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Harmonic Oscillator Realization of the Deformed Bogoliubov (p,q)-Transformations without First Finite Fock States

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Considering a simple generalization of the (p, q)-deformed boson oscillator algebra, which leads to a two-parameter deformed bosonic algebra in an infinite dimensional subspace of the harmonic oscillator Hilbert space without first finite Fock states, we establish a new harmonic oscillator realization of the deformed boson operators based on the Bogoliubov (p, q)-transformations. We obtain exact expressions for the transformation coefficients and show that they depend on arbitrary functions of p and q which can be interpreted as the parameters of the (p, q)-deformed $GL(2,C)$ group. We also examine the existence and structure of the corresponding deformed Fock-space representation for our problem.

§1. Introduction

Since the advent of the theory of parastatistics,[1),2)] there have been many attempts to generalize the canonical commutation relations. Motivations for such work have come from such diverse areas as resonance theory,[3),4)] intermediate statistics[5)] and quantum dissipative systems.[6)] However, the real impetus in the recent studies of deformed commutation relations has been the discovery of quantum groups and algebras.[7),8)] These algebras may be viewed as deformations of classical Lie algebras depending, in general, on one or more parameters. The representation theory of quantum algebras with a single deformation (or quantization) parameter q, has led to the development of q-deformed oscillator algebras (q-deformed bosons).[9)–11)] It has been found that the q-deformed oscillator is a useful tool for quantum field theory since it constitutes a structure more compatible with interactions. The number q, viewed as a convergence parameter can be used to regulate divergences appearing in field theory calculations.

Although one-parameter q-deformations have been mostly studied, the multi-parameter (mainly two-parameter) ones have aroused much interest because they become more flexible when we are dealing with applications to concrete physical models. For example, it has been found that the two-parameter qp-rotor model,[12)] having the symmetry afforded by the two-parameter quantum algebra $\mathrm{U}qp(\mathrm{u}2)$, is useful in rotational spectroscopy of some superdeformed as well as rare earth and actinide deformed nuclei[13)] and rotational spectroscopy of molecules.[14)] The introduction of a second parameter (say p) should permit more flexibility: this is especially appealing for rotational spectroscopy of nuclei that involves two parameters in the variable moment of inertia (VMI) model.[15)] Furthermore, it has been shown[12),13)]

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that the introduction of a second parameter of a “quantum algebra” nature increases the agreement between theory and experiment in a significant way.

By using the language of q-oscillators in describing the representations of quantum algebras with a single deformation parameter q, a two-parameter (p,q)-analogue of the q-boson oscillator has been derived from the study of a (p,q)-deformed $su(2)$ algebra.[16] The (p,q)-oscillator algebra is generated by three elements $\hat{A}, \hat{A}^+$ and $\hat{N}$ obeying the relations

$$\hat{A}\hat{A}^+ - p^{-1}\hat{A}^+\hat{A} = q^{\hat{N}}, \tag{1a}$$

$$\hat{A}\hat{A}^+ - q\hat{A}^+\hat{A} = p^{-\hat{N}}, \tag{1b}$$

$$[\hat{N},\hat{A}] = -\hat{A}, \quad [\hat{N},\hat{A}^+] = \hat{A}^+, \tag{1c}$$

where $\hat{A}, \hat{A}^+$identify, respectively, as the deformed annihilation and creation operators of a (p,q)–oscillator and $\hat{N} = \hat{a}^+\hat{a}$ is the excitation number operator of conventional (non-deformed) oscillator ($[\hat{a},\hat{a}^+] = 1$). The parameters p and q are independent deformation parameters which, in general, may be real or a phase factor. The (p,q)-algebra (1) can be mapped on the one-parameter deformation of the oscillator algebra. Indeed , if instead of $\hat{A}, \hat{A}^+$ one takes

$$\hat{\alpha} = p^{\hat{N}/2}\hat{A}, \quad \hat{\alpha}^+ = p^{(\hat{N}-1)/2}\hat{A}^+ \tag{2}$$

as generators, the relations (1) become

$$\hat{\alpha}\hat{\alpha}^+ - pq\hat{\alpha}^+\hat{\alpha} = 1, \tag{3a}$$

$$\hat{\alpha}\hat{\alpha}^+ - \hat{\alpha}^+\hat{\alpha} = (pq)^{\hat{N}}, \tag{3b}$$

$$[\hat{N},\hat{\alpha}] = -\hat{\alpha}, \quad [\hat{N},\hat{\alpha}^+] = \hat{\alpha}^+, \tag{3c}$$

where only the combination pq appears as a parameter. Furthermore, it should be noted that the relations (1a) and (1b) imply each other and the $q \leftrightarrow p^{-1}$ symmetry generalizes the $q \leftrightarrow q^{-1}$ symmetry of the q-oscillator.

Analogous to the familiar boson realization of the q-oscillator, the relation of $\hat{A}$ and $\hat{A}^+$ to the conventional boson operators $\hat{a}$ and $\hat{a}^+$ is given by

$$\hat{A} = \hat{a}f(\hat{N}) = \hat{a}\sqrt{\frac{[\hat{N}]_{p,q}}{\hat{N}}}, \quad \hat{A}^+ = f(\hat{N})\hat{a}^+ = \sqrt{\frac{[\hat{N}]_{p,q}}{\hat{N}}}\hat{a}^+, \tag{4}$$

