

# Hartree-Fock Theory of Dipolar Fermi Gases



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Falk Wächtler

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Falk Wächtler

Erstgutachter: Prof Dr. Martin Wilkens

Zweitgutachter: Priv.-Doz. Dr. Axel Pelster

# Kurzzusammenfassung

Die experimentellen Realisierungen der ersten Bose-Einstein Kondensate im Jahre 1995 legten den Grundstein für die physikalische Untersuchung von entarteten Quantengasen. Seitdem ist dieses Wissenschaftsfeld ein robuster und schnell wachsender Teil der Tieftemperaturphysik geworden und hat viel Aufmerksamkeit der physikalischen Gemeinschaft auf sich gezogen. Die ersten magnetischen dipolaren Effekte wurden in Bose-Einstein Kondensaten aus Atomen von  $^{52}\text{Cr}$ ,  $^{87}\text{Rb}$ ,  $^7\text{Li}$  und  $^{164}\text{Dy}$  beobachtet. Durch die Verwendung des STIRAP-Verfahrens (stimulierter adiabatischer Raman-Übergang) ist es möglich geworden, die fermionischen Moleküle  $^{40}\text{K}^{87}\text{Rb}$  mit einem Dipolmoment von 0.5 Debye in die Nähe des quantenentarteten Bereiches zu bringen. Mit diesem experimentellen Fortschritt ist es realistisch anzunehmen, dass entartete dipolare Fermigase in Kürze experimentell untersucht werden können. In dieser Diplomarbeit stelle ich die Herleitung einer Theorie dar, die die Dynamik solcher Gase vom kollisionslosen- hin zum hydrodynamischen Regime beschreibt. Der erste Teil behandelt die Herleitung der Bewegungsgleichung, der so genannten Boltzmann-Vlasov-Gleichung, welche die Dynamik des Systems beschreibt und der zweite Teil behandelt die Bestimmung einer approximativen Lösung.

Um eine Gleichung herzuleiten, welche die Zeitentwicklung des Systems beschreibt, führe ich eine Quasi-Phasenraumverteilungsfunktion ein, welche aufgrund einer Gradientenentwicklung zu einer semiklassischen Beschreibung führt und die Gültigkeit dieser Herleitung auf schwache Teilchenwechselwirkungen beschränkt. Deren Effekte werden mit Hilfe von Störungstheorie bestimmt, welche in erster Ordnung zur Molekularfeldbeschreibung mittels Hartree- und Fock-Potenzialen führt. Der Term zweiter Ordnung beschreibt die Zweiteilchen-Wechselwirkung über ein binäres Kollisionsintegral. Aus der Betrachtung von dessen Eigenschaften leiten wir die Erhaltungsgrößen des Systems ab, was zu einer konkreten Form der Verteilungsfunktion im lokalen Gleichgewicht führt, welche den Limes jedes Relaxationsprozesses aufgrund von Kollisionen darstellt. Um das Kollisionsintegral zu vereinfachen, linearisiere ich dieses und diskutiere die mathematische Gültigkeit einer Relaxationszeitnäherung.

Um eine Lösung der Boltzmann-Vlasov Gleichung zu finden, benutze ich einen Skalenansatz und berechne Momente der gesamten Gleichung, was zu gewöhnlichen Differentialgleichungen für die Skalierungsparameter führt. Um diese zu lösen, minimieren wir die Gesamtenergie des Systems, was zu Gleichungen führt, welche die statische Lösung beschreiben. Dabei betrachte ich insbesondere auch den zylindersymmetrischen Fall. Ich berechne die Wirkung der Kollisionen, welche mittels einer Relaxationszeitnäherung beschrieben werden, durch die Bestimmung der Impulssymmetrie des lokalen Gleichgewichts und kopple diese an die Differentialgleichungen mit Hilfe der Erhaltungssätze. Dies führt zu einem geschlossenen Gleichungssystem, das numerisch gelöst werden kann. Durch die Betrachtung der Grenzfälle der Kollisionen, dem kollisionslosen- und dem hydrodynamischen Regime, zeige ich, wie diese Theorie in unterschiedlichen Regimen zu verschiedenen Gleichungen führt und vergleiche die Ergebnisse mit früheren Untersuchungen dieser Grenzfälle. Dies liefert die theoretischen Grundlagen für weitere Forschungen auf diesem Gebiet und könnte in zukünftigen Experimenten mit entarteten dipolaren Fermi-Gasen Anwendung finden.

# Abstract

The experimental realization of the first Bose-Einstein condensates in 1995 created the basis for the physical investigation of quantum degenerate gases. Since then, this research field has become a robust and fast growing branch of low-temperature physics and attracted much attention throughout the physical community. The first magnetic dipolar effects were observed in atomic Bose-Einstein condensates of  $^{52}\text{Cr}$ ,  $^{87}\text{Rb}$ ,  $^7\text{Li}$ , and  $^{164}\text{Dy}$ . By using stimulated Raman adiabatic passage (STIRAP) it was possible to bring the fermionic molecules  $^{40}\text{K}^{87}\text{Rb}$ , with an electric dipole moment of 0.5 Debye, close to quantum degeneracy. With this experimental progress it is realistic to assume that it would soon be possible to investigate degenerate dipolar Fermi gases. In this thesis, I present the derivation of a theory describing the dynamics of such gases from the collisionless until the hydrodynamic regime. The first part deals with the derivation of the underlying equation of motion, the so-called Boltzmann-Vlasov equation, which describes the dynamics of the system and the second part is dedicated to find an approximate solution of it.

In order to derive an equation which governs the time evolution of the system, I introduce a quasi phase-space distribution function which leads, due to a gradient expansion, to a semiclassical description of the system and limits the validity of this derivation to weak interparticle interactions. These effects are determined by means of perturbation theory which leads in first order to a mean-field description by the Hartree and Fock potentials. The second order term describes two-particle interactions via the binary collision integral. By considering its properties, I derive the conserved quantities leading to a concrete form of the local equilibrium distribution function which denotes the limit of any relaxation due to collisional effects. I linearize the collision integral to simplify it and discuss the mathematical validity of a relaxation time approximation.

To find a solution of the Boltzmann-Vlasov equation, I use a scaling ansatz and calculate moments of the whole equation leading to ordinary differential equations for the scaling parameters. In order to solve them, I minimize the total energy of the system and construct the equations which determine the static solution. To this end, I consider in particular the cylinder-symmetric case. Furthermore, I calculate the effect of collisions, which is described within a relaxation time approximation, by determining the momentum symmetry of the local equilibrium and coupling it to the differential equations by means of the conservation laws. This provides a closed set of ordinary differential equations which can be solved numerically. By considering the collisional limits, the collisionless and the hydrodynamic regime, I show how this theory leads to miscellaneous equations of motion in the different regimes and compare these results with previous investigations of these limiting cases. This provides a theoretical background for further theoretical studies and could be useful for future experiments with degenerate dipolar Fermi gases.

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# 1 Introduction

This chapter provides a concise overview of the physics of ultracold quantum systems with a special emphasis on dipolar Fermi gases. Starting in Section 1.1 from the first theoretical and experimental developments, I present in chronological order important historical advances up to the present status of this particular branch of low-temperature physics. In Section 1.2, I review how interparticle interactions affect both static and dynamic properties of Fermi gases. In Section 1.3, I highlight the current theoretical and experimental state of knowledge in the topic of dipolar Fermi gases. Finally, Section 1.4 shows the table of contents of the present thesis.

## 1.1 Bose Gases

Physicists have been working on the topic of low-temperature physics for about 90 years. In 1924 Satyendra Nath Bose derived Planck's quantum radiation law using a novel method for counting states of identical photons [1]. His idea was extended by Albert Einstein in the same year for massive particles [2]. In this way it was discovered that the Maxwell-Boltzmann distribution is no longer valid for microscopic bosonic particles and has to be replaced by another one, which is called Bose-Einstein distribution. It predicts the existence of a new state for macroscopic quantum systems, namely the Bose-Einstein condensate (BEC), in which a large fraction of bosons occupy the quantum mechanical ground state of the system. In a BEC, quantum effects become apparent on a macroscopic scale. Therefore, it is one prominent example for a macroscopic quantum phenomenon. The physics community was skeptical whether a BEC could ever be realized, considering the interaction of the particles. Nonetheless, further theoretical progress was achieved and, in 1956, a common criterion for Bose-Einstein condensation, which is based on the necessity of long-range order, was formulated by O. Penrose and L. Onsager [3].

An important step for experimentally realizing a Bose-Einstein condensate was the development of laser cooling, which cools a gas of atoms or molecules via the interaction with one or more laser fields. The method is based on the detuning of a laser field slightly above an electronic transition frequency for the gas particles. If they move in the opposite direction of the light beam, the laser light is red-shifted due to the Doppler effect, which leads to a higher probability to absorb the photon and, therefore, to an effective cooling due to the direction independence of the spontaneous emission. This method enables physicists to reach the  $\mu\text{K}$  regime.

Evaporative cooling is used to further cool down the gas in order to reach the temperatures necessary for Bose-Einstein condensation. This method removes the particles with the highest energy, thereby lowering the average energy of the particles and providing an effective cooling.

The experimental realization of Bose-Einstein condensates was achieved 70 years after their theoretical prediction in 1995 by Eric Cornell and Carl Wieman at JILA [4] in a rubidium vapor and by Wolfgang Ketterle at MIT [5] in a sodium vapor. In the next few years, most of the theoretical and experimental studies were dedicated to bosonic quantum gases to investigate the consequences of Bose-Einstein condensation and the physical properties of Bose gases. Finally, it was possible to Bose-Einstein condense twelve chemical elements in the last years as shown in Fig. 1.1, i.e., the elements of the first column of the periodic table as hydrogen in 1998 [6], lithium in 1997 [7], potassium in 2001 [8] and cesium in 2003 [9]. The first two-electron BEC

Legend:   
 □ = Hauptgruppen □ = Nebengruppen □ = Edelgase   
 Schale: I, II, IIIa, IVa, Va, VIa, VIIa, VIIIa, Ia, Ib, III, IV, V, VI, VII, VIII

Aggregatzustand unter Normalbedingungen:   
 Fe fest   
 Hg flüssig   
 He gasförmig   
 \* = radioaktives Element

Figure 1.1: Periodic table in which all elements, that were Bose-Einstein condensed, are highlighted in green.

was the rare-element ytterbium in 2003 [10], and recent developments made it possible to Bose condense the alkaline earth metals calcium [11] and strontium [12,13], both in 2009. Another major achievement was the Bose-Einstein condensation of helium in 2001 [14,15]. Additionally, in the search for strong dipolar BEC's, the condensation of chromium in 2005 [16] and dysprosium in 2011 [17] succeeded.

## 1.2 Fermi Gases

In view of the experimental achievements with BECs, the attention of the physics community naturally also turned towards fermionic gases. In contrast to bosons, fermions obey the Pauli exclusion principle, leading to a quite different behavior. One important difference lies in the fact that the s-wave scattering is inhibited for fermions, which has large consequences for cooling mechanisms based on evaporation [18], because thermalization is suppressed in this case. This property makes it hard to achieve low temperatures in Fermi gases. Nevertheless, this can be realized through sympathetic cooling techniques, where two different spin components of the same Fermi gas are used. Alternatively, a Bose gas may be used as a refrigerator. Another important difference between fermionic and bosonic gases is that no phase transition into a BEC occurs in a Fermi gas. Instead, there is a smooth crossover into quantum degeneracy.

These major differences are rooted in quantum statistics, which become prevalent at low temperatures. Hence, ultracold Fermi gases lend themselves to the study of quantum effects in many-body systems. The first important achievements of quantum degeneracy in trapped Fermi gases were obtained in 1999 at JILA [19], where temperatures smaller than the Fermi temperature were reached. As the experimental data did not agree with classical theory, but coincided with theoretical predictions for degenerate Fermi gases, the quantum nature of the sample became apparent.

Interactions play a major role in quantum gases, as one can see in the context of the BCS-BEC crossover. BCS theory, which was proposed by J. Bardeen, L. Cooper and J. R. Schrieffer, was the



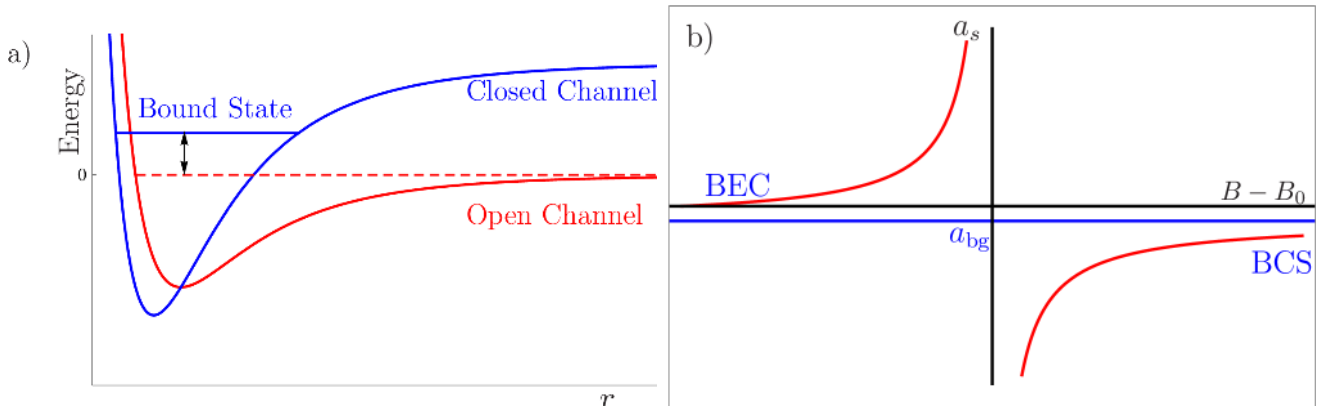


Figure 1.2: a) Schematic representation of the interaction potential with respect to the interparticle distance  $r$ . The red line represents the open channel and the blue one the closed channel. b) Functional behavior of the s-wave scattering length with respect to the applied magnetic field [20].

first microscopic theory of superconductivity, its name being derived from the physicists' initials. The BCS theory explains the effect of superconductivity on the basis that attractively interacting fermions form Cooper pairs which behave like a particle with integer spin. Therefore, these pairs obey Bose-Einstein statistics, despite their fermionic constituents. The condensation of Cooper pairs at the Fermi edge leads to the famous effects of superconductivity and superfluidity. On the other hand, when the interaction between the fermions is repulsive, loosely bound molecules are formed, which are small in comparison to the Cooper pairs, and their condensation leads to superfluidity.

In recent years, the interparticle interaction strength could be changed by means of a Feshbach resonance which enables one to change an attractive interparticle interaction into a repulsive one. To discuss this in detail, consider two particles interacting via an underlying potential as a function of the relative distance  $r$  as is depicted in Fig. 1.2 a). An incoming particle will be scattered back by the barrier of the potential at  $r = 0$  if the potential is independent of the atomic internal degrees of freedom, which is called the open channel. But if the interparticle interaction depends on the internal degrees of freedom, different sub-states will be coupled due to the hyperfine interaction. An example will be the formation of a bound state of two particles due to the collision process between them. This is called the closed channel. A Feshbach resonance occurs when the energy of the bound state is equal to the asymptotic energy of the open channel. The important point is the tunability of this energy difference for the hyperfine interaction by means of an external magnetic field. This enables one to change the s-wave scattering length as shown in Fig. 1.2 b), thereby providing a control over the strength and nature of the interparticle interaction. This represents an ideal testing ground for investigating many-body quantum systems.

In order to characterize Fermi gases, it is important to investigate the elementary excitations of the system, which depend on the interaction. In the nearly collisionless regime, in which only a few scattering events per oscillation of the excitation occur, this corresponds to the zero sound mode, a collective excitation due to the effective mean field provided by a large number of particles. On the other hand, in the case of a large collisional rate, which is called the hydrodynamic regime, the elementary excitation is the first sound mode due to the high collision rate which ensures local equilibrium. The transition between these two regimes was first observed by S. D. Gensemer and D. S. Jin [21] where a change of the density of a gas consisting of  $^{40}\text{K}$  led to a drastic change in the strength of the interparticle interaction.

### 1.3 Dipolar Fermi Gases

The experimental observation of quantum degeneracy leads to the possibility of studying the long-range dipole-dipole interaction in a polarized Fermi gas in a controlled way. The corresponding interaction potential reads

$$V_{\text{int}}(\mathbf{x}) = \frac{C_{\text{dd}}}{4\pi |\mathbf{x}|^3} (1 - 3 \cos^2 \vartheta), \quad (1.1)$$

with  $\vartheta$  being the angle between the polarization direction and the relative position of the dipoles. The constant  $C_{\text{dd}}$  is a useful abbreviation so that Eq. (1.1) holds both for magnetic  $C_{\text{dd}} = \mu_0 m^2$  and for electric dipoles  $C_{\text{dd}} = 4\pi d^2$ , with the magnetic permeability in vacuum  $\mu_0$  and the electric dipole moment  $d$  measured in units of Debye.

The first theoretical article dealing with a dipolar Fermi gas was published in 2001 by K. Góral et al. [22], where a semiclassical theory of fermionic dipoles was used in order to investigate the hydrodynamic excitations of the system [23]. At about the same time, the anisotropic pairing in polarized dipolar gases in homogeneous [24] and harmonically trapped samples [25] was studied. Recently, it was discovered that, due to the dipole-dipole interaction, the momentum distribution of a Fermi gas is deformed from a sphere to an ellipsoid [26]. This important fact was not considered in previous investigations. Moreover, other remarkable studies have been made for trapped dipolar Fermi gases which cover, e.g., stationary solutions, low-lying excitations, and time-of-flight dynamics in the collisionless regime at zero temperature [27] or at finite temperatures [28] and in the hydrodynamic regime [20,29,30].

The experiments in the field of dipolar Fermi gases are not so far advanced, but recently, promising progress was made. Dipolar Fermi gases can be realized in several ways. One method consists of using atoms with large permanent magnetic dipole moments  $m$ , for example the chromium isotope  $^{53}\text{Cr}$  with  $m = 6 \mu_{\text{B}}$ , which has been magneto-optically trapped [31]. A crucial characteristic of these magnetic atoms is the strength of their dipole moments, because the dipole-dipole potential depends quadratically on them. Therefore, dysprosium, which is one of the elements with the largest magnetic dipole moments of ten Bohr magnetons  $\mu_{\text{B}}$ , is the most promising candidate for future experiments. As it was recently possible to condense  $1.5 \times 10^4$   $^{164}\text{Dy}$  atoms below 30 nK [17], it is realistic to assume that fermionic isotopes, such as  $^{163}\text{Dy}$ , which can be trapped simultaneously with bosonic isotopes [32], can soon be brought into the quantum degenerate regime as well.

Another option is to use heteronuclear polar molecules. They are especially suitable due to their large electric dipole moments. Further investigations of this topic led to the possibility to control the interparticle interaction in these systems through applied electric fields [33] similar to a Feshbach resonance. Recently,  $^{40}\text{K}^{87}\text{Rb}$  molecules were brought close to quantum degeneracy by using stimulated Raman adiabatic passage (STIRAP) to convert the molecules into the rovibrational ground state [34]. Further developments made it possible to detect via thermodynamic measurements the anisotropy of the  $^{40}\text{K}^{87}\text{Rb}$ -sample caused by the dipole-dipole interaction [35] as well as both the space and the momentum distribution [36]. However, these molecules are not suitable for further studies at lower temperatures due to their chemical instability. Currently, several groups are trying to cool proper endothermic heteronuclear molecules which are chemically stable, so that polar molecules could serve as a background for testing theoretical predictions as those developed in the present thesis.

## 1.4 This Thesis

The present thesis deals with the physical properties of a Fermi gas interacting via the long-range dipole-dipole interaction within a triaxial trap at zero temperature. In the first part, I present the derivation of the Boltzmann-Vlasov equation describing the time evolution of the system and its properties. The second part is dedicated to find an approximate solution to this equation.

In **Chapter 2**, I present the description of the system in terms of a phase-space distribution function, starting from basic quantum mechanics. The equation of motion for this function will be derived according to perturbation theory, leading in first order to the Hartree and Fock terms and in second order to the collision integral.

**Chapter 3** analyses the physical properties of the collision integral. At first, I will demonstrate how it conserves particle number, momentum and kinetic energy and how any relaxation due to the collision processes leads to a relaxation of the system into a Fermi-Dirac distribution. Then, I will also deal with approximations of the collision integral, i.e., the linearization and the relaxation time approximation, and discuss their validity.

In **Chapter 4**, I present the derivation of an approximate solution of the equation of motion for the distribution function which should be able to describe different collisional regimes, such as the collisionless or the hydrodynamic regime. To this end, I derive the static solution and the momentum symmetry of distribution functions which are invariant under collisions.

**Chapter 5** considers the limiting cases of my approximate solution. For this purpose, I specialize the system into different collisional regimes and trapping geometries to obtain equations which can be checked against previous theoretical work.

The thesis is concluded in **Chapter 6**, where an outlook is given in which I describe applications of the theory and propose further developments, e.g., how a more general equation of motion can be derived and how one can include temperature in this model.



## 2 The Boltzmann-Vlasov Equation

In order to describe an ultracold Fermi gas, I use a semiclassical description in terms of a phase-space distribution function. It describes the dynamics of the system by an equation of motion, which is called Boltzmann-Vlasov equation. I present in this chapter its derivation at zero temperature with all important steps of the calculation.

Consider a Fermi gas, which is contained in a trap and where the constituent fermions can interact via a two-particle interaction. The physical properties of the considered Fermi gas are mathematically contained in the Hamilton operator  $\hat{H}$  of the system which determines the time evolution of quantum states  $|\psi(t)\rangle$  through

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle. \quad (2.1)$$

Moreover, the Hamilton operator of the system decomposes according to

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \quad (2.2)$$

and can be written in terms of the creation and annihilation operators  $\hat{\psi}^\dagger(\mathbf{r})$  and  $\hat{\psi}(\mathbf{r})$ . In turn, these satisfy the anticommutation relations

$$\begin{aligned} \{\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} &= \delta(\mathbf{r} - \mathbf{r}'), \\ \{\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\} &= 0, \\ \{\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} &= 0, \end{aligned} \quad (2.3)$$

where the curly brackets denote the anticommutator  $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ . The one-particle Hamilton operator  $\hat{H}_0$  contains the kinetic energy and the trapping potential  $U(\mathbf{r})$  in the form

$$\hat{H}_0 = \int d^3r \hat{\psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla_{\mathbf{r}}^2}{2m} + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) \quad (2.4)$$

and the interaction Hamilton operator  $\hat{H}_{\text{int}}$  describes two-particle interactions via the potential  $V_{\text{int}}(\mathbf{r} - \mathbf{r}')$  according to

$$\hat{H}_{\text{int}} = \frac{1}{2} \int d^3r d^3r' V_{\text{int}}(\mathbf{r} - \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}). \quad (2.5)$$

In this thesis I only consider the dipole-dipole interaction Eq. (1.1) between the fermions. This is based on the assumption that the van-der-Waals forces between the atoms can be approximated at low temperatures by an effective contact interaction [37–40]. This approximation considers only the s-wave scattering of the potential, which is however forbidden for fermions, due to their asymmetric wave functions [41]. Conveniently, then, there is no need to account for this type of interaction.

## 2.1 Wigner Representation

A classical system is fully described through space and momentum variables and one often uses phase-space distribution functions to describe the dynamics of a many-particle system. In a quantum description, it is possible to consider a quasi-distribution function, the Wigner function, which is defined through the Fourier transform of the correlation function  $G^{(2)}(\mathbf{x}; \mathbf{x}'; t)$  with respect to the relative coordinate  $\mathbf{s} = \mathbf{x} - \mathbf{x}'$  and reads

$$f(\mathbf{r}, \mathbf{p}, t) = \int d^3s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} G^{(2)}\left(\mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t\right). \quad (2.6)$$

This correlation function is defined through the average of a field operator and an adjoint one

$$G^{(2)}(\mathbf{x}; \mathbf{x}'; t) = \left\langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}') \right\rangle_t \quad (2.7)$$

and the quantum average is understood as the trace over the field operators with the underlying density matrix  $\hat{\rho}(t)$

$$\left\langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}') \right\rangle_t = \text{Tr} \left[ \hat{\rho}(t) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}') \right]. \quad (2.8)$$

Calculating the averages with the density matrix is a more general method than using the scalar product of quantum states, because it enables to describe not only pure states but also mixed ones. Therefore, it contains the whole time evolution of the system in the Schrödinger picture.

The Wigner function contains all quantum information of the system and is equivalent to a quantum mechanical wave function [42,43]. Despite of this, it can be negative within small areas of the phase space, which are shielded by the Heisenberg uncertainty relation. Therefore, the Wigner function is only a quasi-distribution function, but the quantum-mechanical expectation values of observables can be obtained by integrating them over the whole phase space, weighted with the Wigner function. In particular, integrating the Wigner function over the space or the momentum variable leads to the respective densities

$$n(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi\hbar)^3} f(\mathbf{x}, \mathbf{p}, t), \quad (2.9)$$

$$n(\mathbf{p}, t) = \int \frac{d^3x}{(2\pi\hbar)^3} f(\mathbf{x}, \mathbf{p}, t), \quad (2.10)$$

which are genuine distribution functions. In my further calculations, I perform gradient expansions which assume that the Wigner function is slowly varying in its spatial argument. Thereby, quantum mechanical information is lost, which yields a semiclassical Wigner function [44].

Note that I will perform all of my further calculations in the Schrödinger picture. Then, the whole time evolution of the system is governed by the density matrix  $\hat{\rho}(t)$  according to the von Neumann equation

$$i\hbar \frac{\partial \hat{\rho}}{\partial t}(t) = \left[ \hat{H}, \hat{\rho}(t) \right], \quad (2.11)$$

where the square brackets denote the commutator  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ . The resulting time evolution of the Wigner function can be computed with the help of Eq. (2.6)–(2.11) by using the cyclic permutation of the trace and the explicit form of the Hamiltonian from Eqs. (2.2), (2.4), and

(2.5):

$$\begin{aligned}
i\hbar\frac{\partial f}{\partial t}(\mathbf{r}, \mathbf{p}, t) = & \int d^3s d^3r' e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \text{Tr} \left( \hat{\rho}(t) \left\{ \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right), \hat{\psi}^\dagger(\mathbf{r}') \left[ -\frac{\hbar^2 \nabla_{\mathbf{r}'}^2}{2m} + U(\mathbf{r}') \right] \hat{\psi}(\mathbf{r}') \right\} \right) \\
& + \frac{1}{2} \int d^3s d^3r' d^3r'' e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \text{Tr} \left\{ \hat{\rho}(t) V_{\text{int}}(\mathbf{r}' - \mathbf{r}'') \right. \\
& \left. \times \left[ \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right), \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}'') \hat{\psi}(\mathbf{r}'') \hat{\psi}(\mathbf{r}') \right] \right\}. \tag{2.12}
\end{aligned}$$

The calculation of the respective terms on the right-hand side of Eq. (2.12) represents the goal of this chapter. The kinetic and trapping terms contain anticommutators of four field operators which lead to averages of two field operators and, through Eq. (2.6) and Eq. (2.7), to Wigner functions. The six operator average in the interaction term is more complicated. It will be necessary to use perturbation theory to compute this term in an appropriate way. For simplicity, I will deal with all these terms individually.

