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HARMONIC MAPPINGS INTO RIEMANNIAN MANIFOLDS WITH NON-POSITIVE SECTIONAL CURVATURE

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1. Introduction.

Let $\mathscr{X}$ be a compact, connected n-dimensional Riemannian manifold of class C^4, with non-void boundary Σ and interior Ω. In terms of local coordinates $x=(x^1,\ldots,x^n)$ on $\mathscr{X}$ the line element is given by

$$d\sigma^2 = \gamma_{\alpha\beta}(x)dx^\alpha dx^\beta ,$$

and the Laplace–Beltrami operator $\Delta_{\mathscr{X}}$ on $\mathscr{X}$ is defined by

$$\Delta_{\mathscr{X}}\varphi = \gamma^{-\frac{1}{2}}D_\beta[\gamma^{\alpha\beta}\sqrt{\gamma}D_\alpha\varphi], \quad \varphi \in C^2(\Omega,\mathsf{R}) ,$$

where $\gamma=\det(\gamma_{\alpha\beta})$.

Furthermore, let $\mathscr{M}$ be a complete connected Riemannian manifold without boundary of dimension $N \geqq 2$, class C^4, and line element

$$ds^2 = g_{ik}(u)du^i du^k .$$

A mapping $U \in C^2(\Omega,\mathscr{M})$ is said to be a *harmonic map* of Ω into $\mathscr{M}$ if in local coordinates it is represented by $u=(u^1,u^2,\ldots,u^N)$ satisfying

$$(1) \qquad \Delta_{\mathscr{X}}u^l+\gamma^{\alpha\beta}\Gamma^l_{ik}(u)D_\alpha u^i D_\beta u^k = 0, \quad 1\leqq l\leqq N .$$

Here Γ^l_{ik} are the Christoffel symbols of $\mathscr{M}$,

$$\Gamma^l_{ik} = g^{lj}\Gamma_{ijk}$$

with

$$\Gamma_{ijk} = \tfrac{1}{2}\left(\frac{\partial g_{jk}}{\partial u^i}-\frac{\partial g_{ik}}{\partial u^j}+\frac{\partial g_{ij}}{\partial u^k}\right).$$

The aim of the present paper is to prove the following

THEOREM. *If $\mathscr{M}$ is simply connected and has non-positive sectional curvature then to any boundary map $\Phi \in C^{1+\alpha}(\Sigma,\mathscr{M})$ with $\alpha>0$ there exists a*

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harmonic map U of $\mathscr{X}$ into $\mathscr{M}$, of class $C^1(\mathscr{X},\mathscr{M}) \cap C^3(\Omega,\mathscr{M})$, such that $U = \Phi$ on Σ.

This result was recently proved by R. S. Hamilton [4] in a complicated and lengthy paper. In fact, Hamilton treats also different boundary value problems, e.g. Neumann's problem, as well as multiply connected manifolds $\mathscr{M}$. His method is to apply the gradient method to the functional E, defined below, so as to get a boundary-initial value problem for a certain parabolic system of non-linear equations. Previously, this approach has been used by Eells and Sampson [2], and Hartman [5], in the case when $\mathscr{X}$ has no boundary, to construct harmonic mappings which are homotopic to a given mapping $\Phi: \mathscr{X} \to \mathscr{M}$.

One of the first papers in this area appearing in the literature is due to Bochner [1]. In a special case, he stated a priori bounds on harmonic surfaces which, as he indicated, would have led to a solution of the Dirichlet problem. However, there seems to be gaps in the proofs of some of these estimates.

Here we want to show that with appropriate modifications Bochner's approach can be used to prove the theorem above in a fairly simple and straightforward way.

So far, all known existence results are based in an essential way on the assumption that the target manifold $\mathscr{M}$ has non-positive sectional curvature. To make our procedure transparent we have also restricted ourselves to this case. However, the curvature assumption is by no means essential for our approach as will be shown in the forthcoming paper [7] of the same authors where also positive sectional curvatures are admitted for $\mathscr{M}$.

2. Preliminaries.

By a well-known theorem due to Hadamard and Cartan (cf. [3, p. 201]) a simply connected complete N-dimensional manifold of nonpositive curvature is diffeomorphic to $\mathbb{R}^N$, the diffeomorphism being given by any normal coordinate system. Hence, by using normal coordinates, mappings $U \in C^s(\Omega,\mathscr{M})$ can be represented by vector functions $u \in C^s(\Omega, \mathbb{R}^N)$. We shall fix a particular such representation, "the standard representation of U", by choosing normal coordinates around an arbitrary but fixed point 0 on $\mathscr{M}$.

