Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 143–238

Hausdorff Convergence of Riemannian Manifolds and Its Applications

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Received June 30, 1989.
This work is partially supported by Inoue foundation and Grant-in-Aid for Scientific Research (No.63740014).

Preface

This is a survey article on the theory of Hausdorff convergence.

Classically, the Hausdorff distance between two closed subsets in a fixed metric space had been defined. In [G7], Gromov gave an abstract version of it. Namely he defined the Hausdorff distance between two (abstract) metric spaces by taking the infimum of the (relative) Hausdorff distances over all ambient spaces into which the two metric spaces are embedded by isometries.[1] There he proved two fundamental results, the precompactness theorem (Theorem 2.2) and the convergence theorem (Theorems 3.2 and 4.1), and applied them to the study of the global structures of Riemannian manifolds. Global Riemannian geometry has long history and fruitful products. Yet, before Gromov, workers of this field do not have so many machineries that the proofs should be done by hand. Hausdorff convergence provides a powerful tool and gives us conceptional and simple arguments which can be applied to many problems in a uniform way. At first sight, its definition looks too naive to be useful. But, the simplicity of the definition makes it possible to be applied in various situations.

After [G7], the studies of Hausdorff convergence have been pursued by many mathematicians and now we have clearer image. Especially, the case of bounded sectional curvature and diameter is well analyzed. We shall discuss mainly this case in Chapters 1 and 2. In section 1, we define Hausdorff convergence and prove its elementary properties. In section 2, We prove the precompactness theorem. (This theorem holds under a weaker assumption on curvature.) In sections 3 and 4, we study the Hausdorff convergence of Riemannian manifolds whose curvatures and diameters are uniformly bounded and whose volumes are uniformly bounded away from 0. In section 5, we discuss the approximation of such Riemannian metrics by those with uniform bound on higher derivatives of the curvature. In section 6, we discuss some generalizations of Hausdorff convergence which we need in Chapter 2 and in the study of noncompact manifolds.

In Chapter 2 we remove the assumption that the volume is uniformly bounded away from 0 and studies collapsing Riemannian manifolds.[2] In section 7, we discuss pseudo fundamental group. This notion is tedious but is useful to understand the local structure of Riemannian manifolds with bounded sectional curvature. Sections 8 and 9 are devoted

[1] The precise definition is in section 1, where we give a little different but equivalent definition.

[2] The definition is given in 10.1.

to the theorem of almost flat manifold due to Gromov. It describes the structure of Riemannian manifold with bounded sectional curvature and small diameter. This is the opposite extreme to the situations in sections 3 and 4. The later concerns the structure of manifolds whose ϵ balls are trivial. These two results are jointed in sections 10,11,12. In section 10 we give various examples of Riemannian manifolds with bounded sectional curvature and small volume. In sections 11 and 12 we prove that the examples in section 10 cover essentially all possible types of such collapsed manifolds. In section 13 we study the structure of the pseudo fundamental group. The first two chapters include proofs or sketches of proofs to most of the results.

Chapter 3 is devoted to the applications and to the results without assuming sectional curvature bound or the diameter bound. As for applications, there are, among others, three types of them, the finiteness theorems, the pinching theorems and the study of ends of complete manifolds. In section 14 and partially in section 16, we discuss the first one, which asserts the finiteness of the number of diffeomorphism classes (homeomorphism classes or homotopy types) containing Riemannian manifolds satisfying given assumptions on curvatures e.t.c. In section 15 and partially in section 16, we discuss the results of the second type, which assert that the two Riemannian manifolds are diffeomorphic (homeomorphic, homotopy equivalent) if they have similar geometric properties. The third one is in section 18.

The generalizations of the results of Chapters 1 and 2 have two directions. One is to relax the assumption on curvatures. We describe some of the researches toward this direction in section 14 (and a bit in section 15.) The other is to remove the assumption on the diameter. The notion of pointed Hausdorff convergence[3] is essential for this purpose. We give various examples and results of this direction in sections 17,18 and 19.

The results on the topics we deal with in Chapter 3 are far from complete. Then, the author tries to give as many examples, conjectures and open problems as possible, rather than to give the proofs of the theorems. We give a sketch of the proof in the case when the argument is simple and typical or when it seems to take a lot of time for the reader to find and comprehend the proof in the literature.

In this article, we discuss various results on global Riemannian geometry. But we treat them from the point of view of Hausdorff convergence. So we can not mention many results which are interesting for its own sake and/or from the point of view of Riemannian geometry itself.

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[3]The definition is in section 6.

We do not mention either some applications of Hausdorff convergence, (those presented in [CG1], [F3], [G7], [KOS] etc.).

This article is of expository nature. But the reader may find some results and examples which are not recorded in the literature. (For example Proposition 11.5, Theorem 13.1.) The part presented in section 8 of the the proof of Theorem 8.1 is new.

We restrict the references to those quoted in the article and then omit many important ones. As for the papers on global Riemannian geometry, the reference in [Sa2] is more complete.

Chapter I. Hausdorff Convergence

§1. Definition and elementary properties

Of course, we are mainly interested in manifolds. But sometimes it is useful to work in wider classes of spaces (as we use weak solusion in the study of partial differential equations). Hence we define our Hausdorff convergence on the class of all compact metric spaces.

Definition 1.1 (Gromov [G7]). Let $\mathcal{MET}$ denote the set of all isometry classes of compact metric spaces. Let X and Y be two elements of $\mathcal{MET}$. A (not necessary continuous) map $\varphi: X \to Y$ is said to be an *ϵ-Hausdorff approximation* if the following two conditions are satisfied.