## 2.2 Kinetic Term

At first, I will calculate the term which contains the one-particle Hamiltonian  $\hat{H}_0$ . After performing the commutator, the kinetic energy contribution in Eq. (2.12) yields the following expression

$$I_K = \int d^3s \frac{-\hbar^2}{2m} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \text{Tr} \left( \hat{\rho}(t) \left\{ \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \left[ \nabla_-^2 \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] - \left[ \nabla_+^2 \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \right] \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right\} \right), \tag{2.13}$$

where  $\nabla_\pm = \nabla_{\mathbf{r} \pm \mathbf{s}/2}$  is an abbreviation. The derivatives can be rewritten into pure derivatives of  $\mathbf{r}$  and  $\mathbf{s}$  by using the chain rule

$$\nabla_{\mathbf{r} \pm \frac{1}{2}\mathbf{s}} = \nabla_{\mathbf{r}} \pm \frac{1}{2} \nabla_{\mathbf{s}}. \tag{2.14}$$

Inserting this into Eq. (2.13) leads to

$$I_K = \int d^3s \frac{\hbar^2}{m} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \nabla_{\mathbf{s}} \cdot \nabla_{\mathbf{r}} \text{Tr} \left[ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right]. \tag{2.15}$$

Using a partial integration, computing the action of  $\nabla_{\mathbf{s}}$  onto the exponential function and taking into account Eqs. (2.6)–(2.8) yields

$$I_K = -i\hbar \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{p}, t). \tag{2.16}$$

Note that the kinetic term Eq. (2.16) was obtained without any approximation.

## 2.3 Trapping Term

In the case of a smooth trapping potential  $U(\mathbf{r})$ , the corresponding term in Eq. (2.12) can be calculated similar to the kinetic term. At first one has to calculate the commutator with the trapping term which leads to

$$I_U = \int d^3s e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \left[ U \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) - U \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \right] \text{Tr} \left\{ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right\}. \tag{2.17}$$

## 2 The Boltzmann-Vlasov Equation

In order to deal with the potentials at different space points, I use the translation operator  $f(\mathbf{x} + \mathbf{a}) = e^{\mathbf{a} \cdot \nabla_{\mathbf{x}}} f(\mathbf{x})$ . This can also be understood as a Taylor expansion of the function  $f(\mathbf{x} + \mathbf{a})$  with respect to  $\mathbf{a}$  at the space point  $\mathbf{x}$ . Using the translation operator yields

$$I_U = \int d^3 s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ (e^{-\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} - e^{\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}}) U(\mathbf{r}) \right] \text{Tr} \left[ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right]. \quad (2.18)$$

To obtain a term proportional to the Wigner function, I have to approximate the exponential functions to first order. Physically, neglecting higher order derivatives of the trapping potential  $U(\mathbf{r})$  within a gradient expansion means that it is assumed to be slowly varying in space. Hence I approximate Eq. (2.18) to

$$I_U \approx - \int d^3 s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \quad \mathbf{s} \cdot \nabla_{\mathbf{r}} U(\mathbf{r}) \text{Tr} \left\{ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right\}. \quad (2.19)$$

Now I can rewrite the factor  $\mathbf{s}$  in the integrand with the help of the exponential function in Eq. (2.19) as a derivative of the momentum  $\mathbf{p}$  and get with Eqs. (2.6)–(2.8)

$$I_U = i\hbar \nabla_{\mathbf{r}} U(\mathbf{r}) \cdot \nabla_{\mathbf{p}} f(\mathbf{r}, \mathbf{p}, t). \quad (2.20)$$

## 2.4 Hierarchical Structure of Quantum Averages

Putting the kinetic and trapping terms (2.16) and (2.20) together and calculating the commutator in the interaction term in Eq. (2.12) yields the intermediate result

$$\begin{aligned} & \frac{\partial f}{\partial t}(\mathbf{r}, \mathbf{p}, t) + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{p}, t) - \nabla_{\mathbf{r}} U(\mathbf{r}) \cdot \nabla_{\mathbf{p}} f(\mathbf{r}, \mathbf{p}, t) = \frac{1}{i\hbar} \int d^3 s \int d^3 x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \\ & \times \left[ V_{\text{int}} \left( \mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2} \right) - V_{\text{int}} \left( \mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] \text{Tr} \left[ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right]. \end{aligned} \quad (2.21)$$

The right-hand side of Eq. (2.21) shows the main problem of deriving an equation that determines the time evolution of the Wigner function: The time derivative of the average of two field operators leads to an average of four field operators, the time derivative of this four-field operator average would lead to a six-field operator average, and so on. Note that this is a hierarchical structure and it is, therefore, impossible to get a closed set of equations without further approximations. Thus, I approximate this term with the help of perturbation theory with respect to the interaction potential, which can only be done in the case of weak interparticle interactions, thereby limiting the validity of the resulting Boltzmann-Vlasov equation. The first-order term leads to the mean-field potentials and the second-order term to the collision integral.

With the help of the Wick contraction, one is able to approximate higher order operator averages as products of two-field operator averages according to

$$\begin{aligned} \left\langle \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) \hat{\psi}(\mathbf{x}_3) \hat{\psi}(\mathbf{x}_4) \right\rangle_t & \approx \left\langle \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}(\mathbf{x}_4) \right\rangle_t \left\langle \hat{\psi}^\dagger(\mathbf{x}_2) \hat{\psi}(\mathbf{x}_3) \right\rangle_t \\ & - \left\langle \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}(\mathbf{x}_3) \right\rangle_t \left\langle \hat{\psi}^\dagger(\mathbf{x}_2) \hat{\psi}(\mathbf{x}_4) \right\rangle_t \\ & = G^{(2)}(\mathbf{x}_1; \mathbf{x}_4; t) G^{(2)}(\mathbf{x}_2; \mathbf{x}_3; t) - G^{(2)}(\mathbf{x}_1; \mathbf{x}_3; t) G^{(2)}(\mathbf{x}_2; \mathbf{x}_4; t). \end{aligned} \quad (2.22)$$

Note that this is only an approximation. It is therefore necessary to understand, what is lost by using it. To explain this in more detail, I consider two- and four-field operator averages. With the help of the abbreviation

$$h(\mathbf{x}) = \frac{-\hbar^2 \nabla_{\mathbf{x}}^2}{2m} + U(\mathbf{x}), \quad (2.23)$$



the dynamics of the correlation functions is described via the following differential equations

$$i\hbar \frac{\partial G^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t)}{\partial t} = [h(\mathbf{x}_2) - h(\mathbf{x}_1)] G^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t) + \int d^3x [V_{\text{int}}(\mathbf{x} - \mathbf{x}_2) - V_{\text{int}}(\mathbf{x} - \mathbf{x}_1)] G^{(4)}(\mathbf{x}_1, \mathbf{x}; \mathbf{x}, \mathbf{x}_2; t), \quad (2.24)$$

$$i\hbar \frac{\partial G^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t)}{\partial t} = [h(\mathbf{x}_4) + h(\mathbf{x}_3) - h(\mathbf{x}_2) - h(\mathbf{x}_1)] G^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t) + [V_{\text{int}}(\mathbf{x}_3 - \mathbf{x}_4) - V_{\text{int}}(\mathbf{x}_2 - \mathbf{x}_1)] G^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t) + \int d^3x [V_{\text{int}}(\mathbf{x} - \mathbf{x}_4) + V_{\text{int}}(\mathbf{x} - \mathbf{x}_3) - V_{\text{int}}(\mathbf{x} - \mathbf{x}_2) - V_{\text{int}}(\mathbf{x} - \mathbf{x}_1)] \times G^{(6)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}; \mathbf{x}, \mathbf{x}_3, \mathbf{x}_4; t). \quad (2.25)$$

Consequently, the equation of motion for an  $n$ -field operator average involves an  $n + 2$  operator average. This hierarchical structure contains, in principle, an infinite number of coupled partial differential equations, where the first one is of first order in the interaction potential and the  $n$ -th differential equation is of order  $n$  in  $V_{\text{int}}$ . Thus, approximately solving this hierarchical structure up to a certain equation corresponds to applying perturbation theory up to this order in the interaction potential. To understand in which case the Wick contraction is exact, I consider at first the simplest case, where no interaction occurs. For this purpose, I introduce two- and four-field operator averages

$$G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t) = \text{Tr} \left[ \hat{\rho}_0(t) \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}(\mathbf{x}_2) \right], \quad (2.26)$$

$$G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t) = \text{Tr} \left[ \hat{\rho}_0(t) \hat{\psi}^\dagger(\mathbf{x}_1) \hat{\psi}^\dagger(\mathbf{x}_2) \hat{\psi}(\mathbf{x}_3) \hat{\psi}(\mathbf{x}_4) \right], \quad (2.27)$$

where the density matrix  $\hat{\rho}_0(t)$  evolves only with respect to the one-particle Hamilton operator  $\hat{H}_0$  according to

$$i\hbar \frac{\partial \hat{\rho}_0}{\partial t} = \left[ \hat{H}_0, \hat{\rho}_0(t) \right]. \quad (2.28)$$

To distinguish them from correlation functions, which evolve with the full Hamilton operator  $\hat{H}$ , I denote them with the index 'zero'. Their differential equations can be obtained from the equations of motion for the general field operator averages Eq. (2.24) and Eq. (2.25) by setting the interaction potential to zero:

$$i\hbar \frac{\partial G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t)}{\partial t} = [h(\mathbf{x}_2) - h(\mathbf{x}_1)] G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t), \quad (2.29)$$

$$i\hbar \frac{\partial G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t)}{\partial t} = [h(\mathbf{x}_4) + h(\mathbf{x}_3) - h(\mathbf{x}_2) - h(\mathbf{x}_1)] G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t). \quad (2.30)$$

Inserting the Wick contraction of  $G_0^{(4)}$ , e.g.

$$G_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4; t) = G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_4; t) G_0^{(2)}(\mathbf{x}_2; \mathbf{x}_3; t) - G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_3; t) G_0^{(2)}(\mathbf{x}_2; \mathbf{x}_4; t), \quad (2.31)$$

into its differential equation Eq. (2.30) and using Eq. (2.29) shows that the Wick contraction is exact in the considered case. The reason for this is that  $\hat{H}_0$  in Eq. (2.4) is quadratic in the field operators. Hence, performing the Wick contraction means neglecting the interaction part  $\hat{H}_{\text{int}}$  in the complete Hamiltonian Eq. (2.2).

But this perturbation theory can not be performed exactly, which means that I have to make an additional approximation. The reason for this is that the Wick contraction only applies exactly to  $G_0^{(2n)}$  and therefore one has to find an exact relation between  $G^{(2n)}$  and  $G_0^{(2n)}$ , which is generally not possible. In the case of  $G^{(2)}$  such a relation can be derived, because of the linearity of the differential equation Eq. (2.24) and the fact that  $G_0^{(2)}$  is the homogeneous solution of it. Then, a relation between these two correlation functions can be derived by using the fact that general solutions of linear differential equations consist of the general homogeneous solution and a particular solution of the inhomogeneous equation. In the case of higher order correlation functions this is not possible, because their differential equations contain two terms proportional to  $G^{(2n)}$ , one due to the one-particle Hamiltonian  $\hat{H}_0$  and a second one originating from the interaction Hamiltonian. For  $G^{(4)}$  these are the first two terms in Eq. (2.25). Thus, the homogeneous solution of this differential equation is not the correlation function which evolves only with the one-particle Hamiltonian  $G_0^{(4)}$ .

## 2.5 Perturbation Theory

In order to achieve an expression for the right-hand side of Eq. (2.21), I have to perform perturbation theory. Eq. (2.11) determines the time evolution of the density matrix  $\hat{\rho}(t)$ . To solve this differential equation approximately, I will iterate the associated integral equation up to first order in the interaction potential. At first, I will derive this integral equation and afterwards solve it approximately. The time evolution due to the one-particle Hamiltonian  $\hat{H}_0$  can be expressed in terms of the time evolution operator  $\hat{U}_0(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$ . To this end, I will reformulate the equation of motion for  $\hat{\rho}(t)$  in order to describe the effects of the one-particle Hamiltonian  $\hat{H}_0$  only in terms of time evolution operators. This can be achieved by using the interaction picture in which the density matrix is defined through the following formula

$$\hat{\rho}(t) = \hat{U}_0(t, t_0)\hat{\rho}_D(t)\hat{U}_0^\dagger(t, t_0). \quad (2.32)$$

The time evolution of  $\hat{\rho}_D(t)$  is then governed by

$$i\hbar \frac{\partial \hat{\rho}_D(t)}{\partial t} = \hat{U}_0^\dagger(t, t_0)[\hat{H}_{\text{int}}, \hat{\rho}(t)]\hat{U}_0(t, t_0). \quad (2.33)$$

The corresponding integral equation can be obtained by integrating the whole differential equation

$$\hat{\rho}_D(t) = \hat{\rho}_D(t_0) - \frac{i}{\hbar} \int_{t_0}^t dt' \hat{U}_0^\dagger(t', t_0)[\hat{H}_{\text{int}}, \hat{\rho}(t')]\hat{U}_0(t', t_0). \quad (2.34)$$

Thus, taking into account Eq. (2.32), I get for the density matrix in the Schrödinger picture

$$\hat{\rho}(t) = \hat{U}_0(t, t_0)\hat{\rho}(t_0)\hat{U}_0^\dagger(t, t_0) - \frac{i}{\hbar} \int_{t_0}^t dt' \hat{U}_0(t, t_0)\hat{U}_0^\dagger(t', t_0)[\hat{H}_{\text{int}}, \hat{\rho}(t')]\hat{U}_0(t', t_0)\hat{U}_0^\dagger(t, t_0). \quad (2.35)$$

The products of the time evolution operators can be rewritten into single time evolution operators. Obviously, a time evolution from  $t'$  to  $t_0$ , i.e.,  $\hat{U}_0^\dagger(t', t_0) = \hat{U}_0(t_0, t')$ , followed by a time evolution from  $t_0$  to  $t$ , i.e.,  $\hat{U}_0(t, t_0)$ , is equivalent to a time evolution from  $t'$  to  $t$ , i.e.,  $\hat{U}_0(t, t')$ . Mathematically, this can be seen by considering the underlying exponential functions of the time evolution operator. Thus, I obtain

$$\hat{U}_0(t, t_0)\hat{U}_0^\dagger(t', t_0) = \hat{U}_0(t, t'). \quad (2.36)$$

The other two time evolution operators can be treated analogously. Inserting this into Eq. (2.35) leads to

$$\hat{\rho}(t) = \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) - \frac{i}{\hbar} \int_{t_0}^t dt' \hat{U}_0(t, t') [\hat{H}_{\text{int}}, \hat{\rho}(t')] \hat{U}_0^\dagger(t, t'). \quad (2.37)$$

Iterating this relation to first order and inserting this into the four-field operator average of Eq. (2.21) leads to

$$\begin{aligned} \text{Tr} \left[ \hat{\rho}(t) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] &= \text{Tr} \left\{ \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) \right. \\ &\times \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) - \frac{i}{\hbar} \int_{t_0}^t \hat{U}_0(t, t') [\hat{H}_{\text{int}}, \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0)] \\ &\left. \times \hat{U}_0^\dagger(t, t') \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) + \dots \right\}. \end{aligned} \quad (2.38)$$

The first term of Eq. (2.38) will lead to the Hartree and Fock mean-field terms, while the second term, which is of second order in the interaction, will lead to the binary collision integral.

## 2.6 Mean-Field Terms

Here, I will derive the mean-field terms of Eq. (2.21). Hence, I only consider the first term of the right-hand side of Eq. (2.38), inserting it into the interaction term of Eq. (2.21). The four-field operator average in this term evolves only with respect to the one-particle Hamiltonian  $\hat{H}_0$ , so it satisfies Eq. (2.30). As discussed earlier, the Wick contraction for this four-operator average is exact, but leads to two-field operator averages which also evolve only via the one-particle Hamiltonian  $\hat{H}_0$ , so I get with the Wick contraction Eq. (2.31)

$$\begin{aligned} \text{Tr} \left\{ \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right\} &= G_0^{(4)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{x}; \mathbf{x}, \mathbf{r} - \frac{\mathbf{s}}{2}; t \right) \\ &= G_0^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right) G_0^{(2)}(\mathbf{x}; \mathbf{x}; t) - G_0^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{x}; t \right) G_0^{(2)} \left( \mathbf{x}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right). \end{aligned} \quad (2.39)$$

Because of the linearity of the differential equation for  $G^{(2)}$  in Eq. (2.24), an exact relation can be obtained between the two-operator averages  $G^{(2)}$  and  $G_0^{(2)}$ , which evolve with and without the interaction Hamiltonian, respectively. A general solution of such a differential equation is the sum of the general solution of the homogeneous differential equation  $G_0^{(2)}$  and a particular solution of the inhomogeneous differential equation. The latter can be derived with the method of varying constants:

$$\begin{aligned} G^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t) &= G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t) \\ &+ \frac{1}{i\hbar} G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t) \int_{t_0}^t dt' \frac{\int d^3x [V_{\text{int}}(\mathbf{x} - \mathbf{x}_2) - V_{\text{int}}(\mathbf{x} - \mathbf{x}_1)] G^{(4)}(\mathbf{x}_1, \mathbf{x}; \mathbf{x}, \mathbf{x}_2; t')}{G_0^{(2)}(\mathbf{x}_1; \mathbf{x}_2; t')}. \end{aligned} \quad (2.40)$$

I approximate  $G_0^{(2)}$  with  $G^{(2)}$ , which is correct up to first order in the interaction. I perform this approximation following Kirkpatrick and Dorfman [45] and note that this means neglecting terms which are of second order in the interaction potential. The importance of these terms for the Boltzmann-Vlasov equation is beyond the scope of this diploma thesis. Thus, I am left with the following result for the left-hand side of Eq. (2.39)

$$\begin{aligned} &\text{Tr} \left\{ \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) \hat{\psi}^\dagger \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right\} \\ &\approx G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right) G^{(2)}(\mathbf{x}; \mathbf{x}; t) - G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{x}; t \right) G^{(2)} \left( \mathbf{x}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right). \end{aligned} \quad (2.41)$$

Inserting this into the the right-hand side of Eq. (2.21) yields

$$I_{\text{int}} \approx I_{\text{H}} + I_{\text{F}}, \quad (2.42)$$

with the Hartree term

$$\begin{aligned} I_{\text{H}} = & \frac{1}{i\hbar} \int d^3s d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ V_{\text{int}} \left( \mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2} \right) - V_{\text{int}} \left( \mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] \\ & \times G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right) G^{(2)}(\mathbf{x}; \mathbf{x}; t) \end{aligned} \quad (2.43)$$

and the Fock term

$$\begin{aligned} I_{\text{F}} = & -\frac{1}{i\hbar} \int d^3s \int d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ V_{\text{int}} \left( \mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2} \right) - V_{\text{int}} \left( \mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] \\ & \times G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{x}; t \right) G^{(2)} \left( \mathbf{x}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right). \end{aligned} \quad (2.44)$$

For simplicity, I treat the mean field terms separately, starting with the Hartree contribution.

### 2.6.1 Hartree Term

At first, I reformulate the interaction potential in Eq. (2.43) with the help of the translation operator

$$\begin{aligned} I_{\text{H}} = & \frac{1}{i\hbar} \int d^3s \int d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left\{ \left[ e^{\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{x}-\mathbf{r}}} - e^{-\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{x}-\mathbf{r}}} \right] V_{\text{int}}(\mathbf{x} - \mathbf{r}) \right\} \\ & \times G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right) G^{(2)}(\mathbf{x}; \mathbf{x}; t). \end{aligned} \quad (2.45)$$

To further evaluate the Hartree term, I have to approximate the exponential functions within the gradient expansion, where the spatial derivatives are considered up to the linear term. Furthermore, the particle density can be obtained according to Eqs. (2.6), (2.7), and (2.9)

$$n(\mathbf{x}, t) = G^{(2)}(\mathbf{x}; \mathbf{x}; t). \quad (2.46)$$

If I neglect higher order derivatives of the interaction potential, the Hartree term takes the form

$$I_{\text{H}} = \nabla_{\mathbf{r}} \left[ \int d^3x V_{\text{int}}(\mathbf{r} - \mathbf{x}) n(\mathbf{x}, t) \right] \cdot \nabla_{\mathbf{p}} f(\mathbf{r}, \mathbf{p}, t). \quad (2.47)$$

### 2.6.2 Fock Term

The derivation of the Fock term is more involved than the Hartree term. With the help of the following abbreviation

$$\Gamma^{\text{Ex}} \left( \mathbf{x}, \mathbf{r} - \frac{\mathbf{s}}{2}, t \right) = -V_{\text{int}} \left( \mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2} \right) G^{(2)} \left( \mathbf{x}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right), \quad (2.48)$$

the Fock term decomposes according to  $I_{\text{F}} = I_{\text{F}_1} - I_{\text{F}_2}$  with the terms

$$I_{\text{F}_1} = \frac{1}{i\hbar} \int d^3s d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \Gamma^{\text{Ex}} \left( \mathbf{x}, \mathbf{r} - \frac{\mathbf{s}}{2}, t \right) G^{(2)} \left( \mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{x}; t \right), \quad (2.49)$$

and

$$I_{F_2} = \frac{1}{i\hbar} \int d^3s d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \Gamma^{\text{Ex}} \left( \mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{x}, t \right) G^{(2)} \left( \mathbf{x}; \mathbf{r} - \frac{\mathbf{s}}{2}; t \right). \quad (2.50)$$

At first, I will treat these two terms separately, starting with  $I_{F_1}$ . For this purpose, I use the translation operator so that the correlation functions  $G^{(2)}$  and  $\Gamma^{\text{Ex}}$  become independent from  $\mathbf{s}$ . Furthermore, I use Fourier transformations with respect to the relative coordinates of  $\Gamma^{\text{Ex}}$  and  $G^{(2)}$  to derive the following form of Eq. (2.49)

$$I_{F_1} = \frac{1}{i\hbar} \int d^3s d^3x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ e^{-\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_1}{(2\pi\hbar)^3} \tilde{\Gamma}^{\text{Ex}} \left( \frac{\mathbf{r} + \mathbf{x}}{2}, \mathbf{p}_1, t \right) e^{-\frac{i}{\hbar} \mathbf{p}_1 \cdot (\mathbf{x} - \mathbf{r})} \right] \\ \times \left[ e^{\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_2}{(2\pi\hbar)^3} f \left( \frac{\mathbf{r} + \mathbf{x}}{2}, \mathbf{p}_2, t \right) e^{-\frac{i}{\hbar} \mathbf{p}_2 \cdot (\mathbf{r} - \mathbf{x})} \right], \quad (2.51)$$

where  $\tilde{\Gamma}^{\text{Ex}}$  is the Fourier transform with respect to  $\mathbf{x} - \mathbf{r}$  of the quantity defined in Eq. (2.48). To obtain a simple  $\mathbf{r}$  dependence of the Wigner function and  $\tilde{\Gamma}^{\text{Ex}}$ , I perform the substitution  $\mathbf{x}' = \mathbf{x} - \mathbf{r}$  and afterwards use the translation operator, which yields

$$I_{F_1} = \frac{1}{i\hbar} \int d^3s d^3x' e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ e^{-\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_1}{(2\pi\hbar)^3} e^{\frac{\mathbf{x}'}{2} \cdot \nabla_{\mathbf{r}}} \tilde{\Gamma}^{\text{Ex}} (\mathbf{r}, \mathbf{p}_1, t) e^{-\frac{i}{\hbar} \mathbf{p}_1 \cdot \mathbf{x}'} \right] \\ \times \left[ e^{\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_2}{(2\pi\hbar)^3} e^{\frac{\mathbf{x}'}{2} \cdot \nabla_{\mathbf{r}}} f (\mathbf{r}, \mathbf{p}_2, t) e^{\frac{i}{\hbar} \mathbf{p}_2 \cdot \mathbf{x}'} \right]. \quad (2.52)$$

The  $\mathbf{x}'$  in the exponential function with the spatial derivative acting on  $\tilde{\Gamma}^{\text{Ex}}$  and  $f$  is proportional to a spatial derivative with respect to  $\mathbf{p}_i$ , with  $i = 1, 2$ , leading to

$$I_{F_1} = \frac{1}{i\hbar} \int d^3s d^3x' e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[ e^{-\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_1}{(2\pi\hbar)^3} \tilde{\Gamma}^{\text{Ex}} (\mathbf{r}, \mathbf{p}_1, t) e^{\frac{i\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overleftarrow{\nabla}_{\mathbf{p}_1}} e^{-\frac{i}{\hbar} \mathbf{p}_1 \cdot \mathbf{x}'} \right] \\ \times \left[ e^{\frac{\mathbf{s}}{2} \cdot \nabla_{\mathbf{r}}} \int \frac{d^3p_2}{(2\pi\hbar)^3} f (\mathbf{r}, \mathbf{p}_2, t) e^{-\frac{i\hbar}{2} \overleftarrow{\nabla}_{\mathbf{r}} \cdot \overleftarrow{\nabla}_{\mathbf{p}_2}} e^{\frac{i}{\hbar} \mathbf{p}_2 \cdot \mathbf{x}'} \right], \quad (2.53)$$

where the arrows over the derivatives denote in which direction they act. With the help of partial integrations, the direction of the momentum derivatives can be changed, so that they act only on the distribution function and  $\tilde{\Gamma}^{\text{Ex}}$ . Due to the substitution, the spatial derivatives of those exponential functions which contain the vector  $\mathbf{s}$  are equal to the negative derivatives with respect to  $\mathbf{x}'$ . Afterwards, their action on the exponential functions can be calculated

$$I_{F_1} = \frac{1}{i\hbar} \int d^3s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \int d^3x' \int \frac{d^3p_1}{(2\pi\hbar)^3} \frac{d^3p_2}{(2\pi\hbar)^3} e^{-\frac{i}{2\hbar} \mathbf{s} \cdot \mathbf{p}_1} \left[ \tilde{\Gamma}^{\text{Ex}} (\mathbf{r}, \mathbf{p}_1, t) e^{-\frac{i\hbar}{2} \overleftarrow{\nabla}_{\mathbf{p}_1} \cdot \overleftarrow{\nabla}_{\mathbf{r}}} \right] \\ \times e^{-\frac{i}{\hbar} \mathbf{x}' \cdot (\mathbf{p}_1 - \mathbf{p}_2)} \left[ e^{\frac{i\hbar}{2} \overleftarrow{\nabla}_{\mathbf{p}_2} \cdot \overleftarrow{\nabla}_{\mathbf{r}}} f (\mathbf{r}, \mathbf{p}_2, t) \right] e^{-\frac{i}{2\hbar} \mathbf{s} \cdot \mathbf{p}_2}. \quad (2.54)$$

It is now possible to perform the  $\mathbf{x}'$  integral, and, with the help of the resulting delta function, also one of the two momentum integrals. The resulting term reads

$$I_{F_1} = \frac{1}{i\hbar} \int d^3s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \int \frac{d^3p_1}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar} \mathbf{s} \cdot \mathbf{p}_1} \left[ \tilde{\Gamma}^{\text{Ex}} (\mathbf{r}, \mathbf{p}_1, t) e^{-\frac{\hbar}{2i} \overleftarrow{\nabla}_{\mathbf{p}_1} \cdot \overleftarrow{\nabla}_{\mathbf{r}} + \frac{\hbar}{2i} \overleftarrow{\nabla}_{\mathbf{p}_1} \cdot \overleftarrow{\nabla}_{\mathbf{r}}} f (\mathbf{r}, \mathbf{p}_1, t) \right]. \quad (2.55)$$

Eq. (2.50) can be treated analogously, yielding

$$I_{F_2} = \frac{1}{i\hbar} \int d^3s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \int \frac{d^3p_1}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar} \mathbf{s} \cdot \mathbf{p}_1} \left[ \tilde{\Gamma}^{\text{Ex}} (\mathbf{r}, \mathbf{p}_1, t) e^{\frac{\hbar}{2i} \overleftarrow{\nabla}_{\mathbf{p}_1} \cdot \overleftarrow{\nabla}_{\mathbf{r}} - \frac{\hbar}{2i} \overleftarrow{\nabla}_{\mathbf{p}_1} \cdot \overleftarrow{\nabla}_{\mathbf{r}}} f (\mathbf{r}, \mathbf{p}_1, t) \right]. \quad (2.56)$$

Now both terms of the Fock contribution are combined. To get an expression of the Fock term, that is similar to the Hartree Term, I approximate the exponential functions within a gradient expansion by considering only derivatives up to first order and get

$$I_F = \int d^3s e^{i\mathbf{p}\cdot\mathbf{s}} \int \frac{d^3p_1}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar}\mathbf{s}\cdot\mathbf{p}_1} \left\{ \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}_1, t) \left[ -\overleftarrow{\nabla}_{\mathbf{r}} \cdot \overleftarrow{\nabla}_{\mathbf{p}_1} + \overrightarrow{\nabla}_{\mathbf{p}_1} \cdot \overrightarrow{\nabla}_{\mathbf{r}} \right] f(\mathbf{r}, \mathbf{p}_1, t) \right\}. \quad (2.57)$$

The last two integrals can now be performed with the result that all momenta  $\mathbf{p}_1$  have to be replaced by momenta  $\mathbf{p}$ . The last step of my calculation is to derive an explicit expression of  $\tilde{\Gamma}^{\text{Ex}}$ . For this, I compute the Fourier transform of  $\Gamma^{\text{Ex}}$  in Eq. (2.48) with respect to its relative coordinate, which yields

$$\begin{aligned} \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}, t) &= - \int d^3s e^{i\mathbf{p}\cdot\mathbf{s}} V_{\text{int}}\left(\mathbf{r} + \frac{\mathbf{s}}{2} - \mathbf{r} + \frac{\mathbf{s}}{2}\right) G^{(2)}\left(\mathbf{r} + \frac{\mathbf{s}}{2}; \mathbf{r} - \frac{\mathbf{s}}{2}; t\right), \\ &= - \int \frac{d^3p'}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}', t) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{p}'), \end{aligned} \quad (2.58)$$

where  $\tilde{V}_{\text{int}}$  is the Fourier transform of the interaction potential. Inserting Eq. (2.58) into Eq. (2.57) concludes the calculation of the Fock term with the result:

$$\begin{aligned} I_F &= \nabla_{\mathbf{p}} \left[ \int \frac{d^3p'}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}', t) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{p}') \right] \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{p}, t) \\ &\quad - \nabla_{\mathbf{r}} \left[ \int \frac{d^3p'}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}', t) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{p}') \right] \cdot \nabla_{\mathbf{p}} f(\mathbf{r}, \mathbf{p}, t). \end{aligned} \quad (2.59)$$

The mean-field terms (2.47) and (2.59) represent one of the two ways in which interparticle interactions enter the Boltzmann-Vlasov equation. On the one hand, they contribute through the effect of collisions, which I explain in the next section; on the other hand, they constitute a mean-field potential. The latter means that the interaction from the whole Fermi gas is described by an averaged potential. Thus, this approximates the interaction between all the particles through an effective one-particle theory in which the particles do not interact with each other, but are rather subject to an additional effective potential which depends on the distribution of the fermions. This dependence is explicitly taken into account by the formulas, as they contain the Wigner function. Note that the  $\mathbf{p}$ -dependence of the mean-field term originates only from the Fock term.

