HARMONIC MAPPINGS IN THE PLANE

PETER DUREN
University of Michigan

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1

Preliminaries

1.1. Harmonic Mappings

A real-valued function $u(x, y)$ is *harmonic* if it satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A one-to-one mapping $u = u(x, y)$, $v = v(x, y)$ from a region D in the xy-plane to a region Ω in the uv-plane is a *harmonic mapping* if the two coordinate functions are harmonic. It is convenient to use the complex notation $z = x + iy$, $w = u + iv$ and to write $w = f(z) = u(z) + iv(z)$. Thus a complex-valued harmonic function is a harmonic mapping of a domain $D \subset \mathbb{C}$ if and only if it is *univalent* (or one-to-one) in D, that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$. Here $\mathbb{C}$ denotes the complex plane.

It must be emphasized that in this book the term "harmonic mapping" will always mean a *univalent* complex-valued harmonic function, except for occasional discussion of higher-dimensional analogues. Some writers use the term in a broader sense that does not require univalence.

A complex-valued function $f = u + iv$ is *analytic* in a domain $D \subset \mathbb{C}$ if it has a derivative $f'(z)$ at each point $z \in D$. The *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are an immediate consequence. Conversely, if f has continuous first partial derivatives and the Cauchy–Riemann equations hold, then f is analytic in D. (See Ahlfors [3] for information about analytic functions.) It follows from the Cauchy–Riemann equations (and from the existence of higher derivatives) that every analytic function is harmonic. A pair of functions (u, v) that satisfy the Cauchy–Riemann equations is said to be a *conjugate pair*, and v is called the *harmonic conjugate* of u. Hence, $-u$ is the harmonic conjugate of v. Strictly speaking, the conjugate function is determined locally only up to an additive constant. In a multiply connected domain the conjugate function need not be single-valued.

An analytic univalent function is called a *conformal mapping* because it preserves angles between curves. In fact, this angle-preserving property characterizes analytic functions among all functions with continuous first partial derivatives and nonvanishing Jacobians, because it implies that the Cauchy–Riemann equations are satisfied.

The object of this book is to study complex-valued harmonic univalent functions whose real and imaginary parts are not necessarily conjugate. As soon as analyticity is abandoned, serious obstacles arise. Analytic functions are preserved under composition, but harmonic functions are not. A harmonic function of an analytic function is harmonic, but an analytic function of a harmonic function need not be harmonic. The analytic functions form an algebra, but the harmonic functions do not. Even the square or the reciprocal of a harmonic function need not be harmonic. The inverse of a harmonic mapping need not be harmonic. The boundary behavior of harmonic mappings may be much more complicated than that of conformal mappings. It will be seen, nevertheless, that much of the classical theory of conformal mappings can be carried over in some way to harmonic mappings.

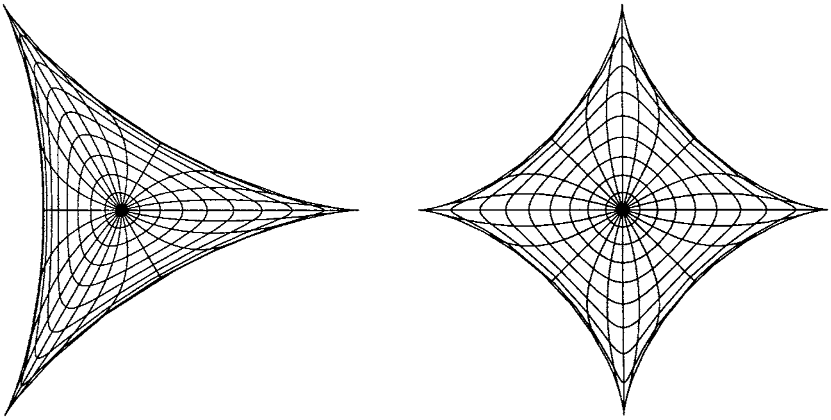
The *Jacobian* of a function $f = u + iv$ is

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x,$$

where the subscripts indicate partial derivatives. If f is analytic, its Jacobian takes the form $J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2$. For analytic functions f, it is a classical result that $J_f(z) \neq 0$ if and only if f is locally univalent at z. Hans Lewy showed in 1936 that this remains true for harmonic mappings. A relatively simple proof will be given in Chapter 2. In view of Lewy's theorem, harmonic mappings are either *sense-preserving* (or *orientation-preserving*) with $J_f(z) > 0$, or *sense-reversing* with $J_f(z) < 0$ throughout the domain D where f is univalent. If f is sense-preserving, then $\overline{f}$ is sense-reversing. Conformal mappings are sense-preserving.

The simplest examples of harmonic mappings that need not be conformal are the *affine mappings* $f(z) = \alpha z + \gamma + \beta\overline{z}$ with $|\alpha| \neq |\beta|$. Affine mappings with $\gamma = 0$ are *linear mappings*. It is important to observe that every composition of a harmonic mapping with an affine mapping is again a harmonic mapping: if f is harmonic, then so is $\alpha f + \gamma + \beta\overline{f}$.

