Harnack inequalities for jump processes

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Abstract. We consider a class of pure jump Markov processes in $\mathbb{R}^d$ whose jump kernels are comparable to those of symmetric stable processes. We establish a Harnack inequality for nonnegative functions that are harmonic with respect to these processes. We also establish regularity for the solutions to certain integral equations.

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1. Introduction.

In this paper we consider pure jump Markov processes in $\mathbb{R}^d$ whose jump structure is comparable to that of symmetric stable processes. Our two main goals are

(1) to establish a Harnack inequality for nonnegative functions that are harmonic with respect to these processes.

(2) to establish regularity for certain integral equations.

Before explaining (1) and (2) in more detail, let us describe the processes we consider. We are interested in pure jump processes that are generalizations of symmetric stable processes. A symmetric stable process Z_t of index α is a Lévy process with no drift, symmetric jump kernel, and no diffusion component, so that

$$\mathbb{E}\, e^{iu\cdot Z_t} = \exp\Big(t\int [e^{iu\cdot h} - 1 - iu\cdot h 1_{(|h|\le 1)}]n(dh)\Big), \tag{1.1}$$

with Lévy measure $n(dh) = c|h|^{-d-\alpha}dh$ for some constant c. Since n is symmetric, $n(-A) = n(A)$ for any set A, and one could omit the term involving $iu \cdot h$ inside the brackets in (1.1). The infinitesimal generator for Z_t is given by

$$\mathcal{L}_0 f(x) = \int_{\mathbb{R}^d - \{0\}} [f(x+h) - f(x) - \nabla f(x)\cdot h 1_{(|h|\le 1)}]n(dh)$$

for suitable functions f.

The infinitesimal behavior of a Lévy process such as Z_t is homogeneous in space: n does not depend on x. In this paper we want to consider pure jump processes in $\mathbb{R}^d$ where the jump structure n does depend on x, but is not too different from that of a symmetric stable process. More precisely, we want to consider processes X_t associated to the operator

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d - \{0\}} [f(x+h) - f(x) - \nabla f(x)\cdot h 1_{(|h|\le 1)}]\frac{a(x,h)}{|h|^{d+\alpha}}, \tag{1.2}$$

where $a(x,-h) = a(x,h)$ and a is uniformly bounded above and below away from 0. The number of jumps of size h of such a process will be comparable to the number of those of a symmetric stable process of index α.

We say that a bounded function $h : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{L}$-harmonic in a domain D if $h(X_t)$ is a martingale up until the time of first exiting the domain. It is easy to see that if h has some smoothness properties and $\mathcal{L}h = 0$ in D, then h will be $\mathcal{L}$-harmonic. A Harnack inequality says that $h(x)/h(y)$ is bounded by a constant independent of h if x, y are in D and h is nonnegative and bounded in $\mathbb{R}^d$ and $\mathcal{L}$-harmonic in D. See Theorem 3.6 for a precise statement.

Harnack inequalities such as that of Moser for divergence form elliptic operators and that of Krylov-Safonov for nondivergence form operators are an extremely important tool in the study of partial differential equations. All existing proofs of these Harnack inequalities use the fact that the operator is local in an essential way. One of our interests in the problem of proving a Harnack inequality for the pure jump case is that the infinitesimal generator is non-local; instead of a differential operator, we have an integral one.

One important application of Harnack inequalities is to establish regularity for the solutions of elliptic PDEs. The techniques that we develop in this paper allow us to obtain regularity for certain integral operators. Consider the equation

$$\mathcal{L}u(x) - \lambda u(x) = -g(x), \qquad x \in \mathbb{R}^d, \quad \lambda \geq 0,$$

where $\mathcal{L}$ is given by (1.2). It is well known that the solution to this equation is given by the λ-resolvent of g:

$$u(x) = S_\lambda g(x) = \mathbb{E}^{\,x} \int_0^\infty e^{-\lambda t} g(X_t) dt.$$

Theorem 4.3 enables us to say that u must be Hölder continuous.

Harnack inequalities are implicit in the work of Chen and Song [CS] for the case of symmetric stable processes. In [BBG] a Liouville property, which is closely related to a Harnack inequality, was proved. In fact, that paper stimulated our interest in this project. It should be mentioned that local regularity-improving L^p estimates for $\mathcal{L}$-harmonic functions for certain non-local operators had been obtained in [T]. For very recent related work on Harnack inequalities see [Ka].

We have not been precise as to what it means for an operator $\mathcal{L}$ to be associated to the process X_t. A natural connection is through the martingale problem formulation. See (2.1) for the definition. Martingale problems for jump processes such as the ones considered here have been studied by [Ba], [K], and especially [H].

Section 2 contains some preliminaries. Section 3 contains the proof of the Harnack inequality, and the regularity is considered in Section 4.