(1.1.1) The ϵ-neighborhood of $\varphi(X)$ in Y is equal to Y.
(1.1.2) For each x,y in X, we have

$$|d(x,y) - d(\varphi(x),\varphi(y))| < \epsilon$$

The *Hausdorff distance*, $d_H(X,Y)$, between X and Y is defined to be the infinimum of the positive numbers ϵ such that there exist ϵ-Hausdorff approximations from X to Y and from Y to X.

Exercise 1.2.

(1.2.1) Suppose that there exists an ϵ-Hausdorff approximation from X to Y. Show

$$d_H(X,Y) < 3\epsilon.$$

(1.2.2) Let $X, Y, Z \in \mathcal{MET}$. Show

$$d_H(X,Z) < 2\{d_H(X,Y) + d_H(Y,Z)\}.$$

Unfortunately $d_H(\cdot,\cdot)$ does not satisfy triangle inequality. But (1.2.2) above shows that it gives a metrizable uniform structure on $\mathcal{MET}$. Then we treat $d_H(\cdot,\cdot)$ as if it is a distance function.

Remark 1.3. $\lim_{n\to\infty} d_H(X_n, X) = 0$ does *not* imply that X_n and X are homeomorphic for large n. For example

$$\lim_{n\to\infty} d_H((M, g_M/n), \text{point}) = 0$$

for every compact Riemannian manifold (M, g_M).

Remark 1.4. The above definition is slightly different from one Gromov introduced in [G7]. It is easy to see that they are equivalent.

Theorem 1.5. *$(\mathcal{MET}, d_H)$ is Hausdorff and complete.*

Proof. First we show that $(\mathcal{MET}, d_H)$ is Hausdorff. Let $X, Y \in \mathcal{MET}$. Assume that $d_H(X, Y) = 0$. We shall prove that X and Y are isometric. By assumption, there exists a $1/n$-Hausdorff approximation $\varphi_n: X \to Y$ for each n. By the compactness of X, we can find a dense countable subset $\{a_m \mid m \in \mathbf{N}\}$ of X. By a standard diagonal procedure we may assume that $\lim_{n\to\infty} \varphi_n(a_m)$ converges to an element $\varphi(a_m)$ of M for each m. It is easy to see that φ can be extended to an isometry from X to Y.

Next we shall prove the completeness. We remark that $(\mathcal{MET}, d_H)$ is separable, because the set $\{X \in \mathcal{MET} \mid \sharp X < \infty\}$[4] is dense in $\mathcal{MET}$. Hence it suffices to consider only Caucy sequences X_n. By taking a subsequence if necessary, we may assume that $d_H(X_n, X_{n+1}) < 4^{-n-2}$. Let $\varphi_{n,n+1}: X_n \to X_{n+1}$ be the 4^{-n-1}-Hausdorff approximation. For $n < m$, we put $\varphi_{n,m} = \varphi_{m-1,m} \circ \cdots \circ \varphi_{n,n+1}$. Then, $\varphi_{n,m}$ is a 2^{-n}-Hausdorff approximation. Let X_∞ be the inductive limit, $\varinjlim X_n$, and X be its completion. Then, it is easy to see that X is a compact metric space and that $\lim_{n\to\infty} d_H(X_n, X) = 0$.

We use also the following finer distance on $\mathcal{MET}$.

Definition 1.6. Let $X, Y \in \mathcal{MET}$ and $f: X \to Y$ be a Lipschitz map. We put

$$\text{dil } f = \sup_{x,y\in X} \frac{d(f(x), f(y))}{d(x,y)}.$$

We define the Lipschitz distance, $d_L(X, Y)$, between X and Y by

$$\exp(d_L(X,Y)) = \inf\{\max\{\text{dil } f, \text{dil } f^{-1}\} \mid f: X \to Y, \text{ is a Lipschitz homeomorphism}\}.$$

[4] $\sharp X$ denotes the order of X.

($d_L(X,Y) < \infty$ if and only if X and Y are Lipschitz homeomorphic.)

Exercise 1.7. Let $X_n, X \in \mathcal{MET}$ satisfy $\lim_{n\to\infty} d_L(X_n, X) = 0$. Show $\lim_{n\to\infty} d_H(X_n, X) = 0$.

Lemma 1.8. *Let $X_n, Y_n, X, Y \in \mathcal{MET}$. We assume*

$$\lim_{n\to\infty} d_H(X_n, X) = \lim_{n\to\infty} d_H(Y_n, Y) = 0.$$

Then

$$d_L(X,Y) \le \liminf_{n\to\infty} d_L(X_n, Y_n).$$

Proof. Put $\mu = \liminf_{n\to\infty} d_L(X_n, Y_n)$. By taking a subsequence if necessary, we can find a Lipschitz homeomorphisms $f_n: X_n \to Y_n$ such that $\operatorname{dil} f_n, \operatorname{dil} f_n^{-1} < e^\mu + 1/n$. Let $\varphi_n: X \to X_n, \psi_n: Y_n \to Y$ be ϵ_n-Hausdorff approximations. (Here $\lim \epsilon_n = 0$). Put $F_n = \psi_n \circ f_n \circ \varphi_n: X \to Y$. Take a dense countable subset X_0 of X. By a standard diagonal procedure, we may assume that $\lim_{n\to\infty} F_n(x)$ converges to an element $F(x)$ of Y for each $x \in X_0$. It is easy to see that F can be extended to a Lipschitz homeomorphism $F: X \to Y$ such that $\operatorname{dil} F, \operatorname{dil} F^{-1} \le e^\mu$. Therefore $d_L(X,Y) \le \mu$, as required.