Another important example is the function $f(z) = z + \frac{1}{2}\overline{z}^2$, which maps the open unit disk $\mathbb{D}$ onto the region inside a hypocycloid of three cusps inscribed in the circle $|w| = \frac{3}{2}$. To verify its univalence, suppose $f(z_1) = f(z_2)$



(a) $n = 2$ (b) $n = 3$

Figure 1.1. Image of mapping $f(z) = z + \frac{1}{n}\overline{z}^n$

for some points z_1 and z_2 in $\mathbb{D}$. Then

$$(\overline{z_1} + \overline{z_2})\,(\overline{z_1} - \overline{z_2}) = 2\,(z_2 - z_1).$$

But this is impossible unless $z_1 = z_2$, because $|z_1 + z_2| < 2$. The same argument shows that $f(z) = z + \frac{1}{n}\overline{z}^n$ is univalent for each $n \geq 2$.

The image of the disk under the mapping $f(z) = z + \frac{1}{n}\overline{z}^n$, as computed by *Mathematica*, is displayed graphically in Figure 1.1 for the cases $n = 2$ and 3. The curves in the figure are images of equally spaced concentric circles and radial segments. In general, the image of the disk under this mapping is bounded by a hypocycloid of $n + 1$ cusps inscribed in the circle $|w| = (n + 1)/n$.

In studying harmonic mappings of simply connected domains in the plane, there is no essential loss of generality in taking the unit disk as the domain of definition. To be more precise, suppose that f is a harmonic mapping of some simply connected domain $\Delta \subset \mathbb{C}$ onto a domain Ω, with $\Delta \neq \mathbb{C}$. The Riemann mapping theorem ensures the existence of a conformal mapping φ of $\mathbb{D}$ onto Δ. Thus the composition $F = f \circ \varphi$ is a harmonic mapping of $\mathbb{D}$ onto Ω. The original mapping is $f = F \circ \psi$, where ψ is the inverse of φ.

1.2. Some Basic Facts

Two simple differential operators appear commonly in complex analysis and are very convenient. They are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

where $z = x + iy$. For a complex-valued function $f(z)$, the equation $\partial f/\partial \overline{z} = 0$ is just another way of writing the Cauchy–Riemann equations. A direct calculation shows that the Laplacian of f is

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}.$$

Thus for functions f with continuous second partial derivatives, is is clear that f is harmonic if and only if $\partial f/\partial z$ is analytic. If f is analytic, then $\partial f/\partial z = f'(z)$, the ordinary derivative.

The operators $\partial/\partial z$ and $\partial/\partial \overline{z}$ are linear, and they have the usual properties of differential operators. For instance, the product and quotient rules hold:

$$\frac{\partial}{\partial z}(fg) = f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z},$$
$$\frac{\partial}{\partial z}\left(\frac{f}{g}\right) = g^{-2}\left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right),$$

and similarly for $\partial/\partial \overline{z}$. The special property

$$\left(\frac{\partial f}{\partial z}\right)^{-} = \frac{\partial \overline{f}}{\partial \overline{z}}$$

connects the two derivatives. The differential

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

can be written as

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z},$$

thus motivating the notation $\partial/\partial z$ and $\partial/\partial \overline{z}$. The subscript notation $f_z = \partial f/\partial z$ and $f_{\overline{z}} = \partial f/\partial \overline{z}$ is often more convenient.

The chain rule for differentiation of composite functions can now be derived (formally). If $w = f(z)$ and $z = g(\zeta)$, then $w = h(\zeta)$, where $h = f \circ g$. Writing

$$dz = \frac{\partial g}{\partial \zeta}\,d\zeta + \frac{\partial g}{\partial \overline{\zeta}}\,d\overline{\zeta}$$

and

$$d\overline{z} = \frac{\partial \overline{g}}{\partial \zeta}\,d\zeta + \frac{\partial \overline{g}}{\partial \overline{\zeta}}\,d\overline{\zeta} = \overline{\frac{\partial g}{\partial \overline{\zeta}}}\,d\zeta + \overline{\frac{\partial g}{\partial \zeta}}\,d\overline{\zeta},$$

one finds after substitution that

$$dh = \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial \zeta}\,d\zeta + \frac{\partial g}{\partial \bar{\zeta}}\,d\bar{\zeta}\right) + \frac{\partial f}{\partial \bar{z}}\left(\frac{\overline{\partial g}}{\partial \bar{\zeta}}\,d\zeta + \frac{\overline{\partial g}}{\partial \zeta}\,d\bar{\zeta}\right).$$

