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Harmonic Functions, Riesz Potentials, and the Lipschitz Spaces of Herz

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Introduction

In the paper [5], T. M. Flett introduced a space of temperatures (solutions of the heat equation) on a half space which is isomorphic to the Lipschitz space $\Lambda(\alpha; p, q)$ of M. H. Taibleson ([16]). As a consequence, many results concerning $\Lambda(\alpha; p, q)$ were proved; in particular, he showed that the topological dual of $\Lambda(\alpha; p, 1)$, $1 \leqq p < \infty$, is isomorphic to $\Lambda(-\alpha; p', \infty)$.

Some time later, R. Johnson ([10]), adopting Flett's idea, defined a space of temperatures which is isomorphic to the Lipschitz space $\Lambda^{\alpha}_{p,q}$ of C. S. Herz ([8]). His method leaned on a theory of Riesz potentials for temperatures. As an application of the theory developed, among other things, a characterization of temperatures whose boundary values are in $\Lambda^{\alpha}_{p,q}$ $(\alpha < 0)$ was given.

In this paper, heavily influenced by [5] and [10], using an integral representation of the Riesz potential $R^{\alpha}f(f \in L^p)$ in [14], we extend the definition of Riesz potential to a class of harmonic functions in a half space. Our first aim is to construct a space of harmonic functions in a half space which is isomorphic to $\Lambda^{\alpha}_{p,q}$. For this purpose, we show that "boundary values" of harmonic functions $u \in \mathscr{H}^*_{n/p-\alpha}$ (see § 5) satisfying

$$\left(\int_0^{\infty}\left[t^{k-\alpha}\int_{R^n}\left|\left(\frac{\partial}{\partial t}\right)^k u(x, t)\right|^p dx\right]^{q/p} t^{-1}dt\right)^{1/q} < \infty \qquad (\alpha \text{ real and } k > \alpha)$$

exist and are tempered distributions if $\alpha < n/p$, whereas the limits are considered as elements of $\mathscr{S}'/\mathscr{P}$ (the space of tempered distributions modulo polynomials) if $\alpha \geqq n/p$. Then, we proceed to characterize these distributions by showing that the map $u \mapsto u(\cdot, 0) \equiv \lim_{t \to 0} u(\cdot, t)$ establishes an isomorphism between the space of those harmonic functions satisfying the above properties and $\Lambda^{\alpha}_{p,q}$; in particular, in case $0 < \alpha < n/p$, a characterization of Poisson integrals of functions in $\Lambda^{\alpha}_{p,q}$ is given. Our other main result concerns the duals of some Lipschitz spaces. As suggested in [10], by studying more specific class of functions, our spaces are better described.

In the study of the space $\Lambda^{\alpha}_{p,q}$, we use the so-called method of Hardy-Littlewood-Taibleson-Flett. This method was intensively employed by M. H. Taibleson ([16]), and later generalized by T. M. Flett ([5], [6]). In contrast to the Gauss-Weierstrass kernel used in [5] and [10], the behaviour at infinity of the Poisson kernel used in our case is not nice; its convolution with an arbitrary tempered distribution may not be defined, and this features the main difference between the present case and [5] and [10]. Our approach is based on Theorem 5.2 whose proof is rather elementary, Theorem 6.1 about the existence of boundary values in the sense of distributions of functions in $\mathscr{H}(\alpha; p, q)$, and Theorem 5.1 about the basic properties of the space $\mathscr{H}(\alpha; p, q)$, of which the most important is the fact that the topological property of the space $\mathscr{H}(\alpha; p, q)$ does not depend on the (Lipschitz) index α, which is shown by using a result of Calderón and Zygmund [2]. These theorems are of some interest of their own. To make the presentation self-contained, most results are proved in details.

The plan of the paper is as follows. § 1 is used to fix notation and to state well-known results. In § 2, a semigroup formula for harmonic functions is studied. In § 3, the Riesz potential is defined and related properties are investigated. § 4 is devoted to the study of the equivalence of various norms. In § 5, the space $\mathscr{H}(\alpha; p, q)$ is defined, and properties of this space are studied. Existence of boundary values (in the sense of distributions) of functions in $\mathscr{H}(\alpha; p, q)$ is proved in § 6. In § 7, relations of $\Lambda^{\alpha}_{p,q}$ and $\mathscr{H}(\alpha; p, q)$ with other spaces in the literature are established. Finally, the duals of $\Lambda^{\alpha}_{p,1}$, $\Lambda^{\alpha}_{1,q}$ and $\lambda^{\alpha}_{p,\infty}$ are investigated in § 8 and § 9 through several lemmas.

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§ 1. Notation and preliminaries

We use R^n to denote the n-dimensional Euclidean space, and for each point $x=(x_1,\dots, x_n)$ we write $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$.

Unless otherwise stated, all functions are supposed to be complex-valued. As usual we use $\mathscr{S}=\mathscr{S}(R^n)$ to denote the space of all rapidly decreasing functions on R^n; $\mathscr{D}$ stands for its subspace consisting of functions with compact supports.

For any positive integer k let Z_k^+ be the set of all ordered k-tuples of non-negative integers, and for each $\mu=(\mu_1,\dots, \mu_k)$ let

$$|\mu| = \mu_1 + \cdots + \mu_k.$$

An element of Z_k^+ is called a multi-index.

If u is a function defined on a subset of R^k, we use $D_i^m u$ to denote the partial derivative of u of order m with respect to the i-th coordinate. Further, for each multi-index $\mu=(\mu_1,\dots, \mu_k)$ we write

$$D^\mu u = D_1^{\mu_1}\cdots D_k^{\mu_k} u.$$

If f is a measurable function defined on R^n, we set

$$\|f\|_p = \left(\int_{R^n} |f(x)|^p dx\right)^{1/p}, \qquad 0 < p < \infty,$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in R^n} |f(x)|,$$

and we define $L^p = L^p(R^n)$, where $1 \leqq p \leqq \infty$, as the space of those measurable functions f for which $\|f\|_p < \infty$, equipped with the norm $\|\cdot\|_p$.

The Fourier transform of a function $f \in L^1$ is given by

$$\hat{f}(x) = \int_{R^n} e^{-2\pi i x \cdot y} f(y) dy, \ x \in R^n,$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

We consider the space R^{n+1} as the Cartesian product $R^n \times R$, so that we can write each $z \in R^{n+1}$ in the form $z = (x, t)$, where $x \in R^n$ and $t \in R$. We denote by Ω the upper half space $R^n \times]0, \infty[$.

We use B to denote a constant, depending on the particular parameters $p, q, \ldots$ concerned in the particular problem in which it appears; if we wish to express the dependency, we write B in the form $B(p, q, \ldots)$. These constants are not necessarily the same on any two occurrences.

For measurable functions u defined on Ω, let

$$M_p(u; t) = \left(\int_{R^n} |u(x, t)|^p dx\right)^{1/p}, \qquad 0 < p < \infty,$$

$$M_\infty(u; t) = \operatorname*{ess\,sup}_{x \in R^n} |u(x, t)|.$$

We also let

$$\|u\|_{p,q} = \left(\int_0^\infty M_p(u; t)^q t^{-1} dt\right)^{1/q}, \qquad 0 < q < \infty,$$

$$\|u\|_{p,\infty} = \operatorname*{ess\,sup}_{t > 0} M_p(u; t).$$

For each measurable function f on R^n, let λ_f be its distribution function, i.e.,

$$\lambda_f(t) = |\{x \in R^n : |f(x)| > t\}| \qquad \text{for} \quad t > 0,$$

where $|E|$ stands for the Lebesgue measure of the set E. The decreasing rearrangement of f is the function f^* with domain $]0, \infty[$ and given by

$$f^*(s) = \inf\{t > 0 : \lambda_f(t) \leqq s\},$$

or equivalently by

$$f^*(s) = \sup\{\operatorname*{ess\,inf}_E |f| : |E| > s\}.$$

The average decreasing rearrangement of f is then defined to be the function f^{**} given by

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(t)dt \qquad (s > 0).$$

Clearly $f^* \leqq f^{**}$.

The Lorentz space $L(p, q)$ where either $1 < p < \infty$, $1 \leqq q < \infty$, or $1 < p \leqq \infty$, $q = \infty$ can be defined as the set of all measurable functions f for which $\|f\|_{pq} < \infty$, where

$$\|f\|_{pq} = \left(\int_0^\infty [s^{1/p} f^{**}(s)]^q s^{-1} ds\right)^{1/q} \qquad (1 < p < \infty,\ 1 \leqq q < \infty),$$

$$\|f\|_{p\infty} = \sup_{s>0} \{s^{1/p} f^{**}(s)\} \qquad (1 < p \leqq \infty).$$

We also define L_* to be the set of all measurable functions f such that $f^{**}(s)$ is finite for all $s > 0$. It is trivial that $f \in L_*$ if and only if

$$\|f\|_* = f^{**}(1) = \int_0^1 f^*(t)dt < \infty.$$

The following properties of $L(p, q)$, L_* and the decreasing rearrangement can be found in [4; pp. 760–761] and [9; p. 262]. Let f and g be measurable functions on R^n.

(i) $\|\cdot\|_{pq}$ and $\|\cdot\|_*$ are norms on $L(p, q)$ and L_* respectively, and $L(p, q)$, L_* are Banach spaces with these norms.

(ii) $$\left|\int_{R^n} f(x)g(x)dx\right| \leqq \int_0^\infty f^*(s)g^*(s)ds.$$

(iii) $$f^{**}(s) = \sup\left\{\frac{1}{|E|}\int_E |f(x)|dx\colon |E| \geqq s\right\}.$$

(iv) If $1 < p < \infty$, $1 \leqq q < \infty$, then

$$\|f\|_{pq} \leqq p'\left(\int_0^\infty [s^{1/p} f^*(s)]^q s^{-1} ds\right)^{1/q} \leqq p'\|f\|_{pq},$$

and if $1 < p \leqq \infty$, then

$$\|f\|_{p\infty} \leqq p' \sup_{s>0} s^{1/p} f^*(s) \leqq p'\|f\|_{p\infty},$$

where $1/p + 1/p' = 1$.

(v) If $1 \leqq p \leqq \infty$, then

$$\|f\|_p = \left(\int_0^\infty f^*(s)^p ds\right)^{1/p} \qquad (1 \leqq p < \infty),$$

$$\|f\|_\infty = \sup_{s>0} f^*(s).$$

Hence if $1 \leqq p \leqq \infty$, then $L^p = L(p, p)$ and $\|\cdot\|_p$ is a norm on $L(p, p)$ equivalent to $\|\cdot\|_{pp}$. Also if $f \in L^1$, then $f \in L_*$ and $\|f\|_* \leqq \|f\|_1$.

(vi) If $1 \leqq q_1 \leqq q_2 \leqq \infty$, then $\|f\|_{pq_2} \leqq \|f\|_{pq_1}$, so that

$$L(p, q_1) \subset L(p, q_2).$$

(vii) If $f \in L(p, q)$, then $f \in L_*$ and

$$\|f\|_* \leqq (p/q)^{1/q}\|f\|_{pq} \quad (q < \infty), \ \|f\|_* \leqq \|f\|_{pq} \quad (q = \infty).$$

(viii) If $1 < p < \infty$, $1 \leqq q < \infty$, then

$$L(p, q)' = L(p', q'),$$

where $L(p, q)'$ stands for the topological dual of $L(p, q)$.

It is trivial that, by (v)–(vii)

$$L(p, q_1) \subset L(p, p) = L^p \subset L(p, q_2) \subset L(p, \infty) \subset L_*$$

whenever $1 \leqq q_1 \leqq p \leqq q_2 \leqq \infty$ and $1 < p \leqq \infty$.

We use P to denote the Poisson kernel on Ω, i.e.,

$$P(x, t) = c_n t/(|x|^2 + t^2)^{(n+1)/2} \qquad \text{for} \quad x \in R^n \quad \text{and} \quad t > 0,$$

where $c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2)$. The following properties of the Poisson kernel are either trivial or proved in [13; pp. 61–62]. Let x, y be in R^n, and s, t be positive numbers.

(P_1) $$P(x, t) > 0.$$

(P_2) $$\int_{R^n} P(x, t)e^{-2\pi i x\cdot y}dx = e^{-2\pi|y|t}.$$

In particular

$$\int_{R^n} P(x, t)dx = 1.$$

(P_3) $P(\cdot, t) \in L^p$, $1 \leqq p \leqq \infty$, and $P(\cdot, t) \in L(p, q)$ for every p and q satisfying the same conditions as in the definition of Lorentz spaces.

(P_4) $P(\cdot, t)*P(\cdot, s) = P(\cdot, s+t)$, where $*$ denotes the convolution operation.

(P_5) Let $\delta > 0$. Then

$$\lim_{t\to 0}\int_{|x|\geqq\delta} P(x, t)dx = 0$$

(P_6) Let $\mu = (\mu_1, \dots, \mu_{n+1}) \in Z^+_{n+1}$, and k be a positive integer. Then

$$|D^\mu P(x, t)| \leqq B(n, \mu)t^{-|\mu|}P(x, t) \leqq Bt^{-n-|\mu|},$$

$$\left(\int_{R^n}|D^\mu P(x, t)|^p dx\right)^{1/p} \leqq B(n, \mu, p)t^{-n-|\mu|+n/p}, \quad 1 \leqq p < \infty,$$

$$\int_{R^n} D_{n+1}^k P(x, t)dx = 0.$$

DEFINITION. Let f be a measurable function on R^n such that $f(x)/(1+|x|^{n+1})$ is integrable. The Poisson integral of f, denoted by u, is the function defined on Ω by

$$u(x, t) = \int_{R^n} P(x - y, t)f(y)dy = \int_{R^n} P(y, t)f(x - y)dy.$$

REMARK. If $f \in L_*$, then the Poisson integral of f is well defined (cf. [4; Theorem 7]).

The following fact will be used in § 9.

LEMMA 1.1. *Let* $1 \leqq p \leqq \infty, f \in L^p$ *and* u *be its Poisson integral. Then* $t \mapsto M_p(u; t)$ *is continuous on* $]0, \infty[$.

PROOF. Let $1 \leqq p < \infty$ first. Fix $t_0 \in]0, \infty[$, and let $t_0/2 < t < 2t_0$. Then

$$\frac{P(x, t)}{P(x, t_0/2)} \leqq \frac{2t}{t_0}\left(\frac{|x|^2 + t_0^2/4}{|x|^2 + t^2}\right)^{(n+1)/2} \leqq \frac{2t}{t_0} \leqq 4$$

and

$$|u(x, t)| \leqq 4P(\cdot, t_0/2)*|f|(x) \in L^p.$$

Lebesgue's dominated convergence theorem yields

$$\lim_{t \to t_0}\int|u(x, t)|^p dx = \int\lim_{t \to t_0}|u(x, t)|^p dx = \int|u(x, t_0)|^p dx.$$

Thus $M_p(u; t) \to M_p(u; t_0)$ as $t \to t_0$. In case $p = \infty$

$$|M_\infty(u; t) - M_\infty(u; t_0)| \leqq \sup_x |u(x, t) - u(x, t_0)|$$

$$\leqq \|f\|_\infty \int |P(y, t) - P(y, t_0)|dy,$$

which tends to 0 again by Lebesgue's dominated convergence theorem.

The following theorem is well-known (cf. [4; Theorem 6], [13; pp. 62–65]).

THEOREM 1.1. *Let* f *be a measurable function on* R^n *with* $\int_{R^n}|f(x)|(1+|x|^{n+1})^{-1}dx < \infty$, *and let* u *be its Poisson integral.*

(i) *u is harmonic in* Ω, *and its partial derivatives of all orders can be calculated by differentiation under the integral sign.*

(ii) $\lim u(x, t) = f(x)$ *as* t *tends to* 0 *for almost every* $x \in R^n$. *Furthermore, if* f *is bounded and uniformly continuous, then the convergence is uniform on* R^n.

(iii) *If* $f \in L^p$, $1 \leqq p < \infty$, *then* $\|u(\cdot, t) - f\|_p \to 0$ *as* $t \to 0$.

Definition. For $0 < \alpha < n$, the Riesz potential of order α of a measurable function f, denoted by $R^\alpha f$, is defined by

$$R^\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{R^n} |x - y|^{\alpha - n} f(y) dy, \qquad x \in R^n,$$

provided that $R^\alpha(|f|) \not\equiv \infty$, where $\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2)/\Gamma(n/2 - \alpha/2)$.

By combining [13; Chapter V, Theorem 1] with Marcinkiewicz interpolation theorem [15; Chapter V, Theorem 3.15] (cf. also [9; Lemma 4.8]), one obtains

Theorem 1.2. *Let* f *be either in* $L^1(p=1)$ *or* $L(p, q)(1 < p < \infty, 1 \leqq q \leqq \infty)$, *and* $0 < \alpha < n/p$.

(i) *The integral defining* $R^\alpha f$ *converges absolutely for almost every* x.

(ii) *If* $1 < p < \infty$ *and* $1/r = 1/p - \alpha/n > 0$, *then*

$$\|R^\alpha f\|_{rq} \leqq B(n, p, \alpha) \|f\|_{pq}.$$

(iii) *If* $p = 1$ *and* $1/r = 1 - \alpha/n$, *then*

$$|\{x: |R^\alpha f(x)| > \lambda\}| \leqq [B(n, \alpha) \|f\|_1/\lambda]^r.$$

Lemma 1.2. *Let* h *be a non-negative, non-increasing function defined on* $]0, \infty[$, α *real, and* $0 < p < q \leqq \infty$. *Then*

$$\left(\int_0^\infty [t^\alpha h(t)]^q t^{-1} dt\right)^{1/q} \leqq B(p, \alpha) \left(\int_0^\infty [t^\alpha h(t)]^p t^{-1} dt\right)^{1/p},$$

where the left hand side is interpreted as $\sup_{t>0} t^\alpha h(t)$ *when* $q = \infty$.

For a proof we refer to Stein [13; Appendices, B. 3] and Johnson [10; Lemma 2].

§2. A semigroup formula for harmonic functions

Hereafter we shall be concerned mostly with harmonic functions satisfying a property which we call "*semigroup formula*".

Theorem 2.1. *If* u *is a harmonic function in* Ω, *then the following two statements are equivalent:*

(i) *For each positive number b, there is a positive number B, possibly depending on b and u, such that*

$$\|u(\cdot, t)\|_* \leqq B \qquad \text{for every} \quad t \geqq b.$$

(ii) *There exists a sequence $\{t_i\}$ tending to 0 such that $\|u(\cdot, t_i)\|_*$ is finite for each i, and*

$$(**) \quad u(x, s+t) = \int_{R^n} P(x-y, t)u(y, s)dy \quad \text{for } s, t \text{ positive}, x \in R^n.$$

The equation (**) is called the semigroup formula hereafter.

PROOF. The implication (ii)⇒(i) is obvious by property (iii) in § 1. Assume that (i) holds. By the subharmonicity of $|u|$ on Ω, we have

$$\begin{aligned} |u(x, t)| &\leqq \frac{\omega_n}{\omega_{n+1}(t/2)} \int_{t/2}^{3t/2} \left\{ \frac{1}{\omega_n (t/2)^n} \int_{|z-x|<t/2} |u(z, s)| dz \right\} ds \\ &\leqq \frac{\omega_n}{\omega_{n+1}(t/2)} \int_{t/2}^{3t/2} u_s^{**}(\omega_n (t/2)^n) ds, \end{aligned}$$

where u_s stands for the partial function $z \mapsto u(z, s)$. Let $0 < \omega_n \delta^n \leqq 1$. Since $u_s^{**}(\tau)$ is a non-increasing function of τ, for $t \geqq 2\delta$ we have

$$\begin{aligned} u_s^{**}(\omega_n (t/2)^n) \leqq u_s^{**}(\omega_n \delta^n) &= \frac{1}{\omega_n \delta^n} \int_0^{\omega_n \delta^n} u_s^*(\tau) d\tau \\ &\leqq \frac{1}{\omega_n \delta^n} \int_0^1 u_s^*(\tau) d\tau = \frac{1}{\omega_n \delta^n} \|u_s\|_*. \end{aligned}$$

Therefore

$$|u(x, t)| \leqq \frac{1}{(\omega_{n+1}/2)\delta^n} \sup_{s \geqq \delta} \|u_s\|_* \qquad \text{for every } x \in R^n \text{ and } t \geqq 2\delta.$$

Hence, from [15; Chapter II, Lemma 2.7], we derive that the semigroup formula holds for u.

COROLLARY 1. *Let u be a harmonic function in Ω and $1 \leqq p \leqq \infty$. Assume that for each positive number b, there is a positive constant B such that $M_p(u;t) \leqq B$ for every $t \geqq b$. Then the semigroup formula holds for u.*

PROOF. This is proved by Theorem 2.1 and property (v) in § 1.