Finally, we remark that we can work in a smaller class of metric spaces than $\mathcal{MET}$.

Definition 1.9. Let X be a metric space, $\ell: [0,T] \to X$ be a continuous map. We say that ℓ is a *minimal geodesic* if $d(\ell(t_1), \ell(t_2)) = |t_1 - t_2|$ holds for each $0 \le t_1 \le t_2 \le T$. We say that X is a *length space* if, for each $p, q \in X$, there exists a minimal geodesic joining them.

Proposition 1.10. *Let $X_i, X \in \mathcal{MET}$. Assume that X_i, $i =$ 1,2,3, ... are length spaces and that $\lim_{i\to\infty} d_H(X_i, X) = 0$. Then X is also a length space.*

The proof is a straightforward application of Ascoli-Arzera's Theorem.

§2. Precompactness theorem

In this section, we shall deal with the following class of Riemannian manifolds.

Definition 2.1. We define the symbol $\mathcal{S}(n,D)$ by

$$\mathcal{S}(n,D)=\left\{ M \;\middle|\; \begin{array}{l} M \text{ is a compact Riemannian manifold.} \\ \dim M = n, \ \ \mathrm{Diam}\, M \leq D \\ \mathrm{Ricci}_M > -(n-1) \end{array} \right\},$$

where dim denotes the dimension, Diam the diameter, and Ricci the Ricci curvature.

The main result of this section is the following :

Theorem 2.2 (Gromov [G7] 5.3). *The set $\mathcal{S}(n,D)$ is precompact in $\mathcal{MET}$ with respect to the Hausdorff distance.*

The proof of Theorem 2.2 is divided into the proofs of the following two Lemmas 2.4 and 2.5.

Definition 2.3. Let $N\colon[0,\infty)\to\mathbf{N}$ be a function. The symbol $\mathcal{MET}(N(\cdot),D)$ denotes the set of all elements X of $\mathcal{MET}$ satisfying the following. For each ϵ, there exists a finite subset Z of X such that the ϵ-neighborhood of Z in X is equal to X and that the order of Z is smaller than $N(\epsilon)$.

Lemma 2.4. *For each $N(\cdot)$ the set $\mathcal{MET}(N(\cdot),D)$ is precompact in $\mathcal{MET}$.*

Lemma 2.5. *We put*

$$N_D(\epsilon)=\frac{\int_0^D \sinh^{n-1} t\,dt}{\int_0^{\epsilon/2} \sinh^{n-1} t\,dt}.$$

Then we have

$$\mathcal{S}(n,D)\subseteq \mathcal{MET}(N_D(\cdot),D)$$

Proof of Lemma 2.4. We shall prove that $\mathcal{S}(n,D)$ is totally bounded. Let $\mathcal{FIN}(N,D)$ denote the set of all elements X of $\mathcal{MET}$ such that the order of X is smaller than N and that the diameter of X is smaller than D. (The elements of $\mathcal{FIN}(N,D)$ are finite sets.)

Sublemma 2.6. *$\mathcal{FIN}(N,D)$ is compact.*

The proof is an easy exercise.

Sublemma 2.7. *For each $X \in \mathcal{MET}(N(\cdot), D)$ and $\epsilon > 0$, there exists $Y \in \mathcal{FIN}(N(\epsilon), D)$ such that $d_H(X, Y) < 3\epsilon$.*

Proof of Sublemma 2.7. By definition, there exists $Y \subset X$, such that $\sharp Y \leq N(\epsilon)$, and that the ϵ-neighborhood of Y is X. Then Y equipped with induced metric is an element of $\mathcal{FIN}(N(\epsilon), D)$, and the inclusion map : $Y \hookrightarrow X$ is an ϵ-Hausdorff approximation. Hence by Exercise (1.2.2), we have $d_H(X, Y) < 3\epsilon$, as required.

It follows immediately from Sublemmas 2.6 and 2.7 that $\mathcal{MET}(N(\cdot), D)$ is totally bounded. Therefore, Theorem 1.5 and the closedness of $\mathcal{MET}(N(\cdot), D)$ imply that $\mathcal{MET}(N(\cdot), D)$ is compact, as required.

For the proof of Lemma 2.4 we need the following theorem. Here and hereafter, we put $B_r(p, M) = \{x \in M | d(p, x) < r\}$.

Theorem 2.8 (Bishop [BC], Cheeger, Gromov [G7] 5.3 bis). *Let X_C be the simply connected n-dimensional Riemannian manifold with $K_{X_C} \equiv C$. Then, for any n-dimensional Riemannian manifold M satisfying* $\mathrm{Ricci}_M \geq C \cdot (n-1)$ *we have*

$$\frac{\mathrm{Vol}(B_{r'}(p, M))}{\mathrm{Vol}(B_r(p, M))} \leq \frac{\mathrm{Vol}(B_{r'}(p_0, X_C))}{\mathrm{Vol}(B_r(p_0, X_C))},$$

where $r' > r > 0$.

The proof of Theorem 2.8 is in [Sa2] p.144.

Proof of Lemma 2.5. We let M be an arbitrary element of $\mathcal{S}(n, D)$. Theorem 2.8 and $B_D(p, M) = M$ imply

$$\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(B_{\epsilon/2}(p, M))} \leq N(\epsilon) \tag{2.9}$$

for each $p \in M$, and $0 < \epsilon/2 < D$.