## 2.7 Collisionless Boltzmann-Vlasov Equation

In this section, I discuss the current status of the derivation of the Boltzmann-Vlasov equation and explain its physical meaning. Taking Eq. (2.12) with Eqs. (2.16), (2.20), (2.47), (2.59) and neglecting the terms of second order in the interaction potential yields

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} \left[ U(\mathbf{r}) + \int d^3x V_{\text{int}}(\mathbf{r} - \mathbf{x}) n(\mathbf{x}, t) + \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}, t) \right] \cdot \nabla_{\mathbf{p}} \right. \\ \left. + \nabla_{\mathbf{p}} \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}, t) \cdot \nabla_{\mathbf{r}} \right\} f(\mathbf{r}, \mathbf{p}, t) = 0, \end{aligned} \quad (2.60)$$

which represents the so called collisionless Boltzmann-Vlasov equation. This semiclassical equation can be interpreted based on its Hamiltonian structure, which becomes clear by introducing the Hamilton function

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{\mathbf{p}^2}{2m} + U(\mathbf{r}) + \int d^3x n(\mathbf{x}, t) V_{\text{int}}(\mathbf{r} - \mathbf{x}) - \int \frac{d^3p'}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}', t) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{p}'). \quad (2.61)$$

Taking into account Eq. (2.58) the collisionless Boltzmann-Vlasov equation (2.60) can then be reformulated with the help of this Hamiltonian as

$$\left[ \frac{\partial}{\partial t} + \nabla_{\mathbf{p}} H(\mathbf{r}, \mathbf{p}, t) \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} H(\mathbf{r}, \mathbf{p}, t) \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) = 0. \quad (2.62)$$

This is in fact the Liouville equation for the distribution function  $f(\mathbf{r}, \mathbf{p}, t)$ , which implies that the function  $H(\mathbf{x}, \mathbf{p}, t)$  denotes really a Hamilton function. Using the Hamilton equations of classical mechanics

$$\frac{d\mathbf{r}}{dt} = \nabla_{\mathbf{p}} H(\mathbf{r}, \mathbf{p}, t), \quad (2.63)$$

$$\frac{d\mathbf{p}}{dt} = -\nabla_{\mathbf{r}} H(\mathbf{r}, \mathbf{p}, t), \quad (2.64)$$

I can reformulate Eq. (2.60) as

$$\frac{df(\mathbf{r}, \mathbf{p}, t)}{dt} = 0, \quad (2.65)$$

with the total derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla_{\mathbf{r}} + \frac{d\mathbf{p}}{dt} \cdot \nabla_{\mathbf{p}}. \quad (2.66)$$

The physical meaning of this equation can be understood by considering a phase-space volume  $d^3r d^3p$  which contains  $f(\mathbf{r}, \mathbf{p}, t) d^3r d^3p$  particles at time  $t$ . Eq. (2.62) governs the change in time of this particle number, which is caused by two different mechanisms. The first one drives particles away from the phase-space volume due to their velocity  $\mathbf{p}/m$ , which is contained in the second term in Eq. (2.66), while the second mechanism involves particles moving out of the considered phase-space volume due to a force field which changes their momenta and is contained in the third term of Eq. (2.66).

Note that, in the case of a contact interaction  $V_{\text{int}}(\mathbf{x}) = g\delta(\mathbf{x})$ , both mean-field terms could be further simplified according to

$$\int d^3x V_{\text{int}}(\mathbf{r} - \mathbf{x}) n(\mathbf{x}, t) = gn(\mathbf{r}, t), \quad (2.67)$$

$$\tilde{\Gamma}^{\text{Ex}} = -gn(\mathbf{r}, t). \quad (2.68)$$

Thus, the mean-field terms vanish completely in Eq. (2.60) or Eq. (2.61), due to the antisymmetric wave functions of the fermions which forbid that they interact via such a contact potential.

## 2.8 Collision Integral

In this section, I derive the collision integral, by following primarily the seminal paper of Kirkpatrick and Dorfman [45]. Since I do not consider delta-type interactions, as these are not allowed for polarized fermions, the concept of collisions is not quite the same as in classical mechanics. At first, I remark that no true point interactions occur in physical systems. Thus, I consider that collisions mean interactions in a finite but small area of space, and on a short time scale.

The derivation of the collision integral is based on some key approximations. On the one hand, I assume that the Wigner function is slowly varying in space, which means that the range of the collisional interaction is much shorter than the length scale over which the distribution function changes significantly. On the other hand, I suppose weak interparticle interactions in order to justify the perturbation theory.

## 2 The Boltzmann-Vlasov Equation

I start with evaluating the second term of the perturbation theory of the time evolution of the density matrix  $\hat{\rho}(t)$  in Eq. (2.38) which leads to the right-hand side of the Boltzmann-Vlasov Equation Eq. (2.21):

$$I_{\text{coll}} = \left(\frac{-i}{\hbar}\right)^2 \int d^3s \int d^3x \int_{t_0}^t dt' e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \left[ V_{\text{int}}\left(\mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2}\right) - V_{\text{int}}\left(\mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2}\right) \right] \quad (2.69)$$

$$\times \text{Tr} \left\{ \hat{U}_0(t, t_0) \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) \left[ \hat{U}_0^\dagger(t, t') \hat{\psi}^\dagger\left(\mathbf{r} + \frac{\mathbf{s}}{2}\right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}\left(\mathbf{r} - \frac{\mathbf{s}}{2}\right) \hat{U}_0(t, t'), \hat{H}_{\text{int}} \right] \right\}.$$

For simplicity, I transform Eq. (2.69) into Fourier-space. For this, I introduce the Fourier transform of the field operators according to

$$\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \hat{a}_{\mathbf{p}}, \quad (2.70)$$

$$\hat{\psi}^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger, \quad (2.71)$$

where the creation and annihilation operators satisfy the anticommutation relations

$$\{\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}\} = \delta_{\mathbf{p}_1, \mathbf{p}_2}, \quad (2.72)$$

$$\{\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger\} = 0, \quad (2.73)$$

$$\{\hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_2}\} = 0. \quad (2.74)$$

After a short calculation, the interaction Hamiltonian Eq. (2.5) reads in terms of the creation and annihilation operators

$$\hat{H}_{\text{int}} = \frac{1}{2V} \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{\mathbf{p}_3} \sum_{\mathbf{p}_4} \tilde{V}_{\text{int}}(\mathbf{p}_1 - \mathbf{p}_4) \delta_{\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4} \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4}. \quad (2.75)$$

To complete the reformulation of Eq. (2.69), I also have to rewrite the other field operators and the interaction potential in Fourier-space

$$I_P = \int_{t_0}^t dt' \int d^3s d^3x e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{s}} \left[ V_{\text{int}}\left(\mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2}\right) - V_{\text{int}}\left(\mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2}\right) \right] \quad (2.76)$$

$$\times \hat{U}_0^\dagger(t, t') \hat{\psi}^\dagger\left(\mathbf{r} + \frac{\mathbf{s}}{2}\right) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}\left(\mathbf{r} - \frac{\mathbf{s}}{2}\right) \hat{U}_0(t, t')$$

$$= \frac{1}{V} \int_{t_0}^t dt' \sum_{\mathbf{q}_1} \sum_{\mathbf{q}_2} \sum_{\mathbf{q}_3} \sum_{\mathbf{q}_4} \tilde{V}_{\text{int}}(\mathbf{q}_2 - \mathbf{q}_3) e^{-\frac{i}{\hbar}\mathbf{r}\cdot(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)}$$

$$\times e^{\frac{i}{\hbar}(t-t')(E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right] \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4},$$

with the energies

$$E_{\mathbf{q}_i} = \frac{\mathbf{q}_i^2}{2m} + U(\mathbf{r}) \quad (2.77)$$

Inserting the previous results into the collision term, Eq. (2.69) leads to the average of a commutator containing eight field operators with the abbreviation  $\sum_{\{\mathbf{x}\}} = \sum_{\mathbf{x}_1} \sum_{\mathbf{x}_2} \sum_{\mathbf{x}_3} \sum_{\mathbf{x}_4}$

$$I_P = -\frac{1}{\hbar^2} \int_{t_0}^t dt' \frac{1}{2V^2} \sum_{\{\mathbf{q}\}} \sum_{\{\mathbf{p}\}} \tilde{V}_{\text{int}}(\mathbf{p}_1 - \mathbf{p}_4) \tilde{V}_{\text{int}}(\mathbf{q}_2 - \mathbf{q}_3) e^{-\frac{i}{\hbar}\mathbf{r}\cdot(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)} e^{\frac{i}{\hbar}(t-t')(E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})}$$

$$\times \delta_{\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4} \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right]$$

$$\times \left\{ \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) \left[ \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4}, \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \right] \hat{U}_0(t, t_0) \right\}. \quad (2.78)$$



The time-evolution operators in the average of Eq. (2.78) describe a time evolution of the eight creation and annihilation operators due to the one-particle Hamiltonian  $\hat{H}_0$ . The next step is the calculation of the commutator

$$\begin{aligned}
 [\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4}, \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4}] &= \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_4} \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_1} - \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{p}_4} \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_2} \\
 &- \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_1} + \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_2} \\
 &- \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_4} \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_3} + \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_4} \\
 &+ \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_4} \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3} - \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_2, \mathbf{q}_4} \\
 &+ \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_1} \delta_{\mathbf{p}_3, \mathbf{q}_2} - \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_2} \delta_{\mathbf{p}_3, \mathbf{q}_1} \\
 &+ \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} - \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3}. \quad (2.79)
 \end{aligned}$$

Instead of calculating all these terms explicitly, I shall illustrate the procedure by calculating the last four terms. Using the Wick contraction, they can be written as

$$\langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \rangle_t^0 \approx \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_4} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_3} \rangle_t^0 - \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_3} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_4} \rangle_t^0, \quad (2.80)$$

where  $\langle \cdot \rangle_t^0 = \text{Tr} [\hat{\rho}_0(t) \cdot]$  denotes the quantum average with respect to the density matrix  $\hat{\rho}_0$ , as

$$\begin{aligned}
 I_T &= \frac{1}{\hbar^2} \int_{t_0}^t dt' \frac{1}{2V^2} \sum_{\{\mathbf{q}\}} \sum_{\{\mathbf{p}\}} \tilde{V}_{\text{int}}(\mathbf{p}_1 - \mathbf{p}_4) \tilde{V}_{\text{int}}(\mathbf{q}_2 - \mathbf{q}_3) e^{-\frac{i}{\hbar} \mathbf{r} \cdot (\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)} e^{\frac{i}{\hbar} (t-t') (E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \\
 &\times \delta_{\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4} \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right] \\
 &\times \left[ \hat{\rho}(t_0) \hat{U}_0^\dagger(t, t_0) (\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_1} \delta_{\mathbf{p}_3, \mathbf{q}_2} - \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \hat{a}_{\mathbf{q}_4} \delta_{\mathbf{p}_4, \mathbf{q}_2} \delta_{\mathbf{p}_3, \mathbf{q}_1} \right. \\
 &\left. + \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} - \hat{a}_{\mathbf{q}_1}^\dagger \hat{a}_{\mathbf{q}_2}^\dagger \hat{a}_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_4} \delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3}) \hat{U}_0(t, t_0) \right] \\
 &= -\frac{1}{\hbar^2} \int_{t_0}^t dt' \frac{1}{2V^2} \sum_{\{\mathbf{q}\}} \sum_{\{\mathbf{p}\}} \tilde{V}_{\text{int}}(\mathbf{p}_1 - \mathbf{p}_4) \tilde{V}_{\text{int}}(\mathbf{q}_2 - \mathbf{q}_3) e^{-\frac{i}{\hbar} \mathbf{r} \cdot (\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)} e^{\frac{i}{\hbar} (t-t') (E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \\
 &\delta_{\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4} \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right] \left[ \left( \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_4} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \rangle_t^0 - \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_3} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_4} \rangle_t^0 \right) \right. \\
 &\times (\delta_{\mathbf{p}_4, \mathbf{q}_1} \delta_{\mathbf{p}_3, \mathbf{q}_2} - \delta_{\mathbf{p}_4, \mathbf{q}_2} \delta_{\mathbf{p}_3, \mathbf{q}_1}) + \left( \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_4} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_3} \rangle_t^0 - \langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{q}_3} \rangle_t^0 \langle \hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{q}_4} \rangle_t^0 \right) \\
 &\left. \times (\delta_{\mathbf{p}_4, \mathbf{q}_1} \delta_{\mathbf{p}_3, \mathbf{q}_2} - \delta_{\mathbf{p}_4, \mathbf{q}_2} \delta_{\mathbf{p}_3, \mathbf{q}_1}) \right]. \quad (2.82)
 \end{aligned}$$

The key point of the derivation of the collision integral is the relation between the average of a creation and an annihilation operator and the Wigner function. To derive the collision integral, I have to approximate this relation by a simple form. To this end, I have to derive at first the exact relation starting with the observation

$$\langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2} \rangle_t^0 = \frac{1}{V} \int d^3x \int d^3x' e^{\frac{i}{\hbar} \mathbf{p}_1 \cdot \mathbf{x}} e^{-\frac{i}{\hbar} \mathbf{p}_2 \cdot \mathbf{x}'} G_0^{(2)}(\mathbf{x}; \mathbf{x}'; t). \quad (2.83)$$

Note that the time evolution of the Wigner function is completely determined by the one-particle Hamilton operator, because of my restriction to perform perturbation theory up to second order. Time evolutions of the correlation functions containing the interaction potential would lead to terms of the order  $O(V_{\text{int}}^3)$  in Eq. (2.82). Inserting the inverted definition of the Wigner function Eq. (2.6) with Eq. (2.7)

$$\left\langle \hat{\psi}^\dagger(\mathbf{x}, t) \hat{\psi}(\mathbf{x}', t) \right\rangle_t^0 \approx \int \frac{d^3p}{(2\pi\hbar)^3} f\left(\frac{\mathbf{x} + \mathbf{x}'}{2}, \mathbf{p}, t\right) e^{-\frac{i}{\hbar} \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \quad (2.84)$$

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and switching from the variables  $\mathbf{x}$  and  $\mathbf{x}'$  to their relative coordinate  $\mathbf{s}$  and their center-of-mass position  $\mathbf{r}$  leads to

$$\langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2} \rangle_t^0 \approx \frac{1}{V} \int d^3r e^{\frac{i}{\hbar} \mathbf{r} \cdot (\mathbf{p}_1 - \mathbf{p}_2)} f \left( \mathbf{r}, \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, t \right). \quad (2.85)$$

In terms of this perturbation theory, this relation is still exact. The phase-space distribution function will be now approximated by assuming that it is only weakly space dependent, so that it can be pulled out of the integral. With the resulting Kronecker delta function, I can conclude

$$\langle \hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2} \rangle_t^0 \approx \delta_{\mathbf{p}_1, \mathbf{p}_2} f(\mathbf{r}, \mathbf{p}_1, t). \quad (2.86)$$

Inserting this approximation into Eq. (2.82) leads with the abbreviation  $f_i = f(\mathbf{r}, \mathbf{q}_i, t)$  to

$$\begin{aligned} I_T &= \frac{1}{\hbar^2} \int dt' \frac{1}{2V^2} \sum_{\{\mathbf{q}\}} \sum_{\{\mathbf{p}\}} \tilde{V}_{\text{int}}(\mathbf{p}_1 - \mathbf{p}_4) \tilde{V}_{\text{int}}(\mathbf{q}_2 - \mathbf{q}_3) e^{-\frac{i}{\hbar} \mathbf{r} \cdot (\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)} e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \\ &\times \delta_{\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4} \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right] [f_3 f_4 (\delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3} \delta_{\mathbf{p}_3, \mathbf{q}_2} \delta_{\mathbf{p}_4, \mathbf{q}_1} \\ &- \delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3} \delta_{\mathbf{p}_3, \mathbf{q}_1} \delta_{\mathbf{p}_4, \mathbf{q}_2} - \delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_2} \delta_{\mathbf{p}_4, \mathbf{q}_1} + \delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_1} \delta_{\mathbf{p}_4, \mathbf{q}_2}) \\ &+ f_1 f_2 (\delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_2} \delta_{\mathbf{p}_4, \mathbf{q}_1} - \delta_{\mathbf{p}_1, \mathbf{q}_3} \delta_{\mathbf{p}_2, \mathbf{q}_4} \delta_{\mathbf{p}_3, \mathbf{q}_1} \delta_{\mathbf{p}_4, \mathbf{q}_2} \\ &- \delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3} \delta_{\mathbf{p}_3, \mathbf{q}_2} \delta_{\mathbf{p}_4, \mathbf{q}_1} + \delta_{\mathbf{p}_1, \mathbf{q}_4} \delta_{\mathbf{p}_2, \mathbf{q}_3} \delta_{\mathbf{p}_3, \mathbf{q}_1} \delta_{\mathbf{p}_4, \mathbf{q}_2})]. \end{aligned} \quad (2.87)$$

The Kronecker delta functions, which contain only one momentum  $\mathbf{p}_i$  and one momentum  $\mathbf{q}_i$ , can now be performed, so that the  $\mathbf{p}$ -sums vanish, yielding

$$\begin{aligned} I_T &= \frac{1}{\hbar^2} \int_{t_0}^t dt' \frac{1}{2V^2} \sum_{\{\mathbf{q}\}} \tilde{V}(\mathbf{q}_2 - \mathbf{q}_3) e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{q}_1} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} \\ &\times \left[ \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_4}{2}} - \delta_{\mathbf{p}, \frac{\mathbf{q}_2 - \mathbf{q}_3 + \mathbf{q}_1 + \mathbf{q}_4}{2}} \right] (f_3 f_4 - f_1 f_2) \\ &\times \left[ \tilde{V}_{\text{int}}(\mathbf{q}_4 - \mathbf{q}_1) - \tilde{V}_{\text{int}}(\mathbf{q}_4 - \mathbf{q}_2) - \tilde{V}_{\text{int}}(\mathbf{q}_3 - \mathbf{q}_1) + \tilde{V}_{\text{int}}(\mathbf{q}_3 - \mathbf{q}_2) \right]. \end{aligned} \quad (2.88)$$

The next step is to eliminate the Kronecker deltas, which contain the momentum  $\mathbf{p}$ , so that the term simplifies to

$$\begin{aligned} I_T &= \frac{1}{\hbar^2} \frac{1}{2V^2} \sum_{\mathbf{q}_2} \sum_{\mathbf{q}_3} \sum_{\mathbf{q}_4} \delta_{\mathbf{p} + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4} (f_3 f_4 - f_1 f_2) \\ &\times \left[ \left( 2\tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4)^2 - 2\tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right) \int_{t_0}^t dt' e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \right. \\ &\left. + \left( 2\tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3)^2 - 2\tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right) \int_{t_0}^t dt' e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \right], \end{aligned} \quad (2.89)$$

where  $f = f(\mathbf{r}, \mathbf{p}, t)$ . The next step of the calculation is to solve the time integral. To this end, I assume that I can consider the limit  $t_0 \rightarrow \infty$ . At first, I will use a Heaviside step function to get an integral with limits  $\pm\infty$ . Afterwards, I will use the Fourier representation of the Heaviside step function

$$\int_{-\infty}^t dt' e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} = \int_{-\infty}^{\infty} dt' e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dy \frac{e^{iy(t-t')}}{y - i\epsilon}. \quad (2.90)$$

After performing the integrals, this leads to

$$\int_{-\infty}^t dt' e^{\frac{i}{\hbar} (t-t')(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})} = i\hbar \lim_{\epsilon \rightarrow 0} \frac{1}{E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4} + i\epsilon}. \quad (2.91)$$

To arrive at a form in which the limit  $\epsilon \rightarrow 0$  can be carried out, I separate the real and imaginary parts from each other

$$\lim_{\epsilon \rightarrow 0} \frac{1}{E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4} \pm i\epsilon} = \lim_{\epsilon \rightarrow 0} \left[ \frac{E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}}{(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})^2 + \epsilon^2} \mp i \frac{\epsilon}{(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})^2 + \epsilon^2} \right]. \quad (2.92)$$

Taking the limit leads to a distributional identity which is called the Sokhatsky-Weierstrass Theorem [46]

$$\lim_{\epsilon \rightarrow 0} \frac{1}{x \pm i\epsilon} = \text{p.v.} \frac{1}{x} \mp i\pi\delta(x), \quad (2.93)$$

where  $\text{p.v.} \frac{1}{x}$  denotes the Cauchy principal value distribution. Thus, I obtain

$$\int_{-\infty}^t dt' e^{\frac{i}{\hbar}(t-t')(E_{\mathbf{p}}+E_{\mathbf{q}_2}-E_{\mathbf{q}_3}-E_{\mathbf{q}_4})} = \hbar \left[ i \text{p.v.} \frac{1}{E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}} + \pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \right]. \quad (2.94)$$

By inserting this calculation into Eq. (2.89), one can observe that the terms proportional to the Cauchy principal value distribution cancel each other. In accordance with the semiclassical approximation, I replace the momentum sums  $(1/V) \sum_{\mathbf{p}_i}$  by the integrals  $\int d^3p_i / (2\pi\hbar)^3$  and the Kronecker delta functions  $V\delta_{\mathbf{p}_i, \mathbf{p}_j}$  by Dirac delta functions  $(2\pi\hbar)^3\delta(\mathbf{p}_i - \mathbf{p}_j)$ , thereby obtaining the final expression of this part of the collision integral

$$I_{\text{T}} = \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} (f_3f_4 - ff_2) 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \times (2\pi\hbar)^3\delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[ \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right]^2. \quad (2.95)$$

The other terms of the collision integral can be calculated in a similar way. This calculation also needs the approximation of a weakly space dependent phase-space distribution function. Note that the Wick contraction, and afterwards the substitutions of  $G^{(i)} = G_0^{(i)}$  for  $i = 2, 4, 6$ , are exact in second order perturbation theory, because the errors are of order  $O(V_{\text{int}}^3)$ .

After a lengthy but straightforward calculation, I finally get for the remaining terms expressions which contain three Wigner functions

$$I_{\text{coll}} - I_{\text{T}} = \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} (ff_2f_3 + ff_2f_4 - ff_3f_4 - f_2f_3f_4) \times 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})(2\pi\hbar)^3\delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \times \left[ \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right]^2. \quad (2.96)$$

Putting all this together leads to the form of the collision integral, which is known in the literature as [47, p. 54]

$$I_{\text{coll}}[f] = \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} [(1-f)(1-f_2)f_3f_4 - ff_2(1-f_3)(1-f_4)] \times 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4})(2\pi\hbar)^3\delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[ \tilde{V}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}(\mathbf{p} - \mathbf{q}_4) \right]^2. \quad (2.97)$$

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The momentum and energy delta functions ensure momentum and energy conservation during the collision process. However, the energy conservation shows the limit of this form of the collision integral, because it is not the total energy that is conserved. Only the trapping and the kinetic energy are conserved. Thus, the interaction energy is neglected, which can only be justified, when I assume a weak two-particle interaction.

Note that the minus sign between the potentials in Eq. (2.97) reflects the fact that I consider fermions and, therefore, I have to deal with anticommutating operators. In the case of a bosonic gas, there would instead appear a plus sign.