COROLLARY 2. *Let u be a harmonic function in Ω. Assume that $0 < p$, $q \leqq \infty$* and *$\alpha \geqq -n/p$. If $C = \|t^\alpha u\|_{p,q} < \infty$, then the semigroup formula holds for u.*

PROOF. We shall prove the corollary only in case $0<q\leqq p<\infty$, because the other cases can be similarly treated. By [3; Lemma 2] we have

$$|u(x,t)|^q \leqq \frac{B(q)}{\omega_{n+1}(t/2)^{n+1}}\int_{t/2}^{3t/2}\int_{|z-x|<t/2}|u(z,s)|^q dz ds.$$

Hölder's inequality implies that

$$\frac{1}{t^{n+1}}\int_{t/2}^{3t/2}\int_{|z-x|<t/2}|u|^q dz ds$$
$$\leqq \frac{1}{t^{n+1}}\int_{t/2}^{3t/2}\left[\int_{|z-x|<t/2}|u|^p dz\right]^{q/p}\left[\int_{|z-x|<t/2}dz\right]^{1-q/p}ds$$
$$\leqq Bt^{-1-nq/p}\int_{t/2}^{3t/2}\left[\int_{R^n}|s^\alpha u|^p dz\right]^{q/p}s^{-1}s^{1-\alpha q}ds \leqq Bt^{-(\alpha+n/p)q}\|s^\alpha u\|_{p,q}^q.$$

Hence $|u(x,t)|\leqq BCt^{-\alpha-n/p}$, and the conclusion follows from Corollary 1.

REMARK. In contrast to temperatures, i.e., solutions of the heat equation (cf. [5; Theorem 4], [10; Lemma 1]), we must take into account the behaviour of $M_p(u;t)$ at infinity in our case. This is due to the lack of a suitable criterion for the uniqueness of the solutions of the Laplace equation.

§3. The Riesz potential

The aim of this section is to define Riesz potential for some classes of harmonic functions and to prove related properties needed later. We adopt the method used by Flett [5] and Johnson [10] in treating temperatures. However, as remarked after Theorem 2.1, when studying harmonic functions, we must require more conditions in order to obtain good results. Most results are proved in details, although many are just the variance of their proofs for temperatures. We begin with the following result due essentially to Stein and Weiss [14; Lemma 6.2].

THEOREM 3.1. *Let f be either in $L^1(p=1)$ or $L(p,q)(1<p<\infty, 1\leqq q\leqq\infty)$, $0<\alpha<n/p$, u be the Poisson integral of f, and $R^\alpha f$ be the Riesz potential of order α of f; this exists for almost all x on account of Theorem* 1.2 (i).

(i) *For almost all x*

$$R^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty u(x,t)t^{\alpha-1}dt.$$

(ii) *The Poisson integral of $R^\alpha f$ is the function $R^\alpha u$ on Ω defined by*

$$R^\alpha u(x,s) = \frac{1}{\Gamma(\alpha)}\int_0^\infty u(x,s+t)t^{\alpha-1}dt.$$

(iii) *For each t positive, the function* $x \mapsto R^\alpha u(x, t)$ *is the Riesz potential of order* α *of the function* $x \mapsto u(x, t)$.

PROOF. By splitting f, we may assume that f is non-negative. With this restriction on f, various applications of Fubini's theorem below are justified. We have

$$\begin{aligned}\int_0^\infty u(x, t)t^{\alpha-1}dt &= \int_{R^n}\left\{\int_0^\infty P(y, t)t^{\alpha-1}dt\right\}f(x-y)dy \\ &= c_n\int_{R^n}\left\{\int_0^\infty |y|^{\alpha-n}t^\alpha(1+t^2)^{-(n+1)/2}dt\right\}f(x-y)dy \\ &= \Gamma(\alpha)R^\alpha f(x).\end{aligned}$$

This gives (i).

To prove (ii), we first observe that the semigroup formula holds for u, i.e.,

$$u(x, s+t) = \int_{R^n} P(x-y, t)u(y, s)dy \qquad \text{for } s, t \text{ positive.}$$

The Poisson integral of $R^\alpha f$ at (x, t) is then given by

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{R^n} P(x-y, t)\int_0^\infty u(y, s)s^{\alpha-1}dsdy \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty u(x, s+t)s^{\alpha-1}ds.\end{aligned}$$

Using the semigroup formula we have

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty P(\cdot, s)*u(\cdot, t)s^{\alpha-1}ds = \frac{1}{\Gamma(\alpha)}\int_0^\infty u(\cdot, s+t)s^{\alpha-1}ds = R^\alpha u(\cdot, t).$$

Thus (iii) is proved, and the proof of the theorem is complete.

DEFINITION A. For any real number b, let $\mathscr{H}_b$ denote the linear space of all harmonic functions u in Ω with the property that if $\mu \in Z_{n+1}^+$, $c>0$, and K is any compact subset of R^n, then there is a positive constant B such that

$$|D^\mu u(x, t)| \leqq Bt^{-(b+|\mu|)} \qquad \text{for every } x \text{ in } K \text{ and } t \geqq c.$$

LEMMA 3.1. *Let b be a non-negative number, and u be a harmonic function in* Ω *with the property that for any positive number c, there is a positive constant B such that*

$$|u(x, t)| \leqq Bt^{-b} \qquad \textit{for every } x \textit{ in } R^n \textit{ and } t \geqq c.$$

Then $u \in \mathscr{H}_b$.

PROOF. By Corollary 1 to Theorem 2.1, the semigroup formula holds for u. Hence it follows from (P_6) that

$$|D^{\mu}u(x, t)| \leqq B(t/2)^{-b}\int_{R^n}|D^{\mu}P(z, t/2)|dz$$
$$\leqq B(t/2)^{-b-|\mu|} \qquad \text{for } t \geqq 2c \text{ and } \mu \in Z_{n+1}^{+}.$$

EXAMPLE. Let u be the Poisson integral of a function in L^p, $1 \leqq p \leqq \infty$. Then $u \in \mathscr{H}_b$, where $b = n/p$.

DEFINITION B. For any u in $\mathscr{H}_b$ and $\alpha < b$, $R^{\alpha}u$ is the function defined on Ω by

(i) $R^0 u = u$;

(ii) if $\alpha > 0$,

$$R^{\alpha}u(x, t) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} u(x, s+t)s^{\alpha-1}ds;$$

(iii) if α is a negative integer, say $\alpha = -m$, then

$$R^{\alpha}u(x, t) = R^{-m}u(x, t) = (-1)^m D_{n+1}^m u(x, t);$$

(iv) if $\alpha = -\beta < 0$ and β is not an integer, then

$$R^{\alpha}u = R^{-\beta}u = R^{m-\beta}(R^{-m}u),$$

where $m = [\beta]+1$ (here $[\gamma]$ stands for the greatest integer not exceeding γ), and $R^{m-\beta}$, R^{-m} are defined by (ii) and (iii).

REMARK. By Theorem 3.1, if u is the Poisson integral of a function f in L^p, $1 \leqq p < \infty$, and $0 < \alpha < n/p$, then for each $s > 0$, $R^{\alpha}u(\cdot, s)$ is the Riesz potential of order α of $u(\cdot, s)$. This inspires us to call $R^{\alpha}u$ in Definition B the Riesz potential of order α of u.

THEOREM 3.2. *Let b be a real number and $u \in \mathscr{H}_b$.*

(i) *If $\alpha < b$, $R^{\alpha}u$ is well-defined and $R^{\alpha}u \in \mathscr{H}_{b-\alpha}$.*

(ii) *If $\beta < b$ and $\alpha + \beta < b$, then*

$$R^{\alpha}(R^{\beta}u) = R^{\alpha+\beta}u.$$

PROOF. The theorem can be proved in the same way as [5; Theorem 8] and [10; Theorem 2].

If $u \in \mathscr{H}_b$ and if $D_{n+1}^k u \equiv 0$ for some positive integer k, then $R^{\alpha}u \equiv 0$ for any $\alpha < b$. Harmonic functions satisfying $D_{n+1}^k u \equiv 0$ for some k are easily classified by the following proposition whose proof is easy (cf. [10; Proposition]).

PROPOSITION A. *Let u be a function defined on Ω, and m be a positive integer. Then*

(i) *u is harmonic in Ω and $D_{n+1}^{2m}u\equiv 0$ if and only if there are C^∞-functions ψ_0, ψ_1 on R^n satisfying $\Delta^m\psi_0=\Delta^m\psi_1\equiv 0$ and*

$$u(x,t)=\sum_{k=0}^{m-1}(-1)^k\frac{\Delta^k\psi_0(x)}{(2k)!}t^{2k}+\sum_{k=0}^{m-1}(-1)^k\frac{\Delta^k\psi_1(x)}{(2k+1)!}t^{2k+1}\ on\ \Omega;$$

(ii) *u is harmonic in Ω and $D_{n+1}^{2m-1}\equiv 0$ if and only if there are C^∞-functions ϕ_0, ϕ_1 on R^n satisfying $\Delta^m\phi_0=\Delta^{m-1}\phi_1\equiv 0$ and*

$$u(x,t)=\sum_{k=0}^{m-1}(-1)^k\frac{\Delta^k\phi_0(x)}{(2k)!}t^{2k}+\sum_{k=0}^{m-2}(-1)^k\frac{\Delta^k\phi_1(x)}{(2k+1)!}t^{2k+1}\ on\ \Omega.$$

COROLLARY. *If $u\in\mathscr{H}_b$, $b>0$, and $D_{n+1}^m u\equiv 0$ for some m, then $u\equiv 0$.*

DEFINITION B*. Define

$$\mathscr{H}=\bigcap_{b>0}\mathscr{H}_b.$$

The following is an immediate consequence of Theorem 3.2 and Definition B*.

THEOREM 3.2*. *Let $u\in\mathscr{H}$ and α, β be real numbers.*
(i) *$R^\alpha u$ is well-defined and $R^\alpha u\in\mathscr{H}$.*
(ii) *$R^\alpha(R^\beta u)=R^{\alpha+\beta}u=R^\beta(R^\alpha u)$.*

LEMMA 3.2. *Let u be a harmonic function in Ω with the property that given $b>0$ and $c>0$, there is a positive constant B such that*

$$|u(x,t)|\leqq Bt^{-b}\qquad \text{for every } x \text{ in } R^n \text{ and } t\geqq c.$$

Then $u\in\mathscr{H}$.

PROOF. This is an easy consequence of Lemma 3.1 and the definition of $\mathscr{H}$.

Let $\mathcal{O}_0$ denote the space of infinitely differentiable functions with compact support in R^n not containing the origin. It is trivial that the Fourier transform can be defined on $\mathcal{O}_0$, and $\hat{\mathcal{O}}_0=\{\hat{f}:f\in\mathcal{O}_0\}\subset\mathscr{S}$.

LEMMA 3.3. *If $f\in\hat{\mathcal{O}}_0$ and u is its Poisson integral, then $u\in\mathscr{H}$.*

PROOF. It is given that

$$u(x,t)=\int_{R^n}P(x-y,t)f(y)dy.$$

Keeping t fixed and taking Fourier transform of both sides, we obtain

$$\hat{u}(\xi,t)=e^{-2\pi|\xi|t}\hat{f}(\xi).$$

For any positive number α, the fact that $\hat{f}=0$ in a neighbourhood of 0 allows us to write

$$\hat{u}(\xi, t) = [|\xi|^\alpha \exp(-2\pi|\xi|t)]\,[|\xi|^{-\alpha}\hat{f}(\xi)] = h_1(\xi)h_2(\xi).$$

An easy computation shows that

$$\|h_1\|_1 = B(n, \alpha)t^{-(\alpha+n)},$$

while $\|h_2\|_\infty < \infty$ is a quantity that depends on f and α. Therefore, it follows that

$$\|\hat{u}(\cdot, t)\|_1 \leqq B(n, \alpha, f)t^{-(\alpha+n)}$$

which implies, by Fourier inverse transform, that

$$\|u(\cdot, t)\|_\infty \leqq B(n, \alpha, f)t^{-(\alpha+n)}$$

and this is valid for an arbitrary $\alpha>0$. Thus by Lemma 3.2 $u \in \mathscr{H}$.

THEOREM 3.3. *Let f be in L^p, $1 \leqq p \leqq \infty$, $\alpha>0$, and let u be the Poisson integral of f. Then for $t>0$*

(i) $M_p(R^{-\alpha}u; t) \leqq B(n, \alpha)\|f\|_p t^{-\alpha}$;

(ii) *furthermore if $1 \leqq p < \infty$, then*

$$M_p(R^{-\alpha}u; t) = o(t^{-\alpha}) \quad \textit{as} \quad t \longrightarrow 0+ .$$

PROOF. We first prove (i) in case α is an integer, say $\alpha=m$. Then

$$R^{-\alpha}u(x, t) = R^{-m}u(x, t) = (-1)^m \int_{R^n} D_{n+1}^m P(y, t) f(x-y)dy,$$

and (i) will follow from (P_6) and Minkowski's inequality (cf. [13; Appendices, A. 1]). Suppose next that α is not an integer, and let $k=[\alpha]+1$. Then for $(x, s) \in \Omega$

$$R^{-\alpha}u(x, s) = \frac{1}{\Gamma(k-\alpha)} \int_0^\infty R^{-k}u(x, s+t)t^{k-\alpha-1}dt.$$

Hence

$$\begin{aligned} M_p(R^{-\alpha}u; s) &\leqq B(\alpha)\int_0^\infty M_p(R^{-k}u; s+t)t^{k-\alpha-1}dt \\ &\leqq B(n, \alpha)\|f\|_p \int_0^\infty (s+t)^{-k}t^{k-\alpha-1}dt \\ &= B(n, \alpha)\|f\|_p s^{-\alpha}\int_0^\infty (1+t)^{-k}t^{k-\alpha-1}dt \end{aligned}$$

which implies (i), because, for $k-\alpha>0$, the last integral is finite.

We shall prove (ii) only when $\alpha=m$, because the general case can be treated in the same manner. Let (x, t) be in Ω. Then by (P_6)

$$R^{-m}u(x, t) = (-1)^m\int_{R^n} D_{n+1}^m P(y, t)[f(x-y)-f(x)]dy,$$

which, together with Minkowski's inequality, implies that

$$t^m M_p(R^{-m}u; t) \leqq t^m\int_{|y|<\delta} |D_{n+1}^m P(y, t)|\, \|f(\cdot-y)-f\|_p dy$$

$$+ t^m\int_{|y|\geqq\delta} |D_{n+1}^m P(y, t)|\, \|f(\cdot-y)-f\|_p dy \qquad (\delta>0).$$

Now for an arbitrary positive number ε, there exists a $\delta>0$ such that $\|f(\cdot-y)-f\|_p<\varepsilon$ if $|y|<\delta$. Let $I_1(t)$ and $I_2(t)$ denote the first term and the second term of the right hand side of the above inequality. Then

$$I_1(t) \leqq B(n, m)\varepsilon$$

and

$$I_2(t) \leqq 2\|f\|_p\int_{|y|\geqq\delta} t^m|D_{n+1}^m P(y, t)|dy$$

$$\leqq B\|f\|_p\int_{|y|\geqq\delta} P(y, t)dy$$

by (P_6). The last integral tends to 0 as t tends to 0 on account of (P_5). Hence (ii) follows.

COROLLARY. *Let $\alpha>0$, $1\leqq p\leqq\infty$, and u be in $\cup_{b>-\alpha}\mathscr{H}_b$. If u satisfies the semigroup formula, then*

$$M_p(R^{-\alpha}u; s+t) \leqq B(n, \alpha)t^{-\alpha}M_p(u; s) \qquad \textit{for all} \quad s, t>0.$$

PROOF. Let s be fixed. We may assume that $M_p(u; s)$ is finite (otherwise the conclusion would be trivial). Then for all $t>0$, by the semigroup formula, we have

$$u(x, s+t) = \int_{R^n} P(x-y, t)u(y, s)dy$$

which implies the corollary by Theorem 3.3.

In the proof of the next theorem, in addition to Lemma 1.2, we shall need the following lemma.

LEMMA 3.4. *Let h be a non-negative, non-increasing function defined on $]0, \infty[$. If there is a $\delta>0$ such that*

$$\int_0^\infty t^{\delta-1}h(t)dt < \infty,$$

then $h(t)=o(t^{-\delta})$ *as* $t\to 0+$ *and* $t\to\infty$.

PROOF. The lemma follows if we note that

$$\infty > \int_0^\infty t^{\delta-1}h(t)dt = \sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k} t^{\delta-1}h(t)dt$$

$$\geqq \delta^{-1}(1-2^{-\delta})\sum_{k=-\infty}^{\infty} h(2^k)2^{\delta k}.$$

THEOREM 3.4. *Let* $1\leqq p\leqq\infty$, $1\leqq q<\infty$, β *be a positive number, and* u *be a harmonic function in* Ω *such that*

$$C = \|t^\beta u\|_{p,q} < \infty.$$

Then for $t>0$, $M_p(u;\,t)\leqq B(q,\beta)Ct^{-\beta}$, *and* $M_p(u;\,t)=o(t^{-\beta})$ *as* $t\to 0+$ *and* $t\to\infty$. *Moreover if* $q<r<\infty$, *then*

$$\|t^\beta u\|_{p,r} \leqq BC.$$

PROOF. By Corollary 2 to Theorem 2.1, the semigroup formula holds for u. It follows that $M_p(u;\,s+t)\leqq\|P(\cdot,s)\|_1 M_p(u;\,t)=M_p(u;\,t)$ so that $M_p(u;\,t)$ is non-increasing in t. Therefore by Lemma 1.2

$$t^\beta M_p(u;\,t) \leqq B(q,\beta)\Big(\int_0^\infty [s^\beta M_p(u;\,s)]^q s^{-1}ds\Big)^{1/q} = B(q,\beta)C.$$

Now if $q<r<\infty$, then, by applying again Lemma 1.2, we obtain

$$\|t^\beta u\|_{p,r} \leqq B\|t^\beta u\|_{p,q} = BC.$$

Finally the o-result will follow if we make use of Lemma 3.4 and the fact that $[M_p(u;\,t)]^q$ is still a non-increasing function in t.

Hereafter we shall extensively use a pair of inequalities due to Hardy which we shall refer to as Hardy's inequality.

Hardy's inequality ([15; Chapter V, Lemma 3.14]). If $q\geqq 1$, $r>0$ and g is a non-negative measurable function defined on $]0,\infty[$, then

(i) $$\Big(\int_0^\infty t^{-r-1}\Big[\int_0^t g(s)ds\Big]^q dt\Big)^{1/q} \leqq (q/r)\Big(\int_0^\infty t^{-r-1}[tg(t)]^q dt\Big)^{1/q},$$

(ii) $$\Big(\int_0^\infty t^{r-1}\Big[\int_t^\infty g(s)ds\Big]^q dt\Big)^{1/q} \leqq (q/r)\Big(\int_0^\infty t^{r-1}[tg(t)]^q dt\Big)^{1/q}.$$

THEOREM 3.5. *Let* $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, α *be a real number,* $\beta>0$, $\beta>\alpha$, *and* u *be a harmonic function in* Ω *such that* $C=\|t^\beta u\|_{p,q}<\infty$.

(i) $u \in \mathscr{H}_{\beta+n/p}$ *and* $\|t^{\beta-\alpha}R^{\alpha}u\|_{p,q} \leqq BC$.

(ii) *If* $1 \leqq q < \infty$, *then* $M_p(R^{\alpha}u;\, t) = o(t^{-(\beta-\alpha)})$ *as* $t \to 0+$ *and* $t \to \infty$.

(iii) *If* $q = \infty$ *and* $M_p(u;\, t) = o(t^{-\beta})$ *as* $t \to 0+$ *(resp.* $t \to \infty$*), then* $M_p(R^{\alpha}u;\, t) = o(t^{-(\beta-\alpha)})$ *as* $t \to 0+$ *(resp.* $t \to \infty$*).*

PROOF. First we shall prove (i). Theorem 3.4, Corollary 2 to Theorem 2.1, Hölder's inequality and (P_6) imply that

$$|u(x,\, t)| \leqq B(q,\, \beta)Ct^{-(\beta+n/p)},$$

which, by Lemma 3.1, shows that $u \in \mathscr{H}_{\beta+n/p}$. Therefore $R^{\alpha}u$ is well-defined. Suppose first that $\gamma = -\alpha > 0$. Then by the corollary of Theorem 3.3 we see that

$$M_p(R^{\alpha}u;\, 2t) \leqq Bt^{\alpha}M_p(u;\, t),$$

which implies that

$$\|t^{\beta-\alpha}R^{\alpha}u\|_{p,q} \leqq BC.$$

Next we shall prove the result for the special case when $\alpha = 1$ and $\beta > 1$. Since

$$R^1u(x,\, s) = \int_0^{\infty} u(x,\, s + t)dt,$$

it follows from Minkowski's inequality and Hardy's inequality that

$$\|s^{\beta-1}R^1u\|_{p,q} \leqq B\left(\int_0^{\infty} [t^{\beta}M_p(u;\, t)]^q t^{-1}dt\right)^{1/q} = BC.$$

To prove the result for $\alpha = \delta > 0$, let γ be the least positive number such that $\gamma + \delta$ is a positive integer. Then by applying (i) in case $\alpha < 0$, we have

$$\|t^{\beta+\gamma}R^{-\gamma}u\|_{p,q} \leqq BC,$$

and hence after repeated applications of (i) in case $\alpha = 1$, we obtain

$$\|t^{\beta-\delta}R^{\delta}u\|_{p,q} \leqq BC.$$

The assertion (ii) then follows from (i) and Theorem 3.4.