Now let M be an arbitrary element of $\mathcal{S}(n, D)$. We put

$$\mathcal{Z} = \{Z \subseteq M \mid d(p, q) > \epsilon, \text{ for each } p, q \in M\}.$$

Let Z be a maximal element of $\mathcal{Z}$. Since $Z \in \mathcal{Z}$, it follows that $B_{\epsilon/2}(p, M)$, $p \in Z$, are disjoint to each other. Hence Formula (2.9) and $B_D(p, M) = M$ imply that $\sharp Z \leq N(\epsilon)$. The maximality of Z in $\mathcal{Z}$ implies that the ϵ-neighborhood of Z is equal to M. Hence, $M \in \mathcal{S}(N_D(\cdot), D)$, as required.

By a similar method we can prove the following :

Proposition 2.10. *Let* $M_i \in \mathcal{S}(n, D)$, $X \in \mathcal{MET}$. *Assume* $\lim_{i\to\infty} d_H(M_i, X) = 0$. *Then the Hausdorff dimension of* X *is equal to or smaller than* n.

When we do not assume $M_i \in \mathcal{S}(n, D)$ the conclusion does not necessary hold. A counter example is given in 18.6–18.11.

§3. Rigidity theorem

In the rest of this chapter, we will mainly study the following class of Riemannian manifolds.

Definition 3.1. We define the symbol $\mathcal{M}(n, D, v)$ by :

$$\mathcal{M}(n, D, v) = \left\{ M \;\middle|\; \begin{array}{l} M \text{ is a compact Riemannian manifold.} \\ \dim M = n, \;\; \text{Diam } M \le D \\ |K_M| \le 1, \;\; \text{Vol } M > v \end{array} \right\},$$

where K_M denotes the sectional curvature. $\mathcal{M}(n, D, 0)$ is denoted by $\mathcal{M}(n, D)$.

In this and the next sections we deal with Gromov's convergence theorem, which we divide into two parts. This section is devoted to the first part. It asserts that the diffeomorphism types are locally constant on $\mathcal{M}(n, D, v)$.

Theorem 3.2 (Gromov [G7] 8.25, Katsuda [Ka1] Theorem 1). *If* $M_i, N \in \mathcal{M}(n, D, v)$ *and if* $\lim_{i\to\infty} d_H(M_i, N) = 0$. *Then* M_i *and* N *are diffeomorphic for sufficiently large* i. *Furthermore we have* $\lim_{i\to\infty} d_L(M_i, N) = 0$.

Remark 3.3. Before Gromov, Shikata [Sh1] proved that $\lim_{i\to\infty} d_L(M_i, N) = 0$ implies that M_i and N are diffeomorphic for large i.

Proof of Theorem 3.2. We first recall the following result :

Theorem 3.4 (Cheeger [C2] Theorem 2.1, Heintze-Karcher [HK] Corollary 2.3.2). *Let* $M \in \mathcal{M}(n, D, v)$. *Then* $\text{inj}(M)$, *the injectivity radius of* M, *satisfies :*

$$\text{inj}(M) \ge \min\left(\pi, \frac{2\pi v}{\text{Vol}(S^n)\cdot \sinh D}\right) \tag{3.4}$$

We omit the proof of Theorem 3.4. We put $\epsilon_i = 2d_H(M_i, N)$. Then there exist ϵ_i-Hausdorff approximations $\varphi_i: N \to M_i$. It is an easy exercise to replace φ_i by measurable maps. Let $L^2(N)$ denote the Hilbert space of all L^2 functions on N. We will embed M_i and N to $L^2(N)$. We take a smooth function $\chi: [0,\infty) \to [0,1]$ satisfying the following :

$$\begin{cases} \chi(t) = 1 & \text{if } t < \mu/3. \\ \chi(t) = 0 & \text{if } t > 2\mu/3. \\ \chi'(t) > 0 & \text{if } 2\mu/3 > t > \mu/3 \end{cases} \tag{3.5}$$

Then we define $I: N \to L^2(N)$ and $I_i: M_i \to L^2(N)$ by :

$$I(p)(q) = \chi(d(p,q)), \tag{3.6.1}$$

$$I_i(p_i)(q) = \chi(d(p_i, \varphi_i(q))). \tag{3.6.2}$$

Here $p, q \in N$ and $p_i \in M_i$.

By definition I_i and I are C^∞-maps.[5] We put

$$\mathcal{N}(N) = \{(p,v) \in N \times L^2(N) \mid v \text{ is perpendicular to } I_*(T_p(N)).\}$$
$$\mathcal{N}_\delta(N) = \{(p,v) \in \mathcal{N}(N) \mid |v| < \delta\}$$
$$\mathrm{Exp}: \mathcal{N}(N) \to L^2(N); (p,v) \mapsto I(p) + v$$

Since I is a C^∞-embedding, it follows that there exists $\delta > 0$ such that Exp restricted to $\mathcal{N}_\delta(N)$ is a diffeomorphism between $\mathcal{N}_\delta(N)$ and $B_\delta(I(N))$, the δ-neighborhood of $I(N)$. Let $\pi: B_\delta(I(N)) \to N$ be the composition of Exp^{-1} and the projection to the first factor.

Lemma 3.7. *If $2\alpha\epsilon_i\sqrt{\mathrm{Vol}(N)} < \delta$, then $I_i(M_i) \subset B_\delta(I(N))$. Here we put $\alpha = \sup|\chi'|$.*

Proof. By definition, $|I(p) - I_i(\varphi_i(p))|_{C^0} < \alpha\epsilon_i$. Hence $\|I(p) - I_i(\varphi_i(p))\|_{L^2} < \delta/2$.