where the symbol $[X]_{p,q}$ stands for $\frac{q^X - p^{-X}}{q-p^{-1}}$. As it is seen, the deformation function $f(\hat{N})$ has no zeros at positive integer eigenvalues of $\hat{N}$ (including zero). So the deformed annihilation operator$\hat{A}$ has a single vacuum state, i.e., $|0\rangle$, like the operator $\hat{a}$. On the other hand for those deformed operators $\hat{A}$'s for which the function $f(\hat{N})$ has zeros at positive integer eigenvalues of $\hat{N}$ there is a set of vacuum states. In this case if we assume that the operator $\hat{A}$ annihilates a set of number states $|n_i\rangle$, $i = 1, 2, 3, \cdots, k$ then we can construct a sector S_i by repeatedly applying $\hat{A}^+$ on the number state $|n_i\rangle$. Thus we have k sectors corresponding to the states that are annihilated by $\hat{A}$. A given sector may turn out to be either finite or infinite dimensional. In particular, the infinite dimensional sectors are of special interest. The reason is that in each infinite dimensional sector it is possible to construct an operator, say $\hat{B}^+$, which is the canonical conjugate of $\hat{A}$, i.e., $[\hat{A}, \hat{B}^+] = 1$. However, in the finite dimensional sectors the construction does not apply. This point plays an important role in the construction of coherent states associated with deformed algebras.

In this pair of papers there are two main goals that we have tried to develop. For the first goal, which corresponds to the present paper, our attempt has been devoted to establish a new harmonic oscillator realization of a two-parameter deformed bosonic oscillator in the form of Bogoliubov (p, q)-transformations in an infinite dimensional subspace of the harmonic oscillator Hilbert space without first finite Fock states, and construct a deformed Fock-space representation for our problem. The second goal, corresponding to the subsequent paper,[17] has been an effort to construct the associated deformed coherent states in the deformed Fock space and analyze their mathematical and quantum statistical properties.

The present paper is organized as follows. In the following section we describe a simple generalization of the (p, q)-deformed oscillator algebra which by a certain transformation leads to a new two-parameter deformed bosonic algebra in an infinite dimensional sector of the harmonic oscillator Hilbert space without first finite Fock states. In §3 we present the Bogoliubov (p, q)-transformations and establish a new harmonic oscillator realization for our deformed boson operators. The results obtained in that section is used as a basis for construction of a deformed Fock-space representation explained in §4. Finally, §5 contains a summary.

§2. Generalized (p, q)-oscillator algebra

We consider the following simple generalization of the relations (1a) and (1b),

$$\hat{A}\hat{A}^+ - p^{-1}\hat{A}^+\hat{A} = q^{\hat{N}}\Phi_1(p, q), \tag{5a}$$

$$\hat{A}\hat{A}^+ - q\hat{A}^+\hat{A} = p^{-\hat{N}}\Phi_2(p, q), \tag{5b}$$

in which p, q are taken to be real and positive and Φ_1, Φ_2 are two arbitrary well-behaved real and positive-valued functions of deformation parameters assuming to obey the following inequalities

$$\begin{aligned} &\Phi_2(p, q) > \Phi_1(p, q)\ ;\ \text{for } Q = pq > 1, \\ &\Phi_2(p, q) < \Phi_1(p, q)\ ;\ \text{for } Q = pq < 1. \end{aligned} \tag{6}$$

Choosing $\Phi_1 = \Phi_2$=1 one recovers relations (1a) and (1b). The harmonic oscillator realization of the generalized (p,q)-oscillator (5) in the simplest form

$$\hat{A} = \hat{a}f(\hat{N}, p, q), \quad \hat{A}^+ = f(\hat{N}, p, q)\hat{a}^+ \tag{7}$$

is

$$\hat{A} = \hat{a}\sqrt{\frac{q^{\hat{N}}\Phi_1(p,q) - p^{-\hat{N}}\Phi_2(p,q)}{\hat{N}(q-p^{-1})}}, \tag{8a}$$

$$\hat{A}^+ = \sqrt{\frac{q^{\hat{N}}\Phi_1(p,q) - p^{-\hat{N}}\Phi_2(p,q)}{\hat{N}(q-p^{-1})}}\hat{a}^+. \tag{8b}$$

As it is clear there are two vacua for the deformed operator $\hat{A}$, namely, the ground state $|0\rangle$ and the number state $|k_0\rangle$ such that

$$k_0 = \frac{1}{\ln Q}\ln\frac{\Phi_2(p,q)}{\Phi_1(p,q)}. \tag{9}$$

It is obvious that according to (9) the conditions (6) guarantee the integer number k_0 be nonnegative. In this manner we have two sectors S_0 and S_{k_0} which are constructed by repeatedly applying $\hat{A}^+$ on $|0\rangle$ and $|k_0\rangle$, respectively. The sector S_0 is of finite dimension spanned by the states $|0\rangle, |1\rangle, \cdots |k_0-1\rangle$ and is invariant with respect to the action of the operators (8a,b). Since, $\hat{A}^+|k_0-1\rangle = 0$ this sector gives a parafermionic-like representation of the algebra (5a, b) of the order k_0. To some extent, the situation is analogous to that taking place for the deformed Heisenberg algebra with reflection (for more details see Ref. 18). On the other hand the infinite dimensional sector S_{k_0}, which is of special interest in our treatment, is spanned by the states $|k_0\rangle, |k_0+1\rangle, \cdots$ and gives a bosonic representation of the algebra (5a,b) without first finite Fock states. Evidently, in this sector the values of the expression under the square root in (8) are non-negative for both the cases $Q<1$, $Q>1$. The usual situation $k_0 = 0$ corresponds to the cases $\Phi_1 = 1 = \Phi_2$ or $\Phi_1 = \Phi_2$. It is trivial that these two cases differ in the energy spectrum of the oscillator. This difference corresponds to different normalization of the commutation relations (5a, b). Moreover, it should be noted that in the absence of deformation $(p,q=1)$ the consistency of the relations (5a) and (5b) requires $\Phi_1 = \Phi_2$ and according to (9) it implies that in this case $k_0 = 0$ (coincidence of the two subspaces S_0 and S_{k_0}).

