# Harmonic sums and polylogarithms at negative multi-indices

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**Abstract**

Extending the Faulhaber's formula, the Bernoulli polynomials and the Eulerian polynomials, we study the multi-indexed harmonic sums and polylogarithms. Our techniques are based on the combinatorics of the noncommutative generating series in the quasi-shuffle[1] Hopf algebra.

## 1 Introduction

In this paper, in order to implement the renormalization of the following divergent polyzetas [11, 12, 6]

$$\sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}, \quad \text{for} \quad s_1,\ldots,s_r\in\mathbb{N},$$

we study, via the combinatorics of noncommutative generating series in the Hopf quasi-shuffle algebra [9, 10], the relations among *harmonic sums* and *polylogarithms*, indexed by the words $y_{s_1}\ldots y_{s_r}$ belonging to the monoid $Y_0^*$, generated by the alphabet $Y_0=\{y_k\}_{k\ge0}$ and among their noncommutative generating series. They are defined as follows

$$\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}}(N):=\sum_{n_1>\ldots>n_r>0}^{N} n_1^{s_1}\ldots n_r^{s_r} \quad\text{and}\quad \mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}(z):=\sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r} z^{n_1},$$

where $r,N\in\mathbb{N}_+$ and $z\in\mathbb{C}$ such that $|z|<1$. In particular, for $r\in\mathbb{N}_+$, we have

$$\mathrm{H}^-_{y_0^r}(N)=\binom{N}{r} \quad\text{and}\quad \mathrm{Li}^-_{y_0^r}(z)=\left(\frac{z}{1-z}\right)^r.$$

Let us introduce also the following noncommutative generating series[2], for $t\in\mathbb{C}$,

$$\mathrm{H}^-(N):=\sum_{w\in Y_0^*}\mathrm{H}^-_w(N)\,w \quad\text{and}\quad \Theta(t):=\sum_{w\in Y_0^*} t^{(w)+|w|}w=\Big(\sum_{y\in Y_0} t^{(y)+1}y\Big)^*, \quad (1)$$

$$\mathrm{L}^-(z):=\sum_{w\in Y_0^*}\mathrm{Li}^-_w(z)\,w \quad\text{and}\quad \Lambda(t):=\sum_{w\in Y_0^*}((w)+|w|)!\,t^{(w)+|w|}w, \quad (2)$$

$$C^-:=1_{Y_0^*}+\sum_{w\in Y_0Y_0^*} C^-_w\,w, \quad\text{where}\quad C^-_w:=\prod_{w=uv,v\neq 1_{Y_0^*}}\frac{1}{(v)+|v|}. \quad (3)$$

[1]The quasi-shuffle product is denoted by $\uplus$ and its coproduct by $\Delta_{\uplus}$.
[2]We denote the length and the weight of $w=y_{s_1}\ldots y_{s_r}\in Y_0^*$ by the numbers $|w|=r$ and $(w)=s_1+\ldots+s_r$, respectively.

## 2 Main results

For harmonic sums, we define, firstly, the *multiple Bernoulli polynomials* $\{B_{y_{n_1}\ldots y_{n_r}}\}_{n_1,\ldots,n_r\in\mathbb{N}}$ by their commutative exponential generating series as follows

$$\sum_{n_1,\ldots,n_r\in\mathbb{N}} B_{y_{n_1}\ldots y_{n_r}}(z)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} = \frac{t_1\ldots t_r e^{z(t_1+\ldots+t_r)}}{\prod_{k=1}^{r}(e^{t_k+\ldots+t_r}-1)}, \quad \text{for} \quad z\in\mathbb{C},$$

or by the following difference equation, for $n_1\in\mathbb{N}_+$,

$$B_{y_{n_1}\ldots y_{n_r}}(z+1) = B_{y_{n_1}\ldots y_{n_r}}(z) + n_1 z^{n_1-1} B_{y_{n_2}\ldots y_{n_r}}(z).$$