With this collision integral the previous intermediate result Eq. (2.60) finally yields the Boltzmann-Vlasov equation in the form

$$\left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} \left[ U(\mathbf{r}) + \int d^3x V(\mathbf{r} - \mathbf{x}) n(\mathbf{x}, t) + \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}, t) \right] \cdot \nabla_{\mathbf{p}} \right. \\ \left. + \nabla_{\mathbf{p}} \tilde{\Gamma}^{\text{Ex}}(\mathbf{r}, \mathbf{p}, t) \cdot \nabla_{\mathbf{r}} \right\} f(\mathbf{r}, \mathbf{p}, t) = I_{\text{coll}}[f](\mathbf{r}, \mathbf{p}, t), \quad (2.98)$$

which completes the derivation.

# 3 Properties of the Collision Integral

This chapter provides an overview of the properties of the collision integral. To this end, I discuss the quantities which are conserved under collisions, demonstrating that any relaxation process, due to collisions, leads to a Fermi-Dirac distribution function. In the last section I deal with approximations of the collision integral which are often used for concrete applications.

## 3.1 Conserved Quantities

The collision integral describes directly the effect of two-particle interactions in the Boltzmann-Vlasov equation, in contrast to the mean-field terms, which describe them indirectly by means of an effective potential. Therefore, the collision integral is nonlinear in the distribution function, which makes it complicated to deal with it. Nevertheless, it is possible to deduce important properties of the collision integral, such as conserved quantities. According to Ref. [47], they are provided by momenta up to second power, i.e., the number of particles, the components of the momentum and the kinetic energy. In order to prove this I have to show that the phase-space average of the conserved quantity with respect to the collision integral vanishes. As an example, I calculate this average for one momentum coordinate. The other conservation laws can then be obtained analogously. Thus, I split the momentum  $p_i$  with  $i = x, y, z$  into four equal terms:

$$\begin{aligned} \int d^3r \int d^3p p_i I_{\text{coll}}[f] &= \int d^3r d^3p \frac{1}{4} (p_i + p_i + p_i + p_i) \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} \\ &\quad \times [(1-f)(1-f_2)f_3f_4 - ff_2(1-f_3)(1-f_4)] 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \\ &\quad \times (2\pi\hbar)^3 \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[ \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right]^2. \end{aligned} \quad (3.1)$$

I rename the integration variables of three of the four momentum terms, yet here one has to be careful. The momentum and energy delta functions, along with the Wigner function terms, are invariant under transformations such as switching  $\mathbf{p}$  and  $\mathbf{q}_2$ . Thus, I switch in the second term  $\mathbf{p}$  and  $\mathbf{q}_2$ . In the third term I switch  $\mathbf{p}$  and  $\mathbf{q}_3$  and also  $\mathbf{q}_2$  and  $\mathbf{q}_4$  to ensure the invariance of the delta functions. In the fourth term, I switch  $\mathbf{p}$  and  $\mathbf{q}_4$  and also  $\mathbf{q}_2$  and  $\mathbf{q}_3$ . With the abbreviation  $\langle p_i \rangle_I = \int d^3r \int d^3p p_i I_{\text{coll}}[f]$ , this leads to

$$\begin{aligned} \langle p_i \rangle_I &= \int d^3r d^3p \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} [(1-f)(1-f_2)f_3f_4 \\ &\quad - ff_2(1-f_3)(1-f_4)] 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) (2\pi\hbar)^3 \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \\ &\quad \times \frac{1}{4} \left\{ p_i \left[ \tilde{V}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}(\mathbf{p} - \mathbf{q}_4) \right]^2 + q_{2,i} \left[ \tilde{V}(\mathbf{q}_2 - \mathbf{q}_3) - \tilde{V}(\mathbf{q}_2 - \mathbf{q}_4) \right]^2 \right. \\ &\quad \left. - q_{3,i} \left[ \tilde{V}(\mathbf{q}_3 - \mathbf{p}) - \tilde{V}(\mathbf{q}_3 - \mathbf{q}_2) \right]^2 - q_{4,i} \left[ \tilde{V}(\mathbf{q}_4 - \mathbf{q}_2) - \tilde{V}(\mathbf{q}_4 - \mathbf{p}) \right]^2 \right\}, \end{aligned} \quad (3.2)$$

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where  $q_{j,i}$  denotes the  $i$ -th component of the vector  $\mathbf{q}_j$ . The momenta in the interaction potential can be switched using the momentum delta function with the result

$$\begin{aligned} \langle p_i \rangle_I &= \int d^3r d^3p \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} [(1-f)(1-f_2)f_3f_4 - ff_2(1-f_3)(1-f_4)] \\ &\quad \times 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \left[ \tilde{V}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}(\mathbf{p} - \mathbf{q}_4) \right]^2 \\ &\quad \times (2\pi\hbar)^3 \frac{1}{4} (p_i + q_{2,i} + q_{3,i} + q_{4,i}) \delta(p_i + q_{2,i} + q_{3,i} + q_{4,i}) \prod_{\substack{j \\ j \neq i}} \delta(p_j + q_{2,j} + q_{3,j} + q_{4,j}). \end{aligned} \quad (3.3)$$

With the distributional identity  $x\delta(x) = 0$  the right-hand side of Eq. (3.3) is equal to zero. The calculations for the particle number  $N$  and the kinetic energy  $\mathbf{p}^2/(2m)$  are performed analogously. Note that the kinetic energy is conserved, because the trapping potential is momentum independent.

Note that the vanishing of the collision integral results only from the momentum-containing delta functions. Hence, a phase-space average quantity which is the product of an arbitrary space-dependent function and either particle number, momentum coordinate, or kinetic energy, is conserved by the collision integral as well.

## 3.2 Local Equilibrium

In this section, I deal with the effect of collisions on the distribution function  $f$ , looking for functions that are not altered by collisions. Physically, the particles interact via collisions and interchange energy and momentum, with a tendency to balance these quantities. This leads to a relaxation of the system into a limiting case in which the collisions will no longer change the distribution function of the system. The limiting function of this relaxation is called local equilibrium function  $f^{\text{le}}(\mathbf{r}, \mathbf{p}, t)$  and satisfies the condition

$$I_{\text{coll}}[f^{\text{le}}] = 0. \quad (3.4)$$

In order to obtain the general form of the local distribution function  $f^{\text{le}}(\mathbf{r}, \mathbf{p}, t)$ , I insert it into the collision integral Eq. (2.97) and demand that the scattering-in and scattering-out terms cancel each other

$$(1 - f^{\text{le}})(1 - f_2^{\text{le}})f_3^{\text{le}}f_4^{\text{le}} - f^{\text{le}}f_2^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) = 0, \quad (3.5)$$

with  $f^{\text{le}} = f^{\text{le}}(\mathbf{r}, \mathbf{p}, t)$  and  $f_i^{\text{le}} = f^{\text{le}}(\mathbf{r}, \mathbf{p}_i, t)$  being abbreviations. Taking the logarithm of Eq. (3.5) leads to

$$\ln \frac{f^{\text{le}}}{1 - f^{\text{le}}} + \ln \frac{f_2^{\text{le}}}{1 - f_2^{\text{le}}} = \ln \frac{f_3^{\text{le}}}{1 - f_3^{\text{le}}} + \ln \frac{f_4^{\text{le}}}{1 - f_4^{\text{le}}}. \quad (3.6)$$

This equation provides me with a new conserved quantity  $f^{\text{le}}/(1 - f^{\text{le}})$ , which I shall denote with  $c$ . Due to the fact that only five fundamental conserved quantities exist, one has to be able to represent this new conserved quantity  $c$  through a linear combination of these five. At first, I reformulate the relation between the equilibrium distribution and  $c$  in the following form

$$f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{-c} + 1}. \quad (3.7)$$

This equation shows that a general equilibrium function has the form of a Fermi-Dirac distribution function and that any distribution function will describe a relaxation into such an equilibrium.

As given by Ref. [47], the expression for  $c$  can be inserted, which leads to

$$f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{\beta^{\text{le}}(\mathbf{r}, t) \left\{ \frac{[\mathbf{p} - m\mathbf{v}^{\text{le}}(\mathbf{r}, t)]^2}{2m} - \mu^{\text{le}}(\mathbf{r}, t) \right\}} + 1}, \quad (3.8)$$

where  $\mu^{\text{le}}$  is the local chemical potential,  $\mathbf{v}^{\text{le}}$  the local velocity field, and  $\beta^{\text{le}} = 1/(k_{\text{B}}T^{\text{le}})$  the inverse local temperature. All of them are still time- and space-dependent quantities.

The space dependence of the chemical potential, the local velocity field and the temperature cannot be fixed by the collision integral. Explicit expressions for them can, however, be deduced by demanding that the whole Boltzmann-Vlasov equation conserves them [47]. Eq. (3.8) implies that the collision integral fixes only the momentum distribution in local equilibrium and, therefore, explains the name *local* equilibrium. This picture is based on the assumption of a weakly space-dependent distribution function which is actually a relation between the range of the collisional interaction and the length scale over which the distribution changes significantly.

The special local equilibrium, which is time independent, denotes the global equilibrium  $f^0$ , in which no dynamics occurs. There are two ways to characterize this global equilibrium. The first is that it is a time-independent local equilibrium

$$I_{\text{coll}}[f^0] = 0 \quad \text{and} \quad \frac{\partial f^0}{\partial t} = 0. \quad (3.9)$$

The second possibility is that it can be defined as the limiting function of the local equilibrium for large times

$$\lim_{t \rightarrow \infty} f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) = f^0(\mathbf{r}, \mathbf{p}). \quad (3.10)$$

Both definitions are equivalent.

## 3.3 Approximations of the Collision Integral

In this section, I derive approximations that allow me to deal with the collision integral in a simpler form. At first, I linearize the collision integral and afterwards I discuss the validity of a relaxation time approximation for it.

### 3.3.1 Linearization of the Collision Integral

To simplify the collision integral, I linearize it in terms of the deviation  $\delta f := f - f^{\text{le}}$  from local equilibrium

$$f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{\beta^{\text{le}}(E_{\mathbf{p}} - \mu^{\text{le}})} + 1}, \quad (3.11)$$

where I described it in the local rest frame in which the local velocity field is zero. I assume that the system is near the thermal equilibrium, so that the distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  can be described as a Fermi-Dirac distribution function containing a deviation  $\Phi(\mathbf{r}, \mathbf{p}, t)$  specifying the difference to local equilibrium

$$f(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{\beta^{\text{le}}(E_{\mathbf{p}} - \mu^{\text{le}}) - \Phi(\mathbf{r}, \mathbf{p}, t)} + 1}. \quad (3.12)$$

A Taylor expansion of the phase-space distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  with respect to the deviation function

$$f(\mathbf{r}, \mathbf{p}, t) \approx f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) + \left. \frac{\partial f}{\partial \Phi} \right|_{\Phi=0}(\mathbf{r}, \mathbf{p}, t) \Phi(\mathbf{r}, \mathbf{p}, t) \quad (3.13)$$

### 3 Properties of the Collision Integral

yields up to first order

$$f(\mathbf{r}, \mathbf{p}, t) \approx f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) + f^{\text{le}}(\mathbf{r}, \mathbf{p}, t)[1 - f^{\text{le}}(\mathbf{r}, \mathbf{p}, t)]\Phi(\mathbf{r}, \mathbf{p}, t). \quad (3.14)$$

Inserting this approximation into the collision integral Eq. (2.97) leads to

$$I_{\text{coll}}[f] \approx I_{\text{lin}}[\Phi] = \frac{1}{2\hbar} \int \frac{d^3q_2}{(2\pi\hbar)^3} \int \frac{d^3q_3}{(2\pi\hbar)^3} \int \frac{d^3q_4}{(2\pi\hbar)^3} 2\pi\delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \\ \times (2\pi\hbar)^3 \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[ \tilde{V}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}(\mathbf{p} - \mathbf{q}_4) \right]^2 F_f \quad (3.15)$$

with the abbreviation

$$F_f = F_{\text{eq}} - \Phi(\mathbf{r}, \mathbf{p}, t) \left[ f^{\text{le}}(1 - f^{\text{le}})(1 - f_2^{\text{le}})f_3^{\text{le}}f_4^{\text{le}} + f^{\text{le}}(1 - f^{\text{le}})f_2^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) \right] \\ - \Phi(\mathbf{r}, \mathbf{q}_2, t) \left[ f_2^{\text{le}}(1 - f_2^{\text{le}})(1 - f^{\text{le}})f_3^{\text{le}}f_4^{\text{le}} + f_2^{\text{le}}(1 - f_2^{\text{le}})f^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) \right] \\ + \Phi(\mathbf{r}, \mathbf{q}_3, t) \left[ f_3^{\text{le}}(1 - f_3^{\text{le}})(1 - f^{\text{le}})(1 - f_2^{\text{le}})f_4^{\text{le}} + f_3^{\text{le}}(1 - f_3^{\text{le}})f^{\text{le}}f_2^{\text{le}}(1 - f_4^{\text{le}}) \right] \\ + \Phi(\mathbf{r}, \mathbf{q}_4, t) \left[ f_4^{\text{le}}(1 - f_4^{\text{le}})(1 - f^{\text{le}})(1 - f_2^{\text{le}})f_3^{\text{le}} + f_4^{\text{le}}(1 - f_4^{\text{le}})f^{\text{le}}f_2^{\text{le}}(1 - f_3^{\text{le}}) \right], \quad (3.16)$$

where

$$F_{\text{eq}} = (1 - f^{\text{le}})(1 - f_2^{\text{le}})f_3^{\text{le}}f_4^{\text{le}} - f^{\text{le}}f_2^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) \quad (3.17)$$

is the equilibrium term and  $f_i^{\text{le}} = f(\mathbf{r}, \mathbf{q}_i, t)$  and  $f^{\text{le}} = f(\mathbf{r}, \mathbf{p}, t)$  are abbreviations. In order to show that the equilibrium term (3.17) has to vanish, I rewrite it according to

$$F_{\text{eq}} = f^{\text{le}}f_2^{\text{le}}f_3^{\text{le}}f_4^{\text{le}} \left[ \frac{1 - f^{\text{le}}}{f^{\text{le}}} \frac{1 - f_2^{\text{le}}}{f_2^{\text{le}}} - \frac{1 - f_3^{\text{le}}}{f_3^{\text{le}}} \frac{1 - f_4^{\text{le}}}{f_4^{\text{le}}} \right]. \quad (3.18)$$

Inserting the Fermi-Dirac distribution function Eq. (3.11) into this equation, I get

$$F_{\text{eq}} = f^{\text{le}}f_2^{\text{le}}f_3^{\text{le}}f_4^{\text{le}} \left\{ e^{\beta^{\text{le}}[E_{\mathbf{p}} + E_{\mathbf{q}_2} - 2\mu^{\text{le}}]} - e^{\beta^{\text{le}}[E_{\mathbf{q}_3} + E_{\mathbf{q}_4} - 2\mu^{\text{le}}]} \right\}. \quad (3.19)$$

The term in the curly brackets in Eq. (3.19) vanishes due to energy conservation, as expected. Thus, I can rewrite terms that are linear in  $\Phi$  in Eq. (3.16), using the property

$$(1 - f^{\text{le}})(1 - f_2^{\text{le}})f_3^{\text{le}}f_4^{\text{le}} = f^{\text{le}}f_2^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) \quad (3.20)$$

yielding

$$F_f = -f^{\text{le}}f_2^{\text{le}}(1 - f_3^{\text{le}})(1 - f_4^{\text{le}}) [\Phi(\mathbf{r}, \mathbf{p}, t) + \Phi(\mathbf{r}, \mathbf{q}_2, t) - \Phi(\mathbf{r}, \mathbf{q}_3, t) - \Phi(\mathbf{r}, \mathbf{q}_4, t)]. \quad (3.21)$$

Inserting  $F_f$  into Eq. (3.15) leads to a linear collision operator which is not as complicated as the full collision integral, but it is still an integral operator.

### 3.3.2 Relaxation Time Approximation

The previous calculation shows that a relaxation time approximation

$$I_{\text{coll}}[f](\mathbf{r}, \mathbf{p}, t) \approx -\frac{\delta f(\mathbf{r}, \mathbf{p}, t)}{\tau} \quad (3.22)$$

cannot be derived from a linearization of the collision integral without further approximations. The first term in Eq. (3.21) has exactly the form of a relaxation time approximation, but the other



non-local terms can not be transformed into such a form. As discussed in Ref. [48], neglecting these non-local terms leads to a relaxation time approximation which yields

$$\begin{aligned}
 I_{\text{coll}}[f](\mathbf{r}, \mathbf{p}, t) &\approx -\delta f \frac{1}{2\hbar} \int \frac{d^3 q_2}{(2\pi\hbar)^3} \int \frac{d^3 q_3}{(2\pi\hbar)^3} \int \frac{d^3 q_4}{(2\pi\hbar)^3} 2\pi \delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \\
 &\quad \times (2\pi\hbar)^3 \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[ \tilde{V}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}(\mathbf{p} - \mathbf{q}_4) \right]^2 \\
 &\quad \times \frac{f^{\text{le}} f_2^{\text{le}} (1 - f_3^{\text{le}}) (1 - f_4^{\text{le}})}{f^{\text{le}} (1 - f^{\text{le}})} (f^{\text{le}} + 1 - f^{\text{le}}). \tag{3.23}
 \end{aligned}$$

Using the property for the equilibrium (3.20) leads to a relaxation time in the following form:

$$\begin{aligned}
 \frac{1}{\tau(\mathbf{r}, \mathbf{p}, t)} &= \frac{1}{2(2\pi)^5 \hbar^7} \int d^3 q_2 \int d^3 q_3 \int d^3 q_4 \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \\
 &\quad \times \left[ \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right]^2 \left[ f_2^{\text{le}} (1 - f_3^{\text{le}}) (1 - f_4^{\text{le}}) + (1 - f_2^{\text{le}}) f_3^{\text{le}} f_4^{\text{le}} \right]. \tag{3.24}
 \end{aligned}$$

Note that Ref. [48] also introduced an additional factor of 2 for the relaxation time (3.24), whose origin is still not clear. However, the validity of the derivation of this relaxation time approximation is questionable. The derivation neglects three of four terms without legitimation, thereby ignoring that  $I_{\text{lin}}$  is, in fact, an integral operator. It is possible to further simplify  $I_{\text{lin}}$  without an approximation by taking into account the eigenvalue problem  $I_{\text{lin}}[\Phi_\lambda] = \lambda \Phi_\lambda$ , where  $\lambda$  denotes the eigenvalue of the eigenvector  $\Phi_\lambda$  [49]. Then, one may expand the deviation

$$\Phi(\mathbf{r}, \mathbf{p}, t) = \sum_{\lambda} A_{\lambda} e^{-\lambda t} \Phi_{\lambda}(\mathbf{r}, \mathbf{p}) \tag{3.25}$$

in a full set of eigenfunctions. The sum in Eq. (3.25) shows that the Boltzmann relaxation kinetics cannot be described in general in terms of a single relaxation time since the eigenvalues  $\lambda$  represent the inverse relaxation times. Thus, this approximation Eq. (3.22) can in general not be a valid description of the relaxation kinetics. It can only be justified, if one eigenvalue  $\lambda_{\text{min}}$  is much smaller than the others, causing the corresponding relaxation time  $\tau_{\text{max}} = 1/\lambda_{\text{min}}$  to be much larger than all the others of the linearized collision operator.

Nevertheless, the relaxation time approximation is successfully used in literature, as for example in Ref. [50], in order to describe the dynamics of a Bose gas with delta-type interaction, or in Ref. [51], which describe a Fermi gas in the hydrodynamic regime.



# 4 Solution of the Boltzmann-Vlasov Equation

The Boltzmann-Vlasov equation determines the time evolution of the Wigner function. In this chapter, I calculate an approximate solution of this nonlinear integro-partial differential equation.

In recent years, approximative solutions of this equation were obtained by rescaling both the coordinates and the momenta of the distribution function. In this way, one gets ordinary differential equations for the scaling parameters, which can then be solved numerically. Most of these studies have so far been limited to either the ballistic or the hydrodynamic regime. The ballistic or collisionless regime is a limiting case in which the average time between two collisions goes to infinity. The hydrodynamic limit is characterized by a high collision rate, which dominates the dynamics of the system, leading to an almost instantaneous relaxation into local equilibrium. Equations for the scaling parameters were achieved in the ballistic regime for a Bose gas with contact interaction within a triaxial harmonic trap [52] and a Fermi gas with dipolar interaction in a cylinder-symmetric trap [27]. A more general rescaling was worked out in Ref. [50], which gives the opportunity to describe not only the ballistic and hydrodynamic regime, but also all intermediate regimes. The purpose of this diploma thesis is to transfer the general ansatz from Ref. [50] to a dipolar Fermi gas. To this end, the collision integral will be treated in the relaxation time approximation, assuming that the collisions lead to an exponential relaxation on the scale of a certain relaxation time, which is related to the average time between two collisions.

To construct a solution of the Boltzmann-Vlasov equation, it will be necessary to proceed in the following three steps. At first, the semiclassical Wigner function in global equilibrium is computed. Afterwards, the scaling parameters for the local equilibrium are introduced, so that the limiting function of the relaxation due to the collisions is known. Finally, it will be possible to compute the scaling parameters to get an approximate expression for the solution of the Boltzmann-Vlasov equation.

## 4.1 Scaling Ansatz

Therefore, I start with the Boltzmann-Vlasov equation for the phase-space distribution function  $f(\mathbf{x}, \mathbf{q}, t)$ , which takes the form

$$\frac{\partial f}{\partial t} + \left\{ \frac{\hbar \mathbf{q}}{m} + \frac{1}{\hbar} \frac{\partial [U(\mathbf{x}) + U_{\text{mf}}(\mathbf{x}, \mathbf{q}, t)]}{\partial \mathbf{q}} \right\} \frac{\partial f}{\partial \mathbf{x}} - \frac{1}{\hbar} \frac{\partial [U(\mathbf{x}) + U_{\text{mf}}(\mathbf{x}, \mathbf{q}, t)]}{\partial \mathbf{x}} \frac{\partial f}{\partial \mathbf{q}} = I_{\text{coll}}[f](\mathbf{x}, \mathbf{q}, t), \quad (4.1)$$

where  $U(\mathbf{x})$  is the trapping potential and  $U_{\text{mf}}(\mathbf{x}, \mathbf{q}, t)$  is the mean-field potential

$$U_{\text{mf}}(\mathbf{x}, \mathbf{q}, t) = \int d^3 x' n(\mathbf{x}', t) V_{\text{int}}(\mathbf{x} - \mathbf{x}') - \int \frac{d^3 q'}{(2\pi)^3} f(\mathbf{x}, \mathbf{q}', t) \tilde{V}_{\text{int}}(\mathbf{q} - \mathbf{q}'). \quad (4.2)$$

Here, the first term denotes the direct Hartree term containing the interaction potential  $V_{\text{int}}$  and the spatial density  $n(\mathbf{x}, t)$ , which is connected to the Wigner function  $f(\mathbf{x}, \mathbf{q}, t)$  through Eq. (2.9), and the second one is the Fock exchange term containing the Fourier transform of

#### 4 Solution of the Boltzmann-Vlasov Equation

the interaction potential  $\tilde{V}_{\text{int}}$ . The distribution function  $f^0(\mathbf{x}, \mathbf{q})$  satisfies the Boltzmann-Vlasov equation in equilibrium

$$\left\{ \frac{\hbar \mathbf{q}}{m} + \frac{1}{\hbar} \frac{\partial [U(\mathbf{x}) + U_{\text{mf}}(\mathbf{x}, \mathbf{q})]}{\partial \mathbf{q}} \right\} \frac{\partial f^0}{\partial \mathbf{x}} - \frac{1}{\hbar} \frac{\partial [U(\mathbf{x}) + U_{\text{mf}}(\mathbf{x}, \mathbf{q})]}{\partial \mathbf{x}} \frac{\partial f^0}{\partial \mathbf{q}} = 0, \quad (4.3)$$

where the effect of collisions vanishes in the static case, i.e.  $I_{\text{coll}}[f_0] = 0$ .

The scaling ansatz to solve the Boltzmann-Vlasov equation approximately assumes that the system is qualitatively near the equilibrium

$$f(\mathbf{x}, \mathbf{q}, t) \rightarrow \Gamma f^0(\mathbf{r}(t), \mathbf{k}(t)), \quad (4.4)$$

with

$$r_i = \frac{x_i}{b_i(t)}, \quad (4.5)$$

$$k_i = \frac{1}{\Theta_i^{\frac{1}{2}}(t)} \left( q_i - \frac{m \dot{b}_i(t) x_i}{\hbar b_i(t)} \right), \quad (4.6)$$

and the normalization factor

$$\Gamma = \frac{1}{\prod_j b_j \Theta_j^{\frac{1}{2}}}. \quad (4.7)$$

The time dependence of the distribution function is governed by the scaling parameters  $b_i$  and  $\Theta_i$ , which denote the time dependent deformations of the space and momentum variables. The second term in Eq. (4.6) describes a gauge transformation of the  $\mathbf{q}$ -vector [53] so that the local velocity field of the rescaled distribution function vanishes. The factor  $\Gamma$  in the scaling was introduced to ensure that the scaling does not change the normalization of the distribution function. Inserting this ansatz Eqs. (4.4)–(4.7) into equation (4.2) yields

$$U_{\text{mf,b}\Theta}(\mathbf{r}, \mathbf{k}, t) = \int d^3 r' n^0(\mathbf{r}') V_{\text{int}}(b, \mathbf{r} - \mathbf{r}') - \frac{1}{\prod_j b_j} \int \frac{d^3 k'}{(2\pi)^3} f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}'), \quad (4.8)$$

where  $n^0(\mathbf{r})$  denotes the spatial density in equilibrium,

$$V_{\text{int}}(b, \mathbf{r} - \mathbf{r}') = V_{\text{int}}[b_x(r_x - r'_x), b_y(r_y - r'_y), b_z(r_z - r'_z)] \quad (4.9)$$

the rescaled interaction potential and

$$\tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') = \tilde{V}_{\text{int}} \left[ \Theta_x^{\frac{1}{2}}(k_x - k'_x), \Theta_y^{\frac{1}{2}}(k_y - k'_y), \Theta_z^{\frac{1}{2}}(k_z - k'_z) \right] \quad (4.10)$$

the rescaled Fourier-transformed interaction potential. Note that Fourier transform and rescaling do not commute. Hence, it is necessary to perform these two operations in the correct order. Substituting the scaling ansatz (4.4)–(4.7) into equation (4.1) leads to

$$\begin{aligned} \dot{\Gamma} f^0(\mathbf{r}, \mathbf{k}) + \Gamma \sum_i \left\{ \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial r_i} \frac{\hbar k_i \Theta_i^{\frac{1}{2}}}{m b_i} - \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial k_i} \left[ k_i \left( \frac{1}{2} \frac{\dot{\Theta}_i}{\Theta_i} + \frac{\dot{b}_i}{b_i} \right) + \frac{r_i m \ddot{b}_i}{\hbar \Theta_i^{\frac{1}{2}}} + \frac{1}{\hbar b_i \Theta_i^{\frac{1}{2}}} \frac{\partial U(b, \mathbf{r})}{\partial r_i} \right] \right\} \\ - \frac{\Gamma}{\hbar} \sum_i \frac{1}{b_i \Theta_i^{\frac{1}{2}}} \left[ \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial k_i} \frac{\partial U_{\text{mf,b}\Theta}}{\partial r_i} - \frac{\partial U_{\text{mf,b}\Theta}}{\partial k_i} \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial r_i} \right] = I_{\text{coll}}[f], \end{aligned} \quad (4.11)$$

where

$$U(b, \mathbf{r}) = U(b_x r_x, b_y r_y, b_z r_z) \quad (4.12)$$

denotes the rescaled trapping potential. Note that I rescale the collision integral later on, when I introduce the relaxation time approximation.