We shall prove the assertion (iii) only when $t \to 0+$; the other case can be similarly treated. First, assume $\alpha < 0$. Then the assertion follows easily from the estimate $M_p(R^{\alpha}u;\, 2t) \leqq Bt^{\alpha}M_p(u;\, t)$ given above. Next, we shall prove the result for the case when $\alpha = 1$ and $\beta > 1$. It follows from Minkowski's inequality that

$$s^{\beta-1}M_p(R^1u;\, s) \leqq s^{\beta-1}\int_0^{\infty} M_p(u;\, s + t)dt$$

$$= s^{\beta-1}\left(\int_0^\delta + \int_\delta^\infty\right) = I_1 + I_2 \qquad (\delta > 0).$$

Let ε be an arbitrarily positive number. By the hypothesis on $M_p(u; t)$, $I_1 < \varepsilon/2$ if δ is small enough and $0 < s < \delta$. Fix such δ. Since $M_p(u; s+t) \leqq \|t^\beta u\|_{p,\infty} \cdot (s+t)^{-\beta}$, it follows that

$$I_2 \leqq \|t^\beta u\|_{p,\infty} s^{\beta-1} \int_\delta^\infty t^{-\beta} dt \leqq B\|t^\beta u\|_{p,\infty} s^{\beta-1} \delta^{1-\beta}.$$

Hence $I_2 < \varepsilon/2$ if $0 < s < \delta_1 < \delta$ and δ_1 is properly chosen. Consequently,

$$s^{\beta-1} M_p(R^1 u; s) < \varepsilon \qquad \text{if} \quad 0 < s < \delta_1.$$

In case $\alpha = \delta > 0$ choose $\gamma > 0$ so that $\gamma + \delta$ is an integer. Applying the above result for $\alpha < 0$ we see that $M_p(R^{-\gamma} u; t) = o(t^{-(\beta+\gamma)})$. Repeated use of the result for $\alpha = 1$ yields $M_p(R^{\gamma+\delta}(R^{-\gamma} u); t) = o(t^{-(\beta+\gamma)+(\gamma+\delta)}) = o(t^{-(\beta-\delta)})$. Thus (iii) is proved.

§4. Auxiliary lemmas

DEFINITION. Let α, b be real numbers satisfying $-\alpha - 1 < b$. For any harmonic function u in $\mathscr{H}_b$, $1 \leqq p \leqq \infty$, and $1 \leqq q \leqq \infty$, let

$$\mathscr{E}^\alpha_{p,q}(u) = \|tR^{-\alpha-1} u\|_{p,q}$$

with infinite values being allowed.

LEMMA 4.1. *Let α, b, u, p, q be as in the above definition. Let γ be a real number such that $\gamma < b$. Then*

$$\mathscr{E}^\alpha_{p,q}(u) = \mathscr{E}^{\alpha+\gamma}_{p,q}(R^\gamma u).$$

PROOF. By Theorem 3.2,

$$R^{-\alpha-1} u = R^{-\alpha-\gamma-1}(R^\gamma u),$$

which implies that

$$\mathscr{E}^\alpha_{p,q}(u) = \|tR^{-\alpha-\gamma-1}(R^\gamma u)\|_{p,q} = \mathscr{E}^{\alpha+\gamma}_{p,q}(R^\gamma u).$$

LEMMA 4.2. *Let $1 \leqq p, q \leqq \infty$, let α, β, b be real numbers such that $b \geqq -\alpha$ and $\beta > \alpha$, and let u be in $\mathscr{H}_b$. Then $\mathscr{E}^\alpha_{p,q}(u)$ is equivalent to $\mathscr{E}^{\alpha,\beta}_{p,q}(u) = \|t^{\beta-\alpha} R^{-\beta} u\|_{p,q}$ in the sense that there exists a positive constant $B(\alpha, \beta, p, q)$ such that*

$$B^{-1}\mathscr{E}^\alpha_{p,q}(u) \leqq \mathscr{E}^{\alpha,\beta}_{p,q}(u) \leqq B\mathscr{E}^\alpha_{p,q}(u).$$

PROOF. This follows immediately from Theorem 3.5 (i).

The following lemma is an easy consequence of Lemma 4.2 and Definition B in § 3.

LEMMA 4.3. *Let α, b be real numbers, $b \geqq -\alpha$, $u \in \mathscr{H}_b$, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, and k be a non-negative integer greater than α. Then $\mathscr{E}^{\alpha}_{p,q}(u)$ is equivalent to $\|t^{k-\alpha}D^k_{n+1}u\|_{p,q}$.*

In view of Corollary 2 to Theorem 2.1 we have

COROLLARY. *Let b, u, p, q be as above. If $b \geqq -\alpha > 0$ and $\mathscr{E}^{\alpha}_{p,q}(u) < \infty$, then the semigroup formula holds for u.*

LEMMA 4.4. *Let $1 \leqq p$, $q \leqq \infty$, α, b, β be real numbers, $b \geqq 0$, $b \geqq -\alpha$, $\beta > \alpha$, and u be in $\mathscr{H}_b$. Then $\mathscr{E}^{\alpha}_{p,q}(u)$ is equivalent to $\sup_{1 \leqq j \leqq n} \mathscr{E}^{\alpha-1,\beta}_{p,q}(D_j u)$.*

PROOF. Assume that $\mathscr{E}^{\alpha,\beta}_{p,q}(u) < \infty$. Since $R^{-\beta}$ commutes with differentiation and the semigroup formula holds for $R^{-\beta}u$ on account of Corollary 2 to Theorem 2.1, if follows that

$$R^{-\beta}(D_j u)(x, t) = D_j(R^{-\beta}u)(x, t)$$

$$= \int_{R^n} D_j P(x - y, t/2) R^{-\beta}u(y, t/2) dy, \qquad j = 1,\dots, n.$$

Then we have

$$M_p(R^{-\beta}(D_j u); t) \leqq Bt^{-1}M_p(R^{-\beta}u; t/2),$$

which implies by the aid of Lemma 4.2 that

$$\mathscr{E}^{\alpha-1,\beta}_{p,q}(D_j u) \leqq B\mathscr{E}^{\alpha,\beta}_{p,q}(u) \leqq B\mathscr{E}^{\alpha}_{p,q}(u), \qquad j = 1,\dots, n.$$

Conversely, assume $\mathscr{E}^{\alpha-1,\beta}_{p,q}(D_j u) < \infty$ for $j = 1,\dots, n$. Since $D_j u \in \mathscr{H}_{b+1}$, the argument above shows that

$$\mathscr{E}^{\alpha-2,\beta}_{p,q}(D_j^2 u) \leqq B\mathscr{E}^{\alpha-1,\beta}_{p,q}(D_j u),$$

and hence $\mathscr{E}^{\alpha-2,\beta}_{p,q}(R^{-2}u) \leqq B \sup_{1 \leqq j \leqq n} \mathscr{E}^{\alpha-1,\beta}_{p,q}(D_j u)$. Therefore the remaining part of the lemma follows from Lemma 4.2 and Lemma 4.1.

COROLLARY. *Let $1 \leqq p$, $q \leqq \infty$, $0 \leqq \alpha < 1$, $b \geqq 0$ and $u \in \mathscr{H}_b$. Then $\mathscr{E}^{\alpha-1,0}_{p,q}(D_j u) \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$ for every $j = 1,\dots, n$.*

PROOF. By Lemma 4.4 $\mathscr{E}^{\alpha-1,1}_{p,q}(D_j u) \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$ for $j = 1,\dots, n$. We apply Lemma 4.2 to obtain $\mathscr{E}^{\alpha-1,0}_{p,q}(D_j u) \leqq B\mathscr{E}^{\alpha-1,1}_{p,q}(D_j u)$ and hence $\mathscr{E}^{\alpha-1,0}_{p,q}(D_j u) \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$.

Let $\alpha > 0$, $1 \leqq p \leqq \infty$, f be a measurable function, and k be the least integer greater than α. Define

$$\Lambda_{p,q}^{\alpha}(f) = \left(\int_{R^n}[\|d^k(h)f\|_p|h|^{-\alpha}]^q|h|^{-n}dh\right)^{1/q}, \qquad 1 \leqq q < \infty,$$

$$\Lambda_{p,\infty}^{\alpha}(f) = \operatorname*{ess\,sup}_{h\in R^n} |h|^{-\alpha}\|d^k(h)f\|_p,$$

where $d(h)f=d^1(h)f=f(\cdot+h)-f$ and $d^k(h)f$ is inductively defined by $d^k(h)f=d(h)(d^{k-1}(h)f)$.

Lemma 4.5. *Suppose $0<\alpha<1$ and $1\leqq p, q\leqq\infty$. Let f be a measurable function whose Poisson integral u exists and assume $u\in\mathscr{H}_b$ for a $b\geqq 0$. Then $\Lambda_{p,q}^{\alpha}(f)$ is equivalent to $\mathscr{E}_{p,q}^{\alpha}(u)$.*

Proof. The proof of this lemma is standard (cf. [10; Lemma 8], [13; Chapter V, §§4–5], [16; Theorem 4]). However we include it here for the sake of easy reference.

Let us show $\mathscr{E}_{p,q}^{\alpha}(u)\leqq B\Lambda_{p,q}^{\alpha}(f)$ first. By Lemma 4.3, it suffices to prove $\|t^{1-\alpha}D_{n+1}u\|_{p,q}\leqq B\Lambda_{p,q}^{\alpha}(f)$. The Poisson kernel satisfies

$$|D_{n+1}P(x, t)| \leqq Bt^{-n-1} \qquad \text{by} \quad (P_6),$$

and

$$|D_{n+1}P(x, t)| \leqq B|x|^{-n-1}.$$

Since $\int_{R^n}D_{n+1}P(x, t)dx=0$,

$$D_{n+1}u(x, t) = \int_{R^n}D_{n+1}P(y, t)f(x + y)dy$$

$$= \int_{R^n}D_{n+1}P(y, t)[f(x + y) - f(x)]dy.$$

Hence by Minkowski's inequality we obtain

$$M_p(D_{n+1}u; t) \leqq \int_{R^n}|D_{n+1}P(y, t)|\,\|f(\cdot + y) - f\|_p dy = \int_{|y|\leqq t} + \int_{|y|>t}$$

$$\leqq Bt^{-n-1}\int_{|y|\leqq t}\|f(\cdot + y) - f\|_p dy$$

$$+ B\int_{|y|>t}|y|^{-n-1}\|f(\cdot + y) - f\|_p dy.$$

In case $q=\infty$, $\|f(\cdot+y)-f\|_p\leqq|y|^{\alpha}\Lambda_{p,\infty}^{\alpha}(f)$ for almost every y in R^n. We obtain immediately $M_p(D_{n+1}u; t)\leqq Bt^{\alpha-1}\Lambda_{p,\infty}^{\alpha}(f)$ so that $\|t^{1-\alpha}D_{n+1}u\|_{p,\infty}\leqq B\Lambda_{p,\infty}^{\alpha}(f)$. Thus $\mathscr{E}_{p,\infty}^{\alpha}(u)\leqq B\Lambda_{p,\infty}^{\alpha}(f)$.

Next let $q<\infty$. Setting $\omega_p(y)=\|f(\cdot+y)-f\|_p=\omega_p(r\sigma)$ with $r=|y|$ and $|\sigma|=1$, we have

$$M_p(D_{n+1}u;\,t) \leqq Bt^{-n-1}\int_0^t\int_S \omega_p(r\sigma)r^{n-1}\,drd\sigma + B\int_t^\infty\int_S \omega_p(r\sigma)r^{-2}\,drd\sigma,$$

where S denotes the unit sphere in R^n. Hence if we set

$$\Omega(r) = \int_S \omega_p(r\sigma)d\sigma,$$

then

$$t^{1-\alpha}M_p(D_{n+1}u;\,t) \leqq Bt^{-n-\alpha}\int_0^t \Omega(r)r^{n-1}dr + Bt^{1-\alpha}\int_t^\infty \Omega(r)r^{-2}dr.$$

Now applying Minkowski's inequality and Hardy's inequality we obtain

$$\|t^{1-\alpha}D_{n+1}u\|_{p,q} = \left(\int_0^\infty [t^{1-\alpha}M_p(D_{n+1}u;\,t)]^q t^{-1}dt\right)^{1/q}$$

$$\leqq B\left(\int_0^\infty [r^{-\alpha}\Omega(r)]^q r^{-1}dr\right)^{1/q}.$$

Then observe by Hölder's inequality that

$$\Omega(r)^q \leqq B\int_S \omega_p(r\sigma)^q d\sigma.$$

Hence it follows that

$$\|t^{1-\alpha}D_{n+1}u\|_{p,q} \leqq B\left(\int_0^\infty\int_S \omega_p(r\sigma)^q r^{-\alpha q}r^{-1}\,\mathrm{d}r\mathrm{d}\sigma\right)^{1/q} = B\Lambda^\alpha_{p,q}(f).$$

Thus $\mathscr{E}^\alpha_{p,q}(u) \leqq B\Lambda^\alpha_{p,q}(f)$.

To prove the converse part, write simply S_j for $\mathscr{E}^{\alpha-1,0}_{p,q}(D_j u)$, $j=1,\dots,n$. By the corollary to Lemma 4.4 we have $S_j \leqq B\mathscr{E}^\alpha_{p,q}(u)$. Next let $h=s\sigma$ with $|h|=s$ and $|\sigma|=1$. Assume $0<s\leqq t$. Then for almost every x and h

$$f(x+h)-f(x) = u(x+h,\,t) - u(x,\,t) + [f(x+h) - u(x+h,\,t)]$$

$$-\,[f(x) - u(x,\,t)] = \int_0^s \frac{\partial}{\partial\lambda}u(x+\lambda\sigma,\,t)d\lambda$$

$$-\int_0^t D_{n+1}u(x+h,\,r)dr + \int_0^t D_{n+1}u(x,\,r)dr.$$

By Minkowski's inequality

$$\|f(\cdot+h)-f\|_p \leqq Bs\sum_j M_p(D_j u;\,t) + 2\int_0^t M_p(D_{n+1}u;r)dr.$$

Set $\omega_1(t,\,p) = \sup_{0<|h|<t}\|f(\cdot+h)-f\|_p$. In case $q=\infty$ $\int_0^t M_p(D_{n+1}u;\,r)dr \leqq Bt^\alpha\mathscr{E}^\alpha_{p,\infty}(u)$ so that

$$t^{-\alpha}\omega_1(t, p) \leqq Bt^{1-\alpha}\sum_j M_p(D_j u;\, t) + B\mathscr{E}^\alpha_{p,\infty}(u)$$

$$\leqq B\sum_j S_j + B\mathscr{E}^\alpha_{p,\infty}(u) \leqq B\mathscr{E}^\alpha_{p,\infty}(u).$$

Thus

$$\Lambda^\alpha_{p,\infty}(f) \leqq \sup_t t^{-\alpha}\omega_1(t, p) \leqq B\mathscr{E}^\alpha_{p,\infty}(u).$$

In case $1 \leqq q < \infty$, Minkowski's inequality and Hardy's inequality give

$$\Lambda^\alpha_{p,q}(f) \leqq \left(\int_0^\infty [t^{-\alpha}\omega_1(t, p)]^q t^{-1} dt\right)^{1/q}$$

$$\leqq B\sum_j \left(\int_0^\infty [t^{1-\alpha} M_p(D_j u;\, t)]^q t^{-1} dt\right)^{1/q}$$

$$+ B\left(\int_0^\infty t^{-\alpha q}\left[\int_0^t M_p(D_{n+1}u;\, r)dr\right]^q t^{-1} dt\right)^{1/q}$$

$$\leqq B\sum_j S_j + B\left(\int_0^\infty [t^{1-\alpha} M_p(D_{n+1}u;\, t)]^q t^{-1} dt\right)^{1/q}$$

$$\leqq B\mathscr{E}^\alpha_{p,q}(u) + B\|t^{1-\alpha}D_{n+1}u\|_{p,q}.$$

The last quantity is equivalent to $\mathscr{E}^\alpha_{p,q}(u)$ by Lemma 4.3, and thus $\Lambda^\alpha_{p,q}(f) \leqq B\mathscr{E}^\alpha_{p,q}(u)$.

§5. Some spaces of harmonic functions

In this section, we shall define several spaces of harmonic functions and study their basic properties.

Definition. Let α be a real number, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$. Define

$$\mathscr{H}(\alpha;\, p, q) = \{u \in \mathscr{H}^*_{n/p-\alpha} \equiv \bigcap_{b<n/p-\alpha} \mathscr{H}_b \colon \mathscr{E}^\alpha_{p,q}(u) < \infty\};$$

$$\mathscr{h}(\alpha;\, p, \infty) = \{u \in \mathscr{H}(\alpha;\, p, \infty) \colon M_p(R^{-\alpha-1}u;\, t)$$

$$= o(t^{-1}) \text{ as } t \longrightarrow 0+ \text{ and } t \longrightarrow \infty\}.$$

Then for $\alpha < n/p$, $\mathscr{E}^\alpha_{p,q}$ is a norm on $\mathscr{H}(\alpha;\, p, q)$ on account of Lemma 4.3 and the corollary to Proposition A in §3. For $\alpha \geqq n/p$, if we identify harmonic functions u satisfying $D^k_{n+1}u \equiv 0$ for some non-negative integer k with the zero element, we still obtain a norm (we shall always assume this identification).

First we give

Lemma 5.1. *Let* $1 \leqq p, q \leqq \infty$, α *be a real number and* $\kappa \in Z^+_{n+1}$ *with*

$|\kappa|=k>\alpha$. *Then* $\|t^{k-\alpha}D^{\kappa}v\|_{p,q}\leqq B\mathscr{E}^{\alpha}_{p,q}(v)$ *for* $v\in\mathscr{H}(\alpha;\, p,\, q)$.

PROOF. This is nothing but Lemma 4.3 in case $\kappa=(0,\dots,0,k)$. So assume $D^{\kappa}=D^{h}_{n+1}D^{\kappa'}$ with $\kappa'_{n+1}=0$ and $|\kappa'|=k-h>0$. Since $D^{\kappa}v\in\mathscr{H}^{*}_{n/p-\alpha+k}$, Theorem 3.2 gives

$$D^{\kappa}v = R^{k-h}(R^{h-k}D^{\kappa}v) = (-1)^{k-h}R^{k-h}(D^{\kappa'}D^{k}_{n+1}v)$$

and hence

$$D^{\kappa}v(x,\, t) = \frac{(-1)^{k-h}}{\Gamma(k-h)}\int_0^{\infty}s^{k-h}D^{\kappa'}D^{k}_{n+1}v(x,\, s+t)s^{-1}ds.$$

By Corollary 2 to Theorem 2.1 and Lemma 4.3 the semigroup formula holds for $D^{k}_{n+1}v$. Using this fact, (P_6) and Minkowski's inequality, we have

$$M_p(D^{\kappa}v;\, t) \leqq B\left\{\int_0^{2t}s^{k-h}(s+t/2)^{h-k}M_p(D^{k}_{n+1}v;\, t/2)s^{-1}ds\right.$$

$$\left. + \int_{2t}^{\infty}s^{k-h}(s/2)^{h-k}M_p(D^{k}_{n+1}v;\, s/2)s^{-1}ds\right\}.$$

Therefore, we derive from Minkowski's inequality and Hardy's inequality that $\|t^{k-\alpha}D^{\kappa}v\|_{p,q}\leqq B\|t^{k-\alpha}D^{k}_{n+1}v\|_{p,q}\leqq B\mathscr{E}^{\alpha}_{p,q}(v)$.

LEMMA 5.2. *Let* $1\leqq p,\, q\leqq\infty$, *and* α *and* γ *be real numbers such that* $\gamma<n/p-\alpha$. *Then* R^{γ} *is an isometric isomorphism of* $\mathscr{H}(\alpha;\, p,\, q)$ $(\mathscr{L}(\alpha;\, p,\, \infty)$ *resp.) onto* $\mathscr{H}(\alpha+\gamma;\, p,\, q)$ $(\mathscr{L}(\alpha+\gamma;\, p,\, \infty)$ *resp.). Moreover, if* $\alpha<n/p$, *then its inverse is* $R^{-\gamma}$.

PROOF. It follows from Lemma 4.1 that R^{γ} is an isometric homomorphism and that it is an isometric isomorphism with inverse $R^{-\gamma}$ if $\alpha<n/p$. Hence, it is sufficient to prove that R^{γ} is onto in case $\alpha\geqq n/p$. Let m be the smallest non-negative integer such that $\alpha-n/p-m<0$. Since $R^{\gamma}=R^{\gamma+m}\circ R^{-m}$ by Theorem 3.2, and since $R^{\gamma+m}$ is an isomorphism of $\mathscr{H}(\alpha-m;\, p,\, q)(\mathscr{L}(\alpha-m;\, p,\, \infty)$ resp.) onto $\mathscr{H}(\alpha+\gamma;\, p,\, q)(\mathscr{L}(\alpha+\gamma;\, p,\, \infty)$ resp.) by what has been just proved, the desired result follows if we can show that the mapping $D^{m}_{n+1}=(-1)^{m}R^{-m}$ from $\mathscr{H}(\alpha;\, p,\, q)(\mathscr{L}(\alpha;\, p,\, \infty)$ resp.) into $\mathscr{H}(\alpha-m;\, p,\, q)(\mathscr{L}(\alpha-m;\, p,\, \infty)$ resp.)) is onto.