Thus,

$$\frac{\partial h}{\partial \zeta} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \zeta} + \frac{\partial f}{\partial \bar{z}}\frac{\overline{\partial g}}{\overline{\partial \bar{\zeta}}} \qquad \text{and} \qquad \frac{\partial h}{\partial \bar{\zeta}} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{\zeta}} + \frac{\partial f}{\partial \bar{z}}\frac{\overline{\partial g}}{\overline{\partial \zeta}}.$$

The Jacobian of a function $f = u + iv$ can be expressed as

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2.$$

Consequently, f is locally univalent and sense-preserving wherever $|f_z(z)| > |f_{\bar{z}}(z)|$, and sense-reversing where $|f_z(z)| < |f_{\bar{z}}(z)|$. Note that $f_z(z) \neq 0$ wherever $J_f(z) > 0$. For sense-preserving mappings $w = f(z)$ one sees that

$$(|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz|.$$

These sharp inequalities have the geometric interpretation that f maps an infinitesimal circle onto an infinitesimal ellipse with

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

as the ratio of the major and minor axes. The quantity $D_f = D_f(z)$ is called the *dilatation* of f at the point z. Clearly, $1 \leq D_f(z) < \infty$. A sense-preserving homeomorphism f is said to be *quasiconformal*, or *K-quasiconformal*, if $D_f(z) \leq K$ throughout the given region, where K is a constant and $1 \leq K < \infty$. The 1-quasiconformal mappings are simply the conformal mappings.

It is often more convenient to consider the ratio $\mu_f = f_{\bar{z}}/f_z$, called the *complex dilatation* of f. Thus, $0 \leq |\mu_f(z)| < 1$ if f is sense-preserving. It may be observed that $D_f(z) \leq K$ if and only if $|\mu_f(z)| \leq (K-1)/(K+1)$. It follows that a sense-preserving homeomorphism is quasiconformal if and only if its complex dilatation μ_f is bounded away from 1 in the given region: $|\mu_f(z)| \leq k < 1$. The mapping f is conformal if and only if $\mu_f = 0$. For the general theory of quasiconformal mappings the books by Lehto and Virtanen [1] and Ahlfors [1] are recommended.

In the theory of harmonic mappings, the quantity $\nu_f = \overline{f_{\bar{z}}}/f_z$, known as the *second complex dilatation*, turns out to be more relevant than the first complex dilatation μ_f. Since $|\nu_f| = |\mu_f|$, it is again clear that f is quasiconformal if and only if $|\nu_f(z)| \leq k < 1$.

Now let f be a complex-valued function defined in a domain $D \subset \mathbb{C}$ having continuous second partial derivatives. Suppose that f is locally univalent in D, with Jacobian $J_f(z) > 0$. Let $\omega = \nu_f = \overline{f_{\bar{z}}}/f_z$ be its second complex dilatation; then $|\omega(z)| < 1$ in D. Differentiating the equation $\overline{f_{\bar{z}}} = \omega f_z$ with respect to $\bar{z}$, one finds

$$\overline{f_{\bar{z}z}} = f_{z\bar{z}}\omega + f_z\omega_{\bar{z}}.$$

Now if f is harmonic in D, then $f_{z\bar{z}} = \frac{1}{4}\Delta f = 0$ there. Thus it follows that $\omega_{\bar{z}} = 0$ in D, so that ω is analytic. Conversely, if ω is analytic, then $\overline{f_{\bar{z}z}} = f_{z\bar{z}}\omega$. But since $|\omega(z)| < 1$, this implies that $f_{z\bar{z}} = 0$, and f is harmonic. Thus, f is harmonic if and only if ω is analytic. In particular, the second complex dilatation ω of a sense-preserving harmonic mapping f is always an analytic function of modulus less than one. This function ω will be called the *analytic dilatation* of f, or simply the dilatation when the context allows no confusion. Note that $\omega(z) \equiv 0$ if and only if f is analytic.

The analytic dilatation has some nice properties. For instance, if f is a sense-preserving harmonic mapping with analytic dilatation ω and it is followed by an affine mapping $A(w) = \alpha w + \gamma + \beta\overline{w}$ with $|\beta| < |\alpha|$, then the composition $F = A \circ f$ is a sense-preserving harmonic mapping with analytic dilatation

$$\frac{\overline{F_{\bar{z}}}}{F_z} = \frac{\overline{\alpha}\omega + \overline{\beta}}{\beta\omega + \alpha}.$$

For a proof, use the chain rule to calculate

$$F_z = A_w f_z + A_{\overline{w}}\overline{f_{\bar{z}}} = \alpha f_z + \beta\overline{f_{\bar{z}}},$$

$$F_{\bar{z}} = A_w f_{\bar{z}} + A_{\overline{w}}\overline{f_z} = \alpha f_{\bar{z}} + \beta\overline{f_z}.$$

Thus,

$$\frac{\overline{F_{\bar{z}}}}{F_z} = \frac{\overline{\alpha}\,\overline{f_{\bar{z}}} + \overline{\beta} f_z}{\beta\overline{f_{\bar{z}}} + \alpha f_z} = \frac{\overline{\alpha}\omega + \overline{\beta}}{\beta\omega + \alpha}.$$