The Fock-space representation of the generalized (p,q)-oscillator (5) in the sector S_{k_0}can be easily constructed. With $\{|n>_{p,q} \equiv |k_0+m>_{p,q}; m = 0,1,2,..\}$ as the complete orthonormal set of number eigenstates, one finds

$$\begin{aligned}\hat{A}\,|n\rangle_{p,q} &= \sqrt{q^{k_0}\Phi_1(p,q)[n-k_0]_{p,q}}\,|n-1\rangle_{p,q}\\ &= \sqrt{p^{-k_0}\Phi_2(p,q)[n-k_0]_{p,q}}\,|n-1\rangle_{p,q}\,,\end{aligned} \tag{10a}$$

$$\begin{aligned}\hat{A}^+\,|n\rangle_{p,q} &= \sqrt{q^{k_0}\Phi_1(p,q)[n-k_0+1]_{p,q}}\,|n+1\rangle_{p,q}\\ &= \sqrt{p^{-k_0}\Phi_2(p,q)[n-k_0+1]_{p,q}}\,|n+1\rangle_{p,q}\,,\end{aligned} \tag{10b}$$

$$\hat{A}\hat{A}^+ = q^{k_0}\Phi_1(p,q)[\hat{N}-k_0+1]_{p,q} = p^{-k_0}\Phi_2(p,q)[\hat{N}-k_0+1]_{p,q}, \tag{10c}$$

$$\hat{A}^+\hat{A} = q^{k_0}\Phi_1(p,q)[\hat{N}-k_0]_{p,q} = p^{-k_0}\Phi_2(p,q)[\hat{N}-k_0]_{p,q}, \tag{10d}$$

$$
\begin{aligned}
|n\rangle_{p,q} \equiv |k_0+m\rangle_{p,q} &= \left(q^{k_0}\Phi_1(p,q)\right)^{-m/2} \frac{(\hat{A}^+)^m}{\sqrt{[m]_{p,q}!}}\,|k_0\rangle \\
&= \left(p^{-k_0}\Phi_2(p,q)\right)^{-m/2} \frac{(\hat{A}^+)^m}{\sqrt{[m]_{p,q}!}}\,|k_0\rangle \\
&= \sqrt{\frac{k_0!}{(k_0+m)!}}(\hat{a}^+)^m\,|k_0\rangle\,,
\end{aligned} \tag{10e}
$$

with $[m]_{p,q}! = [m]_{p,q}[m-1]_{p,q}[m-2]_{p,q}....1$ and $[0]_{p,q}! = 1$. The corresponding number operator $\hat{N}_{p,q}$ reads as

$$
\begin{aligned}
\hat{N}_{p,q} &= \sum_{r=1}^{\infty} \frac{(1-Q)^r}{1-Q^r} p^{-\frac{r(r+1)}{2}+r(\hat{N}-k_0)} \left(q^{k_0}\Phi_1(p,q)\right)^{-r} (\hat{A}^+)^r(\hat{A})^r, \\
&\hat{N}_{p,q}\,|k_0+m\rangle_{p,q} = m\,|k_0+m\rangle_{p,q}\,.
\end{aligned} \tag{10f}
$$

With the choice $k_0 = 0$ ($\Phi_1 = 1 = \Phi_2$) the relations (10) reduce to the corresponding relations for usual (p,q)-deformed oscillator.[16)]

Now we consider the following transformation in the sector S_{k_0}

$$
\hat{\beta} = p^{\frac{(\hat{N}-k_0)}{2}}\Phi_2^{-1/2}(p,q)\hat{A}, \quad \hat{\beta}^+ = p^{\frac{(\hat{N}-k_0-1)}{2}}\Phi_2^{-1/2}(p,q)\hat{A}^+, \tag{11}
$$

which can be regarded as a generalization of the transformation (2) . There are other ways of transforming the operators. The transformation (11) is the version that we will be using for our purposes. Then according to (5) the two new operators $\hat{\beta}$ and $\hat{\beta}^+$ should satisfy the following commutation rules

$$
\hat{\beta}\hat{\beta}^+ - Q\hat{\beta}^+\hat{\beta} = p^{-k_0}, \tag{12a}
$$

$$
\hat{\beta}\hat{\beta}^+ - \hat{\beta}^+\hat{\beta} = p^{-k_0}Q^{\hat{N}-k_0}, \tag{12b}
$$

and

$$
\hat{\beta}\hat{\beta}^+ = p^{-k_0}\left\{\hat{N}-k_0+1\right\}_Q, \quad \hat{\beta}^+\hat{\beta} = p^{-k_0}\left\{\hat{N}-k_0\right\}_Q, \tag{13}
$$

where $\{X\}_Q = \frac{1-Q^X}{1-Q}$ with $Q = pq$. Therefore, the relations (12a,b) define a two-parameter deformed boson algebra in a subspace of Hilbert space without first finite Fock states. It is evident that for $k_0 = 0$ (coincidence of the two subspaces S_0 and S_{k_0}) the relations (12a) and (12b) reduce to one-parameter deformed algebras (3a) and (3b), respectively. By using the relations (8) and (9) the operators $\hat{\beta}$ and $\hat{\beta}^+$ can be expressed in terms of non-deformed boson operators $\hat{a}$ and $\hat{a}^+$,