For this purpose, let $v\in\mathscr{H}(\alpha-m;\, p,\, q)$. For each $\kappa\in Z^{+}_{n+1}$ with $|\kappa|=m$, set

$$v_{\kappa}(x,\, t) = (-1)^{m}R^{m}D^{\kappa}v(x,\, t)$$

$$= \frac{(-1)^{m}}{\Gamma(m)}\int_0^{\infty}D^{\kappa}v(x,\, s+t)s^{m-1}ds \qquad \text{for} \quad (x,\, t)\in\Omega.$$

Since $v\in\mathscr{H}^{*}_{n/p-\alpha+m}$, Theorem 3.2 implies that $v_{\kappa}\in\mathscr{H}^{*}_{n/p-\alpha+m}$ and $v_{(0,\dots,0,m)}=v$. Before proceeding on with the proof, we need some more notation. For any polynomial $Q(x, t)=\sum_\mu c_\mu x_1^{\mu_1}\cdots x_n^{\mu_n}t^{\mu_{n+1}}$, we denote by $Q(D)$ the differential operator defined by $\sum_\mu c_\mu D^\mu$ with $\mu=(\mu_1,\dots, \mu_n, \mu_{n+1})$. For each $\kappa\in Z_{n+1}^+$ with $|\kappa|=m$, let Q_κ be a homogeneous polynomial in (x, t) such that $\sum_{|\kappa|=m}x_1^{\kappa_1}\cdots x_n^{\kappa_n}t^{\kappa_{n+1}}Q_\kappa(x, t)$ is divisible by $x_1^2+\cdots+x_n^2+t^2$, i.e., $\sum_{|\kappa|=m}x_1^{\kappa_1}\cdots x_n^{\kappa_n}t^{\kappa_{n+1}}Q_\kappa(x, t)=(x_1^2+\cdots+x_n^2+t^2)Q'(x, t)$ and Q' is a polynomial in (x, t). Then, it follows from the definition of $v_\kappa(|\kappa|=m)$ and the assumption on $\{Q_\kappa\}_{|\kappa|=m}$ that

$$\begin{aligned}&\sum_{|\kappa|=m}Q_\kappa(D)v_\kappa(x, t)\\&=\frac{(-1)^m}{\Gamma(m)}\int_0^\infty[\sum_{|\kappa|=m}Q_\kappa(D)D^\kappa v(x, s+t)]s^{m-1}ds\\&=\frac{(-1)^m}{\Gamma(m)}\int_0^\infty[Q'(D)\Delta v(x, s+t)]s^{m-1}ds=0\qquad\text{for all}\quad (x, t)\in\Omega.\end{aligned}$$

Hence, it follows from a result of Calderón and Zygmund [2; Chapter II, Theorem 2] that there is a function u harmonic in Ω such that $D^\kappa u=v_\kappa$ for every κ with $|\kappa|=m$. If $|\mu|=m-1$, then

$$D^\mu u(x, t)=\int_c^t D_{n+1}D^\mu u(x, s)ds+D^\mu u(x, c)$$

for $t\geqq c>0$. Let K be a compact set in R^n and $b<n/p-\alpha+m-1$. Since $D_{n+1}D^\mu u\in\mathscr{H}^*_{n/p-\alpha+m}$ and $\alpha-n/p-m+1\geqq 0$, we have

$$|D^\mu u(x, t)|\leqq Bt^{-b}\qquad\text{for all } x\in K \text{ and } t\geqq c.$$

Thus $D^\mu u\in\mathscr{H}^*_{n/p-\alpha+m-1}$ if $|\mu|=m-1$. By continuing such computations, we derive that $u\in\mathscr{H}^*_{n/p-\alpha}$. Further, if k is a positive integer greater than $\max(\alpha, m)$, then $D_{n+1}^k u=D_{n+1}^{k-m}(D_{n+1}^m u)=D_{n+1}^{k-m}v_{(0,\dots,0,m)}=D_{n+1}^{k-m}v$. Therefore, $\|t^{k-\alpha}D_{n+1}^k u\|_{p,q}=\|t^{(k-m)-(\alpha-m)}D_{n+1}^{k-m}v\|_{p,q}<\infty$ by Lemma 4.3. This lemma again implies that $u\in\mathscr{H}(\alpha; p, q)$. Moreover, if $v\in\not{h}(\alpha-m; p, \infty)$, then $M_p(D_{n+1}^k u; t)=M_p(D_{n+1}^{k-m}v; t)=o(t^{-(k-\alpha)})$ as $t\to 0+$ and $t\to\infty$, and $u\in\not{h}(\alpha; p, \infty)$ by Theorem 3.5 (iii). The proof of the lemma is thus complete.

REMARK. If $\alpha-n/p$ is not a non-negative integer, then we can replace $\mathscr{H}^*_{n/p-\alpha}$ by $\mathscr{H}_{n/p-\alpha}$ in the definition of $\mathscr{H}(\alpha; p, q)$. In fact, let m be the smallest non-negative integer such that $\alpha-n/p-m<0$, $u\in\mathscr{H}(\alpha; p, q)$ and $\kappa\in Z_{n+1}^+$ with $|\kappa|\geqq m$. In case $|\kappa|=k>\alpha$, $\|t^{k-\alpha}D^\kappa u\|_{p,q}\leqq B\mathscr{E}^\alpha_{p,q}(u)$ by Lemma 5.1 so that $D^\kappa u\in\mathscr{H}_{n/p-\alpha+k}$ by Theorem 3.5 (i). In case $\alpha\geqq|\kappa|\geqq m$, take an integer $l>\max(|\kappa|, \alpha)$ and set $\mu=\kappa+(0,\dots, 0, l)$. Since $|\mu|>\alpha$, $D^\mu u\in\mathscr{H}_{n/p-\alpha+|\mu|}$ as observed above. From Theorem 3.2 follows $D^\kappa u=(-1)^l R^l(D^\mu u)\in\mathscr{H}_{n/p-\alpha+|\kappa|}$. Thus $D^\kappa u\in\mathscr{H}_{n/p-\alpha+|\kappa|}$ in all cases. From this and the fact that $\alpha-n/p-m+1>0$ if $m\geqq 1$, we easily conclude that $u\in\mathscr{H}_{n/p-\alpha}$. However, the following example suggests that we may not expect this in the other cases: Let $n=1$, $\alpha=1/p$, 1

$\leqq p \leqq \infty$ and $u(x, t)=\log((x^2+t^2)^{1/2})$. Then $u \notin \mathscr{H}_0$ but $u \in \mathscr{H}(1/p; p, \infty)$.

The basic properties of the spaces $\mathscr{H}(\alpha; p, q)$ lie in the following theorem.

THEOREM 5.1. *Let $1 \leqq p, q \leqq \infty$ and α be a real number.*

(i) *$\mathscr{H}(\alpha; p, q)$($\mathscr{A}(\alpha; p, \infty)$ resp.) is a Banach space with norm $\mathscr{E}^{\alpha}_{p,q}$ ($\mathscr{E}^{\alpha}_{p,\infty}$ resp.).*

(ii) *If $1 \leqq q_1 \leqq q_2 < \infty$, then*

$$\mathscr{H}(\alpha; p, q_1) \subset \mathscr{H}(\alpha; p, q_2) \subset \mathscr{A}(\alpha; p, \infty) \subset \mathscr{H}(\alpha; p, \infty),$$

and each inclusion mapping is continuous.

(iii) *If β is a real number such that $\beta > \alpha$, then $\mathscr{E}^{\alpha;\beta}_{p,q}$ is an equivalent norm on $\mathscr{H}(\alpha; p, q)$; moreover $u \in \mathscr{A}(\alpha; p, \infty)$ if and only if $u \in \mathscr{H}(\alpha; p, \infty)$ and $M_p(R^{-\beta}u; t)=o(t^{-(\beta-\alpha)})$ as $t \to 0+$ and $t \to \infty$.*

(iv) *If k is a non-negative integer greater than α, then $\sup_{|\kappa|=k}\|t^{k-\alpha}D^{\kappa}u\|_{p,q}$ is an equivalent norm on $\mathscr{H}(\alpha; p, q)$.*

(v) *The spaces $\mathscr{H}(\alpha; p, q)$, where p, q are fixed and α varies, are isomorphic to one another. The same conclusion holds for the spaces $\mathscr{A}(\alpha; p, \infty)$.*

PROOF. (ii) follows easily from Theorem 3.4.

(iii) is an easy consequence of Lemma 4.2 and Theorem 3.5.

(iv) is derived from Lemmas 4.3 and 5.1.

To prove (v), let δ be another real number and let k be a non-negative integer greater than δ. It then follows from Lemma 5.2 that R^{-k} is an isometric isomorphism of $\mathscr{H}(\delta; p, q)$($\mathscr{A}(\delta; p, \infty)$ resp.) onto $\mathscr{H}(\delta-k; p, q)$($\mathscr{A}(\delta-k; p, \infty)$ resp.); denote its inverse by $(R^{-k})^{-1}$. This lemma again implies that $R^{\delta-\alpha-k}$ is an isometric isomorphism of $\mathscr{H}(\alpha; p, q)$($\mathscr{A}(\alpha; p, \infty)$ resp.) onto $\mathscr{H}(\delta-k; p, q)$ ($\mathscr{A}(\delta-k; p, \infty)$ resp.). Consequently, $(R^{-k})^{-1}\circ R^{\delta-\alpha-k}$ is an isometric isomorphism of $\mathscr{H}(\alpha; p, q)$($\mathscr{A}(\alpha; p, \infty)$ resp.) onto $\mathscr{H}(\delta; p, q)$($\mathscr{A}(\delta; p, \infty)$ resp.).

Finally, we turn to the proof of (i). On account of part (v), it is no loss of generality to assume that $\alpha < n/p$. Let $\{u_j\}$ be a Cauchy sequence in $\mathscr{H}(\alpha; p, q)$ and set $v_j = R^{-\alpha-1}u_j$. Then $\|t(v_j - v_k)\|_{p,q} \to 0$ as $j, k \to \infty$. Since the semigroup formula holds for $v_j - v_k$ by Corollary 2 to Theorem 2.1, it follows from Lemma 1.2 that

$$|v_j(x, t) - v_k(x, t)| \leqq Bt^{-(1+n/p)}\|t(v_j - v_k)\|_{p,q}.$$

Hence, $\lim_{j\to\infty} v_j = v$ exists and is harmonic in Ω, and repeated applications of Fatou's lemma imply that $\|tv\|_{p,q} < \infty$ and $\|t(v_j - v)\|_{p,q} \to 0$ as $j \to \infty$. Then it follows from Theorem 3.5 that $v \in \mathscr{H}_{1+n/p}$. Let $u = R^{\alpha+1}v$ (which has a sense because $\alpha < n/p$). Then we conclude that $u \in \mathscr{H}(\alpha; p, q)$ and $\mathscr{E}^{\alpha}_{p,q}(u_j - u) = \|tR^{-\alpha-1}(u_j - u)\|_{p,q} \to 0$ as $j \to \infty$. Moreover, if $q = \infty$ and $\{u_j\} \subset \mathscr{A}(\alpha; p, \infty)$, then $\|tR^{-\alpha-1}(u_j - u)\|_{p,\infty} = \sup_{t>0} tM_p(R^{-\alpha-1}u_j - R^{-\alpha-1}u; t) \to 0$ as $j \to \infty$, and

$M_p(R^{-\alpha-1}u_j;\ t)=o(t^{-1})$ as $t\to 0+$ and $t\to\infty$ for each $j=1,2,\dots$. From these facts, we easily see that $M_p(R^{-\alpha-1}u;\ t)=o(t^{-1})$ as $t\to 0+$ and $t\to\infty$, and hence $u\in\mathscr{A}(\alpha;\ p,\ \infty)$. The proof of the theorem is now complete.

REMARK. The mapping $(R^{-k})^{-1}\circ R^{\delta-\alpha-k}$ in the proof of part (v) does not depend on $k>\delta$. For if h is a non-negative integer, then repeated applications of Theorem 3.2 and Lemma 5.2 give

$$\begin{aligned}(R^{-h-k})^{-1}\circ R^{-h-k+\delta-\alpha} &= (R^{-k})^{-1}\circ(R^{-h})^{-1}\circ R^{-h}\circ R^{\delta-\alpha-k}\\ &= (R^{-k})^{-1}\circ R^{\delta-\alpha-k}.\end{aligned}$$

Further, if $u\in\mathscr{H}(\alpha;\ p,\ q)$ and $u\in\mathscr{H}_b$ with a $b>\delta-\alpha$ (in particular, if $u\in\mathscr{H}$ or $\delta<n/p$), then $(R^{-k})^{-1}\circ R^{\delta-\alpha-k}(u)=R^k(R^{\delta-\alpha-k}u)=R^{\delta-\alpha}u$ by Theorem 3.2, where $R^{\delta-\alpha}u$ is defined by Definition B in § 3. Therefore, it is reasonable and consistent to denote the isomorphism $(R^{-k})^{-1}\circ R^{\delta-\alpha-k}$ by $R^{\delta-\alpha}$. There should be no ambiguity in using this notation as we shall always state explicitly when it is used; otherwise, Riesz potential (for harmonic functions) is understood in the sense of Definition B in § 3. With this newly adopted notation, the proof of part (v) implies that R^{γ} is an isometric isomorphism of $\mathscr{H}(\alpha;\ p,\ q)$ onto $\mathscr{H}(\alpha+\gamma;\ p,q)$ for all real α and γ, and $R^{-\gamma}$ is the inverse of R^{γ}. To prove the latter statement, let k be a non-negative integer greater than $\max(\alpha,\ \alpha+\gamma)$. Then $R^{\gamma}=(R^{-k})^{-1}\circ R^{\gamma-k}=(R^{-k})^{-1}\circ R^{\gamma}\circ R^{-k}$ by Theorem 3.2 and Lemma 5.2. Therefore, it follows that $(R^{\gamma})^{-1}=(R^{-k})^{-1}\circ(R^{\gamma})^{-1}\circ R^{-k}=(R^{-k})^{-1}\circ R^{-\gamma}\circ R^{-k}=(R^{-k})^{-1}\circ R^{-\gamma-k}=R^{-\gamma}$ by a similar argument.

Next, we shall give a characterization of Poisson integrals of $L(p,\ q)$-functions, which is a natural extension of a result of Stein and Weiss [14; Lemma 3.6].

PROPOSITION B. *Let* $1<p\leqq\infty$, $1\leqq q\leqq\infty$. *Then, a harmonic function* u *defined in* Ω *is the Poisson integral of an* $f\in L(p,\ q)$ *if and only if* $\sup_{t>0}\|u_t\|_{pq}<\infty$. *Furthermore, under this condition, one has*

$$\|f\|_{pq}=\sup_{t>0}\|u_t\|_{pq}.$$

PROOF. Let f be in $L(p,\ q)$ and u be its Poisson integral. It follows from property (iii) in § 1 that $u_t^{**}(s)\leqq f^{**}(s)$ for all $s,\ t>0$. Consequently, $\sup_{t>0}\|u_t\|_{pq}\leqq\|f\|_{pq}$.

Conversely, suppose that $\sup_{t>0}\|u_t\|_{pq}<\infty$. If $p=\infty$, then $q=\infty$ and the proposition is just the above quoted result of Stein and Weiss; therefore, we may consider only the case $p<\infty$. Assume first that $q>1$. Since $\{u_t\}_{t>0}$ is bounded in $L(p,\ q)$ and $L(p,\ q)=L(p',\ q')'$ by property (viii) of Lorentz spaces, Alaoglu's theorem implies that $\{u_t\}_{t>0}$ is relatively w^*-compact in $L(p,\ q)$. Hence there are a sequence $\{t_i\}$ tending to 0 and an $f\in L(p, q)$ such that $u_{t_i}\to f$ in the w*-topology as $i\to\infty$, which means that

$$\int_{R^n} u(y, t_i)g(y)dy \longrightarrow \int_{R^n} f(y)g(y)dy$$

$$\text{as } i\longrightarrow\infty \qquad \text{for every } g\in L(p', q').$$

Now let $s>0$, let x be in R^n, and define $g(y)=P(x-y, s)$ to be a function of $y\in R^n$; then, $g\in L(p', q')$ by (P_3), so that

$$\int_{R^n} u(y, t_i)P(x-y, s)dy \longrightarrow \int_{R^n} f(y)P(x-y, s)dy \qquad \text{as} \quad i\longrightarrow\infty.$$

On the other hand, property (vii) in § 1 and Theorem 2.1 imply that the semigroup formula holds for u. Therefore, it follows that $u(x, s)=\int_{R^n} P(x-y, s)f(y)dy$.

To prove the equality in norms, let $\{t_i\}$ be a decreasing sequence of positive numbers tending to 0. Then $u(x, t_i)\to f(x)$ for almost every $x\in R^n$ as $i\to\infty$ by Theorem 1.1 (ii). Set $f_i(x)=\inf_{j\geqq i}|u(x, t_j)|$. Then $f_i(x)\uparrow|f(x)|$ for almost every $x\in R^n$, so that $f_i^*(t)\uparrow f^*(t)$ as $i\to\infty$ for every $t>0$ (cf. [15; Chapter V, Lemma 3.5]). Hence, it follows that

$$\sup_{t>0}\|u_t\|_{pq} \geqq \|u_{t_i}\|_{pq} \geqq \|f_i\|_{pq} \uparrow \|f\|_{pq}.$$

To prove the result in case $q=1$, let k be a fixed positive number greater than 1. Then, by property (vi) in § 1, $\sup_{t>0}\|u_t\|_{pk}<\infty$. Therefore, by what has been proved, u is the Poisson integral of an $f\in L(p, k)$. The fact that $f\in L(p, 1)$ and the equality in norms holds follows in the same way as for $q>1$.

Motivated by Stein and Weiss [14; p. 30] and the above proposition we give the following definition.

DEFINITION. For $1<p\leqq\infty$, $1\leqq q\leqq\infty$, let $H(p, q)$ be the linear space of all functions u harmonic in Ω such that $\|u\|_{H(p,q)}=\sup_{t>0}\|u_t\|_{pq}<\infty$ with norm $\|\cdot\|_{H(p,q)}$. In accordance with [14; p. 30], for $1<p<\infty$ we shall write H^p for $H(p, p)$.

From the above proposition, it follows that $H(p, q)$ is isometrically isomorphic to $L(p, q)$.

REMARK. If $1<p<\infty$, then, by using the above proposition and Marcinkiewicz interpolation theorem [15; Chapter V, Theorem 3.15], one can derive that $H(p, q)$ will not change if one uses $(n+1)$-tuple of harmonic functions satisfying the system of generalized Cauchy-Riemann equations in its definition.

THEOREM 5.2. *Let* $\alpha>0$, $1\leqq p<\infty$, $1/r=1/p-\alpha/n>0$, *and* $1\leqq q\leqq\infty$. *If* u

is a harmonic function in $\mathscr{H}_b(b>0)$ *with* $\mathscr{E}^{\alpha}_{p,q}(u)<\infty$, *then* u *is the Poisson integral of a unique* $f\in L(r, q)$. *Furthermore, there exists a positive constant* $B=B(\alpha, p, q)$ such that

$$\|f\|_{rq} \leqq B\mathscr{E}^{\alpha}_{p,q}(u).$$

PROOF. Assume first that $q<\infty$. Let δ be the positive number given by $1/\delta=n/r=n/p-\alpha$. Note that by Theorem 3.2, $R^{\alpha+1}(R^{-\alpha-1}u)=u$, so that we have the following integral representation of u:

$$u(x, \eta) = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} t^{\alpha}R^{-\alpha-1}u(x, \eta+t)dt \qquad \text{for} \quad (x, \eta)\in\Omega,$$

which, by property (iii) in § 1, implies that

$$u^{**}_{\eta}(s) \leqq \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} t^{\alpha}(R^{-\alpha-1}u)^{**}(s, \eta+t)dt,\ s>0.$$

Therefore

$$\begin{aligned}
&\left(\int_0^{\infty}[s^{1/\delta}u^{**}_{\eta}(s^n)]^q s^{-1}ds\right)^{1/q}\\
&\leqq \frac{1}{\Gamma(\alpha+1)}\left(\int_0^{\infty}\left[s^{1/\delta}\int_0^{s} t^{\alpha}(R^{-\alpha-1}u)^{**}(s^n, \eta+t)dt\right]^q s^{-1}ds\right)^{1/q}\\
&\quad+\frac{1}{\Gamma(\alpha+1)}\left(\int_0^{\infty}\left[s^{1/\delta}\int_s^{\infty} t^{\alpha}(R^{-\alpha-1}u)^{**}(s^n, \eta+t)dt\right]^q s^{-1}ds\right)^{1/q}\\
&= I_1+I_2.
\end{aligned}$$

Since $\|tR^{-\alpha-1}u\|_{p,q}=\mathscr{E}^{\alpha}_{p,q}(u)<\infty$, the semigroup formula holds for $R^{-\alpha-1}u$ by Corollary 2 to Theorem 2.1. Hence

$$\begin{aligned}
(R^{-\alpha-1}u)^{**}(s^n, \eta+t) &= \sup_{|E|\geqq s}\frac{1}{|E|}\int_E\left|\int_{R^n}R^{-\alpha-1}u(x-y, t)P(y, \eta)dy\right|dx\\
&\leqq \|R^{-\alpha-1}u(\cdot, t)\|_{\infty}.
\end{aligned}$$

The semigroup formula then implies that

$$(R^{-\alpha-1}u)^{**}(s^n, \eta+t) \leqq \|R^{-\alpha-1}u(\cdot, t)\|_{\infty} \leqq B(p)t^{-n/p}M_p(R^{-\alpha-1}u;\ t/2),$$

which, together with Hardy's inequality, gives the estimation $I_2\leqq B\mathscr{E}^{\alpha}_{p,q}(u)$. On the other hand, by Hölder's inequality

$$\begin{aligned}
\frac{1}{|E|}\int_E|R^{-\alpha-1}u(x, \eta+t)|dx &\leqq |E|^{-1/p}M_p(R^{-\alpha-1}u;\ \eta+t)\\
&\leqq s^{-n/p}M_p(R^{-\alpha-1}u;\ \eta+t)\\
&\leqq s^{-n/p}M_p(R^{-\alpha-1}u;\ t)
\end{aligned}$$

if $|E| \geqq s^n$. Hence $(R^{-\alpha-1}u)^{**}(s^n, \eta+t) \leqq s^{-n/p} M_p(R^{-\alpha-1}u; t)$. By using Hardy's inequality again, we conclude that $I_1 \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$. Combining the above estimates, we derive that $\sup_{\eta>0} \|u_\eta\|_{rq} \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$, which, by Proposition B, implies that u is the Poisson integral of a unique $f \in L(r, q)$ and $\|f\|_{rq} \leqq B\mathscr{E}^{\alpha}_{p,q}(u)$. Finally, we observe that trivial modification works for the case $q=\infty$.