The analytic dilatation also behaves well under precomposition. Let f be a sense-preserving harmonic mapping of a simply connected domain D onto a region Ω, with analytic dilatation ω. Let ψ map a domain Δ conformally onto D. Then the composition $F = f \circ \psi$ maps Δ harmonically onto Ω and has analytic dilatation $\omega \circ \psi$. To see this, simply use the chain rule to calculate $F_\zeta = f_z\psi'$ and $F_{\bar{\zeta}} = f_{\bar{z}}\overline{\psi'}$. Thus, the analytic dilatation of F is

$$\frac{\overline{F_{\bar{\zeta}}(\zeta)}}{F_\zeta(\zeta)} = \frac{\overline{f_{\bar{z}}(\psi(\zeta))}}{f_z(\psi(\zeta))} = \omega(\psi(\zeta)).$$

In a similar way, the first complex dilatation $\mu = f_{\bar{z}}/f_z$ shows a true invariance property. If f is followed by a conformal mapping φ and $F = \varphi \circ f$, then

F has the same complex dilatation μ. Indeed, the chain rule gives $F_z = \varphi' f_z$ and $F_{\overline{z}} = \varphi' f_{\overline{z}}$, so that $F_{\overline{z}}/F_z = f_{\overline{z}}/f_z$.

In a simply connected domain $D \subset \mathbb{C}$, a complex-valued harmonic function f has the representation $f = h + \overline{g}$, where h and g are analytic in D; this representation is unique up to an additive constant. For a proof, recall that f_z is analytic if f is harmonic, and let $h' = f_z$, where h is analytic in D. Now let $g = \overline{f} - \overline{h}$ and observe that

$$g_{\overline{z}} = \overline{f_z} - \overline{h_z} = 0 \qquad \text{in} \quad D$$

by the definition of h. Thus, g is analytic in D. The uniqueness of the representation depends on the fact that a function both analytic and anti-analytic must be constant. (An *anti-analytic function* is defined as the conjugate of an analytic function.) If f is real-valued, the representation reduces to $f = h + \overline{h} = \text{Re}\{2h\}$, where $2h$ is the analytic completion of f, unique up to an additive imaginary constant. In a multiply connected domain, the representation $f = h + \overline{g}$ is valid locally but may not have a single-valued global extension.

For a harmonic mapping f of the unit disk $\mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \overline{g}$ is then unique and is called the *canonical representation* of f.

1.3. The Argument Principle

First recall the classical argument principle for analytic functions and its elegant proof. Let D be a domain bounded by a rectifiable Jordan curve C, oriented in the positive or "counterclockwise" direction. Let f be analytic in D and continuous in $\overline{D}$, with $f(z) \neq 0$ on C. The *index* or *winding number* of the image curve $f(C)$ about the origin is $I = (1/2\pi)\Delta_C \arg f(z)$, the total change in the argument of $f(z)$ as z runs once around C, divided by 2π. Let N be the total number of zeros of f in D, counted according to multiplicity. The argument principle asserts that $N = I$.

The customary proof begins with the observation that f'/f has a simple pole with residue n wherever f has a zero of order n, so the residue theorem gives

$$N = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Delta_C \log f(z) = I.$$

(Actually, since the derivative $f'(z)$ need not be defined on C, the curve of integration should be slightly contracted.) As an application, it can be seen that if f is analytic in D and continuous in $\overline{D}$, and if it carries C in a sense-preserving manner onto a Jordan curve Γ bounding a domain Ω, then f maps D

univalently onto Ω. In other words, univalence on the boundary implies univalence in the interior.

Because the argument principle has so many important applications, it will be very useful to have a generalization to complex-valued harmonic functions. In fact, the theorem is essentially of topological nature and may be generalized in various ways to arbitrary continuous mappings. However, it is desirable both to avoid the complications of topological degree theory and to develop a precise extension of the argument principle to "sense-preserving" harmonic functions. The proof for analytic functions suggests that the structure of harmonic functions may allow an elementary approach to a more general form of the theorem, and this turns out to be the case.

A complex-valued harmonic function f, not identically constant, will be classified as sense-preserving in a domain D if it satisfies a Beltrami equation of the second kind, $\overline{f_{\bar{z}}} = \omega f_z$, where ω is an analytic function in D with $|w(z)| < 1$. Since the Jacobian is $J_f = |f_z|^2 - |f_{\bar{z}}|^2$, this implies in particular that $J_f(z) > 0$ wherever $f_z(z) \neq 0$. If $f(z_0) = 0$ at some point z_0 in D, the order of the zero can be defined in terms of the canonical decomposition $f = h + \overline{g}$. Write the power-series expansions of h and g as

$$h(z) = a_0 + \sum_{k=n}^{\infty} a_k (z - z_0)^k, \qquad g(z) = b_0 + \sum_{k=m}^{\infty} b_k (z - z_0)^k,$$

where $n \geq 1$, $m \geq 1$, and $a_n \neq 0$, $b_m \neq 0$. (Here it is tacitly assumed that f is not analytic.) Actually, $b_0 = -\overline{a_0}$ because $f(z_0) = 0$. The sense-preserving property of f takes the equivalent form $g' = \omega h'$, with $|\omega(z)| < 1$. From this it follows that $m > n$, or that $m = n$ and $|b_n| < |a_n|$. In either case, we will say that f has a zero of *order* n at z_0.