$$
\hat{\beta} = \hat{a}p^{-\frac{k_0}{2}}\sqrt{\frac{Q^{\hat{N}}\Phi_1(p,q)-\Phi_2(p,q)}{\hat{N}(Q-1)\Phi_2(p,q)}} = \hat{a}p^{-\frac{k_0}{2}}\sqrt{\frac{Q^{\hat{N}-k_0}-1}{\hat{N}(Q-1)}}, \tag{14a}
$$

$$
\hat{\beta}^+ = p^{-\frac{k_0}{2}}\sqrt{\frac{Q^{\hat{N}}\Phi_1(p,q)-\Phi_2(p,q)}{\hat{N}(Q-1)\Phi_2(p,q)}}\hat{a}^+ = p^{-\frac{k_0}{2}}\sqrt{\frac{Q^{\hat{N}-k_0}-1}{\hat{N}(Q-1)}}\hat{a}^+. \tag{14b}
$$

§3. Deformed Bogoliubov (p,q)-transformations in the subspace S_{k_0}

In this section our main objective is to generalize (14) and find a representation for the deformed oscillator (12) in terms of usual harmonic oscillator in the form

$$\hat{\beta}' = \hat{a}u(\hat{N}-k_0) + v(\hat{N}-k_0)\hat{a}^+, \tag{15a}$$

$$\hat{\beta}'^+ = u^*(\hat{N}-k_0)\hat{a}^+ + \hat{a}v^*(\hat{N}-k_0), \tag{15b}$$

where the functions u and v are to be determined. It is easily seen that with this representation, the commutation relations (12a) and (12b) cannot be satisfied simultaneously and so they should be treated separately. Strictly speaking, the representation (15) for the algebra (12) results in two different types of oscillator realization (two different forms of u and v).

We first consider the deformed algebra (12a) for the operators $\hat{\beta}'$ and $\hat{\beta}'^+$, i.e.,

$$\hat{\beta}'\hat{\beta}'^+ - Q\ \hat{\beta}'^+\hat{\beta}' = p^{-k_0}. \tag{16a}$$

Using (14a,b) one can write (15) in the form

$$\hat{\beta}' = \hat{\beta}\tilde{u}(\hat{N}-k_0) + \tilde{v}(\hat{N}-k_0)\hat{\beta}^+, \tag{16b}$$

$$\hat{\beta}'^+ = \tilde{u}^*(\hat{N}-k_0)\hat{\beta}^+ + \hat{\beta}\tilde{v}^*(\hat{N}-k_0), \tag{16c}$$

i.e.,

$$\begin{pmatrix} \hat{\beta}' \\ \hat{\beta}'^+ \end{pmatrix} = \begin{pmatrix} \tilde{u}(\hat{N}-k_0+1) & \tilde{v}(\hat{N}-k_0) \\ \tilde{v}^*(\hat{N}-k_0+1) & \tilde{u}^*(\hat{N}-k_0) \end{pmatrix} \begin{pmatrix} \hat{\beta} \\ \hat{\beta}^+ \end{pmatrix}, \tag{16d}$$

with

$$\tilde{u}(\hat{N}-k_0) = p^{\frac{k_0}{2}}\sqrt{\frac{\hat{N}(Q-1)\Phi_2(p,q)}{Q^{\hat{N}}\Phi_1(p,q)-\Phi_2(p,q)}}u(\hat{N}-k_0), \tag{17a}$$

$$\tilde{v}(\hat{N}-k_0) = p^{\frac{k_0}{2}}\sqrt{\frac{\hat{N}(Q-1)\Phi_2(p,q)}{Q^{\hat{N}}\Phi_1(p,q)-\Phi_2(p,q)}}v(\hat{N}-k_0). \tag{17b}$$

The transformation (16d), which can be considered as the Bogoliubov (p,q)-transformation defined in subspace S_{k_0}, acts on the two-dimensional quantum space of vectors $(\hat{\beta}, \hat{\beta}^+)$ satisfying (12a) and preserve this property for $(\hat{\beta}', \hat{\beta}'^+)$ [Eq. (16a)]. So we can interpret (16d) as an element of the (p,q)-deformed $GL(2,C)$ group. However this (p,q)-deformed group is not related to the quantum group $GL_{p,q}(2,C)$ as the quantities $\tilde{u}, \tilde{v}, \tilde{u}^*, \tilde{v}^*$ are commuting operators while the elements of $GL_{p,q}(2,C)$ group have non-trivial commutation relations.[19)]

In order to determine u and v of the representation (15) we proceed in the following way. By substituting (15a,b) in (16a) we find the following set of equations

$$F(\hat{N}-k_0+1) - QF(\hat{N}-k_0) + G(\hat{N}-k_0) - QG(\hat{N}-k_0+1) = p^{-k_0}, \tag{18a}$$

$$u(\hat{N}-k_0)v^*(\hat{N}-k_0+1) = Qv^*(\hat{N}-k_0)u(\hat{N}-k_0+1), \tag{18b}$$

$$v(\hat{N}-k_0+1)u^*(\hat{N}-k_0) = Qu^*(\hat{N}-k_0+1)v(\hat{N}-k_0), \tag{18c}$$

where

$$F(\hat{N}-k_0) = \hat{N}u^*(\hat{N}-k_0)u(\hat{N}-k_0),$$
$$G(\hat{N}-k_0) = \hat{N}v^*(\hat{N}-k_0)v(\hat{N}-k_0). \tag{18d}$$

From (18b,c) we find

$$\frac{G(\hat{N}-k_0)}{F(\hat{N}-k_0)} = Q^{2(\hat{N}-k_0)}W(p,q), \tag{19}$$

where $W(p,q)$ is an arbitrary function of deformation parameters. Therefore equation (18a) takes the form

$$F(\hat{N}-k_0+1)\left(1-Q^{2(\hat{N}-k_0+2)}\tilde{W}(p,q)\right)-QF(\hat{N}-k_0)\left(1-Q^{2(\hat{N}-k_0)}\tilde{W}(p,q)\right) = p^{-k_0}, \tag{20}$$

where $\tilde{W}(p,q) = Q^{-1}W(p,q)$.