COROLLARY. *$\mathscr{H}(\alpha; p, q) \subset H(r, q)$, and the inclusion mapping is continuous.*

REMARK 1. As we shall see in §7 (cf. Corollary (ii) to Theorem 7.4), our theorem provides an easy proof of a theorem of Herz (cf. [8; Theorem 5]). Herz's proof leans on an important inversion formula which allows one to recover $f \in \hat{\mathscr{O}}_0$ from its difference. In fact, our proof is to some extent modelled after his.

REMARK 2. The above method can be used to treat temperatures (solutions of the heat equation) and one can obtain similar results.

THEOREM 5.3. *Let $\alpha>0$, $1<p<\infty$, $1/r=1/p-\alpha/n>0$,* and *$1<q\leqq\infty$. Then*

$$H(p, q) \subset \mathscr{H}(-\alpha; r, q),$$

and the inclusion mapping is continuous.

PROOF. Let u be the Poisson integral of an $f \in L(p, q)$. Then by [4; Theorems 8, 9] $u \in \mathscr{H}_{\alpha+n/r}$, and $\|t^\alpha u\|_{r,q} \leqq B\|f\|_{pq} = B\|u\|_{H(p,q)}$. Hence, the theorem follows from Lemma 4.3.

THEOREM 5.4. *Let $1\leqq p<r\leqq\infty$, $1\leqq q\leqq\infty$, α be a real number and $\delta=(1/p-1/r)n$. Then*

$$\mathscr{H}(\alpha; p, q) \subset \mathscr{H}(\alpha-\delta; r, q),$$

and the inclusion mapping is continuous.

PROOF. First observe that $n/p-\alpha=n/r-(\alpha-\delta)$. Let u be in $\mathscr{H}(\alpha; p, q)$. Let β be a real number greater than α. Theorem 5.1 (iii) implies that $\mathscr{E}^{\alpha;\beta}_{p;q}(u)$ is equivalent to $\mathscr{E}^{\alpha}_{p,q}(u)$. Consequently, the semigroup formula holds for $R^{-\beta}u$ by Corollary 2 to Theorem 2.1. Using Young's inequality (see [13; Appendices, A. 2]), we have

$$M_r(R^{-\beta}u; t) \leqq M_p(R^{-\beta}u; t/2) M_h(P; t/2) \quad (1/h = 1-\delta/n).$$

Property (P_6) in §1 yields $M_h(P; t/2) \leqq Bt^{-\delta}$. Hence $M_r(R^{-\beta}u; t) \leqq Bt^{-\delta} M_p(R^{-\beta}u; t/2)$. Therefore

$$\begin{aligned} \mathscr{E}^{\alpha-\delta,\beta}_{r,q}(u) &= \left(\int_0^\infty [t^{\beta-\alpha+\delta} M_r(R^{-\beta}u; t)]^q t^{-1} dt\right)^{1/q} \\ &\leqq B\left(\int_0^\infty [t^{\beta-\alpha} M_p(R^{-\beta}u; t)]^q t^{-1} dt\right)^{1/q} = B\mathscr{E}^{\alpha;\beta}_{p;q}(u), \end{aligned}$$

from which we obtain the desired result after making use of Theorem 5.1 (iii) again.

§6. Boundary values

In this section, we shall study boundary values (in the sense of distributions) of functions in the space $\mathscr{H}(\alpha; p, q)$ defined in the preceding section.

First, we prepare some lemmas.

LEMMA 6.1. *Let* $1 \leqq p, q \leqq \infty$, α *be a real number and* u *be in* $\mathscr{H}(\alpha; p, q)$. *For* $s>0$, *let* $u^{(s)}(x, t)=u(x, s+t)$ *for all* $(x, t) \in \Omega$.

(i) $u^{(s)} \in \mathscr{H}(\alpha; p, q)$ *and* $\mathscr{E}^{\alpha}_{p,q}(u^{(s)}) \leqq \mathscr{E}^{\alpha}_{p,q}(u)$.

(ii) *If* $q<\infty$ *or* $q=\infty$ *and* $M_p(R^{-\alpha-1}u; t)=o(t^{-1})$ *as* $t \to 0+$ *and* $t \to \infty$, *then* $u^{(s)} \to u$ *in* $\mathscr{H}(\alpha; p, q)$ *as* $s \to 0+$.

PROOF. Since $u \in \mathscr{H}(\alpha; p, q)$, $\mathscr{E}^{\alpha}_{p,q}(u)=\|tR^{-\alpha-1}u\|_{p,q}<\infty$. Hence the semigroup formula holds for $R^{-\alpha-1}u$ by Corollary 2 to Theorem 2.1. Therefore, Minkowski's inequality gives

$$M_p(R^{-\alpha-1}u^{(s)}; t) \leqq M_p(R^{-\alpha-1}u; t),$$

which implies that $\mathscr{E}^{\alpha}_{p,q}(u^{(s)}) \leqq \mathscr{E}^{\alpha}_{p,q}(u)$.

We turn to the proof of (ii). For $q<\infty$,

$$\mathscr{E}^{\alpha}_{p,q}(u^{(s)}-u)=\left(\int_0^\infty t^{q-1}[M_p(R^{-\alpha-1}(u^{(s)}-u); t)]^q dt\right)^{1/q}.$$

For each fixed $t>0$, $(x, s) \mapsto R^{-\alpha-1}u(x, s+t)$ is the Poisson integral of the function $x \mapsto R^{-\alpha-1}u(x, t)$. Therefore, $M_p(R^{-\alpha-1}(u^{(s)}-u); t) \to 0$ as $s \to 0+$ by Theorem 1.1 (ii), (iii) (the uniform continuity of $R^{-\alpha-1}u(\cdot, t)$ in case $p=\infty$ follows from the relation $R^{-\alpha-1}u(\cdot, t)=P(\cdot, t/2)*R^{-\alpha-1}u(\cdot, t/2)$ with $P(\cdot, t/2) \in L^1$ and $R^{-\alpha-1}u(\cdot, t/2) \in L^\infty$). Hence (ii) is concluded by Lebesgue's dominated convergence theorem ($q<\infty$) or the hypothesis on the order of $M_p(R^{-\alpha-1}u; t)$ ($q=\infty$) if one notes that $M_p(R^{-\alpha-1}(u^{(s)}-u); t) \leqq 2M_p(R^{-\alpha-1}u; t)$ for every $s>0$.

LEMMA 6.2. *Let* $1 \leqq p \leqq \infty$, f *be a function in* L^p *and* u *be its Poisson integral.*

(i) *If* $\alpha>0$, *then* $u^{(s)} \in \mathscr{H}(\alpha; p, q)$ *and* $\mathscr{E}^{\alpha}_{p,q}(u^{(s)}) \leqq Bs^{-\alpha}\|f\|_p$ *for all* $s>0$ *and* $1 \leqq q \leqq \infty$.

(ii) *If* $\alpha=0$, *then* $u^{(s)} \in \mathscr{H}(0; p, \infty)$ *and* $\mathscr{E}^{0}_{p,\infty}(u^{(s)}) \leqq B\|f\|_p$ *for all* $s>0$.

(iii) *If* k *is a positive integer with* $2k>\alpha>0$, *and* f *is a* C^∞-*function whose derivatives of all orders vanish at infinity and belong to* L^p, *then* $u \in \mathscr{H}(\alpha; p, q)$ *and* $\mathscr{E}^{\alpha}_{p,q}(u) \leqq B(\|f\|_p+\|\Delta^k f\|_p)$ $(1 \leqq q \leqq \infty)$.

PROOF. (i) and (ii) follow easily from Theorem 3.3. Next we have by Theorem 3.4 and Lemma 4.3

$$\mathscr{E}^{\alpha}_{p,q}(u) \leqq B\mathscr{E}^{\alpha}_{p,1}(u) \leqq B\int_0^\infty t^{2k-\alpha}M_p(D^{2k}_{n+1}u;\ t)t^{-1}dt.$$

Split the integral into $\int_0^1$ and $\int_1^\infty$ and denote them by I_1 and I_2 respectively. By Theorem 3.3 $M_p(D^{2k}_{n+1}u;\ t) \leqq B\|f\|_p t^{-2k}$ so that $I_2 \leqq B\|f\|_p$. Next

$$D^{2k}_{n+1}u(x,\ t) = \int_{R^n} D^{2k}_{n+1}P(x-y,\ t)f(y)dy = (-1)^k\int \Delta^k_y P(x-y,\ t)f(y)dy$$

$$= (-1)^k\int P(x-y,\ t)\Delta^k f(y)dy$$

so that $M_p(D^{2k}_{n+1}u;\ t) \leqq \|\Delta^k f\|_p$. Hence $I_1 \leqq B\|\Delta^k f\|_p$. Now (iii) follows.

LEMMA 6.3. *Let* $1 \leqq p,\ q \leqq \infty$, α *be a positive number and* k *be a positive integer such that* $2k > \alpha$. *Define*

$$< u,\ v >_k = < u,\ v > = \frac{1}{\Gamma(2k)}\int_0^\infty\int_{R^n} t^{2k-1}u(x,\ t/2)R^{-2k}v(x,t/2)dxdt$$

for all u *in* $\mathscr{H}(-\alpha;\ p',\ q')$ *and all* v *in* $\mathscr{H}(\alpha;\ p,\ q)$.

(i) $< \cdot,\ \cdot >$ *is a continuous bilinear form on* $\mathscr{H}(-\alpha;\ p',\ q') \times \mathscr{H}(\alpha;\ p,\ q)$.

(ii) *If* $u \in \mathscr{H}(-\alpha;\ p',\ q')$ *and* v *is the Poisson integral of a* $\phi \in \mathscr{S}$, *then*

$$< u,\ v > = \lim_{s\to 0+}\int_{R^n} u(y,\ s)\phi(y)dy.$$

Moreover, if $<u,\ w> = 0$ *for every* w *which is the Poisson integral of a function in* $\mathscr{S}$, *then* $u \equiv 0$.

PROOF. It follows from Hölder's inequality that

$$\Gamma(2k)|< u,\ v >| \leqq \int_0^\infty t^{2k}M_{p'}(u;\ t/2)M_p(R^{-2k}v;\ t/2)t^{-1}dt.$$

Therefore, Lemma 4.3 and Hölder's inequality imply that

$$|< u,\ v >| \leqq B\mathscr{E}^{-\alpha}_{p',q'}(u)\mathscr{E}^{\alpha}_{p,q}(v).$$

The bilinearity of $< \cdot,\ \cdot >$ is obvious. Hence (i) follows.

We turn to the proof of (ii). Let ϕ be in $\mathscr{S}$ and v be its Poisson integral. Lemmas 6.2 (iii) and 6.1 imply that v and $v^{(s)}$ are in $\mathscr{H}(\alpha;\ p,\ q)$ for every $s > 0$. Further, by the semigroup formula and Fubini's theorem we have

$$< u, v^{(s)} > = \frac{1}{\Gamma(2k)} \int_0^\infty t^{2k-1} \left\{ \int_{R^n} u(y, t/2) R^{-2k} v(y, s + t/2) dy \right\} dt$$

$$= \frac{1}{\Gamma(2k)} \int_0^\infty t^{2k-1} \left\{ \int_{R^n} u(y, s/2 + t) R^{-2k} v(y, s/2) dy \right\} dt.$$

Denote the last integral on R^n by I. Since v is harmonic in Ω,

$$I = \int_{R^n} u(y, s/2 + t) D_{n+1}^{2k} v(y, s/2) dy = (-1)^k \int_{R^n} u(y, s/2 + t) \Delta_y^k v(y, s/2) dy.$$

Integration by parts and the harmonicity of u imply

$$I = (-1)^k \int_{R^n} \Delta_y^k u(y, s/2 + t) v(y, s/2) dy = \int_{R^n} R^{-2k} u(y, s/2 + t) v(y, s/2) dy.$$

Hence, by using Fubini's theorem and theorem 3.2 (ii) we obtain

$$< u, v^{(s)} > = \frac{1}{\Gamma(2k)} \int_{R^n} \left\{ \int_0^\infty t^{2k-1} R^{-2k} u(y, s/2 + t) dt \right\} v(y, s/2) dy$$

$$= \int_{R^n} u(y, s/2) v(y, s/2) dy$$

$$= \int_{R^n} u(y, s/2) \left\{ \int_{R^n} P(y - x, s/2) \phi(x) dx \right\} dy = \int_{R^n} u(x, s) \phi(x) dx.$$

Note that various applications of Fubini's theorem above are easily justified. Lemma 6.1 (ii) and the continuity of the bilinear form $< \cdot, \cdot >$ then give

$$<u, v> = \lim_{s \to 0+} <u, v^{(s)}> = \lim_{s \to 0+} \int_{R^n} u(x, s) \phi(x) dx.$$

Before proving the remaining part of (ii), we observe that, for any $s>0$ there is a sequence $\{u_j\}$ in $\mathscr{H}\mathscr{S}$, the set of all Poisson integrals of functions in $\mathscr{S}$, such that $u_j \to P^{(s)}$ in $\mathscr{H}(\alpha; p, q)$. To prove this observation, take a sequence $\{\phi_j\}$ in $\mathscr{D}$ with the property that $\phi_j \to P(\cdot, s)$ and $\Delta^k \phi_j \to \Delta^k P(\cdot, s)$ in L^p. If u_j is the Poisson integral of ϕ_j, then it follows from Lemma 6.2 (iii) that $u_j \to P^{(s)}$ in $\mathscr{H}(\alpha; p, q)$. Now assume that $<u, w> = 0$ for every $w \in \mathscr{H}\mathscr{S}$. For $(x, s) \in \Omega$ let $v = P^{(x,s)}$. Then the above observation implies that $<u, v> = 0$. Further, repeated applications of integration by parts and Theorem 3.2 give

$$<u, v> = \frac{1}{\Gamma(2k)} \int_0^\infty t^{2k-1} \left\{ \int_{R^n} u(y, t/2) R^{-2k} v(y, t/2) dy \right\} dt$$

$$= \frac{1}{\Gamma(2k)} \int_0^\infty t^{2k-1} \left\{ \int_{R^n} R^{-2k} u(y, t/2) v(y, t/2) dy \right\} dt$$

$$= \frac{1}{\Gamma(2k)} \int_0^\infty t^{2k-1} R^{-2k} u(x, s + t) dt = u(x, s)$$

so that $u \equiv 0$. The proof of (ii) is thus completed.

REMARK. It follows from (ii) that $\langle \cdot, \cdot \rangle$ does not depend on k.

Before stating the main result in this section, we give some definitions. Let $\mathscr{S}_0$ be the linear space of all functions ϕ in $\mathscr{S}$ such that $D^\kappa \hat{\phi}(0)=0$ for all $\kappa \in Z_n^+$. It is obvious that $\mathscr{S}_0$ is a closed subspace of $\mathscr{S}$ and hence a locally convex Fréchet space; $\mathscr{S}_0$ is also a Montel space so that it is reflexive. Therefore, it follows that $\mathscr{S}_0$ is dense in its dual $\mathscr{S}_0'=\mathscr{S}'/\mathscr{P}$, the space of tempered distributions modulo polynomials. For a real number α, the Riesz potential of order α (cf. § 1) can be generalized as follows:

$$(R^\alpha \phi)^\wedge(\xi) = (2\pi|\xi|)^{-\alpha}\hat{\phi}(\xi) \qquad \text{for} \quad \phi \in \mathscr{S}_0 \quad \text{and} \quad \xi \in R^n.$$

It is then obvious that R^α is an isomorphism of $\mathscr{S}_0$ onto $\mathscr{S}_0$ with inverse $R^{-\alpha}$ and $R^\alpha \circ R^\beta = R^{\alpha+\beta}$ for all real numbers α and β. We note that $R^{-2}\phi = -\Delta\phi$ for $\phi \in \mathscr{S}_0$, because $(\Delta\phi)^\wedge(\xi) = -(2\pi|\xi|)^2\hat{\phi}(\xi)$.

For $T \in \mathscr{S}_0'$ define $R^\alpha T$ by $(R^\alpha T)(\phi) = T(R^\alpha \phi)$, where $\phi \in \mathscr{S}_0$. Then R^α has the same properties as R^α defined on $\mathscr{S}_0$. Let us see that the definition coincides with the above one for $\psi \in \mathscr{S}_0$. Let $R^\alpha \psi$ be the one defined above. If we regard ψ as $T \in \mathscr{S}_0'$, then

$$\begin{aligned}
(R^\alpha T)(\phi) &= \int_{R^n} \psi R^\alpha \phi dx = \int_{R^n} \psi(x)dx \int_{R^n} e^{2\pi i x\cdot y}(2\pi|y|)^{-\alpha}\hat{\phi}(y)dy \\
&= \int_{R^n} (2\pi|y|)^{-\alpha}\hat{\phi}(y)dy \int_{R^n} e^{2\pi i x\cdot y}\psi(x)dx \\
&= \int_{R^n} (2\pi|y|)^{-\alpha}\hat{\phi}(y)\hat{\psi}(-y)dy = \int_{R^n} (R^\alpha\psi)^\wedge(-y)\hat{\phi}(y)dy \\
&= \int_{R^n} R^\alpha\psi(y)\phi(y)dy
\end{aligned}$$

by Parseval's formula. Thus $R^\alpha T = R^\alpha \psi$ if $T = \psi$.

THEOREM 6.1. *Let $1 \leqq p$, $q \leqq \infty$, α be a real number and u be in $\mathscr{H}(\alpha; p, q)$.*

(i) *If $\alpha < n/p$, then $\lim_{t\to 0} u(\cdot, t) \equiv u(\cdot, 0)$ exists in the sense of tempered distributions, and $u \mapsto u(\cdot, 0)$ is a continuous linear map of $\mathscr{H}(\alpha; p, q)$ into $\mathscr{S}'$.*

(ii) *If $\alpha \geqq n/p$, then $\lim_{t\to 0} u(\cdot, t) \equiv u(\cdot, 0)$ exists in $\mathscr{S}_0'$, and $u \mapsto u(\cdot, 0)$ is a continuous linear map of $\mathscr{H}(\alpha; p, q)$ into $\mathscr{S}_0'$.*

PROOF. First, we shall prove (i) in case $\alpha < 0$. Since $\mathscr{S}$ is a (locally convex) Fréchet space, to see the existence of $u(\cdot, 0)$, by the Banach-Steinhaus theorem it is sufficient to show that $\lim_{t\to 0}\int_{R^n} u(x, t)\phi(x)dx \equiv u(\cdot, 0)(\phi)$ exists for any $\phi \in \mathscr{S}$. Let v be the Poisson integral of $\phi \in \mathscr{S}$. Then, it follows from Lemma

6.3 (i) and (ii) that $u(\cdot, 0)(\phi) = \langle u, v \rangle$. To verify the continuity of the map $u \mapsto u(\cdot, 0)$, let us take a sequence $\{u_j\}$ in $\mathscr{H}(\alpha; p, q)$ such that $u_j \to u$ in $\mathscr{H}(\alpha; p, q)$ as $j \to \infty$. Then

$$|u_j(\cdot, 0)(\phi) - u(\cdot, 0)(\phi)| = |\langle u_j - u, v \rangle|.$$

The last term tends to 0 as j tends to ∞ by Lemma 6.3 (i). This completes the proof of (i) in case $\alpha < 0$. Assume next that $0 \leqq \alpha < n/p$. Let r be a positive number such that $p < r \leqq \infty$ and $\delta = (1/p - 1/r)n > \alpha$. It then follows from Theorem 5.4 that $\mathscr{H}(\alpha; p, q) \subset \mathscr{H}(\alpha - \delta; r, q)$, which implies (i) in this case after making use of the corresponding result in case $\alpha < 0$.