The  $\Theta$ -equations can be derived by multiplying Eq. (4.11) with  $k_i^2$  and integrating over the whole phase space, which leads to

$$\begin{aligned} \frac{\dot{\Theta}_i}{\Theta_i} + 2\frac{\dot{b}_i}{b_i} - \frac{1}{\hbar \langle k_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} k_i^2 \sum_j \frac{1}{b_j \Theta_j^{\frac{1}{2}}} \left( \frac{\partial U_{\text{mf}, b\Theta}}{\partial r_j} \frac{\partial f^0}{\partial k_j} - \frac{\partial U_{\text{mf}, b\Theta}}{\partial k_j} \frac{\partial f^0}{\partial r_j} \right) \\ = \frac{1}{\Gamma \langle k_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} k_i^2 I_{\text{coll}}[f], \end{aligned} \quad (4.13)$$

where  $\langle \cdot \rangle^0 = \int d^3 r d^3 k / (2\pi)^3 \cdot f^0(\mathbf{r}, \mathbf{k})$  denotes the phase-space average weighted with the equilibrium distribution function  $f^0$ . Since  $f^0(\mathbf{r}, \mathbf{k})$  and the interaction potential are even functions in both coordinates and momenta, the interaction term in Eq. (4.13) vanishes. This becomes clear in the context of a Fermi gas without a dipole-dipole interaction which has the form of a Fermi-Dirac distribution. In the limit of zero temperature, this Fermi-Dirac distribution degenerates into a Heaviside function with quadratic dependence on space and momentum parameters, leading to an even distribution. The effect of an interaction, which is described by an even potential, will not change this qualitative behavior. Hence, the  $\Theta$ -equations (4.13) simplify to

$$\frac{\dot{\Theta}_i}{\Theta_i} + 2\frac{\dot{b}_i}{b_i} = \frac{1}{\Gamma \langle k_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} k_i^2 I_{\text{coll}}[f]. \quad (4.14)$$

Multiplying the whole equation (4.11) instead with  $r_i k_i$  and integrating again over the phase space leads to a differential equation for the scaling parameter  $b_i(t)$ :

$$\begin{aligned} -\Gamma \frac{\hbar \Theta_i^{\frac{1}{2}}}{m b_i} \langle k_i^2 \rangle^0 + \frac{\Gamma m \ddot{b}_i}{\hbar \Theta_i^{\frac{1}{2}}} \langle r_i^2 \rangle^0 + \frac{\Gamma}{\hbar b_i \Theta_i^{\frac{1}{2}}} \int \frac{d^3 r d^3 k}{(2\pi)^3} r_i \frac{\partial U(b, \mathbf{r})}{\partial r_i} f^0(\mathbf{r}, \mathbf{k}) \\ - \frac{\Gamma}{\hbar} \int \frac{d^3 r d^3 k}{(2\pi)^3} r_i k_i \sum_j \frac{1}{b_j \Theta_j^{\frac{1}{2}}} \left( \frac{\partial f^0}{\partial k_j} \frac{\partial U_{\text{mf}, b\Theta}}{\partial r_j} - \frac{\partial f^0}{\partial r_j} \frac{\partial U_{\text{mf}, b\Theta}}{\partial k_j} \right) = 0. \end{aligned} \quad (4.15)$$

Note that the zero of the right-hand side of Eq. (4.15) stems from the fact that  $r_i k_i$  is a conserved quantity of the collision integral, as argued at the end of Section 3.2.

The equations for the scaling parameters  $b_i$  in Eq. (4.15) can be simplified by using Fourier transforms to rewrite the mean-field terms in a more compact form. Due to the  $k$ -independence of the Hartree term, only three mean-field potential terms appear in Eq. (4.15). For simplicity, I treat them separately, beginning with the Hartree term in which I perform at first a partial integration with respect to the  $k_i$ -derivative

$$\begin{aligned} T^{\text{H}} &= -\frac{1}{m} \int \frac{d^3 r d^3 k}{(2\pi)^3} r_i k_i \sum_j \frac{\Theta_i^{\frac{1}{2}}}{b_j \Theta_j^{\frac{1}{2}}} \frac{\partial f^0}{\partial k_j} \frac{\partial}{\partial r_j} \int d^3 r' n^0(\mathbf{r}') V_{\text{int}}(b, \mathbf{r} - \mathbf{r}') \\ &= \frac{1}{b_i m} \int d^3 r d^3 r' r_i \frac{\partial V_{\text{int}}(b, \mathbf{r} - \mathbf{r}')}{\partial(r_i - r'_i)} n^0(\mathbf{r}) n^0(\mathbf{r}'). \end{aligned} \quad (4.16)$$

Splitting  $r_i = (r_i + r'_i)/2$  at the right-hand side of Eq. (4.16) and renaming the integration variables in one of these two terms leads to

$$T^{\text{H}} = \frac{1}{2m b_i} \int d^3 r d^3 r' (r_i - r'_i) \frac{\partial V_{\text{int}}(b, \mathbf{r} - \mathbf{r}')}{\partial(r_i - r'_i)} n^0(\mathbf{r}) n^0(\mathbf{r}'). \quad (4.17)$$

#### 4 Solution of the Boltzmann-Vlasov Equation

At last, Fourier transforms of the densities can be used to decrease the number of integrals

$$T^H = \frac{1}{2mb_i} \int \frac{d^3k}{(2\pi)^3} \tilde{W}_i(b, \mathbf{k}) \tilde{n}^0(\mathbf{k}) \tilde{n}^0(-\mathbf{k}), \quad (4.18)$$

where  $\tilde{n}^0$  denotes the Fourier transform of the particle density and

$$\tilde{W}_i(b, \mathbf{k}) = F \left[ r_i \frac{\partial V_{\text{int}}(b, \mathbf{r})}{\partial r_i} \right] \quad (4.19)$$

uses the abbreviation  $F[\cdot]$  which denotes the Fourier transform of the argument. Note that in  $\tilde{W}_i$  the order of Fourier transforming and rescaling is opposite to the potential in the Fock term in Eq. (4.10). The Fock exchange terms will be treated in a similar way. Partial integrations lead to

$$\begin{aligned} & \frac{1}{m \prod_l b_l} \int \frac{d^3r d^3k}{(2\pi)^3} r_i k_i \sum_j \frac{\Theta_i^{\frac{1}{2}}}{b_j \Theta_j^{\frac{1}{2}}} \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial k_j} \frac{\partial}{\partial r_j} \int \frac{d^3k'}{(2\pi)^3} f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') \\ & - \frac{1}{m \prod_l b_l} \int \frac{d^3r d^3k}{(2\pi)^3} r_i k_i \sum_j \frac{\Theta_i^{\frac{1}{2}}}{b_j \Theta_j^{\frac{1}{2}}} \frac{\partial f^0(\mathbf{r}, \mathbf{k})}{\partial r_j} \frac{\partial}{\partial k_j} \int \frac{d^3k'}{(2\pi)^3} f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') \\ & = \frac{1}{m \prod_l b_l b_i} \left\{ \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} \frac{\partial}{\partial r_i} [r_i f^0(\mathbf{r}, \mathbf{k})] f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') \right. \\ & \quad \left. + \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') k_i \frac{\partial \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}')}{\partial k_i} \right\}. \quad (4.20) \end{aligned}$$

After performing the product rule, the first term leads to two terms. After a partial integration and a renaming of the integration variables, one of these terms coincides exactly with the original term

$$\begin{aligned} & \frac{1}{m \prod_j b_j b_i} \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} \frac{\partial}{\partial r_i} [r_i f^0(\mathbf{r}, \mathbf{k})] f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') \\ & = \frac{1}{m \prod_j b_j b_i} \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}') \\ & \quad - \frac{1}{m \prod_j b_j b_i} \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} \frac{\partial}{\partial r_i} [r_i f^0(\mathbf{r}, \mathbf{k})] f^0(\mathbf{r}, \mathbf{k}') \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}'). \quad (4.21) \end{aligned}$$

The second term on the right-hand side of Eq. (4.20) can be split into two equal terms. After renaming the integration variables of one of these two terms, I get

$$\begin{aligned} T^F & = \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') k_i \frac{\partial \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}')}{\partial k_i} \\ & = \int \frac{d^3r d^3k d^3k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \frac{k_i - k'_i}{2} \frac{\partial \tilde{V}_{\text{int}}(\Theta, \mathbf{k} - \mathbf{k}')}{\partial k_i}. \quad (4.22) \end{aligned}$$

This can be rewritten using Fourier transforms

$$T^F = \frac{1}{2} \int \frac{d^3r d^3r' d^3k d^3k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') r'_i \left\{ \frac{\partial}{\partial r'_i} e^{\left[ -i \sum_i r'_i (k_i - k'_i) \Theta_i^{\frac{1}{2}} \right]} \right\} V_{\text{int}}(\mathbf{r}'). \quad (4.23)$$

Putting all these together, the differential equations for the parameters  $b_i$  yield

$$\begin{aligned} \ddot{b}_i + \frac{1}{mb_i \langle r_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} r_i \frac{\partial U(b, \mathbf{r})}{\partial r_i} f^0(\mathbf{r}, \mathbf{k}) - \frac{\hbar^2 \langle k_i^2 \rangle^0 \Theta_i}{m^2 b_i \langle r_i^2 \rangle^0} + \frac{1}{2mb_i \langle r_i^2 \rangle^0} \left[ \int \frac{d^3 k}{(2\pi)^3} \tilde{W}_i(b, \mathbf{k}) \right. \\ \left. \times \tilde{n}^0(\mathbf{k}) \tilde{n}^0(-\mathbf{k}) - \frac{1}{\prod_j b_j} \int \frac{d^3 r d^3 k d^3 k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}') \right] = 0, \end{aligned} \quad (4.24)$$

where  $\tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}')$  is the rescaled Fourier transform of  $r_i \frac{\partial V_{\text{int}}}{\partial r_i}$

$$\tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}') = \left( \int d^3 r e^{-i\mathbf{r} \cdot \mathbf{q}} r_i \frac{\partial V_{\text{int}}}{\partial r_i} \right) \Big|_{q_x = \Theta_x^{\frac{1}{2}}(k_x - k'_x), q_y = \Theta_y^{\frac{1}{2}}(k_y - k'_y), q_z = \Theta_z^{\frac{1}{2}}(k_z - k'_z)}. \quad (4.25)$$

In order to further simplify this equation, I consider the concrete case of a Fermi gas within an anisotropic harmonic trap

$$U(b, \mathbf{r}) = \frac{m}{2} \sum_i \omega_i^2 b_i^2 r_i^2. \quad (4.26)$$

Thus, the trapping term can be rewritten as

$$\frac{1}{mb_i \langle r_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} r_i \frac{\partial U(b, \mathbf{r})}{\partial r_i} f^0(\mathbf{r}, \mathbf{k}) = \frac{1}{mb_i \langle r_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} m \omega_i^2 b_i^2 r_i^2 f^0(\mathbf{r}, \mathbf{k}) = \omega_i^2 b_i, \quad (4.27)$$

so that Eq. (4.24) reduces to

$$\begin{aligned} \ddot{b}_i + \omega_i^2 b_i - \frac{\hbar^2 \langle k_i^2 \rangle^0 \Theta_i}{m^2 b_i \langle r_i^2 \rangle^0} + \frac{1}{2mb_i \langle r_i^2 \rangle^0} \left[ \int \frac{d^3 k}{(2\pi)^3} \tilde{W}_i(b, \mathbf{k}) \tilde{n}^0(\mathbf{k}) \tilde{n}^0(-\mathbf{k}) \right. \\ \left. - \frac{1}{\prod_j b_j} \int \frac{d^3 r d^3 k d^3 k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}') \right] = 0. \end{aligned} \quad (4.28)$$

The integro-differential equations cannot be solved in the form of Eq. (4.14) and Eq. (4.28), because the global equilibrium distribution  $f^0$  is still unknown and I have not yet dealt with the collision integral. To be consistent with my previous calculations, I would have to insert the scaling ansatz into the whole collision integral as well and evaluate it with the ansatz (4.4)–(4.7) for the distribution function to obtain an expression for the scaling parameters without any integrals. However, it is not possible to do this due to the nonlinear and complex structure of the collision integral. Therefore, I have to resort to a simplified procedure. At first, I will derive the equations determining the global equilibrium distribution function. Afterwards, I will describe the effect of collisions by means of a relaxation time approximation.

## 4.2 Static Solution

In the last section, I derived differential equations for the respecting scaling parameters. These equations still contain some integrals over the Wigner function in equilibrium, the dipole-dipole interaction potential, and the collision integral. The dependence of the equilibrium Wigner function stems from the scaling ansatz, which assumes that  $f(\mathbf{x}, \mathbf{q}, t)$  is qualitatively not very different from the equilibrium distribution function  $f^0$ . Hence, knowing  $f^0(\mathbf{r}, \mathbf{k})$  is mandatory for solving the Boltzmann-Vlasov equation with the scaling ansatz. This function is, in principle, defined through the partial differential Eq. (4.3).

Due to the fact that general solutions for partial differential equations are hard to achieve, I will use a variational ansatz for determining the equilibrium Wigner function, yet I have to be careful to construct this ansatz in a self-consistent way. In phase space, the equilibrium distribution function will rapidly decrease to zero outside a certain closed surface, because I consider an ultracold Fermi gas. Therefore, I use an ansatz for the semiclassical Wigner function which resembles the correct form of a non-interacting Fermi gas at zero temperature

$$f^0(\mathbf{r}, \mathbf{k}) = \Theta \left( 1 - \sum_j \frac{r_j^2}{R_j^2} - \sum_j \frac{k_j^2}{K_j^2} \right). \quad (4.29)$$

Here the variational parameters  $R_i$  and  $K_i$  denote the extension of the Fermi surface in equilibrium in both coordinate and momentum space. The dipole-dipole interaction will stretch the momentum distribution in the direction of the polarization, which is taken into account by an anisotropy of the momentum parameters  $K_i$ . With this ansatz, the normalization of  $f^0(\mathbf{r}, \mathbf{k})$  leads to

$$N = \int d^3r \int \frac{d^3k}{(2\pi)^3} f^0(\mathbf{r}, \mathbf{k}) = \frac{1}{48} \bar{R}^3 \bar{K}^3, \quad (4.30)$$

where the bar denotes geometric averaging, i.e.,  $\bar{R} = (R_x R_y R_z)^{\frac{1}{3}}$ . To render the ansatz (4.29) self-consistent, the parameters  $R_i$  and  $K_i$  have to minimize the total energy of the system. This leads together with the particle number conservation Eq. (4.30) equations which fix these parameters.

### 4.2.1 Total energy

The Hamiltonian of the collisionless Boltzmann-Vlasov equation (2.61) describes an effective one-particle theory and, therefore, determines a one-particle energy. Instead, I have to minimize the total energy of the many-body system, which is given in the realm of the Hartree-Fock theory as the sum of the kinetic Energy  $E_{\text{kin}}$ , the trapping Energy  $E_{\text{tr}}$ , the direct Hartree energy  $E_{\text{d}}$ , and the Fock exchange energy  $E_{\text{ex}}$  [20,29,30]:

$$E_{\text{kin}} = \int d^3r \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 \mathbf{k}^2}{2m} f(\mathbf{r}, \mathbf{k}, t), \quad (4.31)$$

$$E_{\text{tr}} = \int d^3r \int \frac{d^3k}{(2\pi)^3} \frac{m}{2} \left( \sum_j \omega_j^2 r_j^2 \right) f(\mathbf{r}, \mathbf{k}, t), \quad (4.32)$$

$$E_{\text{d}} = \frac{1}{2} \int d^3r \int d^3r' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} V_{\text{int}}(\mathbf{r} - \mathbf{r}') f(\mathbf{r}, \mathbf{k}, t) f(\mathbf{r}', \mathbf{k}', t), \quad (4.33)$$

$$E_{\text{ex}} = -\frac{1}{2} \int d^3r \int d^3r' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} V_{\text{int}}(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}'} f(\mathbf{r}, \mathbf{k}, t) f(\mathbf{r}, \mathbf{k}', t). \quad (4.34)$$

This Hartree-Fock energy  $E_{\text{tot}} = E_{\text{kin}} + E_{\text{tr}} + E_{\text{d}} + E_{\text{ex}}$  neglects the collisional effects, hence it is only valid, when the collisions vanish. Because I will calculate the energy in the static case this condition is fulfilled. Note that an intriguing relation exists between the effective one-particle Hamiltonian Eq. (2.61) and the total Hartree-Fock energy of the system  $E_{\text{tot}}$ , which reads

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{\delta E_{\text{tot}}(\mathbf{r}, \mathbf{p}, t)}{\delta f(\mathbf{r}, \mathbf{p}, t)}, \quad (4.35)$$

where  $\delta/\delta f(\mathbf{r}, \mathbf{p}, t)$  denotes a functional derivative with respect to the semiclassical Wigner function. This equation was found by a comparison between  $E_{\text{tot}}(\mathbf{r}, \mathbf{p}, t)$  and  $H(\mathbf{r}, \mathbf{p}, t)$ , but I do not have a clear derivation yet.



A lengthy calculation is required to evaluate the integrals (4.31)–(4.34) in global equilibrium for the ansatz Eq. (4.29), but a detailed computation of all 4 terms can be found in Appendix B. The resulting total energy turns out to be

$$E_{\text{tot}} = \frac{N}{8} \sum_j \frac{\hbar^2 K_j^2}{2m} + \frac{N}{8} \frac{m}{2} \sum_j \omega_j^2 R_j^2 - \frac{48N^2 c_0}{8\bar{R}^3} f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) + \frac{48N^2 c_0}{8\bar{R}^3} f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right), \quad (4.36)$$

where  $c_0 = 2^{10} C_{\text{dd}} / (3^4 \cdot 5 \cdot 7 \cdot \pi^3)$  and  $f(x, y)$  is the anisotropy function defined through the integral

$$f(x, y) = -\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \left[ \frac{3x^2 y^2 \cos^2\vartheta}{(y^2 \cos^2\phi + x^2 \sin^2\phi) \sin^2\vartheta + x^2 y^2 \cos^2\vartheta} - 1 \right], \quad (4.37)$$

which can also be represented in terms of elliptic integrals [20]

$$f(x, y) = 1 + 3xy \frac{E(\varphi, k) - F(\varphi, k)}{(1 - y^2)\sqrt{1 - x^2}}, \quad (4.38)$$

where  $\varphi = \arcsin\sqrt{1 - x^2}$  is an abbreviation,  $F(\varphi, k)$  is the elliptic integral of first kind

$$F(\varphi, k) = \int_0^{\sin\varphi} du \frac{1}{\sqrt{(1 - u^2)(1 - k^2 u^2)}} \quad (4.39)$$

and  $E(\varphi, k)$  is the elliptic integral of second kind

$$E(\varphi, k) = \int_0^{\sin\varphi} du \frac{\sqrt{1 - k^2 u^2}}{\sqrt{1 - u^2}}. \quad (4.40)$$

The effect of the dipole-dipole interaction is only described by the anisotropy function, which is not only valid for Fermi gases but for all dipolar systems, i.e., dipolar Bose gases [54–57]. The properties of the anisotropy function are described in more detail in Ref. [20, Appendix B].

## 4.2.2 Variational Parameters in Momentum Space

To render my ansatz self-consistent, the variational parameters  $R_i$  and  $K_i$  are determined by minimizing the total energy. By doing so, I acknowledge that they are not independent of each other, as they are coupled via Eq. (4.30), which accounts for the particle number conservation. Hence, I introduce a Lagrange multiplier  $\mu$  and add the particle number conservation to Eq. (4.36). A minimization of the resulting quantity  $E_{\text{tot}} - \mu N$  with respect to the momentum parameters  $K_i$  leads to

$$\frac{N\hbar^2 K_x}{8m} - \frac{48N^2 c_0}{8\bar{R}^3} \frac{K_z}{K_x^2} f_1\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} \bar{R}^3 K_y K_z = 0, \quad (4.41)$$

$$\frac{N\hbar^2 K_y}{8m} - \frac{48N^2 c_0}{8\bar{R}^3} \frac{K_z}{K_y^2} f_2\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} \bar{R}^3 K_x K_z = 0, \quad (4.42)$$

$$\frac{N\hbar^2 K_z}{8m} + \frac{48N^2 c_0}{8\bar{R}^3} \frac{1}{K_x} f_1\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) + \frac{48N^2 c_0}{8\bar{R}^3} \frac{1}{K_y} f_2\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} \bar{R}^3 K_x K_y = 0, \quad (4.43)$$

where  $f_{1/2}(x, y)$  denotes the derivative of  $f(x, y)$  with respect to the first/second variable. Eqs. (4.41)–(4.43) depend explicitly on the Lagrange multiplier  $\mu$ . To get an expression for  $\mu$

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I sum up all three momentum equations, so that any terms containing the anisotropy function cancel each other, which leads to the following expression of the Lagrange multiplier

$$\mu = \frac{1}{12} \sum_j \frac{\hbar^2 K_j^2}{2m}. \quad (4.44)$$

Inserting this into the Eqs. (4.41)–(4.43) leads to the following static equations for the parameters  $K_j$

$$\frac{\hbar^2 K_x^2}{2m} = \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{2m} + \frac{48Nc_0}{2\bar{R}^3} \frac{K_z}{K_x} f_1 \left( \frac{K_z}{K_x}, \frac{K_z}{K_y} \right), \quad (4.45)$$

$$\frac{\hbar^2 K_y^2}{2m} = \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{2m} + \frac{48Nc_0}{2\bar{R}^3} \frac{K_z}{K_y} f_2 \left( \frac{K_z}{K_x}, \frac{K_z}{K_y} \right), \quad (4.46)$$

$$\frac{\hbar^2 K_z^2}{2m} = \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{2m} - \frac{48Nc_0}{2\bar{R}^3} \frac{K_z}{K_x} f_1 \left( \frac{K_z}{K_x}, \frac{K_z}{K_y} \right) - \frac{48Nc_0}{2\bar{R}^3} \frac{K_z}{K_y} f_2 \left( \frac{K_z}{K_x}, \frac{K_z}{K_y} \right). \quad (4.47)$$

These equations are not independent of each other, but show the influence of the dipole-dipole interaction on the momentum distribution in phase space. Without the anisotropy function in Eqs. (4.45)–(4.47), they yield a spherically symmetric momentum distribution, which is called in condensed matter physics a Fermi sphere [58]. But through the effect of the dipole-dipole interaction, the Fermi sphere will be deformed to an ellipsoid [26]. Note that the derived equations for the momentum parameters are only influenced by the kinetic and the Fock energy term.

By considering the symmetries of the total energy (4.36), I can obtain a simple relation between the variational parameters for the  $x$  and  $y$  directions. As a consequence of the symmetry of the anisotropy function  $f(x, y) = f(y, x)$  [20], the total energy possesses the same symmetry between the  $x$ - and  $y$ -direction in  $K$ -space. This implies that the momentum distribution of a dipolar Fermi gas in global equilibrium remains cylinder-symmetric even in the case of a triaxial trap [20,29,30], because the trap geometry does not influence the exchange term in the total energy. Thus, I can conclude  $K_x = K_y$ , which also enables me to simplify the relation to  $K_z$  by taking into account

$$\lim_{y \rightarrow x} x f_1(x, y) = \lim_{y \rightarrow x} y f_2(x, y) = -1 + \frac{(2 + x^2) f_s(x)}{2(1 - x^2)}, \quad (4.48)$$

where  $f_s(x) = f(x, x)$  denotes the cylinder-symmetric anisotropy function

$$f_s(x) = \frac{3x^2}{1 - x^2} \left( 1 - \frac{\text{Artanh}\sqrt{1 - x^2}}{\sqrt{1 - x^2}} \right) + 1. \quad (4.49)$$

Subtracting Eq. (4.45) from Eq. (4.47) and inserting Eq. (4.48) yields

$$\frac{\hbar^2 K_z^2}{2m} - \frac{\hbar^2 K_x^2}{2m} = -\frac{3}{2} \frac{48N^2 c_0}{\prod_j R_j} \left[ -1 + \frac{(2K_x^2 + K_z^2) f_s \left( \frac{K_z}{K_x} \right)}{2(K_x^2 - K_z^2)} \right]. \quad (4.50)$$

This equation shows the stretching of the momentum distribution due to the dipole-dipole interaction.

### 4.2.3 Variational Parameters in Coordinate Space

The derivation of the equations which determine the spatial parameters with minimal total energy is analogous to the calculation of the momentum parameters. To this end, I minimize  $E_{\text{tot}} - \mu N$  with respect to the space parameters  $R_i$ , which leads to

$$\begin{aligned} \frac{N}{8}m\omega_x^2 R_x + \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_x} f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_z} f_1\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \\ - \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_x} f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} R_y R_z \bar{K}^3 = 0, \end{aligned} \quad (4.51)$$

$$\begin{aligned} \frac{N}{8}m\omega_y^2 R_y + \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_y} f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_z} f_2\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \\ - \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_y} f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} R_x R_z \bar{K}^3 = 0, \end{aligned} \quad (4.52)$$

$$\begin{aligned} \frac{N}{8}m\omega_z^2 R_z + \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_z} f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) + \frac{48N^2c_0}{8\bar{R}^3} \frac{R_x}{R_z^2} f_1\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \\ + \frac{48N^2c_0}{8\bar{R}^3} \frac{R_y}{R_z^2} f_2\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - \frac{48N^2c_0}{8\bar{R}^3} \frac{1}{R_z} f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{\mu}{48} R_x R_y \bar{K}^3 = 0. \end{aligned} \quad (4.53)$$

To get equations which are independent of the Lagrange multiplier  $\mu$ , I sum up Eqs. (4.51)–(4.53), which yields

$$\mu = \frac{1}{12} \frac{m}{2} \sum_i \omega_i^2 R_i^2 + \frac{48Nc_0}{8\bar{R}^3} \left[ f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) \right]. \quad (4.54)$$

This differs from Eq. (4.44) but represents an equivalent expression for  $\mu$ . Inserting Eq. (4.54) into the Eqs. (4.51)–(4.53) for the spatial parameters leads to

$$\omega_x^2 R_x^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48Nc_0}{m\bar{R}^3} \left[ f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{R_x}{R_z} f_1\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \right] = 0, \quad (4.55)$$

$$\omega_y^2 R_y^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48Nc_0}{m\bar{R}^3} \left[ f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) - f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) - \frac{R_y}{R_z} f_2\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \right] = 0, \quad (4.56)$$

$$\begin{aligned} \omega_z^2 R_z^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48Nc_0}{m\bar{R}^3} \left[ f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \right. \\ \left. - f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right) + \frac{R_x}{R_z} f_1\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) + \frac{R_y}{R_z} f_2\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right) \right] = 0. \end{aligned} \quad (4.57)$$

Thus, I get with the cylinder-symmetric momentum distribution in total five independent Eqs. (4.30), (4.50), and (4.55)–(4.57) for the scaling parameters which determine the static solution.