To solve Eq. (20) we first determine the solution $F_0(\hat{N}-k_0)$ to the corresponding homogeneous equation

$$F_0(\hat{N}-k_0+1)\left(1-Q^{2(\hat{N}-k_0+2)}\tilde{W}(p,q)\right) = QF_0(\hat{N}-k_0)\left(1-Q^{2(\hat{N}-k_0)}\tilde{W}(p,q)\right). \tag{21}$$

This equation is of the general form

$$F_0(\hat{N}-k_0+1)H(\hat{N}-k_0+m) = g(p,q)F_0(\hat{N}-k_0)H(\hat{N}-k_0), \tag{22}$$

where $g(p,q)$ and $H(\hat{N}-k_0+m)$, $(m=1,2,\cdots)$ are known functions while $F_0(\hat{N}-k_0)$ is the function to be determined. In Eq. (21) we have

$$m=2, \quad g(p,q)=Q \quad \text{and} \quad H(\hat{N}-k_0) = \left(1-Q^{2(\hat{N}-k_0)}\tilde{W}(p,q)\right). \tag{23}$$

If F_1 and F_2 are solutions of (22) then

$$\frac{F_1(\hat{N}-k_0+1)}{F_2(\hat{N}-k_0+1)} = \frac{F_1(\hat{N}-k_0)}{F_2(\hat{N}-k_0)} = P(\hat{N}-k_0,p,q) \tag{24}$$

and thus $P(\hat{N}-k_0,p,q)$ is a periodic function, $P(\hat{N}-k_0,p,q) = P(\hat{N}-k_0+1,p,q)$. It means that the general solution of (22) is a product of a special solution and an arbitrary periodic function. We can search for this general solution in the form

$$F_0(\hat{N}-k_0) = \left(\prod_{i=0}^{m-1} R(\hat{N}-k_0+i)\right)P(\hat{N}-k_0,p,q). \tag{25}$$

Putting (25) in (22) gives

$$R(\hat{N}-k_0+m)H(\hat{N}-k_0+m) = g(p,q)R(\hat{N}-k_0)H(\hat{N}-k_0), \tag{26}$$

which after simplification yields

$$R(\hat{N}-k_0) = \frac{g(p,q)^{\frac{\hat{N}-k_0}{m}}}{H(\hat{N}-k_0)}\tilde{P}(\hat{N}-k_0,p,q), \tag{27}$$

where $\tilde{P}(\hat{N}-k_0+m,p,q)=\tilde{P}(\hat{N}-k_0,p,q)$ is an arbitrary periodic function and we can put this function to unity without loss of generality. Using (23), (25) and (27) we find

$$F_0(\hat{N}-k_0)=\frac{Q^{\hat{N}-k_0+1/2}P(\hat{N}-k_0,p,q)}{\left(1-Q^{2(\hat{N}-k_0)}\tilde{W}(p,q)\right)\left(1-Q^{2(\hat{N}-k_0-1)}\tilde{W}(p,q)\right)}. \tag{28}$$

Now we seek for the general solution of (20). For this purpose we propose

$$F(\hat{N}-k_0)=F_0(\hat{N}-k_0)Y(\hat{N}-k_0), \tag{29}$$

where $Y(\hat{N}-k_0)$ is to be determined. Putting (29) in (20) yields

$$Y(\hat{N}-k_0+1,p,q)=Y(\hat{N}-k_0,p,q)+\frac{p^{-k_0}\left(1-Q^{2(\hat{N}-k_0+1)}\tilde{W}(p,q)\right)}{P(\hat{N}-k_0,p,q)Q^{\hat{N}-k_0+3/2}}. \tag{30}$$

With the use of standard techniques the solution to (30) is found to be

$$\begin{aligned}Y(\hat{N}-k_0,p,q)=Y(0,p,q)+\frac{p^{-k_0}}{P(\hat{N}-k_0,p,q)Q^{3/2}}&\Big(Q^{-(\hat{N}-k_0-1)}\\ &-Q^2\tilde{W}(p,q)\Big)\left\{\hat{N}-k_0\right\}_Q.\end{aligned} \tag{31}$$

Using (31), (29) and (18d) we find that the condition $u(\hat{N}-k_0)\Big|_{\hat{N}=k_0}<\infty$ leads to $Y(0,p,q)=0$. Thus by making use of (31), (29) and (28) we arrive at the following solution to Eq. (20)

$$F(\hat{N}-k_0)=\frac{p^{-k_0}\left(1-Q^{(\hat{N}-k)_0}W(p,q)\right)\left\{\hat{N}-k_0\right\}_Q}{\left(1-Q^{2(\hat{N}-k_0)+1}W(p,q)\right)\left(1-Q^{2(\hat{N}-k_0)-1}W(p,q)\right)}, \tag{32}$$

and from (19) we have

$$G(\hat{N}-k_0)=Q^{2(\hat{N}-k_0)}W(p,q)F(\hat{N}-k_0). \tag{33}$$

Finally, by using (18d) we arrive at the following expressions for u and v

$$u(\hat{N}-k_0)=\left|u(\hat{N}-k_0)\right|e^{i\zeta(p,q,\hat{N}-k_0)}, \tag{34a}$$