Now we turn to the proof of (ii). Let k be a positive integer such that $2k > \alpha$. Since $R^{-2k}u \in \mathscr{H}(\alpha - 2k; p, q)$ by Lemma 5.2, (i) implies that $\lim_{t \to 0} \int_{R^n} R^{-2k}u(x, t) \psi(x)dx = (R^{-2k}u)(\cdot, 0)(\psi)$ exists for any $\psi \in \mathscr{S}$. Given $\phi \in \mathscr{S}_0$, there is $\psi \in \mathscr{S}_0$ such that $\Delta^k \psi = \phi$, because $(-1)^k \Delta^k = R^{-2k}$ (recall $R^{-2}\phi = -\Delta\phi$ for $\phi \in \mathscr{S}_0$) and R^{-2k} is an isomorphism of $\mathscr{S}_0$ onto $\mathscr{S}_0$ as stated before Theorem 6.1. Then

$$\int_{R^n} u(x, t)\phi(x)dx = \int_{R^n} u(x, t)\Delta^k\psi(x)dx = (-1)^k \int_{R^n} R^{-2k}u(x, t)\psi(x)dx.$$

This shows that $\lim_{t \to 0} \int_{R^n} u(x, t)\phi(x)dx$ exists for any $\phi \in \mathscr{S}_0$, and $u(\cdot, 0)(\Delta^k\psi) = (-1)^k(R^{-2k}u)(\cdot, 0)(\psi)$. We have also

$$u(\cdot, 0)(\Delta^k\psi) = (-1)^k u(\cdot, 0)(R^{-2k}\psi) = (-1)^k R^{-2k}(u(\cdot, 0))(\psi).$$

Hence $R^{-2k}(u(\cdot, 0)) = (R^{-2k}u)(\cdot, 0)$. On account of Lemma 5.2 $u \mapsto R^{-2k}u$ is continuous, and since $-2k + \alpha < 0$, $R^{-2k}u \mapsto (R^{-2k}u)(\cdot, 0)$ is continuous so that $u \mapsto (R^{-2k}u)(\cdot, 0) = R^{-2k}(u(\cdot, 0))$ is continuous. Finally $R^{-2k}(u(\cdot, 0)) \mapsto u(\cdot, 0)$ is continuous because R^{2k} is an isomorphism of $\mathscr{S}_0'$ onto $\mathscr{S}_0'$ as observed before Theorem 6.1. Thus $u \mapsto u(\cdot, 0)$ is continuous.

REMARK 1. The map $u \mapsto u(\cdot, 0)$ in both (i) and (ii) of Theorem 6.1 is one to one. The assertion for (i) follows easily from Lemma 6.3 (ii). To prove the result for (ii), assume that $u(\cdot, 0)(\phi) = 0$ for all $\phi \in \mathscr{S}_0$. It then follows from the proof of (ii) that $(R^{-2k}u)(\cdot, 0)(\phi) = 0$ for all $\phi \in \mathscr{S}_0$. On account of Lemma 5.2 and Theorem 5.4, we may assume that $R^{-2k}u \in \mathscr{H}(\beta; r, \infty)$ for a $\beta < 0$ and $1 < r \leqq \infty$. Since $\hat{\mathscr{O}}_0 \subset \mathscr{S}_0$, $0 = (R^{-2k}u)(\cdot, 0)(\phi) = \langle R^{-2k}u, v \rangle$ for every $\phi \in \hat{\mathscr{O}}_0$, where v is the Poisson integral of ϕ. Lemma 6.3 (i) and Theorem 7.2 then imply that $\langle R^{-2k}u, v \rangle = 0$ for every $v \in \mathscr{H}(-\beta; r', 1)$. Hence $R^{-2k}u = 0$ by Lemma 6.3 (ii), and by our identification in case $\alpha \geqq n/p$ (see § 5) u is the zero element in $\mathscr{H}(\alpha; p, q)$.

REMARK 2. Let $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$, $0 < \alpha < n/p$ and u be in $\mathscr{H}(\alpha; p, q)$. Theorem 5.2 implies that u is the Poisson integral of an $f \in L(r, q)$ with $1/r = 1/p - \alpha/n$. We shall show $u(\cdot, 0) = f$. Let $\phi(x, t)$ be the Poisson integral of $\phi \in \mathscr{S}$, and set $\psi_t(x) = \phi(x, t) - \phi(x)$. Then, from Fubini's theorem, it follows that

$$\left|\int_{R^n} u(x, t)\phi(x)dx - \int_{R^n} f(x)\phi(x)dx\right| = \left|\int_{R^n} \psi_t(x)f(x)dx\right|$$

$$\leqq \int_0^\infty \psi_t^{**}(s)f^{**}(s)ds \leqq \|f\|_{r\infty}\int_0^\infty s^{1/r'}\psi_t^{**}(s)\frac{ds}{s} = \|\psi_t\|_{r'1}\|f\|_{r\infty}.$$

First observe that, for every $s > 0$, $\psi_t^{**}(s) \leqq \|\psi_t\|_\infty \to 0$ as $t \to 0$ by Theorem 1.1 (ii). Since $\psi_t^{**} \leqq \psi^{**}(\cdot, t) + \phi^{**} \leqq 2\phi^{**}$ and $\phi \in L(r', 1)$, $\|\psi_t\|_{r'1} \to 0$ as $t \to 0$ by Lebesgue's dominated convergence theorem. Hence $\int_{R^n} u(x, t)\phi(x)dx$ tends to $\int_{R^n} f(x)\phi(x)dx \equiv f(\phi)$ as $t \to 0$.

REMARK 3. Assume $0 < p \leqq 1$ and $\alpha < 0$. Let u be a harmonic function in Ω such that $|u(x, t)| \leqq Bt^{-b}$ $(b > 0)$ for every $x \in R^n$ and $t > 0$, and suppose that $\sup_{t>0} t^{-\alpha}M_p(u; t) < \infty$. Then, by imitating the method used in [3; Lemma 4], we shall show that $\lim_{t\to 0} u(\cdot, t) = \bar{\mathscr{F}}(\hat{u}_0)$, where $\hat{u}_0(\xi) = \hat{u}(\xi, \delta)e^{2\pi|\xi|\delta}$ $(\delta > 0)$ and $\bar{\mathscr{F}}$ denotes the inverse Fourier transform.

To prove it we may assume $p = 1$, because if $M_p(u; t) \leqq Bt^\alpha$ with $0 < p < 1$, then

$$M_1(u; t) = \int_{R^n} |u(x, t)|^{1-p}|u(x, t)|^p dx \leqq Bt^{-b(1-p)}M_p(u; t)^p$$

$$\leqq Bt^{-b(1-p)+\alpha p},$$

where $-b(1-p) + \alpha p < 0$. By taking Fourier transforms of

$$u(x, t + \delta) = \int_{R^n} P(x - y, t)u(y, \delta)dy = \int_{R^n} P(x - y, \delta)u(y, t)dy$$

we have $e^{-2\pi|\xi|t}\hat{u}(\xi, \delta) = e^{-2\pi|\xi|\delta}\hat{u}(\xi, t)$. Hence

$$|\hat{u}_0(\xi)e^{-2\pi|\xi|t}| = |\hat{u}(\xi, \delta)e^{2\pi|\xi|(\delta - t)}|$$

$$= |\hat{u}(\xi, t)| \leqq \int_{R^n} |u(x, t)|dx \leqq Bt^\alpha.$$

If, in particular, $t = 1/|\xi|$ $(\xi \neq 0)$, then $|\hat{u}_0(\xi)| \leqq B|\xi|^{-\alpha}$ follows. For $\phi \in \mathscr{S}$

$$\int_{R^n} u(x, t)\phi(x)dx = \int_{R^n} \hat{u}(\xi, t)\hat{\bar{\phi}}(\xi)d\xi = \int_{R^n} \hat{u}_0(\xi)e^{-2\pi|\xi|t}\hat{\bar{\phi}}(\xi)d\xi,$$

where $\check{\phi}(x)=\phi(-x)$. Since $\hat{u}_0$ is continuous and $|\hat{u}_0(\xi)|\leqq B|\xi|^{-\alpha}(\xi\neq 0)$, Lebesgue's dominated convergence theorem can be applied and shows that the last integral tends to $\int_{R^n}\hat{u}_0\hat{\check{\phi}}d\xi$ as $t\to 0$. Thus

$$u(\cdot,0)(\phi)\equiv\lim_{t\to 0}\int_{R^n}u(x,t)\phi(x)dx=\int_{R^n}\hat{u}_0\hat{\check{\phi}}d\xi=\bar{\mathscr{F}}(\hat{u}_0)(\phi)$$

and hence $u(\cdot,0)=\bar{\mathscr{F}}(\hat{u}_0)$.

§7. Lipschitz spaces

Following Herz ([8]), we define the spaces $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ as follows.

DEFINITION C. (a) $\mathscr{H}^*$ is the set of Poisson integrals of functions in $\hat{\mathcal{O}}_0$.

(b) $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ is the closure of $\mathscr{H}^*$ in $\mathscr{H}(\alpha;\,p,\,q)$.

If $q=\infty$, then one needs only to take the closure in $\mathscr{A}(\alpha;\,p,\,\infty)$ on account of Theorem 5.1 (i), (ii).

In [8; §1], Herz defined the space $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ as the completion of $\hat{\mathcal{O}}_0$ for the norm $\Lambda^{\alpha}_{p,q}$ in case $\alpha>0$, and showed that it is continuously injected in $\mathscr{S}'_0=\mathscr{S}'/\mathscr{P}$. He also proved that R^{γ} is an isomorphism of $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ onto $\boldsymbol{\Lambda}^{\alpha+\gamma}_{p,q}$ if $\alpha>0$ and $\alpha+\gamma>0$ [8; Proposition 6.1], and he then defined $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ for $\alpha\leqq 0$ so that R^{γ} is an isomorphism of $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ onto $\boldsymbol{\Lambda}^{\alpha+\gamma}_{p,q}$ for all real α and γ [8; p. 316], where R^{γ} is the generalized Riesz potential (on $\mathscr{S}'/\mathscr{P}$) defined before Theorem 6.1. (Similar spaces have been studied by Peetre [12] in which interpolation properties are investigated.)

Before establishing the relation between $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ and $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$, we need

LEMMA 7.1. *Let α be a real number, $f\in\mathscr{S}_0$ and u be its Poisson integral. Then*

$$R^{\alpha}u(\cdot,t)=P_t*R^{\alpha}f\qquad(t>0).$$

PROOF. By an argument similar to the proof of Lemma 3.3 we can see that $u\in\mathscr{H}$. Hence $R^{\alpha}u$ makes sense for every real α. First we consider the case $0<\alpha<n$. It is easy to check that $f\in L(p,q)$ for any p, $1<p<\infty$, and q, $1\leqq q<\infty$. Hence by Theorem 3.1 (ii) $R^{\alpha}u(\cdot,t)=P_t*R^{\alpha}f$. If $\alpha\geqq n$, then take an integer $k>0$ so that $\alpha/k<n$. We apply k times the result in the case $0<\alpha<n$ and obtain the required relation. If $\alpha=-2m<0$, an even integer, then

$$R^{-2m}u(\cdot,t)=(-1)^m P_t*\Delta^m f=P_t*R^{-2m}f$$

(recall that $R^{-2}\phi=-\Delta\phi$ for $\phi\in\mathscr{S}_0$). Finally, let $0>\alpha\neq -2m$ and take an integer l so that $2l+\alpha>0$. Since $R^{\alpha}u=R^{2l+\alpha}(R^{-2l}u)$, the desired result follows from the two cases already considered.

REMARK. For any real number γ, R^{γ} is a linear bijection of $\mathscr{H}^*$ onto $\mathscr{H}^*$. This follows easily from Lemma 7.1 and the fact that R^{γ} is an isomorphism of $\hat{\mathcal{O}}_0$ onto $\hat{\mathcal{O}}_0$. Next, observe that, for $u\in\mathscr{H}^*$, $R^{\gamma}u$ defined by Definition **B** in § 3 coincides with the one defined in Remark to Theorem 5.1. Therefore, R^{γ} in the sense of Remark to Theorem 5.1 is an isometric isomorphism of $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ onto $\mathscr{H}\boldsymbol{\Lambda}^{\alpha+\gamma}_{p,q}$ for all real α and γ, and $1\leqq p,\ q\leqq\infty$.

LEMMA 7.2. *Let $1\leqq p,\ q\leqq\infty$, α be real, and u be the Poisson integral of $\psi\in\mathscr{S}_0$. Then $\mathscr{E}^{\alpha}_{p,q}(u)$ is equivalent to $\Lambda^{\alpha}_{p,q}(\psi)$.*

PROOF. The assertion is true if $0<\alpha<1$ by Lemma 4.5. In the general case, take γ so that $0<\alpha+\gamma<1$. It follows from Lemma 7.1 that $R^{\gamma}u$ is the Poisson integral of $R^{\gamma}\psi$. Hence, the lemma in case $0<\alpha<1$ implies that $\mathscr{E}^{\alpha+\gamma}_{p,q}(R^{\gamma}u)$ is equivalent to $\Lambda^{\alpha+\gamma}_{p,q}(R^{\gamma}\psi)$. Since $\mathscr{E}^{\alpha}_{p,q}(u)=\mathscr{E}^{\alpha+\gamma}_{p,q}(R^{\gamma}u)$ by Lemma 4.1, and $\Lambda^{\alpha+\gamma}_{p,q}$ $(R^{\gamma}\psi)$ is equivalent to $\Lambda^{\alpha}_{p,q}(\psi)$, $\mathscr{E}^{\alpha}_{p,q}(u)$ is equivalent to $\Lambda^{\alpha}_{p,q}(\psi)$.

Now we give

THEOREM 7.1. *If $1\leqq p,\ q\leqq\infty$ and α is real, then $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ is isomorphic to $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ (and also isomorphic to the space $\mathscr{T}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ of Johnson* [10; p. 310]). *Moreover, an isomorphism is given by the operation of taking boundary values of functions in $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$.*

PROOF. Let u be in $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$, and $\{u_j\}$ be in $\mathscr{H}^*$ such that $\mathscr{E}^{\alpha}_{p,q}(u_j-u)\to 0$ as $j\to\infty$. On account of Lemma 7.2 $\{u_j(\cdot,0)\}$ is a Cauchy sequence in $\boldsymbol{\Lambda}^{\alpha}_{p,q}$. Therefore there is an $f\in\boldsymbol{\Lambda}^{\alpha}_{p,q}$ such that $u_j(\cdot,0)\to f$ *in* $\boldsymbol{\Lambda}^{\alpha}_{p,q}$. Hence $u\mapsto f$ is a bounded linear map of $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ into $\boldsymbol{\Lambda}^{\alpha}_{p,q}$. Similarly we see that $f\mapsto u$ is also a bounded linear map of $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ into $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$ and we conclude that it is an isomorphism. On the other hand, Theorem 7.1 implies that $u(\cdot,0)$ exists and $u_j(\cdot,0)\to u(\cdot,0)$ in $\mathscr{S}'_0$. Since $u_j(\cdot,0)\to f$ in $\mathscr{S}'_0$, $u(\cdot,0)=f$ and the proof of the theorem is complete.

Next we prove

THEOREM 7.2. *Let $1\leqq p<\infty$, and α be a real number.*

(i) *If $1\leqq q<\infty$, then $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}=\mathscr{H}(\alpha;\ p,\ q)$.*

(ii) *If $0<\alpha<n/p$ and $1\leqq q\leqq\infty$, then a harmonic function u is the Poisson integral of an $f\in\boldsymbol{\Lambda}^{\alpha}_{p,q}$ if and only if $u\in\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,q}$.*

PROOF. First, assume that $0<\alpha<\min(1,\ n/p)$ and $u\in\mathscr{H}(\alpha;\ p,\ q)$. Theorem 5.2 and Lemma 4.5 imply that u is the Poisson integral of an $f\in L(r,\ q)$ $(1/r=1/p-\alpha/n)$, and $\Lambda^{\alpha}_{p,q}(f)\leqq B\mathscr{E}^{\alpha}_{p,q}(u)<\infty$. Therefore, it follows from

Theorem 0 of [8] that $f \in \boldsymbol{\Lambda}_{p,q}^{\alpha}$. By Remark 2 to Theorem 6.1 $f = u(\cdot, 0)$. Thus $u \in \mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha}$. To prove the result for general α, let γ be a real number such that $0 < \alpha + \gamma < \min(1, n/p)$, and let u be in $\mathscr{H}(\alpha; p, q)$. Then $R^{\gamma}u \in \mathscr{H}(\alpha+\gamma; p, q)$ by Lemma 5.2. Since $\mathscr{H}(\alpha+\gamma; p, q) = \mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha+\gamma}$ by what has been just proved, there is a sequence $\{v_j\}$ in $\mathscr{H}^*$ such that $\mathscr{E}_{p,q}^{\alpha+\gamma}(v_j - R^{\gamma}u) \to 0$. Setting $u_j = R^{-\gamma}v_j \in \mathscr{H}^*$, we obtain by Lemma 4.1 $\mathscr{E}_{p,q}^{\alpha}(u_j - u) = \mathscr{E}_{p,q}^{\alpha+\gamma}(R^{\gamma}(u_j - u)) = \mathscr{E}_{p,q}^{\alpha+\gamma}(v_j - R^{\gamma}u) \to 0$. Hence $u \in \mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha}$. The proof of (i) thus complete.

To prove (ii), assume that $0 < \alpha < n/p$ and $1 \leqq q \leqq \infty$. Theorem 7.1 implies that $u \in \mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha}$ if and only if $u(\cdot, 0) \in \boldsymbol{\Lambda}_{p,q}^{\alpha}$. It then follows from Theorem 5.2 that u is the Poisson integral of $u(\cdot, 0)$. The proof of the theorem is now complete.

In a manner similar to that in [10; §6], we shall use the space $\mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha}$ to give new proofs to many inclusion relations of Lipschitz spaces of Herz.

THEOREM 7.3. *Let $1 \leqq p \leqq \infty$, and α be a real number.*

(i) *If $1 \leqq q_1 \leqq q_2 \leqq \infty$, then $\mathscr{H}\boldsymbol{\Lambda}_{p,q_1}^{\alpha} \subset \mathscr{H}\boldsymbol{\Lambda}_{p,q_2}^{\alpha} \subset \mathscr{H}\boldsymbol{\Lambda}_{p,\infty}^{\alpha}$.*

(ii) *If $p < r \leqq \infty$, $\delta = (1/p - 1/r)n$, and $1 \leqq q \leqq \infty$, then*

$$\mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha} \subset \mathscr{H}\boldsymbol{\Lambda}_{r,q}^{\alpha-\delta}.$$

In each case the inclusion mapping is continuous.

COROLLARY (cf. [8; Theorem 3], [10; Theorem 8]). *The same results hold for the spaces $\boldsymbol{\Lambda}_{p,q}^{\alpha}$.*

PROOF OF THEOREM 7.3. (i) follows immediately from Theorem 5.1 (ii), whereas (ii) follows from Theorem 5.4.

THEOREM 7.4. *Let $\alpha > 0$ and $1/r = 1/p - \alpha/n > 0$.*

(i) *If $1 < p < \infty$, then*

$$H(p, q) \subset \mathscr{H}\boldsymbol{\Lambda}_{r,q}^{-\alpha} \qquad (1 < q < \infty).$$

(ii) *If $1 \leqq p < \infty$, then*

$$\mathscr{H}\boldsymbol{\Lambda}_{p,q}^{\alpha} \subset H(r, q) \qquad (1 \leqq q \leqq \infty).$$

In each case the inclusion mapping is continuous.

By taking boundary values of functions of the above families, we obtain

COROLLARY. (i) *Let p, r, q, and α be as in* (i) *of Theorem* 7.4. *Then* $L(p, q) \subset \boldsymbol{\Lambda}_{r,q}^{-\alpha}$(cf. [8; Theorem 5 and Proposition 7.1], [10; Theorem 9]).

(ii) *Let p, r, q and α be as in* (ii) *of Theorem* 7.4. *Then* $\boldsymbol{\Lambda}_{p,q}^{\alpha} \subset L(r, q)$ (cf. [8; Theorem 5]).

PROOF OF THEOREM 7.4. (i) is a consequence of Theorems 5.3 and 7.2, whereas (ii) follows from Corollary to Theorem 5.2.

THEOREM 7.5. (i) *If* $1<p\leqq 2$, *then* $H^p \subset \mathscr{H}\Lambda^0_{p,2}$.
(ii) *If* $2\leqq p<\infty$, *then* $\mathscr{H}\Lambda^0_{p,2}\subset H^p$.
In each case the inclusion mapping is continuous.

COROLLARY. (i) *If* $1<p\leqq 2$, *then* $L^p\subset \Lambda^0_{p,2}$ (cf. [8; Lemma 8.2], [7; Theorem 6]).
(ii) *If* $2\leqq p<\infty$, *then* $\Lambda^0_{p,2}\subset L^p$ (cf. [7; Theorem 6]).

PROOF OF THEOREM 7.5. Let u be in H^p. Note that $\mathscr{E}^0_{p,2}(u)$ is equivalent to $\|tD_{n+1}u\|_{p,2}$. For $p\leqq 2$, Minkowski's inequality gives

$$\|tD_{n+1}u\|_{p,2} \leqq \left(\int_{R^n}\left[\int_0^\infty |D_{n+1}u(x,t)|^2 t dt\right]^{p/2} dx\right)^{1/p}$$

$$\leqq \left(\int_{R^n}\left[\int_0^\infty |\nabla u(x,t)|^2 t dt\right]^{p/2} dx\right)^{1/p},$$

where $|\nabla u(x,t)|^2=\sum_{i=1}^{n+1}|D_i u(x,t)|^2$. Then, by [3; Theorem 9, Corrllary 3], we derive that $\mathscr{E}^0_{p,2}(u)\leqq B\|u\|_{H^p}$.

To prove (ii), let $u\in\mathscr{H}\Lambda^0_{p,2}$. Then, $u\in\mathscr{H}_{n/p}$ and by the corollary to Lemma 4.4 one has

$$\mathscr{E}^{-1,0}_{p,2}(D_j u)\leqq B\mathscr{E}^0_{p,2}(u) \qquad \text{for} \quad j=1,\dots,n.$$

It follows that

$$\left(\int_0^\infty\left[\int_{R^n}|\nabla u(x,t)|^p dx\right]^{2/p} t dt\right)^{1/2}\leqq B\mathscr{E}^0_{p,2}(u),$$

which, together with Minkowski's inequality, implies that

$$\left(\int_{R^n}\left[\int_0^\infty |\nabla u(x,t)|^2 t dt\right]^{p/2} dx\right)^{1/p}\leqq B\mathscr{E}^0_{p,2}(u).$$

Hence, by applying again the above quoted result of Fefferman and Stein, one concludes that $\|u\|_{H^p}\leqq B\mathscr{E}^0_{p,2}(u)$.