For any C^1 map $U: \Omega \to \mathscr{M}$ we define $d^V: \mathscr{X} \to \mathbb{R}$ by

$$d^V(x) = [\operatorname{dist}(U(x), V)]^2, \quad V \in \mathscr{M},\ x \in \Omega .$$

Moreover we introduce the "energy density" function of U, $e\colon \Omega \to \mathsf{R}$, which in local coordinates in $\mathscr{X}$ is given by

$$e = g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k$$

where $u=u(x)$ is any normal representation for U. The energy functional of U is then given by

$$E(U) = \int_{\mathscr{X}} e\,dV ,$$

dV being the area element of $\mathscr{X}$.

The defining equations for the harmonic mappings, (1), are the Euler–Lagrange equations of this generalized Dirichlet integral.

For mappings u from Ω or Σ into R^N we shall use norms of the type

$$|u|_{C^1(\Omega)}, \qquad |u|_{C^{1+\alpha}(\Sigma)} .$$

These are defined in the usual manner, using an arbitrary, but fixed, finite atlas of $\mathscr{X}$ and Σ respectively, any two different such atlases yielding equivalent norms.

3. Lemmata.

In what follows we shall always assume that $\mathscr{M}$ has non-positive sectional curvature.

LEMMA 1. *Let $U \in C^2(\Omega,\mathscr{M})$ be such that in some normal coordinate system on $\mathscr{M}$*

$$\Delta_{\mathscr{X}} u^l + t\gamma^{\alpha\beta}\Gamma^l_{ik}(u)D_\alpha u^i D_\beta u^k = 0, \quad 1 \leqq l \leqq N . \tag{2}$$

Then if $t \in [0,1]$ we have

$$\Delta_{\mathscr{X}} |u|^2 = \Delta_{\mathscr{X}} \textstyle\sum_1^N (u^j)^2 \geqq 2te .$$

In particular, if $U \in C^2(\Omega,\mathscr{M})$ is a harmonic map then for every $V \in \mathscr{M}$

$$\Delta_{\mathscr{X}} d^V \geqq 2e .$$

PROOF OF LEMMA 1.

$$\begin{aligned}\Delta_{\mathscr{X}}|u|^2 &= 2u^j\Delta_{\mathscr{X}}u^j + 2\gamma^{\alpha\beta}D_\alpha u^j D_\beta u^j\\ &= 2t\{\delta_{ik} - \Gamma^l_{ik}(u)u^l\}\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k + 2(1-t)\gamma^{\alpha\beta}D_\alpha u^j D_\beta u^j.\end{aligned}$$

Obviously, the last term here is non-negative.

Since the Γ^l_{ik} are calculated with respect to normal coordinates on $\mathscr{M}$, we have (cf. [6, p. 211, formulas (50)–(52)])

$$g_{ik}(u)u^k = g^{ik}(u)u^k = u^i,$$
$$\Gamma_{ijk}(u)u^j + \Gamma_{ikj}(u)u^j = \delta_{ik} - g_{ik} .$$

The first of these formulas implies

$$\Gamma^l_{ik}(u)u^l = \Gamma_{ijk}(u)u^j .$$

Hence, the second one is equivalent to

$$\delta_{ik} - \Gamma^l_{ik}(u)u^l = g_{ik}(u) + \Gamma_{ikj}(u)u^j .$$

Therefore we obtain

$$\Delta_{\mathscr{X}} |u|^2 \geqq 2t\{g_{ik}(u) + \Gamma_{ikj}(u)u^j\}\gamma^{\alpha\beta} D_\alpha u^i D_\beta u^k .$$

Using Rauch's comparison theorem and the fact that $\mathscr{M}$ has non-positive sectional curvature one derives the inequality

$$0 \leqq \Gamma_{ikj}(u)u^j \xi^i \xi^k \quad \text{for all } \xi \in \mathsf{R}^N ,$$

cf. [6, Lemma 6]. This concludes the proof of the first part of the lemma if we note that the trace $A_{\alpha\beta}B^{\alpha\beta}$ of the product of two positive semi-definite matrices $(A_{\alpha\beta})$, $(B^{\alpha\beta})$ is non-negative provided that one of them is symmetric.

The second part of the Lemma is immediate since a harmonic mapping satisfies (2) with $t=1$ and normal coordinates around any point $V \in \mathscr{M}$, remembering that then $d^V = |u|^2$.