As an immediate consequence of the structural formula, it can be inferred that the zeros of a sense-preserving harmonic function are isolated. Indeed, if $f(z_0) = 0$, then for $0 < |z - z_0| < \delta$ it is possible to write

$$f(z) = h(z) + \overline{g(z)} = a_n (z - z_0)^n \{1 + \psi(z)\},$$

where

$$\psi(z) = (\overline{b_m}/a_n)(\bar{z} - \overline{z_0})^m (z - z_0)^{-n} + \cdots .$$

But it is clear that $|\psi(z)| < 1$ for z sufficiently close to z_0, since $m \geq n$ and $|b_n/a_n| < 1$ if $m = n$. Hence $f(z) \neq 0$ elsewhere near z_0, and the zeros of f are isolated. Observe that the sense-preserving hypothesis is essential, because the zeros of a harmonic function are not always isolated. For example, the function $f(z) = z + \bar{z} = 2x$ vanishes at every point on the imaginary axis.

The argument principle for harmonic functions can now be formulated as a direct generalization of the classical result for analytic functions.

Theorem. *Let f be a sense-preserving harmonic function in a Jordan domain D with boundary C. Suppose f is continuous in $\overline{D}$ and $f(z) \neq 0$ on C. Then $\Delta_c \arg f(z) = 2\pi N$, where N is the total number of zeros of f in D, counted according to multiplicity.*

Proof. Suppose first that f has no zeros in D, so that $N = 0$ and the origin lies outside $f(D \cup C)$. A fact from topology says that in this case $\Delta_c \arg f(z) = 0$, which proves the theorem. To prove the topological fact, let ϕ be a homeomorphism of the closed unit square S onto $D \cup C$ with $\phi : \partial S \to C$ a homeomorphism. Then the composition $F = f \circ \phi$ is a continuous mapping of S onto the plane with no zeros, and we want to prove that $\Delta_{\partial S} \arg F(z) = 0$. Begin by subdividing S into finitely many small squares S_j on each of which the argument of $F(z)$ varies by at most $\pi/2$. Then $\Delta_{\partial S_j} \arg F(z) = 0$ and so

$$\Delta_{\partial S} \arg F(z) = \sum_j \Delta_{\partial S_j} \arg F(z) = 0,$$

where the first equality relies on the cancellation of contributions from the ∂S_j except on ∂S.

Next suppose that f does have zeros in D. Because the zeros are isolated and f does not vanish on C, there are only a finite number of distinct zeros in D. Denote them by z_j for $j = 1, 2, \ldots, \nu$. Let γ_j be a circle of radius $\delta > 0$ centered at z_j, where δ is chosen so small that the circles γ_j all lie in D and do not meet each other. Join each circle γ_j to C by a Jordan arc λ_j in D. Consider the closed path Γ formed by moving around C in the positive direction while making a detour along each λ_j to γ_j, running once around this circle in the negative (clockwise) direction, then returning along λ_j to C. This curve Γ contains no zeros of f, and so $\Delta_\Gamma \arg f(z) = 0$ by the case just considered. But the contributions of the arcs λ_j along Γ cancel out, so that

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z),$$

where each of the circles γ_j is now traversed in the positive direction. This formula reduces the global problem to a local one. (The same reduction is often used to prove the residue theorem.)

Suppose now that f has a zero of order n at a point z_0. Then, as observed earlier, f has the local form

$$f(z) = a_n(z - z_0)^n\{1 + \psi(z)\}, \qquad a_n \neq 0,$$

where $|\psi(z)| < 1$ on a sufficiently small circle γ defined by $|z - z_0| = \delta$. This shows that

$$\Delta_\gamma \arg f(z) = n \Delta_\gamma \arg \{z - z_0\} + \Delta_\gamma \arg \{1 + \psi(z)\} = 2\pi n.$$

Therefore, if f has zeros of order n_j at the points z_j, the conclusion is that

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z) = 2\pi \sum_{j=1}^{\nu} n_j = 2\pi N,$$

which proves the theorem. The result admits an obvious extension to multiply connected domains, just as for analytic functions. ■

Several corollaries are worthy of note. First of all, there is a direct extension of Rouché's theorem to sense-preserving harmonic functions. Specifically, if p and $p + q$ are sense-preserving harmonic functions in D, continuous in $\overline{D}$, and $|q(z)| < |p(z)|$ on C, then p and $p + q$ have the same number of zeros inside D. As in the standard proof for analytic functions, the inequality on C implies that neither p nor $p + q$ has a zero on C and that the images of C under the two functions have the same winding numbers about the origin. Thus the harmonic version of Rouché's theorem follows from the harmonic version of the argument principle.

