphi(X,Y) = (\nabla_X^{\varphi^{-1}TN} d\varphi)(Y) = \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla^M_X Y), \forall X, Y \in \Gamma(TM),$$

with ∇^M *and* ∇^N *respectively denoting the Levi-Civita connections on* (M, g) *and* (N, h)*.*

v) φ *is horizontally weakly conformal at* $p \in M$ *if there is a real number* $\Lambda(x)$ *(called square dilation), such that*

$$g(d_p\varphi^*(u), d_p\varphi^*(v)) = \Lambda(p)h(u,v),\ \forall u, v \in T_{\varphi(p)}N.$$

Moreover, φ *is called horizontally weakly conformal (on* M *) if it is horizontally weakly conformal at every point* $p \in M$*.*

Remark. Despite the Riemannian case, where $E(\varphi; D) \geq 0$, and the equality holding if and only if φ is constant, in the semi-Riemannian case, we may have $E(\varphi; D) \leq 0$ and we also may have nonconstant maps whose energy is vanishing.

Note that in the Riemannian case the square dilation (given in [18]) is positive, but this is no longer true for the semi-Riemannian case (defined in [20]).

We recall from [20] the following characterization:

Proposition 5.2 *Let* $\varphi : (M^m, g) \to (N^n, h)$ *be a* C^1*-map between semi-Riemannian manifolds and let* $p \in M$*. Then* φ *is horizontally weakly conformal at* p *with square dilation* $\Lambda(p)$ *if and only if in any local coordinates* $(y^1, \ldots, y^n)$ *on a neighborhood of* $\varphi(p)$*, the following relation holds good:*

$$g(\mathrm{grad}_g\, \varphi^\alpha, \mathrm{grad}_g\, \varphi^\beta) = \Lambda(p)h^{\alpha\beta}(\varphi(p)), \quad \alpha, \beta = \overline{1,n}. \tag{5.1}$$

The result of Eells and Sampson [19] given in the Riemannian case extends to the semi-Riemannian one, as follows:

Theorem 5.3 *Any smooth map* $\varphi : (M^m, g) \to (N^n, h)$ *between semi-Riemannian manifolds is harmonic if and only if*

$$\tau(\varphi) = 0, \tag{5.2}$$

that is, the following Euler–Lagrange system is satisfied:

$$g^{ij}\Big(\frac{\partial^2\varphi^\alpha}{\partial x^i \partial x^j} -^M\ \Gamma^k_{ij}\frac{\partial \varphi^\alpha}{\partial x^k} +^N\ \Gamma^\alpha_{\beta\sigma}\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\sigma}{\partial x^j}\Big) = 0, \tag{5.3}$$

where $i, j, k = \overline{1,m}$, $\alpha, \beta, \sigma = \overline{1,n}$*, and* ${}^M\Gamma^k_{ij}$ *(resp.* ${}^N\Gamma^\alpha_{\beta\sigma}$*) denote the Christoffel symbols of* M *(resp.* N*).*

The main notion that we deal with in Section 6 is the following:

Definition 5.4 *[20] A C^2-map $\varphi : (M,g) \to (N,h)$ between semi-Riemannian manifolds is a harmonic morphism if for any harmonic function f defined on an open set V of N, with $\varphi^{-1}(V)$ nonempty, the pull-back $\varphi^* f = f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$.*

As in the Riemannian case we recall the following characterization result:

Theorem 5.5 *[20] A C^2-map between semi-Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.*

Definition 5.6 *A C^1-map $\varphi : (M,g) \to (N,h)$ between semi-Riemannian manifolds is called nondegenerate if g restricted to subspace $\operatorname{Ker} d_p\varphi$ is nondegenerate for all $p \in M$.*

Theorem 5.7 *[20] Let $\varphi : (M,g) \to (N,h)$ be a C^2-map between connected semi-Riemannian manifolds of equal dimension n.*

Case 1 ($n = 2$): φ is a harmonic morphism if and only if φ is horizontally weakly conformal.

Case 2 ($n > 2$): If φ is a harmonic morphism, then φ is horizontally weakly conformal with constant square dilation. As a partial converse, if φ is nondegenerate and horizontally weakly conformal with constant square dilation, then φ is a harmonic morphism.

6. Quadratic maps on Walker spaces

By a quadratic map, we mean a homogeneous polynomial of degree 2.