2. Preliminaries.

Let us begin by describing more carefully the processes we wish to consider. A probability measure $\mathbb{P}$ on the space $D[0,\infty)$ is a solution to the martingale problem for $\mathcal{L}$ started at x if $X_t(\omega) = \omega(t)$ are the coordinate maps, $\mathcal{F}_t$ is the σ-field generated by the cylindrical sets, and

(a) we have $\mathbb{P}(X_0 = x) = 1$, and

(b) for each $f \in C^2$ such that f and all its first and second partial derivatives are bounded we have that

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s) ds \tag{2.1}$$

is a $\mathbb{P}$-martingale,

where
$$\mathcal{L}f(x) = \int_{\mathbb{R}^d-\{0\}} [f(x+h) - f(x) - \nabla f(x) \cdot h 1_{(|h|\le 1)}] n(x,h)dh. \tag{2.2}$$

The symmetry assumption we will impose on n will make the presence of the ∇f term have no effect; moreover we could replace the $1_{(|h|\le 1)}$ term by $1_{(|h|\le M)}$ with any $M > 0$ whatsoever.

We assume that $(\mathbb{P}^x, X_t)$ is a strong Markov process with state space $\mathbb{R}^d$ such that for each x the probability measure $\mathbb{P}^x$ is a solution to the martingale problem for $\mathcal{L}$ started at x.

Throughout this paper we make the following assumption.

Assumption 2.1. *(a) For all x and h we have $n(x,-h) = n(x,h)$.*

(b) There exist constants $\kappa \in (0,1)$ and $\alpha \in (0,2)$ such that for all x and h we have

$$\frac{\kappa}{|h|^{d+\alpha}} \le n(x,h) \le \frac{\kappa^{-1}}{|h|^{d+\alpha}}. \tag{2.3}$$

We assume in Section 3 that such a process is given, but we comment that with the assumption of some smoothness in x for $n(x,h)$, we have the existence of such processes by the results of [Ba] or [K].

The proof of the following scaling property is an easy change of variables argument.

Proposition 2.2. *Suppose $(\mathbb{P}^x, X_t)$ is as above, $a > 0$, and $Y_t = aX_{a^{-\alpha}t}$. Define $\mathbb{Q}^x = \mathbb{P}^{x/a}$. Then $(\mathbb{Q}^x, Y_t)$ is a strong Markov process. We have $\mathbb{Q}^x(Y_0 = x) = 1$ and if $f \in C^2$, then $f(Y_t) - f(Y_0) - \int_0^t \widetilde{\mathcal{L}} f(Y_s)ds$ is a $\mathbb{Q}^x$-martingale, where $\widetilde{\mathcal{L}} f(x) = \int [f(x+h) - f(x) - \nabla f(x) \cdot h 1_{(|h|\le 1)}] \widetilde{n}(x,h)dh$ and $\widetilde{n}$ satisfies (2.3) with the same values of κ and α.*

Proof. Because $(\mathbb{P}^x, X_t)$ is strong Markov and Y_t is a constant multiple of a time change of X_t, then $(\mathbb{Q}^x, Y_t)$ is strong Markov. That $\mathbb{Q}^x(Y_0 = 1) = 1$ is clear. Let

$$\widetilde{n}(y,k) = a^{-(d+\alpha)} n(a^{-1}y, a^{-1}k), \qquad \widetilde{\mathcal{L}} f(y) = \int_{\mathbb{R}^d-\{0\}} [f(y+k) - f(y)] \widetilde{n}(y,k)dk.$$

Clearly $\widetilde{n}$ satisfies (2.3) with the same values of κ and α. Let $f \in C^2$ and set $g(x) = f(ax)$. Then

$$g(X_{a^{-\alpha}t}) - g(X_0) - \int_0^{a^{-\alpha}t} \mathcal{L} g(X_s)ds$$

is a martingale, hence so is

$$g(X_{a^{-\alpha}t}) - g(X_0) - \int_0^t a^{-\alpha} \mathcal{L} g(X_{a^{-\alpha}s})ds.$$

Consequently

$$f(Y_t) - f(Y_0) - \int_0^t a^{-\alpha}\mathcal{L}g(a^{-1}Y_s)ds$$

is also a martingale.

It remains to check that $a^{-\alpha}\mathcal{L}g(a^{-1}y) = \widetilde{\mathcal{L}}f(y)$. This follows because

$$\begin{aligned}
a^{-\alpha}\mathcal{L}g(a^{-1}y) &= a^{-\alpha}\int [g(a^{-1}y+k) - g(a^{-1}y)]n(a^{-1}y,k)dk \\
&= a^{-\alpha}\int [f(y+ak) - f(y)]n(a^{-1}y,k)dk \\
&= a^{d}\int [f(y+ak) - f(y)]\widetilde{n}(y,ak)dk \\
&= \int [f(y+h) - f(y)]\widetilde{n}(y,h)dh \\
&= \widetilde{\mathcal{L}}f(y).
\end{aligned}$$

□

We will also need the following fact.

Proposition 2.3. *Suppose A and B are Borel sets that are a positive distance from each other. Then*

$$\sum_{s\le t} 1_{(X_{s-}\in A, X_s\in B)} - \int_0^t 1_A(X_s)\int_B n(X_s, u - X_s)du\, ds$$

is a $\mathbb{P}^x$-martingale for each x.