LEMMA 2. *For given $\mathscr{X}$ and $\mathscr{M}$ let $U \in C^3(\Omega,\mathscr{M})$ be harmonic. Then there exists a constant $\tau \geqq 0$, depending only on $\mathscr{X}$, such that*

$$\Delta_{\mathscr{X}}\{e + \tau d^0\} \geqq 0 \quad \text{on } \Omega .$$

PROOF. We shall make use of the following differential inequality derived by Bochner [1] in the special case of a flat manifold $\mathscr{X}$, and, in general, by Eells and Sampson [2, p. 123]:

$$\tfrac{1}{2}\Delta_{\mathscr{X}} e \geqq f_1 + f_2$$

where

$$f_1 = -\gamma^{\alpha\beta}\gamma^{\mu\nu} R_{ijkl}(u) D_\alpha u^i D_\mu u^j D_\beta u^k D_\nu u^l$$

and

$$f_2 = g_{ik}(u) P^{\alpha\beta} D_\alpha u^i D_\beta u^k .$$

Here R_{ijkl} stands for the Riemannian curvature tensor of $\mathscr{M}$:

$$R_{ijkl} = g_{lh} R^h_{ijk}, \quad R^h_{ijk} = \frac{\partial \Gamma^h_{jk}}{\partial u^i} - \frac{\partial \Gamma^h_{ik}}{\partial u^j} + \Gamma^h_{il}\Gamma^l_{jk} - \Gamma^h_{jl}\Gamma^l_{ik}$$

and $P^{\alpha\beta}$ denotes the Ricci curvature tensor on $\mathscr{X}$:

$$P^{\alpha\beta} = \gamma^{\alpha\sigma}\gamma^{\mu\nu} P^\beta_{\sigma\mu\nu}$$

where $P^\beta_{\sigma\mu\nu}$ is the Riemann curvature tensor on $\mathscr{X}$.

Since the sectional curvature of $\mathscr{M}$ is assumed non-positive, we have $f_1 \geqq 0$.

On the other hand, using a compactness argument one sees that there is a number τ such that $|f_2| \leqq \tau e$. In view of Lemma 1 the proof is complete.

4. An a priori estimate.

LEMMA 3. *For given $\mathscr{X}$ and $\mathscr{M}$ there is a function $k(\alpha, M)$ defined for $\alpha \in (0,1)$ and $M > 0$ such that if $U \in C^3(\Omega, \mathscr{M}) \cap C^1(\mathscr{X}, \mathscr{M})$ is a harmonic map with boundary values $U|_\Sigma = \Phi$ such that* $\operatorname{dist}(\Phi(\Sigma), 0) \leqq M$ *then*

$$\sup_{\mathscr{X}} d^0 \leqq M^2 \tag{3}$$

and

$$\sup_{\mathscr{X}} e \leqq k(\alpha, M)(1 + |\varphi|^2_{C^{1+\alpha}(\Sigma, \mathbf{R}^N)}) \tag{4}$$

where φ is the standard representation for Φ.

PROOF. The first inequality follows immediately from the maximum principle since, by Lemma 1, d^0 is a subharmonic function and, by assumption, $\sup_\Sigma d^0 \leqq M^2$.

To prove the second inequality we notice that by Lemma 2 and the maximum principle

$$\sup_{\mathscr{X}} e \leqq \sup_{\mathscr{X}} \{e + \tau d^0\} \leqq \sup_\Sigma \{e + \tau d^0\} \leqq \sup_\Sigma e + \tau M^2,$$

whence we see that it is enough to estimate $\sup_\Sigma e$.

Let $x_0 \in \Sigma$ and $l \in T_{x_0}\mathscr{X}$ be a unit vector and pointing outwards such that

$$\sup_\Sigma e = e(x_0) \leqq (N+1) \left\| \frac{\partial U}{\partial l}(x_0) \right\|^2_{\mathscr{M}},$$

where $\| \ \|_{\mathscr{M}}$ is the norm in $T\mathscr{M}$ induced by the metric in $\mathscr{M}$.

Excluding the trivial case when $\partial U(x_0)/\partial l$ is zero we walk a distance of $\frac{1}{2}$ on the geodesic ray from $U_0 = U(x_0)$ in the direction of $\partial U(x_0)/\partial l$, to arrive at V_0. An easy calculation then shows that

$$\frac{\partial d^{V_0}}{\partial l}(x_0) = - \left\| \frac{\partial U}{\partial l}(x_0) \right\|_{\mathscr{M}}.$$

On the other hand, consider the Riesz decomposition for the subharmonic function d^{V_0},

$$d^{V_0} = h + s$$

where $\Delta_{\mathscr{X}} h=0$ with $h=d^{V_0}$ on Σ and where $\Delta_{\mathscr{X}} s \geqq 0$ with $s=0$ on Σ. Since l is pointing outwards, the maximum principle implies that $-\partial s/\partial l \leqq 0$ and hence

$$-\frac{\partial d^{V_0}}{\partial l}(x_0) \leqq -\frac{\partial h}{\partial l}(x_0) .$$

However, in view of the well-known Schauder estimates one easily realizes that

$$|h|_{C^1(\mathscr{X}, \mathrm{R})} \leqq k'(\alpha, M)|h|_{C^{1+\alpha}(\Sigma, \mathrm{R})} \leqq (N+1)^{-1} k(\alpha, M)(1+|\varphi|_{C^{1+\alpha}(\Sigma, \mathrm{R}^N)}) ,$$

and the statement of the lemma is proved.