The n-dimensional semi-Euclidean space $(\mathbb{R}^n_r, h)$ of index r carries the metric h given by the matrix $(h_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$, with $h_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}$, where

$$\varepsilon_\alpha = \begin{cases} -1, & \alpha = \overline{1,r}, \\ 1, & \alpha = \overline{r+1,n}. \end{cases}$$

We extend here the notion of quadratic map from the context of Euclidean spaces (see [2, 3, 30, 31]) and semi-Euclidean spaces (see [24] and the references therein) to the following:

Definition 6.1 *By a quadratic map $\varphi : (\mathbb{R}^4, q) \to (\mathbb{R}^n_r, h)$ from a 4-dimensional Walker space to the n-dimensional semi-Euclidean space of index r, we mean a homogeneous polynomial of degree 2 given by*

$$\varphi(X) = (XA_1X^t, \ldots, XA_nX^t), \tag{6.1}$$

where X^t and $A_\alpha = (a^\alpha_{ij})_{\alpha=\overline{1,n},\ i,j=\overline{1,4}}$ denote respectively the transpose of the row $X = (x,y,z,t)$ and n symmetric real matrices of order 4, called the component matrices of φ.

Notation 6.2 *From now on, by X we mean any $(x,y,z,t) \in \mathbb{R}^4$, and we denote by $R^{(k)}$ the k-th column of an arbitrary matrix R.*

We characterize harmonic quadratic maps defined on Walker manifolds by the following:

Proposition 6.3 *The quadratic map $\varphi : (\mathbb{R}^4, q) \to (\mathbb{R}^n_r, h)$ given by (6.1) is harmonic if and only if there exist some C^1-functions $K_\alpha(x, y, z, t)$, $\alpha = \overline{1, n}$, which satisfy the following relations:*

$$\begin{cases} X[a(A_\alpha + A_\alpha^t)^{(1)} + c(A_\alpha + A_\alpha^t)^{(2)} - 2(A_\alpha + A_\alpha^t)^{(3)}] = (K_\alpha)_y, \\ X[c(A_\alpha + A_\alpha^t)^{(1)} + b(A_\alpha + A_\alpha^t)^{(2)} - 2(A_\alpha + A_\alpha^t)^{(4)}] = -(K_\alpha)_x, \end{cases}$$

for $\alpha = \overline{1, n}$, where Notation 6.2 *is used, $(K_\alpha)_x$ and $(K_\alpha)_y$ denote the partial derivatives of $K_\alpha(x, y, z, t)$ w. r. t. x and y, and the real matrices A_α, $\alpha = \overline{1, n}$ are not necessarily symmetric.*

Proof The map φ given by (6.1) is harmonic if and only if for any $\alpha = \overline{1, n}$, the component function φ^α given by $\varphi^\alpha(X) = XA_\alpha X^t$ is harmonic, that is, $\Delta\varphi^\alpha = 0$. By applying Theorem 4.1 we obtain that φ is harmonic if and only if for each $\alpha = \overline{1, n}$ there exists a C^1-function H_α such that:

$$\begin{cases} a\varphi_x^\alpha + c\varphi_y^\alpha - 2\varphi_z^\alpha = (H_\alpha)_y, \\ c\varphi_x^\alpha + b\varphi_y^\alpha - 2\varphi_z^\alpha = -(H_\alpha)_x. \end{cases} \tag{6.2}$$

By computing the partial derivatives of φ^α, $\alpha = \overline{1, n}$, we obtain:

$$\frac{\partial\varphi^\alpha}{\partial x} := (\varphi^\alpha)_x = X(A_\alpha + A_\alpha^t)Q^{-1}Q^{(1)} = X(A_\alpha + A_\alpha^t)^{(1)}. \tag{6.3}$$

We proceed in a similar way to compute φ_y^α, φ_z^α, and φ_t^α. Then we replace all these values in (4.2) and denote $K_\alpha = H_\alpha/2$, $\alpha = \overline{1, n}$ to complete the proof. □

Our aim now is to characterize in the context of Walker manifolds those quadratic maps that are horizontally weakly conformal.