For any $w\in y_sY_0^*, s>1$, we have $B_w(1)=B_w(0)$. Then let us define also[3] $b_w := B_w(0)$ and $\beta_w(z) := B_w(z)-b_w$. In the same way, for polylogarithms, we extend the Eulerian polynomials [5] as follows, for any $w\in Y_0^*$,

$$A_w^-(z) := \begin{cases} 1 & \text{if} \quad w=y_0, \\ \sum_{k=0}^{s_1-1} A_{s_1,k} z^k & \text{if} \quad w=y_{s_1}\in Y_0-\{y_0\}, \\ \sum_{i=0}^{s_1}\binom{s_1}{i} A_{y_i}^-(z) A_{y_{s_1+s_2-i}y_{s_3}\ldots y_{s_r}}^-(z) & \text{if} \quad w=y_{s_1}\ldots y_{s_r}\in Y_0Y_0^*, z\in\mathbb{C}, \end{cases}$$

where $A_{s_1,k}$ are Eulerian numbers. We obtain then the following results

1. For any $w\in Y_0^*, N\in\mathbb{N}_+$ and $z\in\mathbb{C}$ such that $\mid z\mid<1$, $\mathrm{Li}_w^-(z)$ and $\mathrm{H}_w^-(N)$ are polynomials, with rational coefficients on $(1-z)^{-1}$ and $N$ respectively, of valuation 1 and of degree $(w)+|w|$. Moreover,

$$\lim_{N\to+\infty}\frac{\mathrm{H}_w^-(N)}{N^{(w)+|w|}} = \lim_{z\to 1}\frac{(1-z)^{(w)+|w|}}{((w)+\mid w\mid)!}\mathrm{Li}_w^-(z) = C_w^- \in\mathbb{Q}.$$

2. For any $n_1,\ldots,n_r,N\in\mathbb{N}_+$, we have[4]

$$\beta_{y_{n_1}\ldots y_{n_r}}(N) = \sum_{k=1}^{r}(\prod_{i=1}^{k} n_i) b_{y_{n_{k+1}}\ldots y_{n_r}} \mathrm{H}_{y_{n_1-1}\ldots y_{n_k-1}}^-(N-1),$$

$$\mathrm{H}_{y_{n_1}\ldots y_{n_r}}^-(N) = \frac{\beta_{y_{(n_1+1)}\ldots y_{(n_r+1)}}(N+1) - \sum_{k=1}^{r-1} b'_{y_{(n_{k+1}+1)}\ldots y_{(n_r+1)}}\beta_{y_{(n_1+1)}\ldots y_{(n_k+1)}}(N+1)}{\prod_{i=1}^{r}(n_i+1)}. \quad (4)$$

where $b'_{n_k} = b_{n_k}$ and $b'_{n_k\ldots n_r} = b_{n_k\ldots n_r} - \sum_{j=0}^{r-1-k} b_{n_{k+j+1}\ldots n_r} b'_{n_k\ldots n_{k+j}}$ for any $1\le k\le r$.

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[3] The number $b_w$ is also called *multiple Bernoulli number* which differ from those defined in [3, 8] or in [11, 12].

[4] The identity (4) extends the Johann Faulhaber's formula (obtained for $r=1$) [4].

3. For any $w = y_{s_1} \ldots y_{s_r} \in Y_0^*$, we get on the one hand[5]

$$\mathrm{Li}_w^-(z) = (\theta_0^{s_1+1}\iota_1) \ldots (\theta_0^{s_{r-1}+1}\iota_1)\,\mathrm{Li}_{y_{s_r}}^-(z) = \left(\frac{z}{1-z}\right)^{|w|} \frac{A_w^-(z)}{(1-z)^{(w)}},$$

and on the other hand

(a) If $r = 1$, then we have $\mathrm{Li}_{y_{s_1}}^-(z) = \sum_{k=1}^{s_1} S_2(s_1,k)k!\frac{z^k}{(1-z)^{k+1}}$, where $\{S_2(s_1,k)\}_{1\le k\le s_1}$ are the Stirling numbers of second kind. Hence,

$$\forall k \in \mathbb{N}_+, \qquad \frac{1}{(1-z)^k} = \frac{(-1)^{k+1}}{1-z} + \sum_{j=2}^{k} \frac{(-1)^{k+j} S_1(k,j)}{k!}\,\mathrm{Li}_{y_{j-1}}^-(z),$$

where $\{S_1(s_1,k)\}_{1\le k\le s_1}$ are the Stirling numbers of first kind.