Proof. Let $f \in C^2$ with $f = 0$ on A and $f = 1$ on B. Let M_t^f denote the martingale in (2.1). Then $\int_0^t 1_A(X_{s-})dM_t^f$ is also a martingale under $\mathbb{P}^x$, since the stochastic integral with respect to a martingale is a martingale. Since $f(X_t)-f(X_0) = \sum_{s\le t}[f(X_s)-f(X_{s-})]$, this says that

$$\sum_{s\le t}[1_A(X_{s-})(f(X_s) - f(X_{s-}))] - \int_0^t 1_A(X_{s-})\mathcal{L}f(X_s)ds$$

is a martingale. Since $X_{s-} \neq X_s$ for only countably many values of s, then

$$\sum_{s\le t}[1_A(X_{s-})(f(X_s) - f(X_{s-}))] - \int_0^t 1_A(X_s)\mathcal{L}f(X_s)ds \tag{2.4}$$

is also a martingale. Now if $x \in A$, then $f(x)$ and $\nabla f(x)$ are both equal to 0, and so

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d-\{0\}} f(x+h)n(x,h)dh = \int_{\mathbb{R}^d-\{0\}} f(u)n(x,u-x)du\,.$$

Note $n(x,h)$ is integrable over h in the complement of any neighborhood of the origin. Because A and B are a positive distance from each other, the sum on the left of (2.4) is actually a finite sum. With these facts we can pass to a limit to see that

$$\sum_{s\le t}[1_A(X_{s-})(1_B(X_s)-1_B(X_{s-})]-\int_0^t 1_A(X_s)\int_B n(X_s,u-X_s)du\,ds$$

is a martingale, which is equivalent to what we wanted to prove. □

Remark 2.4. By taking limits, it is not necessary to assume that A and B are a positive distance apart, but only that they are disjoint.

We let $B(x,r)$ denote the ball of radius r centered at x. We use $|A|$ to denote the Lebesgue measure of A. Set

$$\tau_A=\inf\{t>0: X_t\notin A\},\qquad T_A=\inf\{t>0: X_t\in A\}.$$

The letter c with subscripts will denote finite positive constants whose values are unimportant and which may have different values in different places.

3. Harnack inequality.

We begin this section by proving a tightness result.

Proposition 3.1. *There exists c_1 depending only on κ and not x such that*

$$\mathbb{P}^x(\sup_{s\le t}|X_s-X_0|>1)\le c_1 t.$$

Proof. Let f be a C^2 function taking values in $[0,1]$ such that $f(0)=0$ and $f(y)=1$ if $|y|\ge 1$. Let $f_x(y)=f(x+y)$. By the Taylor expansion of f_x,

$$|(f_x(z+h)-f_x(z))+(f_x(z-h)-f_x(z))|\le c_2|h|^2. \tag{3.1}$$

Since n is symmetric, this and Assumption 2.1(b) imply

$$\begin{aligned}|\mathcal{L}f_x(z)|&\le\Big|\int_{|h|\le 1}[f_x(z+h)-f_x(z)]n(z,h)dh\Big|+\Big|\int_{|h|>1}[f_x(z+h)-f_x(z)]n(z,h)dh\Big|\\&\le c_3\int_{|h|\le 1}|h|^2 n(z,h)dh+c_4\int_{|h|>1}n(z,h)dh\\&\le c_5.\end{aligned}$$

We now use (2.1) to write

$$\mathbb{E}^x f_x(X_{\tau_{B(x,1)}\wedge t})-f_x(x)=\mathbb{E}^x\int_0^{\tau_{B(x,1)}\wedge t}\mathcal{L}f_x(X_s)ds\le c_5 t.$$

If X_t exits $B(x,1)$ before time t then $f_x(X_{\tau_{B(x,1)}\wedge t}) = 1$, and so the left hand side is greater than $\mathbb{P}^x(\tau_{B(x,1)} \le t)$. $\square$

Lemma 3.2. *Let $\varepsilon > 0$. There exists c_1 depending only on ε such that if $x \in \mathbb{R}^d$ and $r > 0$, then*

$$\inf_{z\in B(x,(1-\varepsilon)r)} \mathbb{E}^{\,z}\tau_{B(x,r)} \ge c_1 r^\alpha.$$

Proof. By scaling we may assume $r = 1$. By Proposition 3.1 and scaling, if $z \in B(x, 1-\varepsilon)$

$$\mathbb{P}^z(\tau_{B(x,1)} \le \varepsilon^\alpha t) \le \mathbb{P}^z(\sup_{s\le\varepsilon^\alpha t} |X_s - X_0| \ge \varepsilon) \le c_2 t.$$

Thus

$$\mathbb{E}^{\,z}\tau_{B(x,1)} \ge \varepsilon^\alpha t \mathbb{P}^z(\tau_{B(x,1)} \ge \varepsilon^\alpha t) \ge \varepsilon^\alpha t(1 - c_2 t)\,.$$

Taking $t = 1/(2c_2)$ yields a uniform lower bound. $\square$

Lemma 3.3. *There exists c_1 such that* $\sup_z \mathbb{E}^{\,z}\tau_{B(x,r)} \le c_1 r^\alpha$.