Proposition 6.4 *A quadratic map $\varphi : (\mathbb{R}^4, q) \to (\mathbb{R}^n_r, h)$ given by (6.1) is horizontally weakly conformal if and only if the following two relations are satisfied:*

$$XA_\alpha Q^{-1}A_\beta X^t = 0,\ \alpha \neq \beta, \tag{6.4}$$

$$X(\varepsilon_\alpha A_\alpha Q^{-1}A_\alpha - \varepsilon_\beta A_\beta Q^{-1}A_\beta)X^t = 0,\ \alpha, \beta = \overline{1, n}. \tag{6.5}$$

Proof By a straightforward computation, (5.1) yields

$$\operatorname{grad}\varphi^\alpha = 2XA_\alpha Q^{-1}L^t,$$

where we use the notation $L := (\partial_x, \partial_y, \partial_z, \partial_t)$, which gives $L^tL = Q$, from which we obtain

$$q(\operatorname{grad}_q\varphi^\alpha, \operatorname{grad}_q\varphi^\beta) = 4XA_\alpha Q^{-1}(L^tL)Q^{-1}A_\beta X^t$$

$$= 4XA_\alpha Q^{-1}QQ^{-1}A_\beta X^t = 4XA_\alpha Q^{-1}A_\beta X^t,\ \alpha, \beta = \overline{1, n}.$$

Then (5.1) can be rewritten as:

$$4XA_\alpha Q^{-1}A_\beta X^t = \Lambda\varepsilon_\alpha\delta_{\alpha\beta},\ \alpha, \beta = \overline{1, n}, \tag{6.6}$$

which is equivalent to the identities of quadratic forms (6.4) and (6.5). □

Corollary 6.5 *Let φ be a quadratic map as in* Definition 6.1, *such that the Walker metric q has constant coefficients on $\mathbb{R}^4$. Then φ is horizontally weakly conformal if and only if the following two relations are satisfied:*

$$A_\alpha Q^{-1} A_\beta + A_\beta Q^{-1} A_\alpha = 0, \quad \alpha \neq \beta, \tag{6.7}$$

$$\varepsilon_\alpha A_\alpha Q^{-1} A_\alpha = \varepsilon_\beta A_\beta Q^{-1} A_\beta, \quad \alpha, \beta = \overline{1, n}. \tag{6.8}$$

Proof We use here Proposition 6.4. From (6.4), by taking into account that the matrix $A_\alpha Q^{-1} A_\beta + A_\beta Q^{-1} A_\alpha$ is symmetric and that Q has constant entries, we obtain (6.7). By applying (6.5), the symmetry of quadratic forms, and the constancy of Q, we obtain (6.8), which completes the proof. □

As a generalization of Proposition 6.3, if we drop out the condition of symmetry put on the matrices A_α, and based on this property always satisfied by the matrices $A_\alpha + A_\alpha^t$, $\alpha = \overline{1, n}$, we obtain:

Proposition 6.6 *A quadratic map $\varphi : (\mathbb{R}^4, q) \to (\mathbb{R}^n_r, h)$ given by* (6.1) *is horizontally weakly conformal if and only if the following two relations are satisfied:*

$$X(A_\alpha + A_\alpha^t) Q^{-1} (A_\beta + A_\beta^t) X^t = 0, \ \alpha \neq \beta, \tag{6.9}$$

$$X[\varepsilon_\alpha (A_\alpha + A_\alpha^t) Q^{-1} (A_\alpha + A_\alpha^t) - \varepsilon_\beta (A_\beta + A_\beta^t) Q^{-1} (A_\beta + A_\beta^t)] X^t = 0, \tag{6.10}$$

where $\alpha, \beta = \overline{1, n}$, and the matrices A_α, $\alpha = \overline{1, n}$ are not necessarily symmetric.

Lemma 6.7 *For a quadratic map φ given as in* Definition 6.1, *the assertions below are equivalent:*

(*) φ is nondegenerate,

(**) the following system on $U = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ admits only the trivial solution:

$$\begin{cases} X(A_\alpha + A_\alpha^t) U^t = 0, \ \alpha = \overline{1, n}, \\ UQ = 0, \end{cases}$$

where the matrices A_α, $\alpha = \overline{1, n}$ are not necessarily symmetric.

The proof comes directly from Definition 5.6, and the relation

$$d_p \varphi^\alpha = X(A_\alpha + A_\alpha^t),$$

where p is an arbitrary point in $\mathbb{R}^4$, whose coordinates are denoted by $X = (x, y, z, t)$.