On the other hand, for each $q \in M_i$, we can find $p \in N$ such that $d(\varphi_i(p), q) < \epsilon_i$. Hence $\|I_i(q) - I_i(\varphi_i(p))\|_{L^2} < \delta/2$. Therefore $\|I_i(q) - I(p)\|_{L^2} < \delta$ as required.

Lemma 3.7 implies that, for large i the map $f_i = \pi \circ I_i$ is well defined.

[5] $I_i(p_i)(q)$ is not necessary a continuous function of q. But it is a C^∞ function of p_i, because of (3.4) and (3.5).

The next step is to show that f_i is of maximal rank. Recall that π is a fibration and that

$$(3.8)\quad \begin{aligned}&\{w \in T_{I_i(p_i)}(L^2(N)) \mid \pi_*(w) = 0\}\\ &\qquad = \{w \mid w \text{ is perpendicular to } I_*(T_{f_i(p_i)}(N))\}.\end{aligned}$$

Hence, for our purpose, it suffices to show that $I_{i*}(T_{p_i}(M_i))$ is transversal to the right hand side of (3.8), for sufficiently large i and small μ. This fact follows from the following :

Lemma 3.9. *For each $v \in T_{f_i(p_i)}(N)$, there exists $v_i \in T_{p_i}(M_i)$ such that*[6]

$$\|I_*(v) - I_{i*}(v_i)\|_{L^2} < \pi/10.$$

Proof. Put

$$\ell\colon [0,1) \to N; t \mapsto \exp_{f_i(p_i)}(tv).$$

Then, we have

$$I_*(v)(q) = \frac{d}{dt}\chi\left(d(\ell(t), q)\right)\Big|_{t=0}.$$

(Recall $I_*(v)(q) \in L^2(N)$.) Hence, if q is not a cut point of $f_i(p_i)$, then we have

$$I_*(v)(q) = \frac{d\chi}{dt}\Big|_{t=d(f_i(p_i),q)} \cdot \cos\theta(q).$$

Here $\theta(q)$ denotes the angle at $f_i(p_i)$ between ℓ and the minimal geodesic connecting $f_i(p_i)$ and q_i. (See Figure 3.10.)

Let ℓ_i be the minimal geodesic connecting p_i and $\varphi_i(\ell(\mu/2))$.[7] Put $v_i = \frac{D\ell_i}{dt}\big|_{r=0}$. If $\varphi_i(q)$ is not a cut point of p_i then we have

$$I_{i*}(v_i)(q) = \frac{d\chi}{dt}\Big|_{t=d(p_i,\varphi_i(q))} \cdot \cos\theta_i(q),$$

where $\theta_i(q)$ denotes the angle between ℓ_i and the minimal geodesic connecting p_i and $\varphi_i(q)$. (See Figure 3.10.)

Here we need the following :

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[6] $\pi/10$ in the formula below is not essential. Any small number goes.

[7] Recall $d(p_i, \varphi_i(\ell(\mu/2))) < \mu/2 + 2\epsilon_i + \delta < \text{inj}(M_i)$.

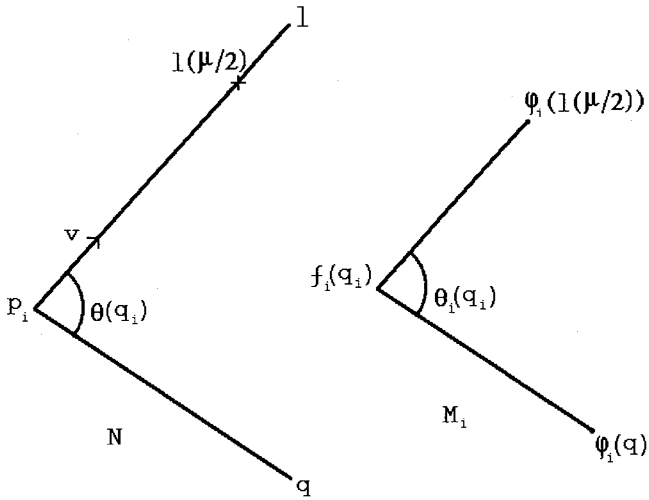

Fig. 3.10

Theorem 3.11 (Toponogov [To1] (See [CE] Chapter 2 or [Sa2] p.139)). *Let X_C be as in Theorem 2.8 and M be a Riemannian manifold satisfying $a \le K_M \le b$. Let $p, q, r \in M$ satisfy $d(p,q), d(p,r) < \mathrm{inj}_M$. Choose $p_1, q_1, r_1 \in X_a$ and $p_2, q_2, r_2 \in X_b$ such that $d(p,q) = d(p_1.q_1) = d(p_2,q_2), d(q,r) = d(q_1,r_1) = d(q_2,r_2), d(r,p) = d(r_1,p_1) = d(r_2,p_2)$. Then we have*

$$\angle q_1 p_1 r_1 \le \angle qpr \le \angle q_2 p_2 r_2$$

It follows from Theorem 3.11 that, if μ is small enough, we have

$$|\theta - \theta_i| < \frac{\alpha}{10\sqrt{\mathrm{Vol}(V)}}.$$

for each q satisfying $\mu/4 < d(f_i(p_i), q) < 3\mu/4$. Hence

$$|I_*(v)(q) - I_*(v_i)(q)| < \frac{\pi}{10\sqrt{\mathrm{Vol}(V)}}$$

for such q. On the other hand, if $d(f_i(p_i), q) \ge 3\mu/4$ or $d(f_i(p_i), q) \le \mu/4$, then

$$\left.\frac{d\chi}{dt}\right|_{t=d(f_i(p_i),q)} = \left.\frac{d\chi}{dt}\right|_{t=d(p_i,\varphi_i(q))} = 0.$$

Therefore

$$\|I_*(v) - I_{i*}(v_i)\|_{L^2} < \pi/10,$$

as required. The proof of Lemma 3.9 is now completed.