Proof. By scaling, we may suppose $r = 1$. Let S be the time of the first jump larger than 2. We want to show there exists $c_2 \in (0, \frac{1}{2})$ such that $\mathbb{P}^z(S \le 1) > c_2$ for all z. For z such that $\mathbb{P}^z(S \le 1) \ge \frac{1}{2}$, there is nothing to show. So suppose z is such that $\mathbb{P}^z(S \le 1) < \frac{1}{2}$. By an argument similar to that in Proposition 2.3,

$$\sum_{s\le t} 1_{(|X_s - X_{s-}|>2)} - \int_0^t \int_{(|h|>2)} n(X_s, h)dh$$

is a martingale. Then by optional stopping and by the lower bounds on n

$$\begin{aligned}\mathbb{P}^z(S \le 1) &= \mathbb{E}^{\,z} \sum_{s\le S\wedge 1} 1_{(|X_s - X_{s-}|>2)}\\ &= \mathbb{E}^{\,z} \int_0^{S\wedge 1} \int_{(|h|>2)} n(X_s, h)dh\, ds\\ &\ge c_3 \mathbb{E}^{\,z}(S \wedge 1) \ge c_3 \mathbb{P}^z(S > 1) \ge c_3/2.\end{aligned}$$

Letting $c_2 = (1 \wedge c_3)/2$, we have $\mathbb{P}^z(S \le 1) \ge c_2$.

If there is a jump larger than 2 before time 1, then $\tau_{B(x,1)} \le 1$. So

$$\sup_z \mathbb{P}^z(\tau_{B(x,1)} > 1) \le 1 - c_2.$$

Let θ_t be the usual shift operator for Markov processes. By the Markov property,

$$\begin{aligned}\mathbb{P}^z(\tau_{B(x,1)} > m+1) &\le \mathbb{P}^z(\tau_{B(x,1)} > m, \tau_{B(x,1)} \circ \theta_m > 1)\\ &= \mathbb{E}^z\Big[\mathbb{P}^{X_m}(\tau_{B(x,1)} > 1); \tau_{B(x,1)} > m\Big]\\ &\le (1-c_2)\mathbb{P}^z(\tau_{B(x,1)} > m).\end{aligned}$$

By induction, $\mathbb{P}^z(\tau_{B(x,1)} > m) \le (1-c_2)^m$, which implies that $\tau_{B(x,1)}$ has moments of all orders. □

Next we show X_t will hit sets of positive Lebesgue measure with positive probability.

Proposition 3.4. *Suppose $A \subset B(x,1)$. There exists c_1 not depending on x or A such that*

$$\mathbb{P}^y(T_A < \tau_{B(x,3)}) \ge c_1|A|, \qquad y \in B(x,2).$$

Proof. Fix $y \in B(x,2)$. Write τ for $\tau_{B(x,3)}$. If X_t is in A for some t less than time τ with probability larger than $1/4$, we are done, so assume $\mathbb{P}^y(T_A < \tau) \le 1/4$. Using Proposition 3.1 and scaling, choose t_0 small enough so that the probability that τ occurs before time t_0 is also less than $1/4$. Note that T_A cannot equal τ because $A \subset B(x,1)$. For $|h| \le 4$, $n(X_s,h)$ is bounded below by Assumption 2.1. Hence for $X_s \in B(x,3)$ and $u \in A \subset B(x,1)$, we have $|X_s - u| \le 4$, and consequently $n(X_s, u - X_s)$ is bounded below. So

$$\begin{aligned}\mathbb{P}^y(T_A < \tau) &\ge \mathbb{E}^y \sum_{s \le T_A \wedge \tau \wedge t_0} 1_{(X_{s-} \ne X_s, X_s \in A)}\\ &= \mathbb{E}^y \int_0^{T_A \wedge \tau \wedge t_0} \int_A n(X_s, u - X_s)du\, ds\\ &\ge c_2|A|\mathbb{E}^y(T_A \wedge \tau \wedge t_0).\end{aligned}$$

Now

$$\begin{aligned}\mathbb{E}^y(T_A \wedge \tau \wedge t_0) &\ge \mathbb{E}^y(t_0; T_A \ge \tau \ge t_0)\\ &= t_0\mathbb{P}^y(T_A \ge \tau \ge t_0)\\ &\ge t_0[1 - \mathbb{P}^y(T_A < \tau) - \mathbb{P}^y(\tau < t_0)] \ge t_0/2.\end{aligned}$$