As a consequence of Theorems 5.5 and 5.7, Propositions 6.3 and 6.6, and the assertion (**) from Lemma 6.7, we obtain the following:

Theorem 6.8 *Let φ be a quadratic map as in* Definition 6.1, *where the matrices A_α, $\alpha = \overline{1, n}$ are not necessarily symmetric, and for any point $p \in W$ let the adapted local coordinate matrix be denoted by $X = (x, y, z, t)$. Then:*

(i) φ is a harmonic morphism if and only if there exist some C^1-functions $H_\alpha(x, y, z, t)$, $\alpha = \overline{1, n}$, such that the relations (6.7), (6.9), *and* (6.10) *are satisfied.*

(ii) φ is a harmonic morphism provided that $n=4$, one has (**), and the relation

$$X(A_\beta + A_\beta^t)Q^{-1}(A_\alpha + A_\alpha^t)X^t = \Lambda\varepsilon_\alpha\delta_{\alpha\beta},\ \alpha,\ \beta = \overline{1,n} \tag{6.11}$$

(with constant square dilation Λ) holds.

The results on harmonic maps and morphisms given by quadratic maps between semi-Euclidean spaces obtained by Lu in [24] are extended on Walker manifolds by the last above statements.

7. Quadratic maps between Walker manifolds

In this section we go further, by extending the notion of quadratic map with values in a semi-Euclidean space from Definition 6.1 to the following:

Definition 7.1 *By $\psi : \mathcal{V} \subseteq (W,q) \to (N,h)$ we mean a quadratic map from an adapted local chart $(\mathcal{V},x,y,z,t)$ of a Walker 4-manifold to an adapted local chart of a Walker manifold (N,h), defined as*

$$\psi(X) = (XA_1X^t, \dots, XA_4X^t), \tag{7.1}$$

where X^t and $A_\alpha = (a^\alpha_{ij})_{\alpha,\ i,j=\overline{1,4}}$ denote respectively the transpose of the row $X=(x,y,z,t)$ and four symmetric real matrices of order 4, called the component matrices of ψ.

We use throughout this section all the notations given in the above definition.

Lemma 7.2 *The Euler–Lagrange system of the quadratic map $\psi : \mathcal{V} \subseteq (W,q) \to (N,h)$ given by (7.1) can be written as*

$$\begin{gathered}\operatorname{trace} QA_\alpha + 2(a_x + c_y)XA_\alpha^{(1)} + 2(c_x + b_y)XA_\alpha^{(2)} \\ +4\operatorname{trace}{}^N\Gamma^\alpha_{\beta\sigma}(XA_\beta Q^{-1}A_\sigma X^t) = 0,\ \alpha, = \overline{1,4},\end{gathered} \tag{7.2}$$

where ${}^N\Gamma^\alpha_{\beta\sigma}$ denotes the Christoffel symbols of the Levi-Civita connection of h and Notation 6.2 *is used.*

Proof Since $\psi(X)$ has the expression (7.1), the first term in (5.3) becomes

$$q^{ij}(a^\alpha_{ij}) = \operatorname{trace} QA_\alpha = -aa^\alpha_{11} - ba^\alpha_{22} - 2ca^\alpha_{12} + 2a^\alpha_{13} + 2a^\alpha_{24},\ \forall i,j,\alpha = \overline{1,4}. \tag{7.3}$$

We use the expressions of Γ^k, $k=\overline{1,4}$, from Section 3 and similar to the relation (6.3), we have

$$\begin{gathered}q^{ij}\Gamma^1_{ij}(\psi^\alpha)_x = 2(a_x + c_y)XA_\alpha^{(1)},\ q^{ij}\Gamma^2_{ij}(\psi^\alpha)_y = 2(c_x + b_y)XA_\alpha^{(2)}, \\ q^{ij}\Gamma^3_{ij}(\psi^\alpha)_z = 0,\ q^{ij}\Gamma^4_{ij}(\psi^\alpha)_t = 0,\ i,j,\alpha,\beta = \overline{1,4}.\end{gathered} \tag{7.4}$$

From the sum of the four equalities in (7.4), the expression of the second term in (5.3) becomes:

$$2(a_x + c_y)XA_\alpha^{(1)} + 2(c_x + b_y)XA_\alpha^{(2)},\ \forall \alpha,\beta,\sigma = \overline{1,4}. \tag{7.5}$$

The last term in (5.3) can be written in the form

$$4\operatorname{trace}{}^N\Gamma^\alpha_{\beta\sigma}XA_\beta Q^{-1}A_\sigma X^t,\ \forall\alpha,\beta,\sigma = \overline{1,4}. \tag{7.6}$$

Taking into account (7.3), (7.5), and (7.6) it follows that the Euler–Lagrange system corresponding to ψ takes the form (7.2), and thus the lemma is proved. □