Before stating the next theorem, we need one more definition. Let u be a harmonic function in Ω. The function u is said to be in H^1 if there exist $n+1$ harmonic functions (in Ω) $u_1,\dots,u_n$, $u_{n+1}=u$ which satisfy

$$\begin{cases} \sum_{i=1}^{n+1} D_i u_i = 0, \\ D_i u_j = D_j u_i, \qquad i,j=1,\dots,n+1, \\ \sup_{t>0}\int_{R^n}|F(x,t)|dx<\infty, \end{cases}$$

where $|F(x, t)|^2 = \sum_{i=1}^{n+1} |u_i(x, t)|^2$. It is well-known that $u \in H^1$ if and only if it is the Poisson integral of an L^1-function f whose n Riesz transforms $R_1 f, \ldots, R_n f$ belong to L^1, where

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{|y| > \varepsilon} f(x-y) y_j |y|^{-n-1} dy,\ j = 1, \ldots, n \text{ and } x \in R^n$$

with $c_n = \pi^{-(n+1)/2} \Gamma((n+1)/2)$; it is a common practice to denote also by H^1 the set of boundary values of functions in H^1. (For properties of the space H^1 and related matters, see e.g., [13; Chap. VII, § 3]).

Finally we shall prove two results of some interest mentioned by Johnson.

THEOREM 7.6. (i) $\mathscr{H}\boldsymbol{\Lambda}^0_{1,1} \subset H^1$.
(ii) $H^1 \subset \mathscr{H}\boldsymbol{\Lambda}^0_{1,2}$.
In each case the inclusion mapping is continuous.

COROLLARY. (i) $\boldsymbol{\Lambda}^0_{1,1} \subset H^1$ (cf. [10; p. 314]).
(ii) $H^1 \subset \boldsymbol{\Lambda}^0_{1,2}$ (cf. [11; p. 135]).

PROOF OF THEOREM 7.6. Let u be in $\mathscr{H}\boldsymbol{\Lambda}^0_{1,1}$. Note that $\mathscr{E}^0_{1,1}(u)$ is then equivalent to $\|tD_{n+1}u\|_{1,1}$ by Lemma 4.3. Since $u \in \mathscr{H}_n$, one has

$$u(x, t) = -\int_t^\infty D_{n+1}u(x, s)ds \qquad \text{for} \quad (x, t) \in \Omega.$$

Hence it follows that $u^+(x) = \sup_{t>0}|u(x, t)| \leqq \int_0^\infty |D_{n+1}u(x, s)|ds$. Then, by using [3; Theorem 9, Corollary 2] and the above estimate, one derives that $\|u\|_{H^1} \leqq B\mathscr{E}^0_{1,1}(u)$. Thus (i) is proved.

To prove (ii), let u be in H^1. Then, by using the same method as in the proof of Theorem 7.5 (i), one obtains $\mathscr{E}^0_{1,2}(u) \leqq B\|u\|_{H^1}$.

Before going to the next section, we remark that in case p or q is ∞ and f is a measurable function with $\Lambda^\alpha_{p,q}(f) < \infty$, f belongs to the corresponding Lipschitz space of Herz provided some o-order at 0 and infinity is satisfied (cf. [8; Theorem 0]). Hence, it is reasonable to denote these spaces by $\boldsymbol{\lambda}^\alpha_{p,q}$; so we have $\boldsymbol{\lambda}^\alpha_{p,\infty}$ and $\boldsymbol{\lambda}^\alpha_{\infty,q}$, and we shall adopt these notations hereafter (cf. also [10; p. 311]). The spaces $\boldsymbol{\Lambda}^\alpha_{p,\infty}$ and $\boldsymbol{\Lambda}^\alpha_{\infty,q}$ are defined as follows:

For $1 \leqq p \leqq \infty$ and real α, define

$$\boldsymbol{\Lambda}^\alpha_{p,\infty} = \{u(\cdot, 0): u \in \mathscr{H}(\alpha; p, \infty)\}$$

with the same norm as for u. It then follows from Remark 1 to Theorem 6.1 and this definition that $\boldsymbol{\Lambda}^\alpha_{p,\infty}$ is isomorphic to $\mathscr{H}(\alpha; p, \infty)$. Hence, Theorem 5.1 (v) implies that the spaces $\boldsymbol{\Lambda}^\alpha_{p,\infty}$, where p is fixed and α varies, are isomorphic to one another.

The spaces $\Lambda^{\alpha}_{\infty,q}$ $(1 \leqq q \leqq \infty)$ are similarly defined.

§8. The dual of $\Lambda_{p,1}$

In [8; Proposition 7.1] Herz proved that the dual of $\Lambda^{\alpha}_{p,q}$ $(1<p,\ q<\infty)$ is (isomorphic to) $\Lambda^{-\alpha}_{p',q'}$. The extreme cases, i.e., when either p or q is 1 or ∞, seem to be not completely solved. On the other hand, Johnson also made a remark in [10; p. 315] about some uncertainty at these critical indexes. In this section and the next one, we shall be concerned with the duals of $\Lambda^{\alpha}_{p,1}$ $(1 \leqq p<\infty)$, $\Lambda^{\alpha}_{1,q}$ $(1<q<\infty)$ and $\lambda^{\alpha}_{p,\infty}$ $(1 \leqq p<\infty)$. We shall work instead with the spaces $\mathscr{H}(\alpha;\ p,\ q)$ and $\mathscr{H}\Lambda^{\alpha}_{p,q}$. Our main result in this section is the next theorem whose proof is modelled after [5; §§ 12–14]; however, in various computations we must take into account the behaviour of t at infinity. Hereafter E' stands for the topological dual of the normed vector space E.

THEOREM 8.1. *If α is a real number and $1 \leqq p<\infty$, then $\mathscr{H}(\alpha;\ p,\ 1)'$ is isomorphic to $\mathscr{H}(-\alpha;\ p',\ \infty)$.*

We shall prove the theorem through several lemmas.

LEMMA 8.1. *Let $1 \leqq p,\ q \leqq \infty$, α be a real number and u be a harmonic function in Ω such that $u^{(s)} \in \mathscr{H}(\alpha;\ p,\ q)$ for every $s>0$. For each $(y,\ s) \in \Omega$, let $u^{(y,s)}$ and $D_i u^{(y,s)} (i=1,\dots,\ n+1)$ be the functions with domain Ω given by*

$$u^{(y,s)}(x,\ t) = u(y-x,\ s+t),$$

$$D_i u^{(y,s)}(x,\ t) = D_i u(y-x,\ s+t),$$

and let

$$\phi_{i,h} = \frac{u^{(y+he_i,s)} - u^{(y,s)}}{h} - D_i u^{(y,s)} \qquad (i=1,\dots,\ n).$$

$$\psi_h = \frac{u^{(y,s+h)} - u^{(y,s)}}{h} - D_{n+1} u^{(y,s)},$$

where h is real (and $s+h>0$ in the case of ψ_h) and $\{e_1,\dots,\ e_n\}$ is the natural basis of R^n. Then $\phi_{i,h}$ and ψ_h tend to 0 in $\mathscr{H}(\alpha;\ p,\ q)$ as h tends to 0.

PROOF. Let $(x,\ t)$ be in Ω. Then

$$\begin{aligned}\phi_{i,h}(x,\ t) &= \frac{1}{h}\int_0^h [D_i u(y+\sigma e_i - x,\ s+t) - D_i u(y-x,\ s+t)]d\sigma \\ &= \frac{1}{h}\int_0^h \left\{\int_0^\sigma D_i^2 u(y+\tau e_i - x,\ s+t)d\tau\right\}d\sigma \\ &= \frac{1}{h}\int_0^h (h-\tau) D_i^2 u(y+\tau e_i - x,\ s+t)d\tau.\end{aligned}$$

Hence, it follows from Minkowski's inequality that

$$M_p(\phi_{i,h};\, t) \leqq \frac{1}{|h|}\left|\int_0^h (h-\tau)M_p(D_i^2 u;\, s+t)d\tau\right|$$
$$= (|h|/2)M_p(D_i^2 u;\, s+t) = (|h|/2)M_p(D_i^2 u^{(s)};\, t).$$

Similarly, if k is any positive integer, then

$$M_p(D_{n+1}^k \phi_{i,h};\, t) \leqq (|h|/2)M_p(D_{n+1}^k D_i^2 u;\, s+t)$$
$$= (|h|/2)M_p(D_{n+1}^k D_i^2 u^{(s)};\, t).$$

We shall treat the case $q<\infty$ only, because the case $q=\infty$ can be similarly treated. Let k be a positive integer greater than α. Then by Lemma 4.3 we have

$$[\mathscr{E}_{p,q}^{\alpha}(\phi_{i,h})]^q \leqq B\int_0^\infty [t^{k-\alpha}M_p(D_{n+1}^k\phi_{i,h};\, t)]^q t^{-1}dt$$
$$= B\int_0^1 + B\int_1^\infty \leqq B|h|^q\int_0^1 t^{q(k-\alpha)}M_p(D_{n+1}^k D_i^2 u;\, s)^q t^{-1}dt$$
$$+ B|h|^q\int_1^\infty t^{q(k-(\alpha-2))}M_p(D_{n+1}^k D_i^2 u^{(s)};\, t)^q t^{-1}dt$$
$$= I_1 + I_2.$$

Clearly $I_1=B|h|^q M_p(D_{n+1}^k D_i^2 u;\, s)^q$. Note that $D_i^2 u^{(s)}\in\mathscr{H}(\alpha-2;\, p,\, q)$ for every $s>0$ (see the proof of Lemma 4.4). Since $\mathscr{H}(\alpha-2;\, p,\, q)\subset\mathscr{H}(\alpha-2;\, p,\, \infty)$ by Theorem 5.1 (ii),

$$M_p(D_{n+1}^k D_i^2 u;\, s) = M_p(D_{n+1}^k D_i^2 u^{(s/2)};\, s/2) \leqq Bs^{\alpha-2-k}\mathscr{E}_{p,\infty}^{\alpha-2}(D_i^2 u^{(s/2)})<\infty$$

so that $I_1\to 0$ as $h\to 0$. Moreover, $I_2\leqq B|h|^q\mathscr{E}_{p,q}^{\alpha-2}(D_i^2 u^{(s)})\to 0$ as $h\to 0$. The desired result for $\phi_{i,h}$ is thus proved. The proof for ψ_h can be carried over in the same manner.

LEMMA 8.2. *Let u, p, q and α be as in Lemma 8.1. Let F be a continuous linear functional on $\mathscr{H}(\alpha;\, p,\, q)$ and let $w(y, s)=F(u^{(y,s)})$ for $(y, s)\in\Omega$. Then w is a harmonic function in Ω and is bounded on $R^n\times]c, \infty[$ for each $c>0$*, and *$w(\cdot, s)$ is uniformly continuous on R^n for each $s>0$.*

PROOF. An induction argument based on Lemma 8.1 shows that, for each multi-index $\kappa\in Z_{n+1}^+$, $D^\kappa w(y, s)=F(D^\kappa u^{(y,s)})$. Hence w is harmonic in Ω by the linearity of F. Furthermore, the continuity of F and Lemma 6.1 (i) imply that

$$|w(y, s)| \leqq \|F\|\mathscr{E}_{p,q}^{\alpha}(u^{(y,s)}) \leqq \|F\|\mathscr{E}_{p,q}^{\alpha}(u^{(c)})$$
$$\text{for all } y\in R^n \text{ and } s\geqq c>0.$$

Finally, $w(\cdot, s)=P(\cdot, s/2)*w(\cdot, s/2)$, and, since $P(\cdot, s/2)\in L^1$ and $w(\cdot, s/2)\in L^\infty$, $w(\cdot, s)$ is uniformly continuous on R^n for each $s>0$.

LEMMA 8.3. *Let $1\leqq p, q\leqq\infty$ and α be a positive real number. Let F be a continuous linear functional on $\mathscr{H}(\alpha; p, q)$ and let $u(y, s)=F(P^{(y,s)})$ for all $(y, s)\in\Omega$. Then for each bounded measurable function g on R^n with compact support and $s>0$*

$$\int_{R^n} u(y, s)g(y)dy = F(v^{(s)}),$$

where v is the Poisson integral of g. (Note that $P^{(y,s)}$ and $v^{(s)}$ belong to $\mathscr{H}(\alpha; p, q)$ by Lemma 6.2 (i).)

PROOF. Let K be the compact support of g. For each positive integer m, let $\{K_i^m\}$ be a finite family of mutually disjoint Borel sets $K_i^m\equiv K_i$ whose union equals K, each K_i having diameter less than $1/m$. Let $y_i\in K_i$ and

$$S_m(x, t) = \sum_i P(y_i - x, s + t)\int_{K_i} g(y)dy \quad \text{for} \quad (x, t)\in\Omega.$$

Then S_m is a finite linear combination of $P^{(y_i,s)}$ and hence belongs to $\mathscr{H}(\alpha; p, q)$. We assert that

$$(A) \qquad S_m \longrightarrow v^{(s)} \text{ in } \mathscr{H}(\alpha; p, q) \text{ as } m \longrightarrow \infty.$$

To prove the assertion (A), set

$$\begin{aligned} U_m(x, t) &= S_m(x, t) - v^{(s)}(x, t) = S_m(x, t) - v(x, s + t) \\ &= \sum_i \int_{K_i} [P(x - y_i, s + t) - P(x - y, s + t)]g(y)dy \quad \text{for}\,(x, t)\in\Omega. \end{aligned}$$

It follows from Minkowski's inequality that

$$M_p(U_m; t) \leqq \sum_i \int_{K_i} \|P(\cdot - y_i, s + t) - P(\cdot - y, s + t)\|_p |g(y)| dy.$$

Since $(x, t)\mapsto P(x-y_i, s+t)-P(x-y, s+t)$ is the Poisson integral of $x\mapsto P(x-y_i, s)-P(x-y, s)$, we have

$$\begin{aligned} \|P(\cdot - y_i, s + t) - P(\cdot - y, s +)\|_p &\leqq \|P(\cdot - y_i, s) - P(\cdot - y, s)\|_p \\ &= \|P(\cdot + y - y_i, s) - P(\cdot, s)\|_p. \end{aligned}$$

If $1\leqq p<\infty$, then $\|P(\cdot + y - y_i, s)-P(\cdot, s)\|_p\to 0$ as $m\to\infty$, while if $p=\infty$, the same statement follows from the uniform continuity of $P(\cdot, s)$ on R^n. Consequently, $M_p(U_m; t)$ tends to 0 uniformly in t as m tends to ∞. Now, let k be a positive integer greater than α. Since

$$D^k_{n+1}U_m(x, t) = \sum_i \int_{K_i} [D^k_{n+1}P(x - y_i, s + t) - D^k_{n+1}P(x - y, s + t)]g(y)dy,$$

an argument similar to the above shows that

$$(1) \qquad M_p(D^k_{n+1}U_m; t) \longrightarrow 0 \text{ uniformly in } t \text{ as } m \longrightarrow \infty.$$

Moreover, it follows from (P_6) and Minkowski's inequality that

$$(2) \qquad M_p(D^k_{n+1}U_m; t) \leqq B\|g\|_1(s + t)^{-n-k+n/p}.$$

If $q < \infty$, then

$$\|t^{k-\alpha}D^k_{n+1}U_m\|^q_{p,q} = \int_0^\infty [t^{k-\alpha}M_p(D^k_{n+1}U_m; t)]^q t^{-1}dt$$

$$= \int_0^\lambda + \int_\lambda^\infty = I_1 + I_2 \quad (\lambda > 0).$$

Inequality (2) now implies that $I_2^{1/q} \leq B\|g\|_1\lambda^{-\alpha-n/p'}$ so that I_2 is small if λ is large enough. On the other hand, it follows from (1) that I_1 is small if m is sufficiently large. Hence $\mathscr{E}^\alpha_{p,q}(U_m) \to 0$ as $m \to \infty$ on account of Lemma 4.3. Observe that trivial modification works for the case $q = \infty$. The proof of the assertion (A) is thus complete.

Next, the continuity of F and (A) imply that

$$F(v^{(s)}) = \lim_{m\to\infty} F(S_m) = \lim_{m\to\infty} \sum_i F(P^{(y_i,s)}) \int_{K_i} g(y)dy$$

$$= \lim_{m\to\infty} \sum_i u(y_i, s) \int_{K_i} g(y)dy.$$

Furthermore,

$$\left| \sum_i u(y_i, s) \int_{K_i} g(y)dy - \int_K u(y, s)g(y)dy \right|$$

$$\leqq \sum_i \int_{K_i} |u(y_i, s) - u(y, s)|\, |g(y)|dy \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

because $u(\cdot, s)$ is uniformly continuous on R^n by Lemma 8.2. The proof of the lemma is now complete.

Proof of Theorem 8.1. Since isomorphic spaces have isomorphic duals, Theorem 5.1 (v) enables us to consider only the case $\alpha > 0$. Let u be in $\mathscr{H}(-\alpha; p', \infty)$ and k be a positive integer such that $2k > \alpha$. Then it follows from Lemma

6.3 (i), (ii) that $F_u = \langle u, \cdot \rangle$ is a continuous linear functional on $\mathscr{H}(\alpha; p, 1)$ with $\|F_u\| \leqq B\mathscr{E}_{p',\infty}^{-\alpha}(u)$, and $F_u = 0$ implies $u = 0$. Conversely, assume that F belongs to $\mathscr{H}(\alpha; p, 1)'$. Let $u(y, s) = F(P^{(y,s)})$ for every $(y, s) \in \Omega$. It is harmonic in Ω on account of Lemma 8.2. Denote by v the Poisson integral of a function ψ in $\mathscr{D}$. Then by Lemmas 8.3 and 6.2 (i), we have

$$\left|\int_{R^n} u(y, s)\psi(y)dy\right| = |F(v^{(s)})|$$
$$\leqq \|F\| \mathscr{E}_{p,1}^{\alpha}(v^{(s)}) \leqq B\|F\| s^{-\alpha} \|\psi\|_p.$$

Hence

$$M_{p'}(u; s) = \sup_{\substack{\psi \in \mathscr{D} \\ \|\psi\|_p = 1}} \left|\int_{R^n} u(y, s)\psi(y)dy\right| \leqq B\|F\| s^{-\alpha},$$

which, together with Theorem 3.5 (i) ($q = \infty$), implies $u \in \mathscr{H}_{n/p'+\alpha}$ and together with Lemma 4.3, implies $\mathscr{E}_{p',\infty}^{-\alpha}(u) \leqq B\|F\|$. It follows that $u \in \mathscr{H}(-\alpha; p', \infty)$. We observe that $v \in \mathscr{H}(\alpha; p, 1)$ by Lemma 6.2 (iii), and that $v^{(s)} \to v$ in $\mathscr{H}(\alpha; p, 1)$ by Lemma 6.1 (ii). The continuity of F and Lemma 6.3 (ii) then give

$$F(v) = \lim_{s \to 0+} F(v^{(s)}) = \lim_{s \to 0+} \int_{R^n} u(y, s)\psi(y)dy = F_u(v).$$

By definition $\mathscr{H}^*$ is dense in $\mathscr{H}\Lambda_{p,1}^{\alpha}$ which is equal to $\mathscr{H}(\alpha; p, 1)$ by Theorem 7.2. Given $\psi \in \hat{\mathscr{O}}_0$ there exists a sequence $\{\psi_j\}$ in $\mathscr{D}$ which converges to ψ in the topology of $\mathscr{S}$. Then $P_t * \psi_j$ converges to $P_t * \psi$ in $\mathscr{H}(\alpha; p, 1)$ by Lemma 6.2 (iii). This shows that the set of all Poisson integrals of functions in $\mathscr{D}$ is dense in $\mathscr{H}(\alpha; p, 1)$. We conclude that $F = F_u$. Combining the above results, we derive that the mapping $u \mapsto F_u$ is an isomorphism of $\mathscr{H}(-\alpha; p', \infty)$ onto $\mathscr{H}(\alpha; p, 1)'$. Our theorem is now proved.

Next we obtain

THEOREM 8.2. *If α is a real number and $1 \leqq p < \infty$, then the dual of $\Lambda_{p,q}^{\alpha}$ is isomorphic to $\Lambda_{p',\infty}^{-\alpha}$.*

PROOF. Since the space $\Lambda_{p,1}^{\alpha}$ ($\Lambda_{p',\infty}^{-\alpha}$ resp.) is isomorphic to $\mathscr{H}(\alpha; p, 1)$ ($\mathscr{H}(-\alpha; p', \infty)$ resp.) by Theorems 7.1 and 7.2 (the definition of $\Lambda_{p',\infty}^{-\alpha}$ resp.), the desired result follows easily from Theorem 8.1.