Combining this with the above,

$$\mathbb{P}^y(T_A < \tau) \ge c_2|A|t_0/2.$$

□

Proposition 3.5. *There exist c_1 and c_2 such that if $x \in \mathbb{R}^d, r > 0, z \in B(x,r)$, and H is a bounded nonnegative function supported in $B(x,2r)^c$, then*

$$\mathbb{E}^z H(X_{\tau_{B(x,r)}}) \le c_1\Big(\mathbb{E}^z \tau_{B(x,r)}\Big) \int \frac{H(y)}{|y-x|^{d+\alpha}}dy$$

and

$$\mathbb{E}^z H(X_{\tau_{B(x,r)}}) \geq c_2\Big(\mathbb{E}^z \tau_{B(x,r)}\Big)\int \frac{H(y)}{|y-x|^{d+\alpha}}dy.$$

Proof. Note $H(w)=0$ if $w\in B(x,r)$ and $H(X_{\tau_{B(x,r)}})>0$ only if there is a jump from $B(x,r)$ to $B(x,2r)^c$. By Proposition 2.3 and optional stopping, if $B\subset B(x,2r)^c$

$$\begin{aligned}
\mathbb{E}^z 1_{(X_{t\wedge\tau(B(x,r))}\in B)} &= \mathbb{E}^z\int_0^{t\wedge\tau(B(x,r))}\int_B n(X_s,u-X_s)du\,ds\\
&\leq \mathbb{E}^z\int_0^{t\wedge\tau(B(x,r))}\int_B \frac{c_3}{|u-X_s|^{d+\alpha}}du\,ds\\
&\leq c_4\mathbb{E}^z(t\wedge\tau_{B(x,r)})\int_B\frac{dy}{|y-x|^{d+\alpha}}.
\end{aligned}$$

Letting $t\to\infty$, using monotone convergence on the right and dominated convergence on the left, we have

$$\mathbb{E}^z 1_B(X_{\tau_{B(x,r)}}) \leq c_4\Big(\mathbb{E}^z\tau_{B(x,r)}\Big)\int\frac{1_B(y)}{|y-x|^{d+\alpha}}dy.$$

Using linearity we have the above when 1_B is replaced by a simple function; approximating H by simple functions and taking limits, we have the first inequality in the statement of the proposition.

The proof of the second inequality is exactly similar, using the lower bound for n instead of the upper bound. $\square$

We say a bounded function $h:\mathbb{R}^d\to\mathbb{R}$ is $\mathcal{L}$-harmonic in a domain D if $h(X_{t\wedge\tau_D})$ is a $\mathbb{P}^x$-martingale for all x. It is easy to see that if h is C^2 in D, and $\mathcal{L}h(x)=0$ for $x\in D$, then h will be $\mathcal{L}$-harmonic.

Theorem 3.6. *There exists c_1 such that if h is nonnegative and bounded on $\mathbb{R}^d$ and $\mathcal{L}$-harmonic in $B(x_0,16)$, then*

$$h(x)\leq c_1 h(y), \qquad x,y\in B(x_0,1).$$

Proof. By looking at a constant multiple of h, we may assume $\inf_{B(x_0,1)}h=\frac{1}{2}$. Choose $z_0\in B(x_0,1)$ such that $h(z_0)\leq 1$. We want to show that h is bounded above in $B(x_0,1)$ by a constant not depending on h. We will establish this by contradiction: if there exists a point $x\in B(x_0,1)$ with $h(x)=K$ where K is too large, we can obtain a sequence of points in $B(x_0,2)$ on which h is unbounded.

Let $\varepsilon < \frac{1}{3}$ be chosen so that $|B(0,1-\varepsilon)|/|B(0,1)| \geq \frac{3}{4}$. Using Lemma 3.2, Lemma 3.3, and Proposition 3.5, there exists c_2 such that if $x \in \mathbb{R}^d, r > 0$, and H is a nonnegative function supported on $B(x,2r)^c$, then for $y, z \in B(x,(1-\varepsilon)r)$,

$$\mathbb{E}^z H(X_{\tau(B(x,r))}) \leq c_2 \mathbb{E}^y H(X_{\tau(B(x,r))}). \tag{3.2}$$

By Proposition 3.4 there exists c_3 such that if $A \subset B(x_0,4)$,

$$\mathbb{P}^y(T_A < \tau_{B(x_0,16)}) \geq c_3|A|, \qquad y \in B(x_0,8). \tag{3.3}$$

Also by Proposition 3.4 there exists $c_4 \leq 1$ such that if $x \in \mathbb{R}^d, r > 0$, and $C \subset B(x,r/3)$ with $|C|/|B(x,r/3)| \geq \frac{1}{3}$, then

$$\mathbb{P}^x(T_C < \tau_{B(x,r)}) \geq c_4. \tag{3.4}$$

Let

$$\eta = \frac{c_4}{3}, \qquad \zeta = \frac{1}{3} \wedge (c_2^{-1}\eta). \tag{3.5}$$