Next there is a generalization of Hurwitz's theorem. If f_n are harmonic functions in a domain D that converge locally uniformly, then their limit function f is harmonic. The harmonic version of Hurwitz's theorem asserts that if f and all of the f_n are sense-preserving, then a point z_0 in D is a zero of f if and only if it is a cluster point of zeros of the functions f_n. More precisely, f has a zero of order m at z_0 if and only if each small neighborhood of z_0 (small enough to contain no other zeros of f) contains precisely m zeros, counted according to multiplicity, of f_n for every n sufficiently large. The proof applies Rouché's theorem exactly as in the analytic case, with $p = f$ and $q = f_n - f$.

Finally, sense-preserving harmonic functions have the *open mapping property*: they carry open sets to open sets. In fact, as in the analytic case, a stronger statement can be made. If f is a sense-preserving harmonic function near a point z_0 where $f(z_0) = w_0$, and if $f(z) - w_0$ has a zero of order $n (n \geq 1)$ at z_0, then to each sufficiently small $\varepsilon > 0$ there corresponds a $\delta > 0$ with the following property. For each point $\alpha \in N_\delta(w_0) = \{w : |w - w_0| < \delta\}$, the function $f(z) - \alpha$ has exactly n zeros, counted according to multiplicity, in $N_\varepsilon(z_0)$. The proof appeals to the harmonic version of Rouché's theorem with $p = f - w_0$ and $q = w_0 - \alpha$.

The argument principle for harmonic functions has been essentially known for some time. Various forms of it have been applied in papers on harmonic mappings. However, the elementary proof presented here was found only recently by Duren, Hengartner, and Laugesen [1], who actually obtained a more general form of the theorem. As they pointed out, the proof still applies when $|\omega(z)| > 1$ in some parts of the domain D, so that f is sense-preserving in some regions and sense-reversing in others, provided that none of the zeros are situated at points where $|\omega(z)| = 1$. A zero at a sense-reversing point of f is assigned *negative* order, minus the order of the zero of $\overline{f}$ at the same point. Then a more general version of the theorem says that $\Delta_C \arg f(z)$ is equal to 2π times the sum of the orders of the zeros of f in D.

The classical version of the argument principle applies more generally to *meromorphic* functions and says that the winding number is equal to the number of zeros minus the number of poles, all counted according to multiplicity. Suffridge and Thompson [1] developed a form of the argument principle for harmonic functions that takes account of some kinds of singularities.

A less elementary proof of the argument principle proceeds through an important representation theorem for sense-preserving harmonic functions. This proof relies heavily on the theory of quasiconformal mappings and will only be sketched here. First contract the curve C to reduce the problem to the quasiconformal case where $|\omega(z)| \le k < 1$ in D. Next observe that f satisfies a Beltrami equation of the *first* kind, $f_{\overline{z}} = \mu f_z$, where $\mu = (f_z/\overline{f_z})\omega$. Appeal to standard results about quasiconformal mappings (see Lehto and Virtanen [1] or Ahlfors [1]) to conclude that f has the form $f = F \circ \Phi$, where Φ is a sense-preserving homeomorphism of $\overline{D}$ onto the closure of a Jordan domain Ω, and F is analytic in Ω. In this way the argument principle for sense-preserving harmonic functions is reduced to the classical result for analytic functions.

1.4. The Dirichlet Problem

In this section some facts about harmonic functions are assembled for easy reference in later chapters. Proofs are omitted but can be found in textbooks on complex analysis, for instance by Ahlfors [3] or Nehari [2].

One corollary of Green's theorem in the plane is *Green's identity*:

$$\iint_D (u\Delta v - v\Delta u)\,dx\,dy = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds,$$

where D is a Jordan domain with smooth boundary C, the real-valued functions u and v have continuous second partial derivatives in $\overline{D}$ (the closure of D), $\partial/\partial n$ denotes an outer normal derivative, and ds denotes an element of arclength. If u is harmonic in D, so that its Laplacian $\Delta u = 0$, then by choosing v to be constant we see u can have no net flux across the boundary: $\int_C (\partial u/\partial n)\, ds = 0$. From this it follows that every harmonic function has the *mean-value property*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$$

for each point $z_0 \in D$ and all radii $\rho > 0$ sufficiently small. Conversely, a continuous function with the local mean-value property must be harmonic. From the mean-value property it is a short step to the *maximum principle*: a function harmonic in a domain D cannot have a local maximum or minimum at any point in D unless it is identically constant. Thus, if u is a nonconstant function harmonic in D and continuous in $\overline{D}$, it will attain its maximum and minimum values only on the boundary.