Theorem 7.3 *We assume that the metric coefficients on the target of a quadratic map* $\psi : \mathcal{V} \subseteq (W, q) \to (N, h)$*, given by* (7.1)*, are constant and the almost paracomplex structure* F *given by* (3.4) *is harmonic. Then* ψ *is harmonic if and only if* $QA_\alpha, \alpha = \overline{1,4}$ *are traceless.*

Proof

$$\begin{aligned}\delta F &= \operatorname{trace}(\nabla.F)\cdot = \textstyle\sum_{i=1}^4 \varepsilon_i(\nabla_{e_i}F)e_i = \sum_{i=1}^4 \varepsilon_i[\nabla_{e_i}(Fe_i) - F\nabla_{e_i}e_i]\\ &= \nabla_{e_1}(e_3 + a\partial_x) + \nabla_{e_2}(e_4 + b\partial_y) - \nabla_{e_3}(e_1 + a\partial_x) - \nabla_{e_4}(e_2 + b\partial_y)\\ &\quad - F(\tfrac{a_x}{2}\partial_x + c_x\partial_y + \tfrac{b_y}{2}\partial_y) = (a_x + c_y)\partial_x + (c_x + b_y)\partial_y,\end{aligned}$$

where the trace is computed w. r. t. the metric q, and we have used Lemma 3.6, (3.6), and (3.4). When the coefficients of the Walker metric h are constant, the Christoffel symbols ${}^N\Gamma^\alpha_{\beta\sigma}$ become zero and hence the last term in (5.3) vanishes. Consequently, in this case ψ is harmonic if and only if

$$\operatorname{trace} QA_\alpha + 2(a_x + c_y)XA^{(1)}_\alpha + 2(c_x + b_y)XA^{(2)}_\alpha = 0,$$

for every $\alpha = \overline{1,4}$, where Notation 6.2 is used. Under the assumption that $\delta F = 0$, by applying Lemma 7.2 we complete the proof. □

Remark. The matrices $QA_\alpha, \alpha = \overline{1,4}$ are traceless if and only if

$$-aa^\alpha_{11} - ba^\alpha_{22} - 2ca^\alpha_{12} + 2a^\alpha_{13} + 2a^\alpha_{24} = 0,\ \ \alpha = \overline{1,4}. \tag{7.7}$$

We recall from [35] that in an adapted local chart (which exists around any point of the Walker space (N, h)), the matrix $\widetilde{Q}$ of h and its inverse are respectively of the form

$$\widetilde{Q} = \begin{pmatrix} O & I \\ I & \widetilde{S} \end{pmatrix}, \qquad \widetilde{Q}^{-1} = \begin{pmatrix} -\widetilde{S} & I \\ I & O \end{pmatrix}, \tag{7.8}$$

where $\widetilde{S} = (s_{\alpha\beta})_{\alpha,\beta=1,2}$ is symmetric.

From Proposition 5.2, we obtain the following characterization, which can be proved in a similar way as Proposition 6.4.

Proposition 7.4 *A quadratic map* $\psi : \mathcal{V} \subseteq (W, q) \to (N, h)$ *defined by* (7.1) *is horizontally weakly conformal if and only if there exists a function* Λ *such that the following three relations are satisfied:*

$$4XA_\alpha Q^{-1}A_\beta X^t = \Lambda s_{\alpha\beta}, \qquad \alpha,\ \beta = \overline{1,2}, \tag{7.9}$$

$$A_1Q^{-1}A_3 + A_3Q^{-1}A_1 = A_2Q^{-1}A_4 + A_4Q^{-1}A_2,\ \textit{and} \tag{7.10}$$

$$A_\alpha Q^{-1}A_\beta + A_\beta Q^{-1}A_\alpha = 0, \tag{7.11}$$

with $\alpha,\ \beta = \overline{3,4},$ *or* $(\alpha,\ \beta) \in \{(1,4),(4,1),(2,3),(3,2)\}$.

From Theorem 5.5, Theorem 7.3, and Proposition 7.4 we characterize the harmonic morphisms between local charts of two Walker 4-manifolds by the following:

Theorem 7.5 *Suppose that the metric coefficients on the target of a quadratic map* $\psi : \mathcal{V} \subseteq (W, q) \to (N, h)$ *given by* (7.1) *are constant and the almost paracomplex structure* F *given by* (3.4) *is harmonic. Then* ψ *is a harmonic morphism if and only if the relations* (7.7) *and* (7.9)–(7.11) *are verified.*

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