5. Proof of the theorem.

Let $\Psi=(\Psi^1, \Psi^2, \ldots, \Psi^N)$ be that mapping of $C^1(\mathscr{X}, \mathrm{R}^N)$ into itself which is defined by $\Psi\colon u \to v$ where

$$\begin{aligned} \Delta_{\mathscr{X}} v^l &= -\Gamma^l_{jk} \gamma^{\alpha\beta} D_\alpha u^j D_\beta u^k, \quad 1 \leqq l \leqq N , \\ v^l &= 0 \quad \text{on } \Sigma . \end{aligned}$$

Well known results from potential theory imply that Ψ maps bounded sets in $C^1(\mathscr{X}, \mathrm{R}^N)$ into bounded sets in $C^{1+\beta}(\mathscr{X}, \mathrm{R}^N)$, if $0<\beta<1$, and on account of the Arzela–Ascoli theorem we see that Ψ is a compact mapping.

Now let $h=h(\varphi) \in C^{1+\alpha}(\mathscr{X}, \mathrm{R}^N)$ be the uniquely determined solution of the linear boundary value problem

$$\begin{aligned} \Delta_{\mathscr{X}} h &= 0 \quad \text{in } \Omega , \\ h &= \varphi \quad \text{on } \Sigma , \end{aligned}$$

and consider the functional equation

$$u = \Psi(u)+h, \quad u \in C^1(\mathscr{X}, \mathrm{R}^N) . \tag{5}$$

The Schauder estimates again imply that any solution of (5) is actually of class $C^3(\Omega, \mathrm{R}^N) \cap C^{1+\alpha}(\mathscr{X}, \mathrm{R}^N)$, and hence to prove the theorem we need only find a solution of (5).

Since Ψ is compact we may use the Schauder–Leray degree theory (cf. [8]), and we shall, in fact, prove that the mapping $u \to F_{t,s}(u)$, where

$$F_{t,s} = I - t\Psi - sh$$

has degree one, first for $\{0 \leqq t \leqq 1, s=0\}$ and then for $\{t=1, 0 \leqq s \leqq 1\}$. Here the degree is calculated with respect to a set $A \subset C^1(\mathscr{X}, \mathrm{R}^N)$ and the element $0 \in C^1(\mathscr{X}, \mathrm{R}^N)$, A being defined by

$$A = A(M_0, M_1) = \{u \in C^1(\mathscr{X}, \mathsf{R}^N) \; ; \; d^0(u) < M_0, \, e(u) < M_1\}$$

where M_0 and M_1 are properly chosen positive numbers.

Now, since $F_{0,0}$ has degree one, the homotopy invariance shows that $F_{t,0}$ has degree one if $0 \notin F_{t,0}(\partial A)$ for $0 \leqq t \leqq 1$. However, as above, any solution $u \in C^1(\mathscr{X}, \mathsf{R}^N)$ of $F_{t,0}(u) = 0$ with $u \in \partial A$ is of class $C^3(\Omega, \mathsf{R}^N)$ and is thus a solution of (2) with boundary values zero. Lemma 1 and the maximum principle then show that $u \equiv 0$, contradicting the assumption that $u \in \partial A$, with any choice of M_0 and M_1.

The same argument applies to the set of mappings $F_{1,s}$ and we find that the degree of $F_{1,1}$ is one if $0 \notin F_{1,s}(\partial A)$ for $0 \leqq s \leqq 1$.

To prove that we choose

$$M_0 > \sup_\Sigma |\varphi|^2 \quad \text{and} \quad M_1 > k(\alpha, M_0)(1 + |\varphi|_{C^{1+\alpha}(\Sigma, \mathsf{R}^N)}) \,,$$

k being the function of Lemma 3. Repeating a by now familiar argument we see that any $u \in C^1(\mathscr{X}, \mathsf{R}^N)$ satisfying $u \in \partial A$ and $F_{1,s}(u) = 0$ is actually of class $C^3(\Omega, \mathsf{R}^N)$. Hence u is a harmonic map with boundary values $s\varphi$, and the apriori estimate of Lemma 3 shows that

$$\sup_{\mathscr{X}} d^0(u) < M_0 \quad \text{and} \quad \sup_{\mathscr{X}} e(u) < M_1 \,,$$

contradicting the assumption that $u \in \partial A$.

The theorem is proved.

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