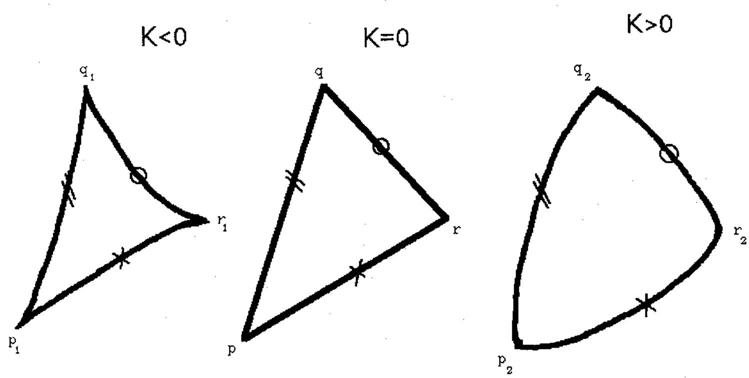

Fig. 3.12

The last step of the proof of Theorem 3.2 is to show that f_i is a diffeomorphism. We have already proved that f_i is of maximal rank. Hence it is a covering map. On the other hand, by construction, f_i is a $10\epsilon_i$-Hausdorff approximation. The following lemma completes the proof of Theorem 3.2.

Lemma 3.13. *f_i is injective.*

Proof. Assume that f_i is not injective. Then, there exists p_i and q_i such that $f_i(p_i) = f_i(q_i)$. Since f_i is a $10\epsilon_i$-Hausdorff approximation, it follows that $d(p_i, q_i) < 10\epsilon_i$. Choose a minimal geodesic ℓ_i in M_i connecting p_i and q_i. Then, $f_i(\ell_i)$ is a closed loop in N which is not 0-homotopic. But this is impossible because $\text{Diam}(f_i(\ell_i)) < 20\epsilon_i < \text{inj}(N)$. We omit the proof of $\lim_{i\to\infty} d_L(M_i, X) = 0$.[8]

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[8] See [Kal]

§4. Convergence theorem

In this section, we study the closure of the set $\mathcal{M}(n, D, v)$ in $\mathcal{MET}$.

Theorem 4.1 (Gromov [G7] 8.28, Peters [Pe2], see also Greene-Wu [GW]). *Let $M_i \in \mathcal{M}(n, \infty, v)$, $(X, d) \in \mathcal{MET}$. Assume $\lim_{i\to\infty} d_H(M_i, X) = 0$. Then*

(4.1.1) *X is a C^∞-manifold,*
(4.1.2) *There exists a metric tensor g_X of $C^{1,\alpha}$-class[9] such that (X, d) is isometric to (X, g_X).*
(4.1.3) *X and M_i are diffeomorphic for large i.*
(4.1.4) $\lim_{i\to\infty} d_L(X, M_i) = 0$.[10]

Sketch of the proof. Remark that the curvature is the second derivative of the metric tensor. Hence we can expect that our assumption $|K_{M_i}| \le 1$ implies the metric tensors of M_i have a uniform bound on C^2-norm. Hence, it is natural that its limit is of $C^{1,1}$-class. But, unfortunately, the situation is not so simple. Indeed the curvature determines the isometry class of Riemannian manifolds but not the metric tensor ! The metric tensors is determined modulo coordinate change. Therefore, to obtain a C^2-bound of the metric tensor from the uniform bound of the curvature, we have to choose a good coordinate system. Namely :

Theorem 4.2 (Jost-Karcher [J], [JK]). *Let $M \in \mathcal{M}(n, \infty)$. Put $\mu = \inf \mathrm{inj}_M$. Then there exists $R = R(n, \mu)$ and $C = C(n, \mu)$, satisfying the following. For each $p \in M$ there exists $h_i: B_R(p, M) \to \mathbf{R}$, $i = 1, \cdots, n$, such that $H = (h_1, \cdots, h_n): B_R(p, M) \to U$ $(\subset \mathbf{R}^n)$ is a diffeomorphism and that $|g_{i,j}|_{C^{1,\alpha}} \le C$, where the $g_{i,j}$ are metric coefficients relative to H and the $C^{1,\alpha}$-norm is taken in the H-coordinates.*

Idea of the proof of Theorem 4.2. (We follow Green-Wu [GW].) At first sight, the geodesic coordinate, (which is a canonical one), is a good candidate for H. But in geodesic coordinate system, only the C^0-bound of the metric tensor is available. The coordinate we use is the harmonic coordinate that is the coordinate system $H = (h_1, \cdots, h_n)$ such that each h_i is a harmonic function. To construct such functions h_i we need to choose boundary conditions. Take an orthonormal frame $(e_i, \cdots, e_n)$