Now suppose there exists $x \in B(x_0,2)$ with $h(x) = K$ for some $K > 2$. Let r be chosen so that

$$|B(x,r/3)| = 2/(c_3\zeta K). \tag{3.6}$$

Note this implies

$$r = c_5 K^{-1/d}. \tag{3.7}$$

Let us write B_r for $B(x,r)$, τ_r for $\tau_{B(x,r)}$ and similarly B_{2r} and τ_{2r}. Let A be a compact set contained in

$$A' = \{w \in B(x,r/3) : h(w) \geq \zeta K\}.$$

By (3.3) and optional stopping,

$$\begin{aligned} 1 \geq h(z_0) &\geq \mathbb{E}^{z_0}[h(X_{T_A \wedge \tau_{B(x_0,16)}}); T_A < \tau_{B(x_0,16)}] \\ &\geq \zeta K \mathbb{P}^{z_0}(T_A < \tau_{B(x_0,16)}) \\ &\geq c_3\zeta K|A|, \end{aligned}$$

hence

$$\frac{|A|}{|B(x,r/3)|} \leq \frac{1}{c_3\zeta K|B(x,r/3)|} \leq \frac{1}{2}.$$

This implies $|A'|/|B(x,r/3)| \leq \frac{1}{2}$. Let C be a compact set contained in $B(x,r/3) - A'$ such that

$$\frac{|C|}{|B(x,r/3)|} \geq \frac{1}{3}. \tag{3.8}$$

Let $H = h1_{B_{2r}^c}$. We claim

$$\mathbb{E}^x[h(X_{\tau_r}); X_{\tau_r} \notin B_{2r}] \le \eta K$$

If not

$$\mathbb{E}^x H(X_{\tau_r}) > \eta K,$$

and by (3.2), for all $y \in B(x, r/3)$,

$$\begin{aligned} h(y) &\ge \mathbb{E}^y h(X_{\tau_r}) \ge \mathbb{E}^y[h(X_{\tau_r}); X_{\tau_r} \notin B_{2r}] \\ &\ge c_2^{-1}\mathbb{E}^x H(X_{\tau_r}) \ge c_2^{-1}\eta K \\ &\ge \zeta K, \end{aligned}$$

contradicting (3.8) and the definition of A'.

Let $M = \sup_{B_{2r}} h(z)$. We then have

$$\begin{aligned} K = h(x) &= \mathbb{E}^x[h(X_{T_C}); T_C < \tau_r] + \mathbb{E}^x[h(X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \in B_{2r}] \\ &\qquad + \mathbb{E}^x[h(X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \notin B_{2r}] \\ &\le \zeta K \mathbb{P}^x(T_C < \tau_r) + M\mathbb{P}^x(\tau_r < T_C) + \eta K \\ &= \zeta K \mathbb{P}^x(T_C < \tau_r) + M(1 - \mathbb{P}^x(T_C < \tau_r)) + \eta K, \end{aligned}$$

or

$$\frac{M}{K} \ge \frac{1 - \eta - \zeta\mathbb{P}^x(T_C < \tau_r)}{1 - \mathbb{P}^x(T_C < \tau_r)}.$$

Using (3.4) and (3.5) there exists $\beta > 0$ such that $M \ge K(1 + 2\beta)$. Therefore there exists $x' \in B(x, 2r)$ with $h(x') \ge K(1 + \beta)$.

Now suppose there exists $x_1 \in B(x_0, 1)$ with $h(x_1) = K_1$. Define r_1 in terms of K_1 analogously to (3.6). Using the above argument (with x_1 replacing x and x_2 replacing x'), there exists $x_2 \in B(x_1, 2r_1)$ with $h(x_2) = K_2 \ge (1 + \beta)K_1$. We continue and obtain r_2 and then x_3, K_3, r_3, etc. Note $x_{i+1} \in B(x_i, 2r_i)$ and $K_i \ge (1 + \beta)^{i-1}K_1$. In view of (3.7), $\sum_i |x_{i+1} - x_i| \le c_6 K_1^{-1/d}$. So if $K_1 > c_6^d$, then we have a sequence $x_1, x_2, \ldots$ contained in $B(x_0, 2)$ with $h(x_i) \ge (1 + \beta)^{i-1}K_1 \to \infty$, a contradiction to h being bounded on $\mathbb{R}^d$. Therefore we cannot take K_1 larger than $c_1 = c_6^d$, and thus $\sup_{B(x_0,1)} h(y) \le c_1$, which is what we wanted to prove. □

Corollary 3.7. *Suppose D is a bounded connected domain and $r > 0$. There exists c_1 depending only on D and r such that if h is nonnegative and bounded in $\mathbb{R}^d$ and $\mathcal{L}$-harmonic in D, then $h(x) \le c_1 h(y)$ if $x, y \in D$ and $\mathrm{dist}\,(x, \partial D)$ and $\mathrm{dist}\,(y, \partial D)$ are both greater than r.*

Proof. We form a sequence $x = y_0, y_1, y_2, \ldots, y_m = y$ such that $|y_{i+1} - y_i| < (a_{i+1} \wedge a_i)/32$, where $a_i = \mathrm{dist}\,(y_i, \partial D)$ and each $a_i < r$. By compactness we can choose M depending only on r so that no more than M points y_i are needed. By scaling and Theorem 3.6, $h(y_i) \le c_2 h(y_{i+1})$ with $c_2 > 1$. So

$$h(x) = h(y_0) \le c_2 h(y_1) \le \cdots \le c_2^m h(y_m) = c_2^m h(y) \le c_2^M h(y).$$

$\square$

4. Regularity.

In this section we obtain some estimates on equicontinuity of resolvents.

