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HARMONIC MEASURES SUPPORTED ON CURVES

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Let Ω_1 and Ω_2 be two disjoint, simply connected domains in the plane, and let ω_1 and ω_2 be harmonic measures associated to Ω_1 and Ω_2. We present necessary and sufficient conditions for ω_1 and ω_2 to be mutually singular.

1. Introduction. Let Γ be a Jordan curve in C and let Ω_1 and Ω_2 be the two simply connected domains complementary to Γ. For each domain Ω_j fix a point $z_j \in \Omega_j$ and let ω_j be the harmonic measure for z_j relative to Ω_j. In this paper we discuss when the two measures are singular, $\omega_1 \perp \omega_2$, i.e. when there are disjoint sets E_1, E_2 such that $\omega_j(E_j) = 1, j = 1, 2$. If Γ is a Jordan arc, Γ^c consists of only one domain Ω, but since Γ has two sides there are two measures ω_1, ω_2 which give the harmonic measure of sets on each of the two sides of Γ. Again it makes sense to ask whether $\omega_1 \perp \omega_2$.

If Γ is a Jordan curve or arc and $z_0 \in \Gamma$ we say that Γ has a tangent at z_0 if there is θ_0 with the property that for all $\varepsilon > 0$ there is $r > 0$ such that whenever $z \in \Gamma$ and $|z - z_0| < r$, either $|\theta_0 - \arg(z - z_0)| < \varepsilon$ or $|\theta_0 + \pi - \arg(z - z_0)| < \varepsilon$. We denote by T the collection of all tangent points on Γ. When Γ is a Jordan curve we also say that $z_0 \in T_1$ if there is a unique θ_0 (mod 2π) with the property that for all $\varepsilon > 0$ there is $r > 0$ such that

$$\{z : 0 < |z - z_0| < r,\ |\theta_0 - \arg(z - z_0)| < \pi/2 - \varepsilon\} \subset \Omega_1.$$

T_1 is called the set of inner tangent points with respect to Ω_1. With T_2 similarly defined one sees that $T = T_1 \cap T_2$. If Γ is a Jordan arc T_1 and T_2 are similarly defined. Finally, we denote one dimensional Hausdorff measure by Λ_1.

THEOREM. *Suppose Γ is a Jordan curve or arc. Then $\omega_1 \perp \omega_2$ if and only if $\Lambda_1(T) = 0$.*

Let $A(\Gamma)$ denote the class of all bounded continuous functions on the Riemann sphere which are holomorphic off Γ. In **[4]** Browder and Wermer proved that $A(\Gamma)$ is a Dirichlet algebra if and only if $\omega_1 \perp \omega_2$.

COROLLARY. *$A(\Gamma)$ is a Dirichlet algebra if and only if $\Lambda_1(T) = 0$.*

We prove the theorem in §2 and make some remarks in §3.

2. Proof of the theorem. We prove the theorem in the case where Γ is a Jordan curve; the modifications needed when Γ is an arc are outlined in §3. First suppose that $\Lambda_1(T) > 0$. It is then an easy matter to find two curves Γ_1, Γ_2 such that each Γ_j is *rectifiable*, $\Gamma_j \subset \overline{\Omega}_j$, and $\Lambda_1(\Gamma_1 \cap \Gamma_2 \cap T) > 0$. Denoting by $\tilde{\Omega}_j$ the component of Γ_j^c contained in Ω_j, we may also assume that $z_j \in \tilde{\Omega}_j$, $j = 1, 2$. Let $\tilde{\omega}_1$, $\tilde{\omega}_2$ be the obvious associated harmonic measures, and let $E = \Gamma_1 \cap \Gamma_2 \cap T$. Since Γ_j is rectifiable, $\tilde{\omega}_j$ is mutually absolutely continuous with respect to Λ_1, $\tilde{\omega}_j \ll \Lambda_1 \ll \tilde{\omega}_j$, and consequently $\tilde{\omega}_1(E), \tilde{\omega}_2(E) > 0$. But by the maximum principle, $\tilde{\omega}_j(E) \leq \omega_j(E)$, $j = 1, 2$. We have thus proven that if $\Lambda_1(T) > 0$, it cannot be that $\omega_1 \perp \omega_2$.

We now assume that $\Lambda_1(T) = 0$ and make the normalizing assumption distance $(z_j, \Gamma) \geq 1$, $j = 1, 2$.

LEMMA 1. *Suppose $z_0 \in \Gamma$ and $D = \{z\colon |z - z_0| \leq r\}$ where $r < 1$. Then*

$$\omega_1(D \cap \Gamma) \cdot \omega_2(D \cap \Gamma) \leq Ar^2$$

where A is independent of z_0, Γ and r.

Proof. This lemma should be credited to Beurling; it is contained in the last section of his thesis [**2**]. For completeness we include a proof. Without loss of generality the component Ω_1 is bounded. The set $\Gamma \setminus D$ can be written as a disjoint collection of open arcs γ_k. For exactly one of these arcs γ_k, call it γ, it is true that $C \setminus \{\gamma \cup D\}$ has a *bounded* component, call it $\hat{\Omega}_1$, containing z_1. Let $\hat{\Omega}_2$ denote the component of $\mathbf{C} \setminus \{\gamma \cup D\}$ containing z_2, and let $\hat{\omega}_j$ be the harmonic measures associated to $\hat{\Omega}_j$ and z_j, $j = 1, 2$. Then by the maximum principle,

$$\omega_j(D \cap \Gamma) \leq \hat{\omega}_j(D \cap \partial\hat{\Omega}_j), \qquad j = 1, 2. \tag{1}$$