[9]This means that the first derivative of the metric tensor is α-Lipschits continuous for each $\alpha \in (0, 1)$.
[10]The last two statements are generalizations of Theorem 3.2

of $T_p(M)$. Put $p_i(r) = \exp_p(re_i)$, $q_i(r) = \exp_p(-re_i)$. Now we define almost linear function $\ell_i: B_{\mu_0}(p, M) \to \mathbf{R}$ by

$$\ell_i(x) = \frac{(d(x, p_i(r(x))))^2 - (d(x, q_i(r(x)))^2}{4r(x)},$$

where $r(x) = d(p, x)$ and $\mu_0 \ll \mu$. Remark that in the case when $X = \mathbf{R}^n$, $p = 0$, $e_i = \frac{\partial}{\partial x_i}$, the function ℓ_i is the i-th coordinate. Let $e_i(x) \in T_x(M)$ denote the parallel transform of e_i along the minimal geodesic. Using comparison theorems we have :

$$|\mathrm{grad}_x\, \ell_i - e_i(x)| < Cr(x)^2 \tag{4.3.1}$$

$$\left|D^2\, \ell_i(x)\right| < Cr(x) \tag{4.3.2}$$

We can use these formulae to show that $L = (\ell_1, \cdots, \ell_n)$ is diffeomorphism. Take $R \ll \mu_0$. Let h_i be the unique solution of

$$\begin{cases} \triangle\, h_i = 0 \\ h_i|_{\partial B_R(p,M)} = \ell_i|_{\partial B_R(p,M)}\,. \end{cases} \tag{4.4}$$

Then, by Formula (4.3.2), we have

$$\begin{cases} |\triangle\,(h_i - \ell_i)| < Cr(x)^2 \\ (h_i - \ell_i)|_{\partial B_R(p,M)} = 0. \end{cases}$$

Applying an elliptic estimate to the above inequality, we obtain :

$$|\mathrm{grad}(h_i - \ell_i)| < CR.$$

This formula implies that $H = (h_1, \cdots, h_n)$ is also a coordinate system. Now using Bochner technique and Nash-Moser type estimate, we obtain a uniform $C^{1,\alpha}$ bound of the metric tensor.

To continue the proof of Theorem 4.1, we need the following result.

Lemma 4.5 (Cheeger [C2], Weinstein [We]). *There exists a positive number $K = K(v, D, r)$ satisfying the following. For each $M \in \mathcal{M}(n, D)$ there exist $p_1, \cdots, p_K$ such that $M = \cup_{i=1}^K B_r(p_i, M)$*

The proof is similar to the proof of Lemma 2.4 and is omitted.

Now choose $r \ll R$, and take $p_{i,k} \in M_i$ such that $\cup_{k=1}^K B_r(p_{i,k}, M_i) = M_i$ and a harmonic coordinate $H_{i,k}: B_R(p_{i,k}, M_i) \to U_{i,k} \subset \mathbf{R}^n$.

Lemma 4.6. *Suppose* $B_{3r}(p_{i,k}, M_i) \cap B_{3r}(p_{i,k}, M_i) \neq \phi$. *Then the map* $H_{k,l;i} = H_{i,l} \circ H_{i,k}^{-1}: U_{i,k} \to \mathbf{R}^n$ *satisfies*

$$|H_{k,l;i}|_{C^{2,\alpha}} < C.$$

This lemma is proved by making use of Theorem 4.2 and Schauder estimate for the harmonic functions. We may assume that $H_{i,k}(B_{3r}(p, M_i)) \supset B_{2r}(0, \mathbf{R}^n) \supset H_{i,k}(B_r(p, M_i))$. We may assume also, by taking a subsequence if necessary, that $\lim_{i\to\infty} H_{k,l;i}|_{B_{2r}(0,\mathbf{R}^n)}$ converges to a $C^{1,\alpha}$-diffeomorphism $H_{k,l}$ with respect to $C^{1,\alpha}$- topology. Let $\sum g_{a,b}^{(i;k)} dx_a dx_b$ be the metric on $B_{2r}(0, \mathbf{R}^n)$ induced by $H_{i,k}$. Then we may assume furthermore that $\sum g_{a,b}^{(i;k)} dx_a dx_b$ converges to a metric tensor $\sum g_{a,b}^k dx_a dx_b$ of $C^{1,\alpha}$-class. Now we patch the Riemannian manifolds $(B_{2r}(0, \mathbf{R}^n), \sum g_{a,b}^k dx_a dx_b)$ by the isometries $H_{a,b}$ and obtain a Riemannian manifold. It is easy to see that this manifold is isometric to X. We have verified (4.1.1) and (4.1.2). We remark that (4.1.3) and (4.1.4) will be verified if we can apply Theorem 3.2 to our situation. In fact, this is possible but a bit delicate because of the fact that the limit metric g_X is not of C^2-class. Another method of the proof of (4.1.3) and (4.1.4) is to use the center of mass technique. The detail of the argument is in [GW], [Pe2].

Remark 4.7. A stronger result than theorem (4.14) holds. Namely we have diffeomorphisms $f_i: X \to M_i$ such that $f_i^*(g_{M_i})$ converges to the Riemannian metric on X with respect to the $C^{1,\alpha}$ topology. (See Kasue [K2]).

Remark 4.8. In sections 3 and 4, we discussed two approaches to the study of Hausdorff convergence in $\mathcal{M}(n, D, v)$. One uses an embedding to the Hilbert space, and the other uses harmonic coordinate and the center of mass technique. Many parts of Theorem 4.1 can be proved also by the first approach. However the author does not know the proof of the $C^{1,\alpha}$-regularity of the limit metric without using harmonic coordinate. Hence the second approach yields stronger conclusions. The first method will come to be essential when we study collapsing Riemannian manifolds in section 12.

§5. Smoothing Riemannian metrics

When studying Hausdorff convergence of Riemannian manifolds, one of the difficulties we meet is that the limit space, $\lim_{n\to\infty} M_n$, is not smooth enough. One of the methods to handle this difficulty is to approximate Riemannian manifolds M_n by uniformly smooth ones, (then, roughly speaking, the limit space comes to be smooth). The following results are useful for this purpose.