§9. The duals of $\Lambda_{1,q}^{\alpha}$ and $\lambda_{p,\infty}^{\alpha}$

In this section we shall consider the many important cases left out in the preceding section. Namely, we shall investigate the duals of $\Lambda_{1,q}^{\alpha}(1 < q < \infty)$

and $\boldsymbol{\lambda}^{\alpha}_{p,\infty}(1 \leqq p < \infty)$. An essential tool in our proof is an operator T^{α} which is similar to a class of operators developed by Herz [8] although our case is basically simpler. Our main result is the following theorem:

THEOREM 9.1. *Let α be a real number.*
(i) *If $1<q<\infty$, then the dual of $\boldsymbol{\Lambda}^{\alpha}_{1,q}$ is isomorphic to $\boldsymbol{\Lambda}^{-\alpha}_{\infty,q'}$.*
(ii) *If $1\leqq p<\infty$, then the dual of $\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ is isomorphic to $\boldsymbol{\Lambda}^{-\alpha}_{p',1}$.*

As in the preceding section, we shall work instead with the spaces $\mathscr{H}(\alpha; p, q)$ and $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,\infty}$ respectively. In fact, Theorem 9.1 follows from the following theorem in the same way as Theorem 8.2 was derived from Theorem 8.1.

THEOREM 9.2. *Let α be a real number.*
(i) *If $1<q<\infty$, then $\mathscr{H}(\alpha; 1, q)'$ is isomorphic to $\mathscr{H}(-\alpha; \infty, q')$.*
(ii) *If $1\leqq p<\infty$, then $\mathscr{H}\boldsymbol{\Lambda}^{\alpha}_{p,\infty}{}'$ is isomorphic to $\mathscr{H}(-\alpha; p', 1)$.*

First, we shall prepare some lemmas. For $\alpha>0$ and $w\in C_K$, the space of all continuous functions with compact supports in Ω, define

$$T^{\alpha}(w)(x, s) = \int_0^{\infty}\int_{R^n} t^{\alpha-1}P(x-y, s+t)w(y, t)dydt$$

for every $(x, s)\in\Omega$. Note that $T^{\alpha}(w)$ is the Poisson integral of the function $y\mapsto \int_0^{\infty}\int_{R^n} t^{\alpha-1}P(y-z, t)w(z, t)dzdt$, which belongs to L^p for every $1\leqq p\leqq\infty$.

LEMMA 9.1. *Let* $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$ *and* $\alpha>0$. *For* $w\in C_K$, *let* $T^{\alpha}(w)$ *be defined as above. Then $T^{\alpha}(w)\in\mathscr{H}(\alpha; p, q)$ and there is a constant B, possibly depending on α, n, p and q, such that*

$$\mathscr{E}^{\alpha}_{p,q}(T^{\alpha}(w)) \leqq B\|w\|_{p,q}.$$

PROOF. Let k be a positive integer greater than α. As observed above $T^{\alpha}(w)$ is the Poisson integral of an L^1-function, and hence $T^{\alpha}(w)\in\mathscr{H}_n$ which is contained in $\mathscr{H}^*_{n/p-\alpha}$. Since

$$D^k_{n+1}T^{\alpha}(w)(x, s) = \int_0^{\infty}\int_{R^n} t^{\alpha-1}D^k_{n+1}P(y, s+t)w(x-y, t)dydt,$$

it follows from Minkowski's inequality and (P_6) that

$$M_p(D^k_{n+1}T^{\alpha}(w); s) \leqq B\int_0^{\infty} t^{\alpha}(s+t)^{-k}M_p(w; t)t^{-1}dt.$$

Hence

$$\begin{aligned}\|s^{k-\alpha}D^k_{n+1}T^{\alpha}(w)\|_{p,q} &\leqq B\left(\int_0^{\infty}\left[s^{k-\alpha}\int_0^{\infty} t^{\alpha}(s+t)^{-k}M_p(w; t)t^{-1}dt\right]^q s^{-1}ds\right)^{1/q}\\ &\leqq B(I_1+I_2),\end{aligned}$$

where

$$I_1 = \left(\int_0^\infty \left[s^{k-\alpha}\int_0^s t^\alpha (s+t)^{-k} M_p(w;\, t)t^{-1}dt\right]^q s^{-1}ds\right)^{1/q}$$

$$\leqq \left(\int_0^\infty \left[s^{-\alpha}\int_0^s t^\alpha M_p(w;\, t)t^{-1}dt\right]^q s^{-1}ds\right)^{1/q} \leqq B\|w\|_{p,q}$$

and

$$I_2 = \left(\int_0^\infty \left[s^{k-\alpha}\int_s^\infty t^\alpha (s+t)^{-k} M_p(w;\, t)t^{-1}dt\right]^q s^{-1}ds\right)^{1/q}$$

$$\leqq \left(\int_0^\infty \left[s^{k-\alpha}\int_s^\infty t^{\alpha-k} M_p(w;\, t)t^{-1}dt\right]^q s^{-1}ds\right)^{1/q} \leqq B\|w\|_{p,q}$$

by Hardy's inequality. Note that trivial modification works for the case $q=\infty$. Therefore, the lemma follows from Lemma 4.3.

REMARK. Note that, at least in the case $1\leqq p<\infty$ and $0<\alpha<n/p$, T^α is a bounded linear operator from $L^{(p,q)}$, the Banach space of all measurable functions w on Ω with $\|w\|_{p,q}<\infty$, into $\mathscr{H}(\alpha;\, p,\, q)$; the proof is similar to the above lemma. (We refer to [1] for properties of the spaces $L^{(p,q)}$.) Also in this case, if we define

$$T_{-\alpha}(u)(x,\, t) = \frac{(-1)^k 2^k}{\Gamma(k)}\, t^{k-\alpha} D_{n+1}^k u(x,\, t)$$

for all $u\in\mathscr{H}(\alpha;\, p,\, q)$, where k is a positive integer greater than α, then $T_{-\alpha}$ is a bounded linear operator from $\mathscr{H}(\alpha;\, p,\, q)$ into $L^{(p,q)}$ by Lemma 4.3, and $T^\alpha T_{-\alpha} = I$, the identity operator on $L^{(p,q)}$. Each of the operators T^α and $T_{-\alpha}$ is a modified version of some class of operators constructed by Herz [8; Propositions 5.1 and 5.2].

LEMMA 9.2. *Let $1\leqq p<\infty$ and α be a positive number. If v is the Poisson integral of an $f\in L^p$ and $u\in\mathscr{H}(-\alpha;\, p',\, \infty)$, then*

$$<u^{(s)},\, v^{(t)}> = \int_{R^n} u(y,\, s)v(y,\, t)dy$$

for all positive numbers s and t.

PROOF. Let $\{\psi_j\}$ be a sequence in $\mathscr{S}$ which converges to f in L^p. Then $v_j^{(t)}\to v^{(t)}$ in $\mathscr{H}(\alpha;\, p,\, 1)$ for each $t>0$ by Lemma 6.2 (i), where v_j is the Poisson integral of ψ_j. However, an easy application of Fubini's theorem, the semigroup formula and Theorem 3.2 imply as in the proof of Lemma 6.3 (ii) that

$$<u^{(s)},\, v_j^{(t)}> = \int_{R^n} u(y,\, s)v_j(y,\, t)dy,$$

which yields the required formula after making use of the continuity of $<\cdot,\cdot>$ and the fact that $v_j(\cdot, t)\to v(\cdot, t)$ in L^p.

LEMMA 9.3. *Let $1\leqq p, q\leqq\infty$, α be a real number and u be a harmonic function in Ω such that $u^{(s)}\in\mathscr{H}(\alpha; p, q)$ for every positive real s and $\sup_{0<s\leqq 1}\mathscr{E}^{\alpha}_{p,q}(u^{(s)})<\infty$. Then $u\in\mathscr{H}(\alpha; p, q)$ and $\mathscr{E}^{\alpha}_{p,q}(u)=\lim_{s\to 0+}\mathscr{E}^{\alpha}_{p,q}(u^{(s)})$.*

PROOF. It is obvious that $u\in\mathscr{H}^*_{n/p-\alpha}$ and $t\mapsto M_p(R^{-\alpha-1}u^{(s)}; t)=M_p(R^{-\alpha-1}u; s+t)$ is non-increasing on $]0, \infty[$ for each $s>0$. It is continuous in virtue of Lemma 1.1. If $q<\infty$, then the lemma follows from Lebesgue's monotone convergence theorem. If $q=\infty$, then

$$\begin{aligned}\lim_{s\to 0+}\mathscr{E}^{\alpha}_{p,\infty}(u^{(s)}) &= \lim_{s\to 0+}\{\sup_{t>0} tM_p(R^{-\alpha-1}u; s+t)\}\\ &= \sup_{s>0}\{\sup_{t>0} tM_p(R^{-\alpha-1}u; s+t)\}\\ &= \sup_{t>0}\{\sup_{s>0} tM_p(R^{-\alpha-1}u; s+t)\}\\ &= \sup_{t>0} tM_p(R^{-\alpha-1}u; t) = \mathscr{E}^{\alpha}_{p,\infty}(u).\end{aligned}$$

LEMMA 9.4. *Let $1\leqq p<\infty$, $1<q\leqq\infty$ and $\alpha>0$. If u is a function in $\mathscr{H}(-\alpha; p', \infty)$ such that*

$$\sup_{\substack{v\in\mathscr{X}^*\\ \mathscr{E}^{\alpha}_{p,q}(v)\leqq 1}} |<u^{(s)}, v>|\leqq C<\infty \qquad \text{for all } s>0,$$

where $<\cdot,\cdot>$ is defined as in Lemma 6.3, then $u\in\mathscr{H}(-\alpha; p', q')$ and $\mathscr{E}^{-\alpha}_{p',q'}(u)\leqq BC$.

PROOF. Let s be a positive number. First, we prove that

$$\|t^{\alpha}u^{(s)}\|_{p',q'} = \sup_{\substack{w\in C_K\\ \|w\|_{p,q}\leqq 1}}\left|\int_0^{\infty}\int_{R^n} t^{\alpha}u(x, s+t)w(x, t)dx\frac{dt}{t}\right|.$$

Denote by $N(u)$ the right hand side of the above equality. On account of [1; §2, Theorem 1] we need only to see that $\|t^{\alpha}u^{(s)}\|_{p',q'}\leqq N(u)$. For each $\lambda>1$, set $E_{\lambda}=\{|x|<\lambda\}\times]1/\lambda, \lambda[$ and

$$u(\lambda)(x, t)=\begin{cases} t^{\alpha}u(x, s+t) & \text{if } (x, t)\in E_{\lambda}\\ 0 & \text{otherwise.}\end{cases}$$

Since $\|t^{\alpha}u^{(s)}\|_{p',q'}=\sup_{\lambda>1}\|u(\lambda)\|_{p',q'}$, it is sufficient to show that $\|u(\lambda)\|_{p',q'}\leqq N(u)$ for every $\lambda>1$. Fix a $\lambda>1$. For any $\varepsilon>0$, by [1; §2, Theorem 1] there exists a measurable function w such that $\|w\|_{p,q}\leqq 1$ and

$$\|u(\lambda)\|_{p',q'} \leqq \left| \iint_{E_\lambda} t^\alpha u(x, s+t) w(x, t) dx \frac{dt}{t} \right| + \varepsilon.$$

Furthermore, we may assume that the support of w is a compact set $C \subset E_\lambda$. For each $0<\delta<1$, let $\psi(\delta)$ be a non-negative C^∞-function with compact support in $\{|x|<\delta\} \times]1-\delta, 1+\delta[$ and $\|\psi(\delta)\|_{1,1}=1$, and define

$$\begin{aligned} w(\delta)(x, t) &= \int_0^\infty \int_{R^n} w(x-y, t/\tau)\psi(\delta)(y, \tau) dy \frac{d\tau}{\tau} \\ &= \int_0^\infty \int_{R^n} w(y, \tau)\psi(\delta)(x-y, t/\tau) dy \frac{d\tau}{\tau} \qquad (x, t) \in \Omega. \end{aligned}$$

Then $w(\delta)$ is a C^∞-function with compact support in E_λ for small δ. Further, $\|w(\delta)\|_{p,q} \leqq \|w\|_{p,q} \|\psi(\delta)\|_{1,1} \leqq 1$ by Minkowski's inequality. On the other hand, if δ is small enough, we have

$$\begin{aligned} \int_0^\infty \int_{R^n} t^\alpha u(x, s+t) w(\delta)(x, t) dx \frac{dt}{t} &= \int_0^\infty \int_{R^n} u(\lambda)(x, t) w(\delta)(x, t) dx \frac{dt}{t} \\ = \iint_C w(y, \tau) \left\{ \int_{1-\delta<t/\tau<1+\delta} \int_{|x-y|<\delta} u(\lambda)(x, t)\psi(\delta)(x-y, t/\tau) dx \frac{dt}{t} \right\} dy \frac{d\tau}{\tau}. \end{aligned}$$

The last term tends to

$$\iint_C u(\lambda)(y, \tau) w(y, \tau) dy \frac{d\tau}{\tau} = \iint_{E_\lambda} u(\lambda)(y, \tau) w(y, \tau) dy \frac{d\tau}{\tau}$$

as $\delta \to 0$, because

$$\int_{1-\delta<t/\tau<1+\delta} \int_{|x-y|<\delta} u(\lambda)(x, t)\psi(\delta)(x-y, t/\tau) dx \frac{dt}{t} \longrightarrow u(\lambda)(y, \tau)$$

uniformly in $(y, \tau) \in C$ as $\delta \to 0$ by the uniform continuity of $u(\lambda)$ on any compact set of E_λ. Consequently,

$$\|u(\lambda)\|_{p',q'} \leqq \left| \int_0^\infty \int_{R^n} t^\alpha u(x, s+t) w(\delta)(x, t) dx \frac{dt}{t} \right| + 2\varepsilon$$

for sufficiently small δ. Hence $\|u(\lambda)\|_{p',q'} \leqq N(u)$, and the required equality follows.

Now, for each $w \in C_K$ the semigroup formula, Fubini's theorem and Lemma 9.2 imply that

$$\begin{aligned} \int_0^\infty \int_{R^n} t^\alpha u(x, s+t) w(x, t) dx \frac{dt}{t} &= \int_{R^n} u(y, s/2) T^\alpha(w)(y, s/2) dy \\ &= \langle u^{(s/2)}, T^\alpha(w)^{(s/2)} \rangle. \end{aligned}$$

By applying Fubini's theorem and the semigroup formula we see that the last quantity is equal to $<u^{(s)}, T^{\alpha}(w)>$. Hence it follows that

$$\|t^{\alpha}u^{(s)}\|_{p',q'} = \sup_{\substack{w\in C_K \\ \|w\|_{p,q}\leq 1}} |<u^{(s)}, T^{\alpha}(w)>|.$$

Fix a $w\in C_K$ such that $\|w\|_{p,q}\leqq 1$ and let $\{v_j\}$ be a sequence in $\mathscr{H}^*$ which converges to $T^{\alpha}(w)$ in $\mathscr{H}\Lambda^{\alpha}_{p,1}$; note that this is equal to $\mathscr{H}(\alpha; p, 1)$ by Theorem 7.2. Then $<u^{(s)}, v_j>\to<u^{(s)}, T^{\alpha}(w)>$ by the continuity of $<\cdot,\cdot>$. Since $\mathscr{E}^{\alpha}_{p,q}(\cdot)\leqq B\mathscr{E}^{\alpha}_{p,1}(\cdot)$ for any $q\geqq 1$ by Theorem 3.4, Lemma 9.1 implies that

$$\begin{aligned}\mathscr{E}^{\alpha}_{p,q}(v_j) &\leqq \mathscr{E}^{\alpha}_{p,q}(v_j - T^{\alpha}(w)) + \mathscr{E}^{\alpha}_{p,q}(T^{\alpha}(w))\\ &\leqq B \qquad \text{for large } j.\end{aligned}$$

Therefore, we derive that $|<u^{(s)}, T^{\alpha}(w)|>\leqq B\sup_{v\in\mathscr{H}^*,\mathscr{E}^{\alpha}_{p,q}(v)\leqq 1}|<u^{(s)}, v>|$. Since w is arbitrary, we obtain

$$\|t^{\alpha}u^{(s)}\|_{p',q'} \leqq B \sup_{\substack{v\in\mathscr{H}^* \\ \mathscr{E}^{\alpha}_{p,q}(v)\leqq 1}} |<u^{(s)}, v>| \leqq BC.$$

Hence, the desired result follows from Lemmas 4.3 and 9.3.

PROOF OF THEOREM 9.2. By a similar reason to that in the proof of Theorem 8.1, we may assume that $\alpha>0$. We shall prove (ii) first. For $u\in\mathscr{H}(-\alpha; p', 1)$ and $v\in\mathscr{H}\Lambda^{\alpha}_{p,\infty}$ define $<u, v>$ as in Lemma 6.3. Then $<\cdot,\cdot>$ is a continuous bilinear form on $\mathscr{H}(-\alpha; p', 1)\times\mathscr{H}\Lambda^{\alpha}_{p,\infty}$, and $<u, v>=0$ for all $v\in\mathscr{H}\Lambda^{\alpha}_{p,\infty}$ implies that $u\equiv 0$ by Lemma 6.3 (ii). Conversely, let F be a bounded linear functional on $\mathscr{H}\Lambda^{\alpha}_{p,\infty}$. Set $u(y, s)=F(P^{(y,s)})$ for all $(y, s)\in\Omega$. By Theorem 7.2 $\mathscr{H}\Lambda^{\alpha}_{p,1}=\mathscr{H}(\alpha; p, 1)$ and by Theorem 7.3 (i) $\mathscr{H}\Lambda^{\alpha}_{p,1}\subset\mathscr{H}\Lambda^{\alpha}_{p,\infty}$ so that $\mathscr{H}(\alpha; p, 1)\subset\mathscr{H}\Lambda^{\alpha}_{p,\infty}$. Hence F may be considered as a bounded linear functional on $\mathscr{H}(\alpha; p, 1)$. Therefore, Lemma 8.3 and the proof of Theorem 8.1 imply that $u\in\mathscr{H}(-\alpha; p', \infty)$ and

$$F(v^{(s)}) = \int_{R^n} u(y, s)\psi(y)dy$$

for every v which is the Poisson of a $\psi\in\mathscr{D}$ and $s>0$. By Lemma 6.3 (ii) the right hand side is equal to $<u^{(s)}, v>$. Given $\phi\in\hat{\mathscr{O}}_0$ there exists a sequence $\{\phi_j\}$ in $\mathscr{D}$ which converges to ϕ in $\mathscr{S}$. Then $P_t*\phi_j\to P_t*\phi$ in $\mathscr{H}(\alpha; p, 1)$ by Lemma 6.2 (iii). Lemma 6.1 (i) implies that $(P_t*\phi_j)^{(s)}\to(P_t*\phi)^{(s)}$ in $\mathscr{H}(\alpha; p, 1)$, and $F((P_t*\phi_j)^{(s)})\to F((P_t*\phi)^{(s)})$ on account of the continuity of F. Since

$$F((P_t*\phi_j)^{(s)}) = <u^{(s)}, P_t*\phi_j> \longrightarrow <u^{(s)}, P_t*\phi>,$$

$F(v^{(s)})=<u^{(s)}, v>$ for all $v\in\mathscr{H}^*$ and $s>0$. Consequently,

$$| < u^{(s)}, v > | \leqq \|F\| \mathscr{E}^{\alpha}_{p,\infty}(v^{(s)}) \leqq \|F\| \mathscr{E}^{\alpha}_{p,\infty}(v)$$

by Lemma 6.1 (i) and

$$\sup_{\substack{v \in \mathscr{H}^* \\ \mathscr{E}^{\alpha}_{p,\infty}(v) \leqq 1}} | < u^{(s)}, v > | \leqq \|F\| < \infty \qquad \text{for all} \quad s > 0,$$

which, by Lemma 9.4, implies that $u \in \mathscr{H}(-\alpha; p', 1)$ and $\mathscr{E}^{-\alpha}_{p',1}(u) \leqq B\|F\|$. Note that $v \in \mathscr{H}^* \subset \mathscr{H}\Lambda^{\alpha}_{p,\infty}$, and $v^{(s)}(\cdot, t) = P_t*(P_s*f)$. By taking the Fourier transform of P_s*f we see that $P_s*f \in \hat{\mathcal{O}}_0$. Hence $v^{(s)} \in \mathscr{H}^*$. By Lemma 6.1 (ii) $u^{(s)} \to u$ in $\mathscr{H}(-\alpha; p, 1)$ and $v^{(s)} \to v$ in $\mathscr{H}\Lambda^{\alpha}_{p,\infty}$. Hence $F(v) = <u, v>$ for every $v \in \mathscr{H}^*$. Since $\mathscr{H}^*$ is dense in $\mathscr{H}\Lambda^{\alpha}_{p,\infty}$, $F(v) = <u, v>$ for every $v \in \mathscr{H}\Lambda^{\alpha}_{p,\infty}$. Thus we have shown that $u \mapsto F_u = <u, \cdot>$ is an isomorphism of $\mathscr{H}(-\alpha; p', 1)$ onto $\mathscr{H}\Lambda^{\alpha}_{p,\infty}{}'$.

Finally, the assertion (i) follows if we replace $\mathscr{H}\Lambda^{\alpha}_{p,\infty}$ and $\mathscr{H}(-\alpha; p', 1)$ in the above proof by $\mathscr{H}(\alpha; 1, q)$ and $\mathscr{H}(-\alpha; \infty, q')$ respectively.

REMARK. We have removed some restriction on p and q imposed by Flett [5]; Flett considered only the duals of $\Lambda(\alpha; p, 1)(1 \leqq p < \infty)$ and $\lambda(\alpha; p, \infty)$ $(1 < p < \infty)$. Further, he also noted that his mothod does not permit to say anything about the dual of $\Lambda(\alpha; 1, q)$ $(1 < q < \infty)$. The method used in this section can be adopted to treat the spaces $\Lambda(\alpha; 1, q)$ and $\lambda(\alpha; 1, \infty)$ as well.

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