Fix $t, r < t < 1$, and let $\gamma_1(t)$ be the unique subarc of $\{z \in \hat{\Omega}_1\colon |z - z_0| = t\}$ which separates D from z_1 in $\hat{\Omega}_1$. Let $\theta_1(t) = \Lambda_1(\gamma_1(t))$. Define in a similar fashion $\gamma_2(t)$ and $\theta_2(t)$ with respect to the domain $\hat{\Omega}_2$. The distortion theorem (see e.g. pp. 76–78 of [**1**]) asserts that

$$\hat{\omega}_j(D \cap \partial\hat{\Omega}_j) \leq A \exp\left\{-\pi \int_r^1 \frac{dt}{\theta_j(t)}\right\}, \qquad j = 1, 2.$$

Since $\theta_1(t) + \theta_2(t) \leq 2\pi t$, inequality (1) yields

$$\omega_1(D \cap \Gamma) \cdot \omega_2(D \cap \Gamma) \leq A^2 \exp\left\{-\pi \int_r^1 \frac{2\,dt}{\pi t}\right\} \leq A^2 r^2.$$

In [**6**] Makarov developed an ingenious and simple method using Plessner's theorem to show that whenever Ω is simply connected there is a set E of full harmonic measure and Hausdorff dimension one. We shall use a slightly sharper version of that result which has been obtained by Pommerenke [**8**]. Let Ω be a Jordan domain and let ω be harmonic measure with respect to Ω. Let E be the collection of all inner tangents with respect to Ω and let $F = \partial\Omega \setminus E$. Then Pommerenke shows that with $\omega^a \equiv \omega|_E$ and $\omega^s \equiv \omega|_F$ one has

(2) $$\omega^a \ll \Lambda_1 \ll \omega^a \quad \text{on } E$$

and

(3) For all $M, r_0 > 0$ there are disks $D_k = D(\zeta_k, r_k)$ where $r_k < r_0$,

$$\omega^s\left(\bigcup_k D_k\right) = \omega^s(F), \text{ and } \omega^s(D_k) \geq M r_k.$$

Let $\omega_j^a = \omega_j|_{T_j}$ and let $\omega_j^s = \omega_j - \omega_j^a$, $j = 1, 2$. Then since $\Lambda_1(T) = \Lambda_1(T_1 \cap T_2) = 0$, condition (2) shows that $\omega_1^a \perp \omega_2^a$. On the other hand, taking M large and applying (3) yields $\omega_1^a \perp \omega_2^s$ and $\omega_1^s \perp \omega_2^a$. It is therefore only necessary to prove $\omega_1^s \perp \omega_2^s$. To this end notice by (3) that there is a set $\tilde{F}$ such that $\omega_1^s(\tilde{F}) = \|\omega_1^s\|$ and such that for all $z \in \tilde{F}$ there are disks $D_n \downarrow z$ such that $z \in D_n$ and

(4) $$\omega_1^s(D_n) \geq M r_n,$$

where r_n is the radius of D_n. But by Lemma 1,

(5) $$\omega_1^s(D_n) \cdot \omega_2^s(D_n) \leq A r_n^2.$$

Taking M larger, we see that (4) and (5) imply $\omega_1^s \perp \omega_2^s$.

3. Remarks. When Γ is not a curve but an arc, the only point that needs modification in the preceding proof is that in Lemma 1 the conclusion must be weakened to $\omega_1(D \cap \Gamma) \cdot \omega_2(D \cap \Gamma) \leq A_{z_0} r^2$, where A_{z_0} depends on z_0, and one also requires that $r \leq r_{z_0} = \min\{|z_0 - \zeta_1|, |z_0 - \zeta_2|\}$, where ζ_1 and ζ_2 are the two endpoints of Γ. That is because the distortion theorem can only be used to conclude

$$\hat{\omega}_j(D \cap \partial\hat{\Omega}) \leq A \exp\left\{-\pi \int_r^{r_{z_0}} \frac{dt}{\theta_j(t)}\right\}.$$

Here $\hat{\Omega}$ is the appropriate domain formed out of Γ and D.

The theorem can be generalized to the case where Γ is not a Jordan curve. Let Ω_1 and Ω_2 be two disjoint, simply connected domains and denote by T_1 and T_2 the respective sets of inner tangent points. Then $\omega_1 \perp \omega_2$ if and only if $\Lambda_1(T_1 \cap T_2) = 0$. The proof of Lemma 1 is then most easily accomplished by the previous argument together with Beurling's theorem: $\omega(E) \leq C \exp\{-\pi\lambda\}$ where λ is the extremal length associated to all paths in a domain Ω joining some disk $K \Subset \Omega$ to $E \subset \partial\Omega$. (See **[7]** for an alternative proof.) A minor modification of the theorem can also be used to prove $\omega_1 \ll \omega_2 \ll \omega_1$ if and only if for all $\varepsilon > 0$ there are rectifiable curves $\Gamma_j \subset \overline{\Omega}_j$ such that $\omega_j(\Gamma_1 \cap \Gamma_2) > 1 - \varepsilon, j = 1, 2$.

It is worth noting that previous authors (see e.g. **[5]**) have used the Browder-Werner theorem to conclude in certain cases $\omega_1 \perp \omega_2$. An interesting problem that remains open is to *construct* one non constant function in $A(\Gamma)$ for a general arc Γ where $\omega_1 \perp \omega_2$.

Added in Proof. See **[3]** for a construction of non constant functions in $A(\Gamma)$.

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