The *Dirichlet problem* is to find a function harmonic in a domain D and continuous in $\overline{D}$ that agrees with a prescribed continuous function on the boundary ∂D. The uniqueness of a solution is an immediate consequence of the maximum principle. Existence of a solution is more difficult to establish, but an elegant proof can be given with the help of subharmonic functions if the boundary is sufficiently nice (see Ahlfors [3], p. 245 ff.). In particular, a solution always exists if D is a Jordan domain. Much more generally, it can be shown that a solution always exists (for every prescribed continuous boundary function) if and only if the boundary of D has no degenerate components. A *degenerate* boundary component is a component consisting of a single point.

When the given domain is a disk, the Dirichlet problem can be solved explicitly. For simplicity, consider the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let φ be an arbitrary continuous function on the interval $[0, 2\pi]$ with $\varphi(0) = \varphi(2\pi)$. Then the *Poisson formula* is

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t)\, \varphi(t)\, dt, \qquad 0 \le r < 1,$$

where

$$P(r, t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}$$

is the *Poisson kernel*. This function u is harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, and $u(e^{it}) = \varphi(t)$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Thus, u solves the Dirichlet problem for the unit disk.

Suppose now that the prescribed function φ is piecewise continuous but has a finite number of jump discontinuities, so that at certain points $\theta \in [0, 2\pi]$

the left- and right-hand limits $\varphi(\theta-)$ and $\varphi(\theta+)$ exist but $\varphi(\theta-) \neq \varphi(\theta+)$. Then the function $u(z)$ given by the Poisson integral is harmonic in $\mathbb{D}$ and has the radial limit

$$\lim_{r\to 1} u(re^{i\theta}) = \frac{1}{2}(\varphi(\theta-) + \varphi(\theta+)).$$

More generally, as the point z in the disk approaches the boundary point $e^{i\theta}$ along a linear segment at an angle $\alpha (0 < \alpha < \pi)$ with the tangent line, it can be shown that $u(z)$ tends to the corresponding weighted average

$$\frac{\alpha}{\pi}\varphi(\theta-) + \left(1 - \frac{\alpha}{\pi}\right)\varphi(\theta+).$$

The Poisson formula can be generalized to an arbitrary Jordan domain D with rectifiable boundary C with the help of *Green's function*. This is the function $G(z,\zeta)$, harmonic in $D\backslash\{\zeta\}$ for each point $\zeta \in D$, for which $G(z,\zeta) + \log|z-\zeta|$ is harmonic at ζ and $G(z,\zeta) = 0$ for $z \in C$. For any function $\varphi(z)$ continuous on C, the function

$$u(\zeta) = -\frac{1}{2\pi}\int_C \varphi(z)\frac{\partial G}{\partial n}(z,\zeta)\,ds, \qquad \zeta \in D,$$

is the solution to the Dirichlet problem. Green's function $G(z,\zeta)$ can be obtained from the solution to a special Dirichlet problem with boundary function $\log|z-\zeta|$.

If D is again a Jordan domain with boundary C, the *harmonic measure* of a closed arc $I \subset C$ is the function $u(z)$ harmonic in D and continuous in $\overline{D}$ except at the endpoints of I, with $u(z) = 1$ on the interior of I and $u(z) = 0$ on $C\backslash I$. For example, if D is the unit disk and I is an arc of the unit circle with endpoints $e^{i\sigma}$ and $e^{i\tau}$, subtending an angle $\theta = \tau - \sigma$ $(0 < \theta < 2\pi)$ at the center of the disk, the harmonic measure of I has the form $u(z) = (1/\pi)(\alpha - (\theta/2))$, where $\alpha = \alpha(z)$ is the angle that I subtends at z.

With the help of the Poisson formula, it is easy to derive *Harnack's inequality*,

$$\frac{R-r}{R+r}u(0) \leq u(z) \leq \frac{R+r}{R-r}u(0), \qquad r = |z|,$$

for a *positive* harmonic function $u(z)$ in the disk $|z| < R$.

1.5. Conformal Mappings

Much of the theory of harmonic mappings is inspired by the classical theory of conformal mappings, a very special case. For later reference, we give here a rapid survey of conformal mappings and related topics. Proofs of theorems and further information can be found in the books by Nehari [2], Ahlfors [3], Pommerenke [1], and Duren [2].

A *domain* is defined to be an open connected set. A domain is *simply connected* if its complement with respect to the extended complex plane $\widehat{\mathbb{C}}$ is connected. A *doubly connected* domain is one whose complement consists of two components.

A function f is said to be a *conformal mapping* of a domain $\Omega \subset \mathbb{C}$ onto a domain D if it is analytic in Ω and *globally univalent* (*i.e.*, one-to-one) and it maps Ω *onto* D, so that $f(\Omega) = D$. An analytic function f is said to be *locally univalent* in Ω if it is univalent in some neighborhood of each point in Ω. A necessary and sufficient condition for local univalence is that $f'(z) \neq 0$ in Ω.