$$v(\hat{N}-k_0)=\left|u(\hat{N}-k_0)\right|Q^{(\hat{N}-k_0)}\sqrt{W(p,q)}e^{i\xi(p,q,\hat{N}-k_0)}, \tag{34b}$$

where

$$\left|u(\hat{N}-k_0)\right|=\sqrt{\frac{p^{-k_0}\left(1-Q^{(\hat{N}-k_0)}W(p,q)\right)\frac{\left\{\hat{N}-k_0\right\}_Q}{\hat{N}}}{\left(1-Q^{2(\hat{N}-k_0)+1}W(p,q)\right)\left(1-Q^{2(\hat{N}-k_0)-1}W(p,q)\right)}} \tag{34c}$$

and $\zeta(p,q,\hat{N}-k_0), \xi(p,q,\hat{N}-k_0)$ are some arbitrary phase factors periodic with respect to $\hat{N}$. The representation (15) may now be written as

$$\hat{\beta}' = \hat{a}\left|u(\hat{N}-k_0)\right| e^{i\zeta(p,q,\hat{N}-k_0)} + Q^{(\hat{N}-k_0)}\sqrt{W(p,q)}\left|u(\hat{N}-k_0)\right| e^{i\xi(p,q,\hat{N}-k_0)}\hat{a}^+, \tag{35a}$$

$$\hat{\beta}'^+ = \left|u(\hat{N}-k_0)\right| e^{-i\zeta(p,q,\hat{N}-k_0)}\hat{a}^+ + \hat{a}Q^{(\hat{N}-k_0)}\sqrt{W(p,q)}\left|u(\hat{N}-k_0)\right| e^{-i\xi(p,q,\hat{N}-k_0)}. \tag{35b}$$

Using (17a,b) and (34) the Bogoliubov (p,q)-transformations (16d) may be written as

$$\begin{pmatrix}\hat{\beta}' \\ \hat{\beta}'+\end{pmatrix} = \begin{pmatrix} \left|\tilde{u}(\hat{N}-k_0+1)\right| e^{i\zeta(p,q,\hat{N}-k_0)} & Q^{(\hat{N}-k_0)}\sqrt{W(p,q)}\left|\tilde{u}(\hat{N}-k_0)\right| e^{i\xi(p,q,\hat{N}-k_0)} \\ Q^{(\hat{N}-k_0+1)}\sqrt{W(p,q)}\left|\tilde{u}(\hat{N}-k_0+1)\right| e^{-i\xi(p,q,\hat{N}-k_0)} & \left|\tilde{u}(\hat{N}-k_0)\right| e^{-i\zeta(p,q,\hat{N}-k_0)} \end{pmatrix}\begin{pmatrix}\hat{\beta} \\ \hat{\beta}+\end{pmatrix} \tag{36a}$$

with

$$\left|\tilde{u}(\hat{N}-k_0)\right| = \sqrt{\frac{1-Q^{(\hat{N}-k_0)}W(p,q)}{\left(1-Q^{2(\hat{N}-k_0)+1}W(p,q)\right)\left(1-Q^{2(\hat{N}-k_0)-1}W(p,q)\right)}}. \tag{36b}$$

Now we consider the deformed algebra (12b) for $\hat{\beta}'$ and $\hat{\beta}'^+$, i.e.,

$$\hat{\beta}'\hat{\beta}'^+ - \hat{\beta}'^+\hat{\beta}' = p^{-k_0}Q^{(\hat{N}-k_0)}. \tag{37a}$$

By employing the same procedure as before we obtain the following expressions for the transformation coefficients

$$u(\hat{N}-k_0) = \sqrt{\frac{p^{-k_0}}{1-W(p,q)}\frac{\{\hat{N}-k_0\}_Q}{\hat{N}}}e^{i\zeta(p,q,\hat{N}-k_0)}, \tag{37b}$$

$$v(\hat{N}-k_0) = \sqrt{\frac{p^{-k_0}W(p,q)}{1-W(p,q)}\frac{\{\hat{N}-k_0\}_Q}{\hat{N}}}e^{i\xi(p,q,\hat{N}-k_0)}, \tag{37c}$$

$$\tilde{u}(\hat{N}-k_0) = \sqrt{\frac{1}{1-W(p,q)}}e^{i\zeta(p,q,\hat{N}-k_0)}, \tag{37d}$$

$$\tilde{v}(\hat{N}-k_0) = \sqrt{\frac{W(p,q)}{1-W(p,q)}}e^{i\xi(p,q,\hat{N}-k_0)}. \tag{37e}$$

Now we would like to give an interpretation for the functions W, ζ and ξ. To this end consider the case in which $p,q \to 1$ $(k_0 = 0)$ and the phase factors ζ and ξ be independent of $\hat{N}$. From (17), (34), (36b) and (37b,c) we find that the two types of solutions for u and v, corresponding to the algebras (16a) and (37a) respectively, become the same and independent of $\hat{N}$,

$$u = \sqrt{\frac{1}{1-W(1,1)}}e^{i\zeta(1,1)}, \quad v = \sqrt{\frac{W(1,1)}{1-W(1,1)}}e^{i\xi(1,1)}. \tag{38}$$

Furthermore, in this case the transformation matrix in (16d) becomes unimodular, $\tilde{u}\tilde{u}^* - \tilde{v}\tilde{v}^* = 1$. Therefore, in the absence of deformation the transformation (16d) stands for $SL(2,C)$ canonical transformation for the ordinary oscillator, where $W(1{,}1)$, $\zeta(1{,}1)$ and $\xi(1{,}1)$ are the parameters of the transformation. Thus it is reasonable to interpret $W(p,q)$, $\zeta(p,q,\hat{N}-k_0)$ and $\xi(p,q,\hat{N}-k_0)$ as the parameters of the (p,q)-deformed $GL(2,C)$ transformation in the subspace S_{k_0}. Furthermore as it is expected for $W=0$ and $\zeta=\xi=0$, the representation (15) coincides with (14).