(b) If $r > 1$ then $\mathrm{Li}_{y_{s_1}\ldots y_{s_r}}^-(z) = \left(\frac{z}{1-z}\right)^{|w|} \sum_{i=r}^{s_1+\ldots+s_r} \sum_{j=0}^{s_1\ldots+s_{r-1}} l_{i,j}\frac{z^{i-1-j}}{(1-z)^i}$, where for $r \le i \le s_1+\ldots+s_r$ and $0 \le j \le s_1+\ldots+s_{r-1}$, $l_{i,j}$ are defined as follows,

$$l_{ij} = \sum_{\substack{1\le k_t\le s_t\\ k_1+\ldots+k_r=i}} \left(\prod_{n=1}^{r}(k_n!S_2(s_n,k_n))\right) \sum_{\substack{0\le t_m\le k_m;\forall m=1,\ldots,r-1\\ t_1+\ldots+t_{r-1}=j}} \prod_{p=1}^{r-1}$$
$$\binom{k_r+\ldots+k_{r-p+1}+p-t_{r-p+1}-\ldots-t_{r-1}}{t_{r-p}}\binom{k_{r-p}+t_{r-p+1}+\ldots t_{r-1}}{k_{r-p}-t_{r-p}}.$$

4. The noncommutative generating series $\mathrm{H}^-$ and $C^-$ are group-like, for $\Delta_{⧢}$, and[6]

$$\lim_{N\to+\infty} \Theta^{\odot-1}(N)\odot \mathrm{H}^-(N) = \lim_{z\to1} \Lambda^{\odot-1}((1-z)^{-1})\odot \mathrm{L}^-(z) = C^-.$$

5. There is a law of algebra $\top$ which is not dualizable in $\mathbb{Q}\langle Y_0\rangle$ [7] such that the following maps are surjective morphisms of algebras

$$\mathrm{H}_\bullet^- : (\mathbb{Q}\langle Y_0\rangle, ⧢) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w^-\}_{w\in Y_0^*}, .), \qquad w \longmapsto \mathrm{H}_w^-,$$
$$\mathrm{Li}_\bullet^- : (\mathbb{Q}\langle Y_0\rangle, \top) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w^-\}_{w\in Y_0^*}, .), \qquad w \longmapsto \mathrm{Li}_w^-.$$

Moreover, $\ker \mathrm{H}_\bullet^- = \ker \mathrm{Li}_\bullet^- = \mathbb{Q}\langle\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}\rangle$.

6. Let $\top' : \mathbb{Q}\langle Y_0\rangle \times \mathbb{Q}\langle Y_0\rangle \longrightarrow \mathbb{Q}\langle Y_0\rangle$ be a law such that $\mathrm{Li}_\bullet^-$ is a morphism for $\top'$ and such that $(1_{Y_0^*}\top'\mathbb{Q}\langle Y_0\rangle) \cap \ker(\mathrm{Li}_\bullet^-) = \{0\}$. Then $\top' = g\circ\top$ where $g \in GL(\mathbb{Q}\langle Y_0\rangle)$ such that $\mathrm{Li}_\bullet^- \circ g = \mathrm{Li}_\bullet^-$.

7. $\{\mathrm{H}_{y_k}^-\}_{k\ge0}$ (resp. $\{\mathrm{Li}_{y_k}^-\}_{k\ge0}$) are $\mathbb{Q}-$ linearly independent.

[5]Here, we use the operators over polylogarithms $\theta_0 : g(z) \longmapsto z\frac{d}{dz}g(z)$ and $\iota_1 : g(z) \longmapsto \int_0^z \frac{g(t)dt}{1-t}$.

[6]Here, the Hadamard product is denoted by $\odot$ and its dual law is denoted by $\Delta_\odot$. The inverses of $\Theta$ and $\Lambda$, for the Hadamard product [13], are denoted respectively by $\Theta^{\odot-1}$ and $\Lambda^{\odot-1}$ (their coefficients do not vanish). One obtains then an Abel like theorem for the noncommutative generating series given in (1),(2) and (3).

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