Theorem 4.1. *If h is bounded on $\mathbb{R}^d$ and $\mathcal{L}$-harmonic in a ball $B(x_0, 2)$, then h is Hölder continuous in $B(x_0, 1)$: there exist c_1 and $\beta > 0$ such that*

$$|h(x) - h(y)| \le c_1 \|h\|_\infty |x-y|^\beta, \qquad x, y \in B(x_0, 1).$$

Proof. By Proposition 3.4 there exists c_2 such that if $x \in \mathbb{R}^d, r > 0$, and $A \subset B(x, r/3)$ with $|A|/|B(x, r/3)| \ge \frac{1}{3}$, then

$$\mathbb{P}^x(T_A < \tau_{B(x,r)}) \ge c_2. \tag{4.1}$$

By Proposition 3.5 and Lemma 3.3 with $H = 1_{B(x,s)^c}$, there exists c_3 such that if $s \ge 2r$, then

$$\mathbb{P}^x(X_{\tau_{B(x,r)}} \notin B(x,s)) \le c_3 r^\alpha / s^\alpha. \tag{4.2}$$

Let

$$\gamma = 1 - \frac{c_2}{4}, \qquad \rho = \frac{1}{3} \wedge \Big(\frac{\gamma}{2}\Big)^{1/\alpha} \wedge \Big(\frac{c_2 \gamma^2}{8c_3}\Big)^{1/\alpha}.$$

By linearity and scaling it suffices to suppose $0 \le h \le M$ on $\mathbb{R}^d$ and h is $\mathcal{L}$-harmonic on $B(x, 1)$. We will show

$$\sup_{B(x,\rho^k)} h - \inf_{B(x,\rho^k)} h \le M\gamma^k \tag{4.3}$$

for all k.

We write B_i for $B(x, \rho^i)$ and τ_i for $\tau_{B(x,\rho^i)}$. Let

$$a_i = \inf_{B_i} h, \qquad b_i = \sup_{B_i} h.$$

Suppose $b_i - a_i \le M\gamma^i$ for all $i \le k$; we want to show

$$b_{k+1} - a_{k+1} \le M\gamma^{k+1}. \tag{4.4}$$

We have $a_k \le h \le b_k$ on B_{k+1}. Let

$$A' = \{z \in B_{k+1} : h(z) \le (a_k + b_k)/2\}.$$

We may suppose $|A'|/|B_{k+1}| \ge \frac{1}{2}$, for if not we look at $M - h$ instead of h. Let A be a compact set contained in A' with $|A|/|B_{k+1}| \ge \frac{1}{3}$. Let $\varepsilon > 0$, pick $y \in B_{k+1}$ with $h(y) \ge b_{k+1} - \varepsilon$, and pick $z \in B_{k+1}$ with $h(z) \le a_{k+1} + \varepsilon$.

By optional stopping

$$\begin{aligned} h(y) - h(z) = \mathbb{E}^{\,y}[h(X_{T_A}) - h(z); T_A < \tau_k] \\ + \mathbb{E}^{\,y}[h(X_{\tau_k}) - h(z); \tau_k < T_A, X_{\tau_k} \in B_{k-1}] \\ + \sum_{i=1}^{\infty} \mathbb{E}^{\,y}[h(X_{\tau_k}) - h(z); \tau_k < T_A, X_{\tau_k} \in B_{k-i-1} - B_{k-i}]. \end{aligned}$$

The first term on the right is bounded by

$$\Big(\frac{a_k + b_k}{2} - a_k\Big)\mathbb{P}^y(T_A < \tau_k).$$

The second term is bounded by

$$(b_k - a_k)\mathbb{P}^y(\tau_k < T_A) = (b_k - a_k)(1 - \mathbb{P}^y(T_A < \tau_k)).$$

Using (4.2) the infinite sum is bounded by

$$\begin{aligned} \sum_{i=1}^{\infty} (b_{k-i-1} - a_{k-i-1})\mathbb{P}^y(X_{\tau_k} \notin B_{k-i}) \\ &\le \sum_{i=1}^{\infty} c_3 M \gamma^{k-i-1} (\rho^k)^\alpha / (\rho^{k-i})^\alpha \\ &= c_3 M \gamma^{k-1} \sum_{i=1}^{\infty} (\rho^\alpha/\gamma)^i \\ &\le 2c_3 M \gamma^{k-2} \rho^\alpha \\ &\le \frac{c_2}{4} M \gamma^k. \end{aligned}$$