Establishing a new type of harmonic oscillator realization of the bosonic (p,q)-deformed algebra (12) in the form of Bogoliubov (p,q)-transformations we are now in a position to seek for the corresponding Fock-space representation, which will be carried out in the next section.

§4. Fock-space representation

According to Ref. 20) the necessary and sufficient conditions for the one-dimensional quantum mechanical problem to allow a Fock-space representation is that there exists a vacuum state $|\mathrm{vac}\rangle$ such that

$$\hat{b}\,|\mathrm{vac}\rangle = 0, \quad \langle\mathrm{vac}|\,\hat{b}\hat{b}^+\,|\mathrm{vac}\rangle > 0, \tag{39a}$$

together with

$$\hat{b}\hat{b}^+ \neq \hat{b}^+\hat{b}, \quad [\hat{b}\hat{b}^+, \hat{b}^+\hat{b}] = 0, \tag{39b}$$

in which $\hat{b}$ and $\hat{b}^+$ are the corresponding annihilation and creation operators, respectively.

On the basis of the above criteria, we now examine the existence of a Fock-space representation for each of the deformed oscillators (16a) and (37a). Assuming $|\psi_0\rangle_Q^{(k_0)}$ be the normalized vacuum state for the oscillator $(\hat{\beta}', \hat{\beta}'^+)$ obeying (16a) it is easily found that all conditions (39) are satisfied and thus the Fock-space representation exists. However, in the case of (37a) if one applies the conditions (39) it is found that the condition $[\hat{\beta}'\hat{\beta}'^+, \hat{\beta}'^+\hat{\beta}'] = 0$ is satisfied only if $p, q = 1$ (the absence of deformation) or $W=0$ (no Bogoliubov transformations). Therefore our conclusion is that for the deformed operators $(\hat{\beta}', \hat{\beta}'^+)$ [given by (15a,b) or (16d)] only in the case of oscillator algebra (16a) one can find a Fock-space representation. Hereafter, we thus abandon (37a) and try to find a Fock-space representation corresponding to the deformed algebra (16a).

We first construct the deformed vacuum state $|\psi_0\rangle_Q^{(k_0)}$. For this purpose we consider the following number state expansion

$$|\psi_0\rangle_Q^{(k_0)} = \sum_{n=k_0}^{\infty} c_n\,|n\rangle. \tag{40}$$

Then the condition $\hat{\beta}'\,|\psi_0\rangle_Q^{(k_0)} = 0$ results in

$$c_{k_0+1} = 0,$$

$$c_{n+1}\sqrt{n+1}u(n-k_0+1)+c_{n-1}\sqrt{n}v(n-k_0)=0 \quad ; \quad n\geq k_0+1, \tag{41}$$

where u and v are given by (34). Straightforward calculation yields

$$|\psi_0\rangle_Q^{(k_0)}=c_{k_0}\sum_{m=0}^{\infty}\frac{\left(-\sqrt{W(p,q)}\;\mathrm{e}^{i(\xi-\zeta)}(\hat{a}^+)^2\right)^m}{2^m m!\;\mathrm{L}(m,k_0,Q)}|k_0\rangle, \tag{42a}$$

with

$$L(m,k_{0,}Q)=$$
$$\prod_{j=1}^{m}\sqrt{\frac{(1-Q^{2j}W(p,q))(1-Q^{4j-3}W(p,q))}{(1-Q^{2j-1}W(p,q))(1-Q^{4j+1}W(p,q))}(1+Q^{-j})(k_0+2j-1)}\frac{\{m\}_{1/Q}!}{\sqrt{(2m)!\{2m\}_Q!}} \tag{42b}$$

and the coefficient c_{k_0} is obtained by imposing the normalization condition ${}_Q^{(k_0)}\langle\psi_0|\psi_0\rangle_Q^{(k_0)}=1$. It is easily seen that if $p,q\rightarrow 1(k_0=0)$ and ζ, ξ be independent of $\hat{N}$ (non-deformed case) then

$$\begin{aligned}\lim_{p,q\rightarrow 1}|\psi_0\rangle_Q^{(k_0)}\equiv|\psi_0\rangle&=c_0\sum_{m=0}^{\infty}\frac{\left(-e^{i(\xi(1,1)-\zeta(1,1))}\sqrt{W(1,1)}(\hat{a}^+)^2\right)^m}{2^m m!}|0\rangle\\&=c_0\exp\left(-\frac{v}{2u}(\hat{a}^+)^2\right)|0\rangle\,.\end{aligned} \tag{43}$$

The state $|\psi_0\rangle$ is indeed the well-known squeezed vacuum state of the boson field.[21)]Therefore it is reasonable to interpret $|\psi_0\rangle_Q^{(k_0)}$ as the two-parameter deformed squeezed vacuum state in the subspace S_{k_0}.