### 4.2.4 Cylinder-Symmetric Case

Before I use the equations derived for the global equilibrium, I specialize these equations to a cylinder-symmetric trapping potential, which means  $\omega_x = \omega_y$ . To emphasize this, I denote both of these directions with the subscript  $\rho$ . Due to the symmetry of the anisotropy function  $f(x, y) = f(y, x)$ , (4.55) and (4.56) become then equivalent equations and, therefore, I conclude  $R_x = R_y =: R_\rho$ . This symmetry of the system can be used to simplify the equations for the

variational parameters  $R_\rho$  and  $R_z$  with the result

$$\omega_\rho^2 R_\rho^2 - \frac{\hbar^2 K_\rho^2}{m^2} - \frac{48Nc_0}{m\bar{R}^3} \left[ \frac{3\frac{R_\rho^2}{R_z^2} f_s \left( \frac{R_\rho}{R_z} \right)}{2\left(1 - \frac{R_\rho^2}{R_z^2}\right)} + \frac{3f_s \left( \frac{K_z}{K_\rho} \right)}{2\left(1 - \frac{K_\rho^2}{K_z^2}\right)} \right] = 0, \quad (4.58)$$

$$\omega_z^2 R_z^2 - \frac{\hbar^2 K_z^2}{m^2} + \frac{48Nc_0}{m\bar{R}^3} \left[ \frac{3f_s \left( \frac{R_\rho}{R_z} \right)}{\left(1 - \frac{R_\rho^2}{R_z^2}\right)} + \frac{3\frac{K_\rho^2}{K_z^2} f_s \left( \frac{K_z}{K_\rho} \right)}{\left(1 - \frac{K_\rho^2}{K_z^2}\right)} \right] = 0, \quad (4.59)$$

where  $\bar{R} = (R_\rho^2 R_z)^{1/3}$ . In the cylinder-symmetric case, only 4 independent equations remain: Eqs. (4.30), (4.50), (4.58), and (4.59).

### 4.3 Integrals with Stationary Distribution

I rescaled the Boltzmann-Vlasov equation and derived the ordinary differential equations (4.14) and (4.28) for the scaling parameters. However, the equations (4.28) for  $b_i$  still contain integrals. In the following, I will solve or at least simplify them.

At first, I write the  $b$ -equations (4.28) in the shorter form

$$\ddot{b}_i + \omega_i^2 b_i - \frac{\hbar^2 \langle k_i^2 \rangle^0 \Theta_i}{m^2 b_i \langle r_i^2 \rangle^0} + \frac{M_i^H(b) + M_i^F(b, \Theta)}{m b_i \langle r_i^2 \rangle^0} = 0, \quad (4.60)$$

where  $M_i^H(b)$  and  $M_i^F(b, \Theta)$  are the integrals from the Hartree (H) and Fock (F) term:

$$M_i^H(b) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \tilde{W}_i(b, \mathbf{k}) \tilde{n}^0(\mathbf{k}) \tilde{n}^0(-\mathbf{k}), \quad (4.61)$$

$$M_i^F(b, \Theta) = -\frac{1}{2 \prod_j b_j} \int \frac{d^3 r d^3 k d^3 k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}'). \quad (4.62)$$

These two integrals are similar to the energy integrals calculated in Appendix B. Hence, some of the previous calculations can be used.

At first, I study the Hartree integral  $M_i^H(b)$ . To this end, I have to find an explicit expression for  $\tilde{W}_i(b, \mathbf{k})$  which was defined in Eq. (4.19)

$$\tilde{W}_i(b, \mathbf{k}) = \int d^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} r_i \frac{\partial V_{\text{int}}(b_x r_x, b_y r_y, b_z r_z)}{\partial r_i}. \quad (4.63)$$

Rescaling of the integration variable and partial integrations lead to

$$\tilde{W}_i(b, \mathbf{k}) = -\frac{1}{\prod_l b_l} \frac{\partial}{\partial k_i} \left[ k_i \tilde{V}_{\text{int}} \left( \frac{k_x}{b_x}, \frac{k_y}{b_y}, \frac{k_z}{b_z} \right) \right]. \quad (4.64)$$

The Fourier transform of the interaction potential is outlined in Appendix A. Inserting the result (A.12) in Eq. (4.64) leads to the following explicit expression for  $\tilde{W}_i(b, \mathbf{k})$

$$\tilde{W}_i(b, \mathbf{k}) = -\frac{C_{\text{dd}}}{3 \prod_j b_j} \frac{\partial}{\partial k_i} \left[ k_i \left( \frac{3\frac{k_z^2}{b_z^2}}{\frac{k_x^2}{b_x^2} + \frac{k_y^2}{b_y^2} + \frac{k_z^2}{b_z^2}} - 1 \right) \right]. \quad (4.65)$$

The Fourier transform of the density  $\tilde{n}^0(\mathbf{k})$  is calculated in Appendix B, its result being given by Eq. (B.23). Using the particle number conservation Eq. (4.30), I can write the Fourier-transformed spatial density as

$$\tilde{n}^0(\mathbf{k}) = 48N \frac{J_3 \left[ \left( R_x^2 k_x^2 + R_y^2 k_y^2 + R_z^2 k_z^2 \right)^{\frac{1}{2}} \right]}{\left( k_x^2 R_x^2 + k_y^2 R_y^2 + k_z^2 R_z^2 \right)^{\frac{3}{2}}}, \quad (4.66)$$

which has manifestly the symmetry property  $\tilde{n}^0(\mathbf{k}) = \tilde{n}^0(-\mathbf{k})$ . Having all ingredients for the Hartree integral (4.61) at hand, I can now write it down

$$M_i^H(b) = -\frac{1}{2} \frac{48^2 N^2 C_{dd}}{3 \prod_j b_j} \int \frac{d^3 k}{(2\pi)^3} \frac{\partial}{\partial k_i} \left[ k_i \left( \frac{3 \frac{k_z^2}{b_z^2}}{\frac{k_x^2}{b_x^2} + \frac{k_y^2}{b_y^2} + \frac{k_z^2}{b_z^2}} - 1 \right) \right] \frac{J_3^2 \left[ \left( R_x^2 k_x^2 + R_y^2 k_y^2 + R_z^2 k_z^2 \right)^{\frac{1}{2}} \right]}{\left( k_x^2 R_x^2 + k_y^2 R_y^2 + k_z^2 R_z^2 \right)^3}. \quad (4.67)$$

Applying the product rule of differentiation leads to two terms which differ only in the derivative with respect to a momentum coordinate, multiplied by the same momentum coordinate

$$\begin{aligned} M_i^H(b) = & -\frac{48^2 N^2 C_{dd}}{6(2\pi)^3 \prod_j b_j} \int d^3 k \left( \frac{3 \frac{k_z^2}{b_z^2}}{\frac{k_x^2}{b_x^2} + \frac{k_y^2}{b_y^2} + \frac{k_z^2}{b_z^2}} - 1 \right) \frac{J_3^2 \left[ \left( R_x^2 k_x^2 + R_y^2 k_y^2 + R_z^2 k_z^2 \right)^{\frac{1}{2}} \right]}{\left( k_x^2 R_x^2 + k_y^2 R_y^2 + k_z^2 R_z^2 \right)^3} \\ & - \frac{48^2 N^2 C_{dd}}{6(2\pi)^3 \prod_j b_j} \int d^3 k k_i \frac{\partial}{\partial k_i} \left( \frac{3 \frac{k_z^2}{b_z^2}}{\frac{k_x^2}{b_x^2} + \frac{k_y^2}{b_y^2} + \frac{k_z^2}{b_z^2}} - 1 \right) \frac{J_3^2 \left[ \left( R_x^2 k_x^2 + R_y^2 k_y^2 + R_z^2 k_z^2 \right)^{\frac{1}{2}} \right]}{\left( k_x^2 R_x^2 + k_y^2 R_y^2 + k_z^2 R_z^2 \right)^3}. \end{aligned} \quad (4.68)$$

Then, I perform the substitution  $u_i = R_i k_i$  and rewrite the derivative in the second term as a derivative of  $b_i R_i$ , using the identity

$$\frac{\partial f}{\partial a} \left( \frac{x}{a} \right) = -\frac{x}{a} \frac{\partial f}{\partial x} \left( \frac{x}{a} \right), \quad (4.69)$$

which is valid for any differentiable function. Thus, I obtain

$$\begin{aligned} M_i^H(b) = & -\frac{48^2 N^2 C_{dd}}{6(2\pi)^3 \prod_j b_j R_j} \int d^3 u \left( \frac{3 \frac{u_z^2}{b_z^2 R_z^2}}{\frac{u_x^2}{b_x^2 R_x^2} + \frac{u_y^2}{b_y^2 R_y^2} + \frac{u_z^2}{b_z^2 R_z^2}} - 1 \right) \frac{J_3^2(|\mathbf{u}|)}{|\mathbf{u}|^6} \\ & + \frac{48^2 N^2 C_{dd}}{6(2\pi)^3 \prod_j b_j R_j} \int d^3 u b_i R_i \frac{\partial}{\partial b_i R_i} \left( \frac{3 \frac{u_z^2}{b_z^2 R_z^2}}{\frac{u_x^2}{b_x^2 R_x^2} + \frac{u_y^2}{b_y^2 R_y^2} + \frac{u_z^2}{b_z^2 R_z^2}} - 1 \right) \frac{J_3^2(|\mathbf{u}|)}{|\mathbf{u}|^6}. \end{aligned} \quad (4.70)$$

Switching to spherical coordinates and taking into account the definition of the anisotropy function Eq. (4.37) as well as [59, (6.574.2)]

$$\int_0^\infty dt J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2\lambda \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)} \quad (4.71)$$

for  $\text{Re}(\nu + \mu + 1) > \text{Re}(\lambda) > 0, \alpha > 0$

leads to

$$M_i^H(b) = \frac{48 N^2 c_0}{8 \prod_j b_j R_j} \left[ f \left( \frac{b_x R_x}{b_z R_z}, \frac{b_y R_y}{b_z R_z} \right) - b_i R_i \frac{\partial}{\partial b_i R_i} f \left( \frac{b_x R_x}{b_z R_z}, \frac{b_y R_y}{b_z R_z} \right) \right]. \quad (4.72)$$

#### 4 Solution of the Boltzmann-Vlasov Equation

An explicit expression for the Fock integral (4.62) can be obtained in a similar way. Using Fourier transforms, one can rewrite it in the following form

$$M_i^F(b, \Theta) = -\frac{1}{2 \prod_j b_j} \int \frac{d^3 k''}{(2\pi)^6} \tilde{W}_i(\Theta, \mathbf{k}'') \int d^3 k' \int d^3 r' \bar{f}^0(\mathbf{k}', \mathbf{r}') \bar{f}^0(-\mathbf{k}', -\mathbf{r}') e^{i\mathbf{r}' \cdot \mathbf{k}''}, \quad (4.73)$$

where  $\bar{f}^0(\mathbf{k}', \mathbf{k})$  denotes the Fourier transform of  $f^0(\mathbf{x}, \mathbf{k})$  with respect to the first variable and  $\tilde{f}^0(\mathbf{x}, \mathbf{x}')$  the Fourier transform with respect to the second variable. The rescaled Fourier transform of the interaction potential as defined in Eq. (4.25) reads

$$\tilde{W}(\Theta, \mathbf{k}'') = -\frac{C_{\text{dd}}}{3} \frac{\partial}{\partial k_i''} \left[ k_i'' \left( \frac{3\Theta_z k_z''^2}{\Theta_x k_x''^2 + \Theta_y k_y''^2 + \Theta_z k_z''^2} - 1 \right) \right]. \quad (4.74)$$

The calculation of the  $\mathbf{r}'$  and  $\mathbf{k}'$  integrals over the double Fourier transformed Wigner functions and the exponential function can be found in Appendix B, its result being given by Eq. (B.39). Putting all this together leads to

$$M_i^F(b, \Theta) = \frac{C_{\text{dd}}}{6 \prod_j b_j} \int \frac{d^3 k}{(2\pi)^6} \frac{\partial}{\partial k_i} \left[ k_i \left( \frac{3\Theta_z k_z^2}{\Theta_x k_x^2 + \Theta_y k_y^2 + \Theta_z k_z^2} - 1 \right) \right] \frac{16\pi\sqrt{2\pi}}{2^{\frac{5}{2}}} \\ \times \int_0^{\frac{\pi}{2}} \frac{dt}{(2\sin t)^6} \bar{R}^3 \bar{K}^3 \frac{\Gamma(\frac{3}{2})}{\Gamma(4)} \left( 4\sin^2 t - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2} - \frac{k_z^2}{K_z^2} \right)^3 \Theta \left( 4\sin^2 t - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2} - \frac{k_z^2}{K_z^2} \right). \quad (4.75)$$

Substituting  $k_i = u_i K_i$ , I get

$$M_i^F(b, \Theta) = \frac{C_{\text{dd}} 16\pi\sqrt{2\pi}\Gamma(\frac{3}{2}) \bar{R}^3 \bar{K}^6}{6 \prod_j b_j (2\pi)^6 \Gamma(4)} \int d^3 u u_i \frac{\partial}{\partial u_i} \left( \frac{3\Theta_z K_z^2 u_z^2}{\Theta_x K_x^2 u_x^2 + \Theta_y K_y^2 u_y^2 + \Theta_z K_z^2 u_z^2} - 1 \right) \\ \times \int_0^{\frac{\pi}{2}} \frac{dt}{(2\sin^2 t)^6} \left( 4\sin^2 t - \sum_l u_l^2 \right)^3 \Theta \left( 4\sin^2 t - \sum_m u_m^2 \right) \\ + \frac{C_{\text{dd}} 16\pi\sqrt{2\pi}\Gamma(\frac{3}{2}) \bar{R}^3 \bar{K}^6}{6 \prod_j b_j (2\pi)^6 \Gamma(4)} \int d^3 u \left( \frac{3\Theta_z K_z^2 u_z^2}{\Theta_x K_x^2 u_x^2 + \Theta_y K_y^2 u_y^2 + \Theta_z K_z^2 u_z^2} - 1 \right) \\ \times \int_0^{\frac{\pi}{2}} \frac{dt}{(2\sin^2 t)^6} \left( 4\sin^2 t - \sum_l u_l^2 \right)^3 \Theta \left( 4\sin^2 t - \sum_m u_m^2 \right). \quad (4.76)$$

After switching into spherical coordinates and rewriting the derivative  $\partial/\partial u_i$  into a derivative with respect to  $\Theta_i^{\frac{1}{2}} K_i$ , I finally get

$$M_i^F(b, \Theta) = -\frac{48N^2 c_0}{8 \prod_j b_j R_j} \left[ f \left( \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_x^{\frac{1}{2}} K_x}, \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_y^{\frac{1}{2}} K_y} \right) + \Theta_i^{\frac{1}{2}} K_i \frac{\partial}{\partial \Theta_i^{\frac{1}{2}} K_i} f \left( \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_x^{\frac{1}{2}} K_x}, \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_y^{\frac{1}{2}} K_y} \right) \right]. \quad (4.77)$$

The Hartree and Fock integrals Eq. (4.72) and Eq. (4.77) are both determined through the anisotropy function Eq. (4.37) and its derivatives. The remaining equilibrium averages over  $k_i^2$  and  $r_i^2$  yield

$$\langle k_i^2 \rangle^0 = \frac{N}{8} K_i^2, \quad (4.78)$$

$$\langle r_i^2 \rangle^0 = \frac{N}{8} R_i^2. \quad (4.79)$$

Having calculated all ingredients now, I insert the Eqs. (4.72) and (4.77)–(4.79) into the equation of motion (4.60) for the scaling parameter  $b_i$ , which leads to

$$\begin{aligned} \ddot{b}_i + \omega_i^2 b_i - \frac{\hbar^2 K_i^2 \Theta_i}{m^2 b_i R_i^2} + \frac{48 N c_0}{m b_i R_i^2 \prod_j b_j R_j} \left[ f \left( \frac{b_x R_x}{b_z R_z}, \frac{b_y R_y}{b_z R_z} \right) - b_i R_i \frac{\partial}{\partial b_i R_i} f \left( \frac{b_x R_x}{b_z R_z}, \frac{b_y R_y}{b_z R_z} \right) \right] \\ - \frac{48 N c_0}{m b_i R_i^2 \prod_j b_j R_j} \left[ f \left( \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_x^{\frac{1}{2}} K_x}, \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_y^{\frac{1}{2}} K_y} \right) + \Theta_i^{\frac{1}{2}} K_i \frac{\partial}{\partial \Theta_i^{\frac{1}{2}} K_i} f \left( \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_x^{\frac{1}{2}} K_x}, \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_y^{\frac{1}{2}} K_y} \right) \right] = 0. \end{aligned} \quad (4.80)$$

## 4.4 Local Equilibrium Distribution

Through the effect of collisions, the Fermi gas relaxes into local equilibrium, which is in general defined by Eq. (3.4). The system is then in local thermodynamical equilibrium, but not in global equilibrium, meaning that, for example, the local velocity field or the density can still vary from point to point. The scaling ansatz assumes that the distribution function is qualitatively not very different from equilibrium. Therefore, I assume, furthermore, that it is appropriate to approximately describe a relaxation from a distribution near equilibrium in terms of only one relaxation time

$$I_{\text{coll}}[f] \approx -\frac{f - f^{\text{le}}}{\tau}, \quad (4.81)$$

where  $\tau$  denotes this relaxation time. For simplicity, we neglect that  $\tau$  may depend on space and momentum as well as time and assume that it is constant. Note that the collision integral defines an infinite number of local equilibrium distribution functions, but the particular distribution function  $f^{\text{le}}$  in the relaxation time approximation is clearly defined, determining the limit function of the relaxation of  $f$ . Therefore, the choice of  $f^{\text{le}}$  is not arbitrary, since it depends on  $f$ . This point is crucial in understanding the structure of this local equilibrium function in the relaxation time approximation Eq. (4.81). It was shown in Section 3.2 that the collisions determine the momentum distribution in local equilibrium, but do not fix any spatial dependence. This implies that the collisions do not affect the spatial dependence of the distribution function.

Inserting Eq. (4.81) into the equations (4.14) for the  $\Theta_i$ 's and rescaling both distribution functions leads to

$$\frac{\dot{\Theta}_i}{\Theta_i} + 2 \frac{\dot{b}_i}{b_i} = -\frac{1}{\tau} \left[ 1 - \frac{\Gamma^{\text{le}}}{\Gamma \langle k_i^2 \rangle^0} \int d^3 r \int d^3 k k_i^2 f^0(\mathbf{r}, \mathbf{k}^{\text{le}}, t) \right], \quad (4.82)$$

where I also rescaled the local equilibrium according to Eqs. (4.4)–(4.7), but with the new scaling parameters  $\Theta_i^{\text{le}}$ , owing to the change of the momentum distribution by the collisions. Furthermore,

$$\Gamma^{\text{le}} = \frac{1}{\prod_j b_j (\Theta_j^{\text{le}})^{\frac{1}{2}}} \quad (4.83)$$

represents the scaling factor for  $f^{\text{le}}$ . As discussed above, the limiting function of the relaxation inherits its spatial dependence from the distribution function  $f$ , so that the spatial scaling parameters  $b_i$  of both functions are equal, as is also concluded in Refs. [50,51].

Through the scaling ansatz, one can derive coupling equations between the rescaled  $k$ -vectors

$$k_i = \left( \frac{\Theta_i^{\text{le}}}{\Theta_i} \right)^{\frac{1}{2}} k_i^{\text{le}}. \quad (4.84)$$

#### 4 Solution of the Boltzmann-Vlasov Equation

Inserting this into the equation of motion for the momentum scaling parameter Eq. (4.82) leads to

$$\dot{\Theta}_i + 2\frac{\dot{b}_i}{b_i}\Theta_i = -\frac{1}{\tau}(\Theta_i - \Theta_i^{\text{le}}). \quad (4.85)$$

The right-hand side shows that dissipation occurs when the system is not in local equilibrium and the relaxation time  $\tau$  has a finite value.

The most important property of the  $\Theta_i^{\text{le}}$ 's is that they are not independent of each other. Rather, the system is in local equilibrium for all directions. Therefore, the different directions of the system are coupled. The only exception is the ballistic regime, in which all directions are independent of each other. Note that this represents the fundamental difference between the hydrodynamic and the ballistic regime.

The system relaxes through the effect of collisions into a state described by the local equilibrium distribution function  $f^{\text{le}}$ , which means that the system only minimizes its energy with respect to the momentum scaling parameters  $\Theta_i^{\text{le}}$ , because the local collisional interactions do not influence the spatial distribution of the Fermi gas. Therefore, I minimize the total energy of the system, whose components vary generally in time, with respect to the momentum scaling parameters. To obtain the corresponding form for the total energy, I rescale the distribution functions according to Eqs. (4.4)–(4.7) in the energy integrals (4.31)–(4.34). Afterwards, I calculate the integrals by means of the ansatz for the global equilibrium function  $f^0$  in Eq. (4.29). The calculation of these integrals is analogous to the computation in the stationary case, but contains the scaling parameters. The solution is given by

$$\begin{aligned} E_{\text{tot}} = & -\frac{N m}{8} \sum_i \frac{b_i^2 R_i^2}{2} + \frac{N}{8} \sum_i \frac{\hbar^2 K_i^2 \Theta_i}{2m} + \frac{N m}{8} \sum_i \frac{\omega_i^2 b_i^2 R_i^2}{2} - \frac{48N^2 c_0}{8 \prod_j b_j R_j} f \left( \frac{b_x R_x}{b_z R_z}, \frac{b_y R_y}{b_z R_z} \right) \\ & + \frac{48N^2 c_0}{8 \prod_j b_j R_j} f \left( \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_x^{\frac{1}{2}} K_x}, \frac{\Theta_z^{\frac{1}{2}} K_z}{\Theta_y^{\frac{1}{2}} K_y} \right), \end{aligned} \quad (4.86)$$

where the first term, which vanishes in global equilibrium, represents the flow energy which is one part of the kinetic energy contribution.

To obtain relations for the scaling parameters in local equilibrium, I have to minimize the quantity  $E_{\text{tot}} - \mu N$  with respect to  $\Theta_i^{1/2} K_i$ . Here,  $\mu$  is also a Lagrange multiplier which is introduced because of the particle number conservation. By summing up the resulting equations, I find again an explicit expression of the Lagrange multiplier  $\mu$

$$\mu = \frac{1}{12} \sum_j \frac{\hbar^2 \Theta_j^{\text{le}} K_j^2}{2m} \quad (4.87)$$



which leads to the equations

$$\frac{\hbar^2 \Theta_x^{\text{le}} K_x^2}{2m} = \frac{1}{3} \sum_j \frac{\hbar^2 \Theta_j^{\text{le}} K_j^2}{2m} + \frac{48Nc_0}{2 \prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x} f_1 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right), \quad (4.88)$$

$$\frac{\hbar^2 \Theta_y^{\text{le}} K_y^2}{2m} = \frac{1}{3} \sum_j \frac{\hbar^2 \Theta_j^{\text{le}} K_j^2}{2m} + \frac{48Nc_0}{2 \prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} f_2 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right), \quad (4.89)$$

$$\begin{aligned} \frac{\hbar^2 \Theta_z^{\text{le}} K_z^2}{2m} &= \frac{1}{3} \sum_j \frac{\hbar^2 \Theta_j^{\text{le}} K_j^2}{2m} - \frac{48Nc_0}{2 \prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x} f_1 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right) \\ &\quad - \frac{48Nc_0}{2 \prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} f_2 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right). \end{aligned} \quad (4.90)$$

These are the coupling equations for the scaling parameters in local equilibrium. I drop the second equation, as it provides no new information. Subtracting the first equation from the third one yields

$$\begin{aligned} \frac{\hbar^2 \Theta_z^{\text{le}} K_z^2}{2m} - \frac{\hbar^2 \Theta_x^{\text{le}} K_x^2}{2m} &= - \frac{48Nc_0}{\prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x} f_1 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right) \\ &\quad - \frac{48Nc_0}{2 \prod_j b_j R_j} \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} f_2 \left( \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x}, \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_y^{\text{le}})^{\frac{1}{2}} K_y} \right). \end{aligned} \quad (4.91)$$

By considering the symmetries of the total energy Eq. (4.86) I can obtain, as discussed in Section 4.2.1, the identity  $\Theta_x^{\text{le}} = \Theta_y^{\text{le}}$ , which also enables us to simplify Eq. (4.91) to

$$\frac{\hbar^2 \Theta_z^{\text{le}} K_z^2}{2m} - \frac{\hbar^2 \Theta_x^{\text{le}} K_x^2}{2m} = - \frac{3}{2} \frac{48Nc_0}{\prod_j b_j R_j} \left\{ -1 + \frac{(2\Theta_x^{\text{le}} K_x^2 + \Theta_z^{\text{le}} K_z^2) f_s \left[ \frac{(\Theta_z^{\text{le}})^{\frac{1}{2}} K_z}{(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x} \right]}{2(\Theta_x^{\text{le}} K_x^2 - \Theta_z^{\text{le}} K_z^2)} \right\}. \quad (4.92)$$

The relations between the scaling parameters can be physically interpreted. Without the dipole-dipole interaction, i.e.  $c_0 = 0$ , the effect of collisions will lead to a spherically symmetric distribution in momentum space which corresponds to an equilibrium with minimized energy. Through the influence of the dipole-dipole interaction, this spherical-symmetric distribution is deformed, however. Note that only the Fock term is responsible for this.

As discussed earlier, the three scaling parameters  $\Theta_i^{\text{le}}$  are coupled via the cylinder-symmetric distribution in momentum space and Eq. (4.92). Hence, there is effectively only one independent scaling parameter in local equilibrium which can be determined using the fact that the collision integral conserves the kinetic energy. The effect of collisions is considered by means of the relaxation time approximation. This method is only consistent with the full collision integral when the approximation has the same conserved quantities. This provides me with a coupling between the distribution function  $f$  and the semiclassical Wigner function in local equilibrium  $f^{\text{le}}$

$$-\frac{1}{\tau} \int d^3x \int \frac{d^3q}{(2\pi\hbar)^3} q_i^2 [f(\mathbf{x}, \mathbf{q}, t) - f^{\text{le}}(\mathbf{x}, \mathbf{q}, t)] = 0. \quad (4.93)$$

By using the scaling ansatz, I can reformulate this equation into the simple form

$$\sum_i \Theta_i^{\text{le}} \langle k_i^2 \rangle^0 = \sum_j \Theta_j \langle k_j^2 \rangle^0. \quad (4.94)$$

This connects the  $\Theta_i$  with the  $\Theta_i^{\text{le}}$ , so that Eqs. (4.80) and Eqs. (4.85) represents now a closed set of equations.

Note that the ansatz Eq. (4.29) ensures that the conservation laws for the momentum and the particle number are consistent with the relaxation time approximation. In the case of the particle conservation, the rescaling can be done without any difficulties, leading to the same term for the local equilibrium and for an arbitrary distribution  $f$ . In the case of the momentum conservation law, I use that the distribution functions are even, so that the integrals vanishes. Hence, the conservation laws for the particle number and the momentum do not provide any new information.