The famous *Riemann mapping theorem* asserts that every simply connected domain $\Omega \subset \mathbb{C}$ with $\Omega \neq \mathbb{C}$ admits a unique conformal mapping f onto the unit disk $\mathbb{D}$ with the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$ for an arbitrarily prescribed point $\zeta \in \Omega$. Because the inverse function is necessarily analytic, it is equivalent to say that $\mathbb{D}$ can be mapped conformally onto Ω. The *Carathéodory extension theorem* says (in a special case) that each conformal mapping of a Jordan domain Ω onto a Jordan domain D can be extended to a homeomorphism of $\overline{\Omega}$ onto $\overline{D}$. This last theorem can be generalized to quasiconformal mappings.

The modern proof of Riemann's theorem is based on the theory of normal families. A collection $\mathscr{F}$ of functions f defined on a domain Ω is said to be a *normal family* if every sequence of functions in $\mathscr{F}$ has a subsequence that converges locally uniformly in Ω, meaning that it converges uniformly in some neighborhood of each point of Ω. In view of the Heine–Borel theorem, locally uniform convergence is the same as uniform convergence on each compact subset of Ω. One can show that $\mathscr{F}$ is a normal family if and only if each sequence of functions in $\mathscr{F}$ has a subsequence that converges uniformly in each compact subset of Ω. To see this, exhaust Ω by a sequence of expanding compacta and apply a diagonalization argument. *A* family $\mathscr{F}$ is said to be *locally bounded* in Ω if the functions in $\mathscr{F}$ are uniformly bounded in some neighborhood of each point of Ω. *Montel's theorem* says that a family of analytic functions is normal if and only if it is locally bounded. The proof essentially uses the Arzela–Ascoli theorem (*cf.* Rudin [1]) that a family is normal if it is equicontinuous and pointwise bounded. An application of the Cauchy integral formula shows that if a family $\mathscr{F}$ of analytic functions is locally bounded, then so is $\mathscr{F}' = \{f' : f \in \mathscr{F}\}$, and equicontinuity follows.

A standard result in complex analysis says that the locally uniform limit of a sequence of analytic functions is again analytic. The locally uniform limit of a sequence of analytic *univalent* functions is either univalent or constant.

The *Carathéodory convergence theorem* (see Duren [2], p. 76) relates the locally uniform convergence of a sequence of univalent analytic functions to a notion of convergence of its sequence of ranges. Let $\{D_n\}$ be a sequence of domains in the complex plane, each containing the origin. If the origin is an interior point of the intersection of the domains D_n, then the *kernel* of the sequence $\{D_n\}$ is defined as the largest domain D containing the origin and having the property that each compact subset of D lies in all but a finite number of the domains D_n. If the origin is not an interior point of the intersection, the kernel is defined as $D = \{0\}$. In either case, the sequence $\{D_n\}$ is said to *converge* to its kernel (written $D_n \to D$) if every subsequence has the same kernel. Now let f_n be a conformal mapping of the unit disk $\mathbb{D}$ onto a domain D_n, with $f_n(0) = 0$ and $f_n'(0) > 0$. Let D be the kernel of $\{D_n\}$. Then the Carathéodory convergence theorem says that $f_n \to f$ locally uniformly in $\mathbb{D}$ if and only if $D_n \to D \neq \mathbb{C}$. In the case of convergence, there are two possibilities. If $D = \{0\}$, then $f = 0$. If $D \neq \{0\}$, then D is a simply connected domain and f maps $\mathbb{D}$ conformally onto D.

The class S consists of all analytic univalent functions in $\mathbb{D}$, normalized so that $f(0) = 0$ and $f'(0) = 1$. Each function $f \in S$ has a power-series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad |z| < 1.$$

The *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

belongs to S and maps the disk onto the entire complex plane minus the portion of the negative real axis from $-\infty$ to $-\frac{1}{4}$. The *Koebe one-quarter theorem* says that the disk $|w| < \frac{1}{4}$ is contained in the range of every function in S. The Koebe function shows that the radius $\frac{1}{4}$ is best possible. *Bieberbach's theorem* asserts that $|a_2| \leq 2$ for every function $f \in S$, with equality only for functions $f(z) = e^{-i\theta} k(e^{i\theta} z)$, rotations of the Koebe function. Bieberbach's theorem gives an easy proof of the Koebe one-quarter theorem and a wealth of other geometric information. It leads to the *Koebe distortion theorem*, which provides the sharp bounds

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad r = |z| < 1$$

for every $f \in S$. Again, equality occurs only for suitable rotations of the Koebe function. The Koebe distortion theorem leads in turn to the *growth theorem*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad r = |z| < 1$$