As usual, the deformed excited states (number states) are generated by repeated application of the deformed creation operator $\hat{\beta'}^+$ on the deformed vacuum state $|\psi_0\rangle_Q^{(k_0)}$,

$$\hat{\beta'}^+|\psi_0\rangle_Q^{(k_0)}=t_1|\psi_1\rangle_Q^{(k_0)},\;\hat{\beta'}^+|\psi_1\rangle_Q^{(k_0)}=t_2|\psi_2\rangle_Q^{(k_0)},\cdots,\hat{\beta'}^+|\psi_{n-1}\rangle_Q^{(k_0)}=t_n|\psi_n\rangle_Q^{(k_0)}\,. \tag{44}$$

We may, therefore, write

$$|\psi_n\rangle_Q^{(k_0)}=(t_1t_2\cdots t_n)^{-1}(\hat{\beta'}^+)^n|\psi_0\rangle_Q^{(k_0)}\,. \tag{45}$$

The matrix elements of $\hat{\beta}'$ and $\hat{\beta'}^+$ in the deformed orthonormal number states $|\psi_n\rangle_Q^{(k_0)}$ are therefore of the form

$${}_Q^{(k_0)}\langle\psi_m|\hat{\beta}'|\psi_n\rangle_Q^{(k_0)}=t_n\delta_{m,n-1},\quad {}_Q^{(k_0)}\langle\psi_m|\hat{\beta}'^+|\psi_n\rangle_Q^{(k_0)}=t_{n+1}\delta_{m,n+1}. \tag{46}$$

Taking matrix elements of (16a) and using (46) we arrive at the following recursion relation for the coefficients t_n,

$$t_{n+1}^2-Qt_n^2=p^{-k_0}, \tag{47}$$

which is readily solved to yield

$$t_n^2 = p^{-k_0}\frac{1-Q^n}{1-Q} \equiv p^{-k_0}\{n\}_Q\,. \tag{48}$$

Therefore we have

$$|\psi_n\rangle_Q^{(k_0)} = \frac{p^{\frac{nk_0}{2}}}{\sqrt{\{n\}_Q!}}\left(\hat{\beta}'^{+}\right)^n |\psi_0\rangle_Q^{(k_0)}\,, \tag{49a}$$

together with

$$\hat{\beta}'^{+}|\psi_n\rangle_Q^{(k_0)} = \sqrt{p^{-k_0}\{n+1\}_Q}\,|\psi_{n+1}\rangle_Q^{(k_0)}\,, \tag{49b}$$

$$\hat{\beta}'|\psi_n\rangle_Q^{(k_0)} = \sqrt{p^{-k_0}\{n\}_Q}\,|\psi_{n-1}\rangle_Q^{(k_0)}\,, \tag{49c}$$

where$\{n\}_Q! = \{n\}_Q\{n-1\}_Q\cdots 1$. Furthermore, it can be shown that the matrix elements of $(\hat{\beta}'^{+})^r\hat{\beta}'^r$ and $\hat{\beta}'^r(\hat{\beta}'^{+})^r$ are respectively given by

$${}_Q^{(k_0)}\langle\psi_m|\,(\hat{\beta}'^{+})^r\hat{\beta}'^r\,|\psi_n\rangle_Q^{(k_0)} = p^{-rk_0}\frac{\{m\}_Q!}{\{m-r\}_Q!}\delta_{m,n}, \tag{50a}$$

$${}_Q^{(k_0)}\langle\psi_m|\,\hat{\beta}'^r(\hat{\beta}'^{+})^r\,|\psi_n\rangle_Q^{(k_0)} = p^{-rk_0}\frac{\{m+r\}_Q!}{\{m\}_Q!}\delta_{m,n}. \tag{50b}$$

Finally, we find an appropriate deformed number operator $\hat{N}_{Q,k_0}$ such that $\hat{N}_{Q,k_0}|\psi_m\rangle_Q^{(k_0)} = m\,|\psi_m\rangle_Q^{(k_0)}$. To this end, we can use the following identity[22)]

$$\sum_{r=1}^{m}\frac{(1-Q)^r}{1-Q^r}\frac{\{m\}_Q!}{\{m-r\}_Q!} = m. \tag{51}$$

In view of this identity it follows from (50a) that

$$\sum_{r=1}^{\infty}\frac{(1-Q)^r}{1-Q^r}\,{}_Q^{(k_0)}\langle\psi_m|p^{rk_0}(\hat{\beta}'^{+})^r(\hat{\beta}')^r\,|\psi_n\rangle_Q^{(k_0)} = m\delta_{m,n}. \tag{52}$$

Therefore, we can write

$$\hat{N}_{Q,k_0} = \sum_{r=1}^{\infty}p^{rk_0}\frac{(1-Q)^r}{1-Q^r}(\hat{\beta}'^{+})^r(\hat{\beta}')^r. \tag{53}$$

One may also check directly, using (16a), that $\hat{N}_{Q,k_0}$ satisfies the following relations

$$[\hat{N}_{Q,k_0},\hat{\beta}'] = -\hat{\beta}', \quad [\hat{N}_{Q,k_0},\hat{\beta}'^{+}] = \hat{\beta}'^{+}. \tag{54}$$

It can be easily shown that for $p,q \to 1 (k_0 = 0), W = 0$ and $\zeta,\xi = 0$, Eq. (53) reduces to the usual expression for the conventional number operator $\hat{N} = \hat{a}^{+}\hat{a}$.

§5. Summary

Introducing a simple generalization of the (p, q)-boson oscillator algebra, we have constructed a two-parameter deformed bosonic algebra in an infinite dimensional subspace of the harmonic oscillator Hilbert space without first finite Fock states. We have established a new harmonic oscillator realization of the deformed boson operators in the form of Bogoliubov (p, q)-transformations. We have also obtained exact expressions for the transformation coefficients and demonstrated the existence of arbitrary functions of p and q which in the limit $p, q \to 1$, are related to the parameters of the $SL(2,C)$ group. Finally, we have examined the existence and structure of the corresponding deformed Fock-space representation for the new harmonic oscillator realization of our deformed boson operators.

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