## 4.5 Discussion

In the previous sections, I have shown how to perform the scaling ansatz on the Boltzmann-Vlasov equation. This ansatz assumes a distribution function which has mathematically the same form as the global equilibrium distribution function  $f^0(\mathbf{x}, \mathbf{q})$ . This was derived with a variational ansatz, whose parameters were determined by minimizing the total energy of the system. By taking moments of the space and momentum variables over the Boltzmann-Vlasov equation, it was possible to derive differential equations for the scaling parameters  $b_i$  in Eq. (4.80) and  $\Theta_i$  in Eq. (4.85). The differential equation for  $b_i$  contained integrals from the Hartree and the Fock term which could be solved if the global equilibrium function is known. The  $\Theta$ -equations contained the moment of the collision integral, which was calculated within the relaxation time approximation. The effect of the local equilibrium function was also calculated via the scaling ansatz. The corresponding scaling parameters  $\Theta_i^{\text{le}}$  were determined through a minimization of the time dependent total energy, reflecting the fact that the global equilibrium is only a static local equilibrium.

The effect of collisions is governed by the equations for  $\Theta_i$ , whereas the equations for  $b_i$  are unaffected by them. Therefore, the equations for the scaling parameters  $b_i$  have the same form in all collisional regimes. Hence, the strength and frequency of the collisions affect the equations for  $\Theta_i$ , which relates them to  $b_i$ . Inserting this relation, which is different for different collisional regimes, fixes the differential equations for  $b_i$  and couples them, if necessary.

To discuss this in more detail, I provide the derivation of the equations of motion for the scaling parameters in the limiting cases. In the collisionless regime, all directions are independent from each other and the relaxation time  $\tau$  goes to infinity. Therefore, the differential equations (4.85) for the scaling parameters  $\Theta_i^{\text{cl}}$  have the form

$$\dot{\Theta}_i^{\text{cl}} + 2 \frac{\dot{b}_i^{\text{cl}}}{b_i^{\text{cl}}} \Theta_i^{\text{cl}} = 0, \quad (4.95)$$

which are solved by  $\Theta_i^{\text{cl}} = (b_i^{\text{cl}})^{-2}$ . Inserting this solution in the equations for the scaling parameters  $b_i^{\text{cl}}$  in (4.80) yields the equations of motion in the collisionless regime. Note that then, the normalization factor (4.7) turns out to be  $\Gamma^{\text{cl}} = 1$ .

In the opposite hydrodynamic limit, the system is always in local equilibrium due to the high collision rate. Hence, the scaling parameters  $\Theta_i^{\text{hy}}$  are equal to the scaling parameters in local equilibrium, and the right-hand side of Eqs. (4.85) vanishes. Thus, no dissipation occurs in the

system but, in contrast to the collisionless regime, the different directions are coupled, as the system is in local thermodynamic equilibrium. Therefore, one has to be careful when taking the hydrodynamic limit  $\Theta_i \rightarrow \Theta_i^{\text{hy}} = \Theta_i^{\text{le}}$  on the left-hand side of these equations. A naive substitution  $\Theta_i = \Theta_i^{\text{hy}}$  leads to three different equations which is obviously wrong. To discuss this in detail, consider the case of a spherically symmetric distribution in  $k$ -space. This would be the case with a delta-type interparticle interaction. In such a case all  $\Theta_i^{\text{hy}}$  are equal and, therefore, Eqs. (4.85) must lead to three equivalent equations in the hydrodynamic limit, which is obviously wrong due to the different  $b_i^{\text{hy}}$ 's in the equations.

To take the hydrodynamic limit in the correct way, I sum up all three equations, which leads to

$$\sum_i \left( \frac{\dot{\Theta}_i}{\Theta_i} + 2 \frac{\dot{b}_i}{b_i} \right) = -\frac{1}{\tau} \sum_i \left( 1 - \frac{\Theta_i^{\text{le}}}{\Theta_i} \right). \quad (4.96)$$

The advantage of this equation is that it does not describe an isolated direction. Carrying out the hydrodynamic limit by substituting  $\Theta_i = \Theta_i^{\text{hy}}$  leads to the vanishing of the right-hand side of this equation.

$$\sum_i \left( \frac{\dot{\Theta}_i^{\text{hy}}}{\Theta_i^{\text{hy}}} + 2 \frac{\dot{b}_i^{\text{hy}}}{b_i^{\text{hy}}} \right) = 0 \quad (4.97)$$

The left-hand side of Eq. (4.97) can be reformulated so that a relation to the normalization factor  $\Gamma$ , which is defined in Eq. (4.7), can be obtained

$$\frac{-2\dot{\Gamma}^{\text{hy}}}{\Gamma^{\text{hy}}} = \sum_i \left( \frac{\dot{\Theta}_i^{\text{hy}}}{\Theta_i^{\text{hy}}} + 2 \frac{\dot{b}_i^{\text{hy}}}{b_i^{\text{hy}}} \right). \quad (4.98)$$

Combining these two equations leads to the simple condition

$$\dot{\Gamma}^{\text{le}} = 0. \quad (4.99)$$

Thus,  $\Gamma^{\text{hy}}$  has to be time independent, which means that the form of the particle number conservation is invariant. The value of  $\Gamma^{\text{hy}}$  can be determined, because its value in the limit  $t \rightarrow \infty$  is the stationary solution and therefore equal to one. Thus, we conclude  $\Gamma^{\text{hy}} = 1$ . So the particle number conservation determines the relation between the scaling parameters  $\Theta_i^{\text{hy}}$  and  $b_i^{\text{hy}}$ . Thus, the error by taking the hydrodynamic limit in the Eqs. (4.85) occurred due to the fact that the scaling parameters are not independent of each other.

This concludes the derivation of the theory, which allows me to describe a dipolar Fermi gas within a triaxial trap. The next step would be the numerical analysis of the system in order to derive the low-lying excitation modes or the time-of-flight dynamics. Due to the experimental progress in cooling down dipolar Fermi gases, it is realistic to assume that this theory could soon be compared with experiments.



# 5 Limiting Cases

This chapter provides a concise overview of the limiting cases of the equations for the scaling parameters. At first, I discuss the limiting case of the ballistic regime and compare these equations with Ref. [27], which describe a dipolar Fermi gas in the collisionless regime within a cylinder-symmetric trap. Afterwards, I consider the hydrodynamic limit and compare the specialized equations with the papers [29,30] which describe a dipolar Fermi gas in the hydrodynamic regime within a triaxial trap.

## 5.1 Collisionless Limit

In this section, I will calculate the equations of motion for the scaling parameters in the case of a cylinder-symmetric trap in the collisionless regime. The purpose is to compare the resulting equations of the theory with the equations derived in Ref. [27]. To this end, it will be necessary to calculate the integrals in terms of the anisotropy function.

The scaling parameters derived in Ref. [27] determine a dipolar Fermi gas, as described above. Hence, I specialize the general equations to this case and derive a simpler form of the differential equations for the scaling parameters. The relaxation time in the collisionless regime goes to infinity, which leads to the relation between the scaling parameters  $b_i^{\text{cl}}$  and  $\Theta_i^{\text{cl}}$  as discussed in the previous chapter to  $\Theta_i^{\text{cl}} = (b_i^{\text{cl}})^{-2}$ .

Due to the cylinder symmetry of the system, the equations of motion for  $b_x^{\text{cl}}$  and  $b_y^{\text{cl}}$  are equal. Hence, I only have to consider differential equations for  $b_z^{\text{cl}}$  and for  $b_x^{\text{cl}}$ . Inserting the relation between  $\Theta_x^{\text{cl}} = (b_x^{\text{cl}})^{-2}$  into the equation of motion for  $b_x^{\text{cl}}$  leads to

$$\begin{aligned} \ddot{b}_x^{\text{cl}} = & -\omega_x^2 b_x^{\text{cl}} + \frac{\hbar^2 K_x^2}{m^2 (b_x^{\text{cl}})^3 R_x^2} - \frac{48 N c_0}{m \prod_j b_j^{\text{cl}} R_j b_x^{\text{cl}} R_x^2} \left[ f \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z}, \frac{b_y^{\text{cl}} R_y}{b_z^{\text{cl}} R_z} \right) - \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} f_1 \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z}, \frac{b_y^{\text{cl}} R_y}{b_z^{\text{cl}} R_z} \right) \right. \\ & \left. - f \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x}, \frac{b_y^{\text{cl}} K_z}{b_z^{\text{cl}} K_y} \right) + \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} f_1 \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x}, \frac{b_y^{\text{cl}} K_z}{b_z^{\text{cl}} K_y} \right) \right]. \end{aligned} \quad (5.1)$$

Due to the cylinder symmetry of the system, the anisotropy functions are symmetric in their arguments. Thus, I can rewrite the derivatives of  $f$  as algebraic terms of the anisotropy function, using the identity Eq. (4.48). After inserting this into Eq. (5.1), the equation of motion for the scaling parameter  $b_x^{\text{cl}}$  reads

$$\begin{aligned} \ddot{b}_x^{\text{cl}} = & -\omega_x^2 b_x^{\text{cl}} + \frac{\hbar^2 K_x^2}{m^2 (b_x^{\text{cl}})^3 R_x^2} - \frac{48 N c_0}{m \prod_j b_j^{\text{cl}} R_j b_x^{\text{cl}} R_x^2} \left\{ \frac{2 \left[ 1 - \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right)^2 \right] - 3 \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right)^2 f_s \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right)}{2 \left[ 1 - \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right)^2 \right]} \right. \\ & \left. - \frac{2 \left[ 1 - \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right)^2 \right] - 3 \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right)^2 f_s \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right)}{2 \left[ 1 - \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right)^2 \right]} \right\}. \end{aligned} \quad (5.2)$$

## 5 Limiting Cases

Inserting the expression for the cylinder-symmetric anisotropy function (4.49) into Eq. (5.2) yields

$$\ddot{b}_x^{\text{cl}} + \omega_x^2 b_x^{\text{cl}} - \frac{\hbar^2 K_x^2}{m^2 (b_x^{\text{cl}})^3 R_x^2} + \frac{48Nc_0}{2m \prod_j b_j^{\text{cl}} R_j b_x^{\text{cl}} R_x^2} \left[ d_x \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right) - d_x \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right) \right] = 0, \quad (5.3)$$

where

$$d_x(y) = \frac{2 - 7y^2 - 4y^4 + 9y^4 \text{Artanh} \left( \sqrt{1 - y^2} \right) / \sqrt{1 - y^2}}{(1 - y^2)^2} \quad (5.4)$$

is an abbreviation. An analogous calculation leads to the equation for the scaling parameter  $b_z^{\text{cl}}$

$$\ddot{b}_z^{\text{cl}} + \omega_z^2 b_z^{\text{cl}} - \frac{\hbar^2 K_z^2}{m^2 (b_z^{\text{cl}})^3 R_z^2} + \frac{48Nc_0}{2m \prod_j b_j^{\text{cl}} R_j b_z^{\text{cl}} R_z^2} \left[ d_z \left( \frac{b_x^{\text{cl}} R_x}{b_z^{\text{cl}} R_z} \right) - d_z \left( \frac{b_x^{\text{cl}} K_z}{b_z^{\text{cl}} K_x} \right) \right] = 0 \quad (5.5)$$

with the abbreviation

$$d_z(y) = \frac{1 + 10y^2 - 2y^4 - 9y^2 \text{Artanh} \left( \sqrt{1 - y^2} \right) / \sqrt{1 - y^2}}{2(1 - y^2)^2}. \quad (5.6)$$

Eqs. (5.3) and (5.5) are the equations of motion, which have to be solved to obtain the time evolution of the Fermi gas in the collisionless regime. It is a lengthy calculation to reformulate these equations into the form given by Ref. [27]. I will only sketch the necessary steps. One has to make both dimensionless the time  $t = \frac{t'}{\bar{\omega}}$  and the space variable  $\langle R_x^2 \rangle = N^{\frac{4}{3}} \frac{\hbar^2}{m\bar{\omega}} \langle R_x'^2 \rangle$  and one has to use the stationary equations for the momentum parameters. After doing this, one has to connect the space and momentum parameters  $R_i$  and  $K_i$  with the stationary parameters used in Ref. [27]. These authors used a special ansatz for a cylinder-symmetric Fermi gas, which reads

$$f^0(\mathbf{r}, \mathbf{k}) = \Theta \left[ k_F^2 - \frac{k_x^2}{\alpha} - \alpha^2 k_z^2 - \frac{\lambda^2 m^2 \bar{\omega}^2}{\hbar^2} \left( \beta x^2 + \frac{z^2}{\beta^2} \right) \right], \quad (5.7)$$

so that the number conservation simplifies to

$$N = \frac{k_F^6 \hbar^3}{48\lambda^3 m^3 \bar{\omega}}. \quad (5.8)$$

A comparison between the two approaches for the equilibrium Wigner function shows the relation between their parameters

$$\frac{1}{R_x^2} = \frac{1}{R_y^2} = \frac{\lambda^2 \beta m^2 \bar{\omega}^2}{k_F^2 \hbar^2}, \quad (5.9)$$

$$\frac{1}{R_z^2} = \frac{\lambda^2 m^2 \bar{\omega}^2}{k_F^2 \hbar^2 \beta^2}, \quad (5.10)$$

$$\frac{1}{K_x^2} = \frac{1}{K_y^2} = \frac{1}{\alpha k_F^2}, \quad (5.11)$$

$$\frac{1}{K_z^2} = \frac{\alpha^2}{k_F^2}. \quad (5.12)$$

Inserting all this into the equations of motion for the scaling parameters Eq. (5.3) and Eq. (5.5), and reformulating these equations, leads to

$$\frac{d^2 b_x^{\text{cl}}}{dt'^2} + \frac{\omega_x^2}{\bar{\omega}^2} b_x^{\text{cl}} - \frac{\omega_x^2}{\bar{\omega}^2 (b_x^{\text{cl}})^3} + \frac{N^{\frac{1}{6}} C_{\text{dd}}}{4\pi \langle R_x'^2 \rangle} \left( \frac{m^3 \bar{\omega}}{\hbar^5} \right)^{\frac{1}{2}} \left\{ \frac{q \left[ d_x \left( \frac{b_x^{\text{cl}}}{b_x^{\text{cl}}} \beta^{-\frac{3}{2}} \right) - d_x \left( \frac{b_x^{\text{cl}}}{b_x^{\text{cl}}} \alpha^{-\frac{3}{2}} \right) \right]}{\prod_j b_j^{\text{cl}} b_x^{\text{cl}}} \right. \\ \left. - \frac{q \left[ d_x \left( \beta^{-\frac{3}{2}} \right) - d_x \left( \alpha^{-\frac{3}{2}} \right) \right]}{\prod_j b_j^{\text{cl}} (b_x^{\text{cl}})^3} \right\} = 0, \quad (5.13)$$

$$\frac{d^2 b_z^{\text{cl}}}{dt'^2} + \frac{\omega_z^2}{\bar{\omega}^2} b_z^{\text{cl}} - \frac{\omega_z^2}{\bar{\omega}^2 (b_z^{\text{cl}})^3} + \frac{N^{\frac{1}{6}} C_{\text{dd}}}{4\pi \langle R_z'^2 \rangle} \left( \frac{m^3 \bar{\omega}}{\hbar^5} \right)^{\frac{1}{2}} \left\{ \frac{q \left[ d_z \left( \frac{b_z^{\text{cl}}}{b_z^{\text{cl}}} \beta^{-\frac{3}{2}} \right) - d_z \left( \frac{b_z^{\text{cl}}}{b_z^{\text{cl}}} \alpha^{-\frac{3}{2}} \right) \right]}{\prod_j b_j^{\text{cl}} b_z^{\text{cl}}} \right. \\ \left. - \frac{q \left[ d_z \left( \beta^{-\frac{3}{2}} \right) - d_z \left( \alpha^{-\frac{3}{2}} \right) \right]}{\prod_j b_j^{\text{cl}} (b_z^{\text{cl}})^3} \right\} = 0, \quad (5.14)$$

where  $q = 1024(3\lambda^3)^{\frac{1}{2}} / (2835\pi^2)$  is an abbreviation. These are exactly the equations of motions as given by Ref. [27]. Thus, the theory is able to reproduce this previous solution of the Boltzmann-Vlasov equation in the collisionless regime. The equations of motion are nonlinear differential equations, which can be solved numerically, as discussed in detail in Ref. [27].

## 5.2 Hydrodynamic Limit

In this section, I compare the differential equations for the scaling parameters with the hydrodynamic limit of Refs. [29,30]. To this end, it is not necessary to calculate these equations, as this was already done in Section 4.4. The equations of motion for the scaling parameters  $b_i^{\text{hy}}$  are given by Eq. (4.80), while for the scaling parameter  $\Theta_i^{\text{hy}} = \Theta_i^{\text{le}}$ , they are given with  $\Theta_x^{\text{hy}} = \Theta_y^{\text{hy}}$  and Eq. (4.92). To compare these equations with the corresponding ones given in Refs. [29,30], I have to note that I introduced the time-dependent scaling parameters  $b_i$  and  $\Theta_i$ , which characterize the dynamics of the system relative to the global equilibrium, whereas the global equilibrium itself is described by time-independent variational parameters. In the ansatz of Refs. [29,30], only one sort of parameters appears, which has to contain the information about both the dynamics and the equilibrium. The first step in Refs. [29,30] is that all one-particle orbitals  $\phi_i(\mathbf{x}, t)$  have the same phase. These orbitals determine the correlation functions  $G^{(2n)}$  in Eq. (2.7) and, therefore, the distribution function. This seems to be equivalent to the gauge transformation of the  $q$ -vector, as discussed in Section 4.1, which leads to a vanishing local velocity field. Thus, reversing the scaling ansatz on the local equilibrium function and neglecting this gauge transformation leads to

$$f^0(\mathbf{x}, \mathbf{q}, t) = \Theta \left[ 1 - \sum_i \left( \frac{x_i^2}{b_i^2(t) R_i'^2} - \frac{q_i^2}{\Theta_i^{\frac{1}{2}}(t) K_i'^2} \right) \right]. \quad (5.15)$$

Comparing this with the ansatz made in Refs. [29,30], I can relate my parameters to those authors' parameters  $R_i'$  and  $K_i'$ , yielding

$$R_i' = R_i b_i^{\text{hy}}, \quad (5.16)$$

$$K_i' = K_i (\Theta_i^{\text{hy}})^{\frac{1}{2}}. \quad (5.17)$$

## 5 *Limiting Cases*

With these relations, I can see that Eqs. (4.80) and (4.92) are equal to those given by Refs. [29,30], and that the same cylinder-symmetrical momentum distribution occurs, enabling me to conclude that the hydrodynamic limit is in perfect agreement with this previous investigation.



## 6 Discussion

I derived in this diploma thesis a theory that is capable to describe a dipolar Fermi gas in different collisional regimes. To this end, I introduced in Chapter 2 a quasi phase-space distribution function and described the dynamics of the system in terms of the equation of motion for it. Moreover, a gradient expansion in the derivation of this kinetic equation, called Boltzmann-Vlasov equation, led in the thermodynamic limit to a semiclassical description of the system. The interparticle interaction was calculated by means of perturbation theory leading in first order to the Hartree- and Fock potentials and in second order to the collision integral. By doing this, I used the approximation that the interaction range of the dipole-dipole interaction is smaller than the length scale in which the semiclassical distribution function changes significantly. The experiment has to show whether this approximation is good or not.

In Chapter 3, I derived that the particle number, the momentum coordinates and the kinetic energy are conserved by collisions and found out that a Fermi-Dirac distribution is the limiting function of any relaxation process due to collisions. Afterwards, I linearized the collision integral and discussed that a relaxation time approximation can not be obtained by a linearization.

Then, I rescaled in Chapter 4 the Boltzmann-Vlasov equation in order to obtain differential equations for the scaling parameters, which can be solved numerically. In order to evaluate the remaining integrals in these equations, I had to determine the global equilibrium distribution function in terms of an energy minimization. The local equilibrium distribution function, which describes the limiting function of the collisional relaxation process, was determined by a separate rescaling. The corresponding spatial and momentum scaling parameters were obtained differently. The collisions lead to a certain momentum distribution which could be determined by an energy minimization with respect to the momentum scaling parameters. The spatial parameters could be determined by the fact that the collisions are described as local interactions. Therefore, the collisions do not change the spatial dependence of the distribution function during the relaxation process.

I discussed in Chapter 5 how this theory works and compared its differential equations with previous results in the topic of ultracold dipolar Fermi gases. Thus, it is now possible to solve the ordinary differential equations for the scaling parameters in order to investigate the effect of collisions on the dipolar Fermi gas. At last, I will give a short overview about possibilities for further investigations of dipolar Fermi gases which are not performed in this diploma thesis.

I derived the equations of motion for the scaling parameters in order to describe the time dependent change in the distribution function, but I did not solve these equations. I would have to do this with a vanishing trapping potential in order to describe the time-of-flight dynamics, which should lead to different results in the collisional regimes. Another important application would be the calculation of the low-lying excitation modes and their dependence on the relaxation time, so that the influence of the collisions on this property becomes clear.

The equations of motion for the scaling parameters describe a dipolar Fermi gas at zero temperature. One can use the Sommerfeld expansion to introduce additional terms into the variational ansatz of the global equilibrium distribution function, in order to describe its temperature dependence. This ansatz would be consistent with the derived equations in this diploma thesis and it should be possible to obtain analytic equations, which would not be possible if one introduces the

full Fermi-Dirac distribution with its temperature dependence.

I derived the Boltzmann-Vlasov equation following the approach of Kirkpatrick and Dorfman [45]. In these calculations, I found second-order terms which are neglected during the derivation of the collision integral in the form of Eq. (2.98). It seems reasonable to calculate these terms in order to determine how they change the dynamics of the system. In particular, this would be interesting as it should soon be possible to compare the theoretical results with experimental data.

The relaxation time approximation is still not clearly derived from the collision integral, which leads to an empirical relaxation time with a yet unknown space and momentum dependence. It is therefore necessary to find a self-consistent derivation of the relaxation time approximation, thereby obtaining an explicit expression for the relaxation time.

My derivation of the Boltzmann-Vlasov equation holds only for a dipolar Fermi gas, but could also be applied to a dipolar Bose gas [60]. In this case, one has to deal with two components, a BEC and a thermal cloud. Ref. [60] discusses that taking the average of the Heisenberg equations of motion for the field operator leads to equations for an expectation value of the field operator which describes the Bose-Einstein condensate. The difference between the field operator and its average yields a new operator, which describes the thermal cloud. Using the derivation of Chapter 2, it should then be possible to construct kinetic equations for the BEC describing a Bose-Einstein condensate in different collisional regimes. This would include collisional interactions among particles of the thermal cloud, as well as collisions between the thermal cloud and the BEC.

# A Fourier Transform of Dipole-Dipole Potential

In this appendix, I present the Fourier transformation of the dipole-dipole interaction potential

$$V_{\text{int}}(\mathbf{x}) = C_{\text{dd}} \frac{1 - 3\cos^2\vartheta}{4\pi |\mathbf{x}^3|}, \quad (\text{A.1})$$

where I consider that the dipole moments are aligned along the  $z$ -axis. In a first step, I will derive a relation between the dipole-dipole and the Coulomb potential. Afterwards, I will Fourier transform this to obtain an explicit expression for the Fourier transformed dipole-dipole potential from the Fourier transformed Coulomb potential. The relation between these two potentials is an operator identity. To this end, I integrate two arbitrary  $x$ -differentials of  $1/(4\pi x)$  over a sphere, where  $x$  denotes the norm of  $\mathbf{x}$ :

$$\int_V \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{4\pi x} d^3x = - \int_V \frac{\partial}{\partial x_i} \frac{x_j}{4\pi x^3} d^3x. \quad (\text{A.2})$$

This simplifies with the Gauss' theorem, but there is no divergence on the right-hand side of Eq. (A.2). Therefore, consider a vector field  $\mathbf{B}(\mathbf{x})$  depending explicitly on the vector  $\mathbf{x}$ , which is of the form  $\mathbf{B}(\mathbf{x}) = g(\mathbf{x})\mathbf{a}$ , with a suitable vector  $\mathbf{a}$  and a function  $g(\mathbf{x})$ . The Gauss' theorem for the vector field  $\mathbf{B}(\mathbf{x})$  can then be reformulated in terms of these new quantities

$$\int_V \nabla g(\mathbf{x}) \cdot \mathbf{a} d^3x = \oint_{\partial V} g(\mathbf{x})\mathbf{a} \cdot d\mathbf{A}, \quad (\text{A.3})$$

where  $\partial V$  is the surface of the volume  $V$  and  $d\mathbf{A}$  is the infinitesimal surface vector. The vector  $\mathbf{a}$  can be pulled out of the integral due to its independence from  $\mathbf{x}$ . A comparison of both sides of the equation leads to the vector identity

$$\int_V \nabla g(\mathbf{x}) d^3x = \oint_{\partial V} g(\mathbf{x}) d\mathbf{A}. \quad (\text{A.4})$$

Inserting Eq. (A.4) into Eq. (A.2) yields

$$\int_V \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{4\pi x} d^3x = - \oint_{\partial V} dA_i \frac{x_j}{4\pi x^3}. \quad (\text{A.5})$$

The direction of the infinitesimal surface vector is directed radially away from the sphere. Hence, it is sensible to solve these integrals in spherical coordinates. An exchange of the indices  $i$  and  $j$  does not change the value of the integral making it necessary to compute only 6 integrals

$$- \oint_{\partial V} dA_i \frac{x_j}{4\pi x^3} = \begin{cases} 0 & i \neq j \\ -\frac{1}{3} & i = j \end{cases}. \quad (\text{A.6})$$

Calculating two equal partial derivatives acting on  $1/(4\pi x)$  without the integral, as done in Eq. (A.2), leads to zero, but this is only valid for  $\mathbf{x} \neq 0$ , implying that one has to be aware

## A Fourier Transform of Dipole-Dipole Potential

of a delta function if  $i = j$ . With  $\hat{x}_i$  as the  $i$ -th component of the unit vector, this leads to the operator identity, which can also be found in Ref. [61],

$$-\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{4\pi x} = \frac{\delta_{ij} - 3\hat{x}_i\hat{x}_j}{4\pi x^3} + \frac{1}{3}\delta_{ij}\delta(\mathbf{x}). \quad (\text{A.7})$$

This equation can be Fourier-transformed, which yields

$$F\left[\frac{\delta_{ij} - 3\hat{x}_i\hat{x}_j}{4\pi x^3}\right] = \frac{k_i k_j}{4\pi} F\left[\frac{1}{x}\right] - \frac{1}{3}\delta_{ij}, \quad (\text{A.8})$$

where  $F[\cdot]$  denotes the Fourier transform. Eq. (A.7) shows the relation between the dipole-dipole and the Coulomb potential. The Fourier transform of the latter can be computed with the Schwinger identity [62]

$$\frac{1}{a^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty d\tau \tau^{\nu-1} e^{-a\tau}, \quad (\text{A.9})$$

which can be proven by substituting the integration variable, with  $\Gamma(\nu)$  denoting the Gamma function. Inserting this into the Fourier transform of the Coulomb potential leads to Gaussian integrals

$$\int d^3x \frac{1}{x} e^{i\mathbf{k}\cdot\mathbf{x}} = \int d^3x \frac{1}{\sqrt{\mathbf{x}^2}} e^{i\mathbf{k}\cdot\mathbf{x}} = \int d^3x \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d\tau \tau^{-\frac{1}{2}} e^{-\tau\mathbf{x}^2} e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (\text{A.10})$$

The solution of the integration yields

$$F\left[\frac{1}{x}\right] = \frac{4\pi}{\mathbf{k}^2}. \quad (\text{A.11})$$

Putting everything together leads to the Fourier transform of the dipole-dipole interaction potential. For the case of dipoles directed along the  $z$ -axis, the solution, which is also given in Ref. [63], reads

$$\tilde{V}_{\text{int}}(\mathbf{k}) = \int d^3x V_{\text{int}}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{C_{\text{dd}}}{3} \left( \frac{3k_z^2}{\mathbf{k}^2} - 1 \right). \quad (\text{A.12})$$

# B Computation of the Energy Integrals

In this appendix, I present the integration of the four energy terms Eqs. (4.31)–(4.34) in global equilibrium. This will lead to an explicit expression for the total energy of the system, which depends then only on the variational parameters  $R_i$  and  $K_i$ . The minimal energy determines the suitable parameters for the scaling ansatz.