Therefore

$$\begin{aligned} h(y) - h(z) &\le \frac{1}{2}(b_k - a_k)\mathbb{P}^y(T_A < \tau_k) + (b_k - a_k)(1 - \mathbb{P}^y(T_A < \tau_k)) + c_2 M \gamma^k/4 \\ &\le (b_k - a_k)[1 - \tfrac{1}{2}\mathbb{P}^y(T_A < \tau_k)] + c_2 M \gamma^k/4 \\ &\le M\gamma^k[1 - (c_2/2)] + c_2 M \gamma^k/4 \\ &= M\gamma^k[1 - (c_2/4)] = M\gamma^{k+1}. \end{aligned}$$

We conclude that

$$b_{k+1} - a_{k+1} \le M\gamma^{k+1} + 2\varepsilon.$$

Since ε is arbitrary, this proves (4.4) and hence (4.3).

If $x, y \in B(x_0, 1)$, let k be the smallest integer such that $|x - y| < \rho^k$. Then $\log|x-y| \ge (k+1)\log\rho$, $y \in B(x, \rho^k)$, and

$$\begin{aligned}|h(y) - h(x)| &\le M\gamma^k = Me^{k\log\gamma}\\ &\le c_4 Me^{\log|x-y|(\log\gamma/\log\rho)} = c_4 M|x-y|^{\log\gamma/\log\rho}.\end{aligned}$$

□

Define

$$S_\lambda g(x) = \mathbb{E}^{\,x}\int_0^\infty e^{-\lambda t} g(X_t)dt.$$

Proposition 4.2. *Suppose g is bounded and has compact support. There exists $c_1 > 2$ and $\beta \in (0,1)$ such that*

$$|S_0 g(x) - S_0 g(y)| \le c_1(\|S_0 g\|_\infty + \|g\|_\infty)(|x-y| \wedge 1)^\beta.$$

Proof. Suppose $|x - y| \le 1$, for otherwise there is nothing to prove. We write

$$S_0 g(x) = \mathbb{E}^{\,x}\int_0^{\tau_{B(x,r)}} g(X_s)ds + \mathbb{E}^{\,x} S_0 g(X_{\tau_{B(x,r)}})$$

and

$$S_0 g(y) = \mathbb{E}^{\,y}\int_0^{\tau_{B(x,r)}} g(X_s)ds + \mathbb{E}^{\,y} S_0 g(X_{\tau_{B(x,r)}}).$$

Taking the difference,

$$|S_0 g(x) - S_0 g(y)| \le 2\|g\|_\infty \sup_z \mathbb{E}^{\,z}\tau_{B(x,r)} + c_2\|S_0 g\|_\infty\Big(\frac{|x-y|}{r}\Big)^\beta,$$

using Theorem 4.1, scaling, and the fact that $z \to \mathbb{E}^{\,z} S_0 g(X_{\tau_{B(x,r)}})$ is $\mathcal{L}$-harmonic inside $B(x,r)$. Taking $r = |x-y|^{1/2}$ and using Lemma 3.3, we obtain our result. □

Theorem 4.3. *Suppose g is bounded and $\lambda > 0$. There exists $c_1 > 0$ and $\beta \in (0,1)$ such that*

$$|S_\lambda g(x) - S_\lambda g(y)| \le c_1\|g\|_\infty(|x-y| \wedge 1)^\beta.$$

Proof. Without loss of generality assume $g \ge 0$. Temporarily assume g has compact support. Let $h = g - \lambda S_\lambda g$. Note $S_0 h \le S_0 g + \lambda S_\lambda S_0 g$, so h is bounded. We have $S_\lambda g = S_0 h$ by the resolvent equation. Since $\|S_\lambda g\|_\infty \le c_2\|g\|_\infty$, then $\|S_0 h\|_\infty + \|h\|_\infty \le c_3\|g\|_\infty$. Our result now follows by Proposition 4.2 if g has compact support. Taking limits allows us to remove this restriction. $\square$

Remark 4.4. The solution to the integral equation

$$\mathcal{L}u(x) - \lambda u(x) = -g(x)$$

is given by $u(x) = S_\lambda g(x)$. So Theorem 4.3 provides a regularity result for the solutions of such integral equations.

Remark 4.5. Theorem 4.3 also has applications to the uniqueness of the martingale problem for $\mathcal{L}$. One important technique in proving uniqueness of martingale problems is perturbation. In the diffusion case, one looks at the operator $\frac{\partial^2}{\partial x_i \partial x_j} U_\lambda$, where U_λ is the resolvent operator for Brownian motion, and one needs to establish L^p bounds for suitable p. The case of L^2 is quite easy, but the L^p estimates for other p rely on singular integrals.

To establish uniqueness for the martingale problem for $\mathcal{L}$, one needs to obtain estimates for an analogous operator. Again the L^2 estimates are relatively straightforward, but the L^p estimates are much harder. However, it is possible to establish uniqueness and to weaken the hypotheses of [K] by using Theorem 4.3 and the L^2 estimates. The argument is rather lengthy, so we do not include it here, but a similar argument is used in [A].

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