I will present all important steps of the calculations in the hope that it will be easier to follow for the reader. For the Hartree and Fock integrals, I will consult the famous book by I. S. Gradshteyn and I. M. Ryzhik [59]. All numbers and the formulas of the integrals used from Ref. [59] will be given whenever they appear.

## B.1 Kinetic and Trapping Energy

The kinetic and trapping energy terms can be integrated in the same way. Hence, when one of these terms is calculated, the other term is given, the only difference being that it has other constants. Thus, I will only calculate in detail the kinetic energy Eq. (4.31) by inserting the ansatz Eq. (4.29):

$$E_{\text{kin}} = \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta \left( 1 - \sum_j \frac{x_j^2}{R_j^2} - \sum_j \frac{k_j^2}{K_j^2} \right). \quad (\text{B.1})$$

It is easier to compute at first the space integrals. Due to the spherical symmetry of the integral it is useful to switch to spherical coordinates and to rescale the space variables according to  $x_i = u_i R_i$ , which yields

$$E_{\text{kin}} = \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \bar{R}^3 \int_0^\infty du 4\pi u^2 \Theta \left( 1 - \sum_j \frac{k_j^2}{K_j^2} - u^2 \right). \quad (\text{B.2})$$

The Heaviside step function is 1 in the range  $|u| \leq \sqrt{1 - \sum_j k_j^2/K_j^2}$ , otherwise it will be 0. This leads to

$$E_{\text{kin}} = \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \bar{R}^3 4\pi \Theta \left( 1 - \sum_j \frac{k_j^2}{K_j^2} \right) \int_0^{\sqrt{1 - \sum_j k_j^2/K_j^2}} du u^2. \quad (\text{B.3})$$

The  $u$ -integral can now be performed

$$E_{\text{kin}} = \int \frac{d^3v}{(2\pi)^3} \frac{4\pi}{3} \bar{R}^3 \bar{K}^3 \frac{\hbar^2 \sum_j K_j^2 v_j^2}{2m} (1 - v_x^2 - v_y^2 - v_z^2)^{3/2} \Theta \left( 1 - \sum_j v_j^2 \right), \quad (\text{B.4})$$

where I used the rescaling  $k_i = K_i v_i$ . The three remaining integrals will be computed step by step. At first I have

$$E_{\text{kin}} = \frac{4\pi}{3(2\pi)^3} \bar{R}^3 \bar{K}^3 \frac{\hbar^2}{m} \int dv_x dv_y \int_0^{\sqrt{1 - v_x^2 - v_y^2}} dv_z \left( \sum_j K_j^2 v_j^2 \right) \left( 1 - \sum_j v_j^2 \right)^{3/2} \Theta(1 - v_x^2 - v_y^2). \quad (\text{B.5})$$

## B Computation of the Energy Integrals

After another rescaling  $v_z = \sqrt{1 - v_x^2 - v_y^2} \cos \vartheta$ , I get

$$E_{\text{kin}} = \frac{4\pi}{3(2\pi)^3} \bar{R}^3 \bar{K}^3 \frac{\hbar^2}{m} \int dv_x dv_y (1 - v_x^2 - v_y^2)^2 \Theta(1 - v_x^2 - v_y^2) \times \int_0^{\pi/2} d\vartheta \sin^4 \vartheta [K_x^2 v_x^2 + K_y^2 v_y^2 + K_z^2 (1 - v_x^2 - v_y^2) \cos^2 \vartheta]. \quad (\text{B.6})$$

The  $\vartheta$ -integral can be performed using the standard integrals

$$\int_0^{\pi/2} d\vartheta \sin^4 \vartheta = \frac{3\pi}{16}, \quad (\text{B.7})$$

$$\int_0^{\pi/2} d\vartheta \sin^4 \vartheta \cos^2 \vartheta = \frac{\pi}{32}. \quad (\text{B.8})$$

The other two integrals can be treated in the same way, so that the rest of the calculation can be done analogously. This leads to the kinetic energy

$$E_{\text{kin}} = \frac{N}{8} \sum_j \frac{\hbar^2 K_j^2}{2m}. \quad (\text{B.9})$$

The computation of the trapping term is analogous. Its solution is given by

$$E_{\text{tr}} = \frac{N}{8} \frac{m}{2} \sum_j \omega_j^2 R_j^2. \quad (\text{B.10})$$

## B.2 Hartree Energy

The main idea behind this calculation is to use a Fourier transform trick so that only 3 integrals have to be computed with all functions in the integral Eq. (4.33). This implies decoupling the distribution functions and the interaction potential with respect to their spatial arguments. Thus, I can rewrite the Hartree term by means of the Fourier transform of the potential according to

$$E_d = \frac{1}{2} \int d^3x \int d^3x' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3k''}{(2\pi)^3} e^{i\mathbf{k}'' \cdot (\mathbf{x} - \mathbf{x}')} \tilde{V}_{\text{int}}(\mathbf{k}'') f^0(\mathbf{x}, \mathbf{k}) f^0(\mathbf{x}', \mathbf{k}') \\ = \frac{1}{2} \int \frac{d^3k''}{(2\pi)^3} \tilde{V}_{\text{int}}(\mathbf{k}'') \int \frac{d^3k}{(2\pi)^3} \tilde{f}^0(-\mathbf{k}'', \mathbf{k}) \int \frac{d^3k'}{(2\pi)^3} \tilde{f}^0(\mathbf{k}'', \mathbf{k}'). \quad (\text{B.11})$$

Thus, in a first step I have to compute the Fourier transform of the Wigner function Eq. (4.29)

$$\tilde{f}^0(-\mathbf{k}'', \mathbf{k}) = \int d^3x e^{i\mathbf{k}'' \cdot \mathbf{x}} \Theta \left[ h(\mathbf{k}) - \sum_j \frac{x_j^2}{R_j^2} \right], \quad (\text{B.12})$$

where  $h(\mathbf{k}) = 1 - \sum_j \frac{k_j^2}{K_j^2}$  is a suitable abbreviation. At first, I have

$$\tilde{f}^0(-\mathbf{k}'', \mathbf{k}) = \int dx dy e^{i(k''_x x + k''_y y)} \int_{-\infty}^{\infty} dz \Theta \left[ h(\mathbf{k}) - \sum_j \frac{x_j^2}{R_j^2} \right] (\cos k''_z z + i \underbrace{\sin k''_z z}_{\text{because of symmetry}=0}). \quad (\text{B.13})$$

After using the fact that the Heaviside step function is 1 for  $|z| \leq R_z \sqrt{h(k) - x^2/R_x^2 - y^2/R_y^2}$ , 0 otherwise, as well as using the rescaling  $z = R_z \sqrt{h(k) - x^2/R_x^2 - y^2/R_y^2} \cos\vartheta$  the integral yields

$$\begin{aligned} \tilde{f}^0(-\mathbf{k}'', \mathbf{k}) &= \int dx dy e^{i(k''_x x + k''_y y)} 2 \int_0^{\pi/2} d\vartheta \sin\vartheta \cos \left\{ k''_z R_z \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right]^{\frac{1}{2}} \cos\vartheta \right\} \\ &\quad \times R_z \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right]^{\frac{1}{2}} \Theta \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right]. \end{aligned} \quad (\text{B.14})$$

This can be integrated using the following formula from [59, (3.715.20)]

$$\int_0^{\pi/2} \cos(z \cos x) \sin^{2\nu} x dx = \frac{\sqrt{\pi}}{2} \left( \frac{2}{z} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(z) \quad \text{for } \text{Re } \nu > -\frac{1}{2}, \quad (\text{B.15})$$

where  $J_\nu(z)$  is a Bessel function of first kind. This leads to

$$\begin{aligned} \tilde{f}^0(-\mathbf{k}'', \mathbf{k}) &= \int dx e^{ik''_x x} \frac{\sqrt{2\pi} R_z}{\sqrt{k''_z R_z}} \int_{-\infty}^{\infty} dy (\underbrace{\cos k''_y y + i \sin k''_y y}_{=0}) \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right]^{\frac{1}{4}} \\ &\quad \times \Theta \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right] J_{\frac{1}{2}} \left\{ k''_z R_z \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right]^{\frac{1}{2}} \right\}. \end{aligned} \quad (\text{B.16})$$

This can be simplified by limiting the  $y$ -integral with the help of the Heaviside function und using an analogous substitution for the  $y$ -variable

$$\begin{aligned} \tilde{f}^0(-\mathbf{k}'', \mathbf{k}) &= \int dx e^{ik''_x x} \frac{\sqrt{2\pi} R_z}{\sqrt{k''_z R_z}} 2 \int_0^{\pi/2} du \sin^{\frac{3}{2}} u R_y \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} \right]^{\frac{3}{4}} \Theta \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} \right] \\ &\quad \times J_{\frac{1}{2}} \left\{ k''_z R_z \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} \right]^{\frac{1}{2}} \sin u \right\} \cos \left\{ k''_y R_y \left[ h(\mathbf{k}) - \frac{x^2}{R_x^2} \right]^{\frac{1}{2}} \cos u \right\}. \end{aligned} \quad (\text{B.17})$$

The  $y$ -integral can now be evaluated using [59, (6.688.2)]

$$\int_0^{\pi/2} dx \sin^{\nu+1} x \cos(\beta \cos x) J_\nu(\alpha \sin x) = 2^{-\frac{1}{2}} \sqrt{\pi} \alpha^\nu (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} J_{\nu+\frac{1}{2}} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} \right] \quad \text{for } \text{Re } \nu > -1. \quad (\text{B.18})$$

After computing the  $y$ -integral and simplifying the resulting term, the  $x$ -integral can be treated in a similar way. Hence, the Fourier-transformed distribution function yields

$$\tilde{f}^0(-\mathbf{k}'', \mathbf{k}) = \frac{(2\pi)^{\frac{3}{2}} \bar{R}^3 h(\mathbf{k})^{\frac{3}{4}} \Theta[h(\mathbf{k})]}{(k''_x{}^2 R_x^2 + k''_y{}^2 R_y^2 + k''_z{}^2 R_z^2)^{\frac{3}{4}}} J_{\frac{3}{2}} \left[ h(\mathbf{k})^{\frac{1}{2}} (k''_x{}^2 R_x^2 + k''_y{}^2 R_y^2 + k''_z{}^2 R_z^2)^{\frac{1}{2}} \right]. \quad (\text{B.19})$$

The next step is the calculation of the  $k$ -integral over the Fourier-transformed Wigner function in Eq. (B.11). This is an even function so that the  $k'$ -integral is exactly the same as the  $k$ -integral. Hence, it is only necessary to compute one of them:

$$\begin{aligned} \int \frac{d^3 k}{(2\pi)^3} \tilde{f}^0(\mathbf{k}'', \mathbf{k}) &= \frac{\bar{R}^3}{(2\pi)^{\frac{3}{2}} (k''_x{}^2 R_x^2 + k''_y{}^2 R_y^2 + k''_z{}^2 R_z^2)^{\frac{3}{4}}} \int d^3 k \left( 1 - \sum_j \frac{k_j^2}{K_j^2} \right)^{\frac{3}{4}} \Theta \left( 1 - \sum_j \frac{k_j^2}{K_j^2} \right) \\ &\quad \times J_{\frac{3}{2}} \left[ \left( 1 - \sum_j \frac{k_j^2}{K_j^2} \right)^{\frac{1}{2}} (k''_x{}^2 R_x^2 + k''_y{}^2 R_y^2 + k''_z{}^2 R_z^2)^{\frac{1}{2}} \right]. \end{aligned} \quad (\text{B.20})$$

## B Computation of the Energy Integrals

To calculate this integral, I perform the substitution  $k_i = K_i u_i$  and use the spherical symmetry of the integrand. Then, the integral yields

$$\int \frac{d^3 k}{(2\pi)^3} \tilde{f}^0(\mathbf{k}'', \mathbf{k}) = \frac{\bar{R}^3 \bar{K}^3}{(2\pi)^{\frac{3}{2}} (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^{\frac{3}{4}}} 4\pi \int_0^1 du u^2 (1-u^2)^{\frac{3}{4}} \times J_{\frac{3}{2}} \left[ (1-u^2)^{\frac{1}{2}} (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^{\frac{1}{2}} \right] \quad (\text{B.21})$$

This integral can be calculated after substituting  $u = \cos\vartheta$  and using [59, (6.683)]

$$\int_0^{\pi/2} d\vartheta J_\mu(a \sin\vartheta) \sin^{\mu+1}\vartheta \cos^{2\rho+1}\vartheta = 2^\rho \Gamma(\rho+1) a^{-\rho-1} J_{\rho+\mu+1}(a) \quad \text{for } \text{Re } \rho, \text{Re } \mu > -1. \quad (\text{B.22})$$

Hence, the integral over the Fourier transformed Wigner function yields

$$\int \frac{d^3 k}{(2\pi)^3} \tilde{f}^0(\mathbf{k}'', \mathbf{k}) = \frac{4\sqrt{\pi}\pi \bar{R}^3 \bar{K}^3}{(2\pi)^{\frac{3}{2}} \sqrt{2} (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^{\frac{3}{2}}} J_3 \left[ (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^{\frac{1}{2}} \right] = \tilde{n}^0(\mathbf{k}''). \quad (\text{B.23})$$

With the previous calculations, I am now able to solve the last integral for the Hartree term. With the Fourier transformed dipole-dipole interaction potential, I get

$$E_d = \int \frac{d^3 k''}{(2\pi)^3} \frac{C_{dd}}{6} \left( \frac{3k_z''}{k''^2} - 1 \right) \frac{16\pi^2 \bar{R}^6 \bar{K}^6}{(2\pi)^3 (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^3} \frac{\pi}{2} J_3^2 \left[ (k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2)^{\frac{1}{2}} \right]. \quad (\text{B.24})$$

After substituting  $k_i'' R_i = u_i$  and switching into spherical coordinates, the integral yields

$$E_d = \frac{C_{dd} 8\pi^3 \bar{R}^3 \bar{K}^6}{6(2\pi)^6} \int_0^\pi d\vartheta \int_0^{2\pi} d\phi \sin\vartheta \times \left( \frac{3\cos^2\vartheta}{(R_z/R_x)^2 \cos^2\phi \sin^2\vartheta + (R_z/R_y)^2 \sin^2\phi \sin^2\vartheta + \cos^2\vartheta} - 1 \right) \int_0^\infty du u^{-4} J_3^2(u). \quad (\text{B.25})$$

With the definition of the anisotropy function Eq. (4.37) and [59, (6.574.2)]

$$\int_0^\infty dt J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)} \quad \text{for } \text{Re}(\nu+\mu+1) > \text{Re}(\lambda) > 0, \alpha > 0. \quad (\text{B.26})$$

the Hartree energy  $E_d$  yields

$$E_d = \frac{-48N^2 c_0}{8\bar{R}^3} f\left(\frac{R_x}{R_z}, \frac{R_y}{R_z}\right). \quad (\text{B.27})$$

with  $c_0 = 2^{10} C_{dd} / (\pi^3 \cdot 3^4 \cdot 5 \cdot 7)$ .

## B.3 Fock Energy

The Fock term represents the most complicated integral of these four energy terms. Nevertheless, it is possible to compute it as well by using Fourier transforms. The trick is to switch the space



and momentum variables of the Wigner functions via two Fourier transforms. Hence, the Fock term can be rewritten in the following form

$$\begin{aligned} E_{\text{ex}} &= -\frac{1}{2} \int d^3x \int d^3x' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} V_{\text{int}}(\mathbf{x}) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} f^0(\mathbf{x}, \mathbf{k}) f^0(\mathbf{x}, \mathbf{k}'), \\ &= -\frac{1}{2} \int d^3x' \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3k''}{(2\pi)^3} \bar{f}^0(\mathbf{k}'', \mathbf{x}') \bar{f}^0(-\mathbf{k}'', -\mathbf{x}') \tilde{V}_{\text{int}}(\mathbf{k}'') e^{i\mathbf{x}'\cdot\mathbf{k}'}, \end{aligned} \quad (\text{B.28})$$

where  $\bar{f}^0(\mathbf{k}', \mathbf{x})$  denotes the Fourier transform of  $f^0(\mathbf{x}, \mathbf{k})$  with respect to the first variable and  $\tilde{f}^0(\mathbf{x}, \mathbf{x}')$  the Fourier transformation with respect to the second variable. The order of the calculation of the integrals is determined by the interaction potential. I assume that the integral over the interaction potential will lead to the anisotropy function exactly as in the calculation of the Hartree term. Thus, I will solve this integral at last, trying to perform at first the other integrals over the Fourier transformed Wigner functions. Hence, the first step is to calculate these two Fourier transforms of the Wigner function. I have already performed the first of these two transforms in the calculations of the Hartree term, where the result is given by Eq. (B.19). Using this result, I get

$$\begin{aligned} \bar{f}^0(-\mathbf{k}'', \mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}^0(-\mathbf{k}'', \mathbf{k}), \\ &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\bar{R}^3 \Theta\left(1 - \sum_j \frac{k_j^2}{K_j^2}\right)^{\frac{3}{4}}}{g(\mathbf{k}'')^{\frac{3}{4}}} \left(1 - \sum_l \frac{k_l^2}{K_l^2}\right)^{\frac{3}{4}} J_{\frac{3}{2}} \left[ \left(1 - \sum_m \frac{k_m^2}{K_m^2}\right)^{\frac{1}{2}} g(\mathbf{k}'')^{\frac{1}{2}} \right], \end{aligned} \quad (\text{B.29})$$

where  $g(\mathbf{k}'') = k_x''^2 R_x^2 + k_y''^2 R_y^2 + k_z''^2 R_z^2$  is an abbreviation. The three  $k$ -integrals will be treated separately, yet all in the same way. Hence, it is only necessary to compute one of them. With the use of the substitution  $k_z = K_z \sqrt{1 - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2} \cos \vartheta}$ , I can rewrite Eq. (B.30) as

$$\begin{aligned} \bar{f}^0(-\mathbf{k}'', \mathbf{x}) &= \frac{\bar{R}^3}{(2\pi)^{\frac{3}{2}}} \frac{1}{g(\mathbf{k}'')^{\frac{3}{4}}} \int dk_x dk_y e^{ixk_x + iyk_y} \Theta\left(1 - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2}\right) \left(1 - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2}\right)^{\frac{5}{4}} \\ &\times 2 \int_0^{\frac{\pi}{2}} d\vartheta \sin^{\frac{5}{2}} \vartheta K_z \cos\left(z K_z \sqrt{1 - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2} \cos \vartheta}\right) J_{\frac{3}{2}} \left[ g(\mathbf{k}'')^{\frac{1}{2}} \left(1 - \frac{k_x^2}{K_x^2} - \frac{k_y^2}{K_y^2}\right)^{\frac{1}{2}} \cos \vartheta \right]. \end{aligned} \quad (\text{B.31})$$

After this substitution, the  $\vartheta$ -integral can be calculated using Eq. (B.18). The other two  $k$ -integrals can be treated in the same way. Then, the Fourier transform reads

$$\bar{f}^0(-\mathbf{k}'', \mathbf{x}) = \frac{\bar{R}^3 \bar{K}^3}{[g(\mathbf{k}'') + z^2 K_z^2 + y^2 K_y^2 + x^2 K_x^2]^{\frac{3}{2}}} J_3 \left\{ [g(\mathbf{k}'') + z^2 K_z^2 + y^2 K_y^2 + x^2 K_x^2]^{\frac{1}{2}} \right\}. \quad (\text{B.32})$$

It is clear that  $\bar{f}^0(\mathbf{k}'', \mathbf{x})$  is an even function, which simplifies further calculations. The next step is to calculate the  $\mathbf{x}'$ -integral in Eq. (B.28). To avoid a quadratic Bessel function, I use the integral representation [59, (6.519.2.2)]

$$\int_0^{\frac{\pi}{2}} J_{2\nu}(2z \sin t) dt = \frac{\pi}{2} J_{\nu}^2(z) \quad \text{for} \quad \text{Re } \nu > -\frac{1}{2}, \quad (\text{B.33})$$

## B Computation of the Energy Integrals

which leads to an integral over a Bessel function

$$J_3^2 \left\{ [x^2 K_x^2 + y^2 K_y^2 + z^2 K_z^2 + g(\mathbf{k}'')]^{\frac{1}{2}} \right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} dt J_6 \left\{ 2 \sin t [x^2 K_x^2 + y^2 K_y^2 + z^2 K_z^2 + g(\mathbf{k}'')]^{\frac{1}{2}} \right\}. \quad (\text{B.34})$$

I will treat the three integrals separately, starting with the  $z$ -integral. After the substitution  $u_z = zK_z$ , I can use [59, (6.726.2)]

$$\begin{aligned} \int_0^\infty (x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu \left( a\sqrt{x^2 + b^2} \right) \cos(cx) dx &= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} (a^2 - c^2)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}} \left( b\sqrt{a^2 - c^2} \right) \\ &\quad \text{for } 0 < c < a, b > 0, \text{Re } \nu > -\frac{1}{2}, \\ &= 0 \quad \text{for } 0 < a < c, b > 0, \text{Re } \nu > -\frac{1}{2}, \end{aligned} \quad (\text{B.35})$$

to obtain

$$\begin{aligned} \int d^3 x' \bar{f}^0(\mathbf{k}'', \mathbf{x}')^2 e^{i\mathbf{k}' \cdot \mathbf{x}'} &= \bar{R}^6 \bar{K}^6 \int dx dy e^{ix'k'_x + iy'k'_y} \frac{4}{\pi K_z} \int_0^{\frac{\pi}{2}} dt \frac{\left( 4 \sin^2 t - \frac{k_z'^2}{K_z^2} \right)^{\frac{11}{4}} \Theta \left( 2 \sin t - \sqrt{\frac{k_z'^2}{K_z^2}} \right)}{(2 \sin t)^6 [x'^2 K_x^2 + y^2 K_y^2 + g(\mathbf{k}'')]^{\frac{11}{4}}} \\ &\quad \times \sqrt{\frac{\pi}{2}} J_{\frac{11}{2}} \left\{ \left( 4 \sin^2 t - \frac{k_z'^2}{K_z^2} \right)^{\frac{1}{2}} [x'^2 K_x^2 + y^2 K_y^2 + g(\mathbf{k}'')]^{\frac{1}{2}} \right\}. \end{aligned} \quad (\text{B.36})$$

The Heaviside function in the last equation ensures that both possible solutions of Eq. (B.35) are covered. The other two integrals will be calculated in the same way. The solution of the  $x'$ -integral reads

$$\begin{aligned} \int d^3 x' \bar{f}^0(\mathbf{k}'', \mathbf{x}')^2 e^{i\mathbf{k}' \cdot \mathbf{x}'} &= \frac{2(2\pi)^{\frac{3}{2}}}{\pi} \bar{R}^6 \bar{K}^3 \int_0^{\frac{\pi}{2}} dt \frac{\left( 4 \sin^2 t - \frac{k_z'^2}{K_z^2} - \frac{k_y'^2}{K_y^2} - \frac{k_x'^2}{K_x^2} \right)^{\frac{9}{4}}}{(2 \sin t)^6 g(\mathbf{k}'')^{\frac{9}{4}}} \\ &\quad \times J_{\frac{9}{2}} \left[ g(\mathbf{k}'')^{\frac{1}{2}} \left( 4 \sin^2 t - \frac{k_z'^2}{K_z^2} - \frac{k_y'^2}{K_y^2} - \frac{k_x'^2}{K_x^2} \right)^{\frac{1}{2}} \right] \Theta \left( 2 \sin t - \sqrt{\frac{k_z'^2}{K_z^2} + \frac{k_y'^2}{K_y^2} + \frac{k_x'^2}{K_x^2}} \right). \end{aligned} \quad (\text{B.37})$$

The next step is to integrate the  $k''$ -integral. Using spherical symmetry, I can make the calculation of this integral relatively short. Substituting  $u_i = k_i'' R_i$  and then transforming these new integration variables into spherical coordinates, I can use [59, (6.561.17)]

$$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}} dx = \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)}{2^{\nu-q} a^{\nu-q+1} \Gamma\left(\nu - \frac{1}{2}q + \frac{1}{2}\right)} \quad \text{for } -1 < \text{Re } q < \text{Re } \nu - \frac{1}{2}, \quad (\text{B.38})$$

which leads to

$$\begin{aligned} \int d^3 k'' \int d^3 x' \bar{f}^0(\mathbf{k}'', \mathbf{x}')^2 e^{i\mathbf{k}' \cdot \mathbf{x}'} &= \frac{8(2\pi)^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \bar{R}^3 \bar{K}^3}{\Gamma(4) 2^{\frac{5}{2}}} \int_0^{\frac{\pi}{2}} dt \frac{\left( 4 \sin^2 t - \frac{k_z'^2}{K_z^2} - \frac{k_y'^2}{K_y^2} - \frac{k_x'^2}{K_x^2} \right)^3}{(2 \sin t)^6} \\ &\quad \times \Theta \left( 2 \sin t - \sqrt{\frac{k_z'^2}{K_z^2} + \frac{k_y'^2}{K_y^2} + \frac{k_x'^2}{K_x^2}} \right). \end{aligned} \quad (\text{B.39})$$

The last step of the calculation of the Fock term is to solve the  $k'$ -integral. Therefore, I substitute  $u_i = k_i' / K_i'$  and afterwards switch to spherical coordinates. The Fourier transform of the

interaction potential is given by Eq. (A.12). Inserting all this, the Fock term yields

$$\begin{aligned}
 E_{\text{ex}} = & - \frac{4C_{\text{dd}}(2\pi)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)}{3 \cdot 2^{\frac{5}{2}}\Gamma(4)} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \left( \frac{3 \cos^2 \vartheta}{\frac{K_x^2}{K_z^2} \sin^2 \vartheta \cos^2 \phi + \frac{K_y^2}{K_z^2} \sin^2 \vartheta \sin^2 \phi + \cos^2 \vartheta} - 1 \right) \\
 & \times \overline{K}^6 \overline{R}^3 \int_0^{\frac{\pi}{2}} \frac{dt}{(2 \sin t)^6} \int_0^{2 \sin t} du u^2 (4 \sin^2 t - u^2)^3.
 \end{aligned} \tag{B.40}$$

The  $\vartheta$ - and  $\phi$ -integrals will lead to the anisotropy function, which is defined by Eq. (4.37), and the  $u$ - and  $t$ -integrals can be solved without any difficulties. With the abbreviation  $c_0 = 2^{10}C_{\text{dd}}/(3^4 \cdot 5 \cdot 7 \cdot \pi^3)$ , the solution of the Fock term reads

$$E_{\text{ex}} = \frac{48N^2c_0}{8\overline{R}^3} f\left(\frac{K_z}{K_x}, \frac{K_z}{K_y}\right). \tag{B.41}$$



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