Theorem 5.1 (Bemelmans-Min'no-Ruh [BMR], Bando [Ba]). *There exists $C(\epsilon) = C(n,\epsilon), r = r(n)$ such that, if $(M,g) \in \mathcal{M}(n,\infty)$ is compact, then there exists a metric g_ϵ on M satisfying*

$$(5.1.1) \qquad |g - g_\epsilon|_{C^0} < \epsilon,$$

$$|\nabla^g - \nabla^{g_\epsilon}|_{C^0} < \epsilon$$

$$(5.1.2) \qquad \left|\nabla^k R(g_\epsilon)\right|_{C^0} < C(\epsilon)\, k!\, r^k.$$

The most general version of the next theorem is a bit complicated. Hence we state somewhat weaker one.

Theorem 5.2 (Cheeger-Gromov [CG1], Abrech [A]). *For each complete Riemannian manifold (M,g) satisfying $|K_g| \le 1$ and a positive number ϵ, there exists a Riemannian metric g_ϵ such that* (5.1.1) *and the following holds for each $p \in M$.*

$$(5.2.1) \qquad |\nabla^{g_\epsilon} R(g_\epsilon)| \le C(n,\epsilon)$$

$$(5.2.2) \qquad g_\epsilon(p) \text{ depends only on } g|_{B_{1/4}(p,M)}.$$

Remark 5.3. These two theorems are almost equivalent. The estimate in Theorem 5.1 is sharper and implies the real analyticity of the metric. On the other hand, the construction of Theorem 5.2 is local and can be applied to a noncompact manifold. It will be useful if we obtain the result joining them.

Remark 5.4. In both theorems, if we assume that a compact group G acts on (M,g) by isometry, then we can choose (M,g_ϵ) which is G invariant, in addition.

Remark 5.5. Abrech proved that Theorem 5.2 holds also for $C^{1,1}$-Riemannian manifold. This gives a kind of converse to Theorem 4.1.

Remark 5.6. First Bemelmans, Min-no and Ruh proved this type of smoothing results. In their version, (5.1.2) was replaced by (5.2.1). Next Cheeger-Gromov found an alternative method and proved Theorem 5.2 under the additional assumption $\mathrm{inj}(M) >$ const. Bando improved the result of Bemelmans-Min'no-Ruh to the form stated above. Abrech improved Cheeger-Gromov's method and proved Theorem 5.2.

Sketch of the proofs. The method of the proofs of Theorems 5.2 and 5.3 are different. First we discuss one for Theorem 5.2.

Consider the following differential equations on $M \times [0, \infty)$:

$$(5.7) \qquad \begin{cases} \dfrac{dg_t}{dt} = -2 \text{ Ricci } g_t \\ g|_{t=0} = g \end{cases}$$

Hamilton [Ha] proved that (5.7) has a unique solution in a neighborhood of $M \times \{0\}$. Our assumption on K_M implies the following :

Lemma 5.8. *There exists a positive number T_n such that if $(M, g) \in \mathcal{M}(n, \infty)$ and if M is compact, then (5.7) has a unique solution on $M \times [0, T_n]$.*

The fact that T_n is independent of M is essential for our purpose. The lemma follows essentially from the following inequality

$$\sup_{p \in M,\, 0<t<t_0} |R(g_t)(p)| < |R(g_0)|_{C^0} \cdot (1 - Ct_0)^{-1},$$

which is proved by applying maximum principle. The detail of the proof is in [BMR]. The second step is to show (5.1.2). This follows from a type of interior regularity estimate of harmonic functions. We omit the proof which can be found in [BMR], [Ba].

Next we discuss the method by Cheeger-Gromov-Abrech. First we assume $\text{inj}_M > \text{const}\ (= \mu)$. We can embed M into $L^2(M)$ by the map I defined by (3.6). Then, roughly speaking, we can find a sufficiently smooth submanifold in a neighborhood of $I(M)$. We can perform this construction in each coordinate chart of M and can patch them together to obtain the desired metric g_ϵ. In the case when we do not assume $\text{inj}(M) > \text{const}$, each small neighborhood of M is a quotient of a Riemannian manifold satisfying $\text{inj} > \text{const}$ by an action of a pseudo fundamental group.[11] Hence, if we can make our construction of the metric g_ϵ equivariant with respect to the (pseudo) group action, then the proof of Theorem 5.2 will be completed. This is what Abrech has done in his paper [A].

[11]See section 7 for the definition of this notion and the explanation of this fact.

Finally we remark that we can *not* expect that the curvature of our metric g_ϵ is uniformly close to that of g. In other words, we can not obtain a metric g_ϵ satisfying $|R(g_\epsilon) - R(g)| < \epsilon$, in addition to (5.1.1) and (5.1.2). In fact if we take a sequence of metrics g_i on S^2 such that $\left|K_{(S^2,g_i)}\right| \leq 1$ but that $\lim_{i\to\infty} K_{(S^2,g_i)}$ is not continuous, it is obvious that we can not obtain such an approximation $g_{i,\epsilon}$. However, it is possible to prove the following by examining the proofs of Theorems 5.1 and 5.2 in detail.

Proposition 5.9. *If we assume $C_1 < K_{(M,g)} < C_2$ in Theorems 5.1 or 5.2, then we obtain a metric g_ϵ satisfying $C_1 - \epsilon < K_{(M,g_\epsilon)} < C_2 + \epsilon$ in addition.*[12]

We close this section with the following remark.

Proposition 5.10. *Let $M_i \in \mathcal{M}(n, D, v)$, $X \in \mathcal{MET}$. Suppose that M_i satisfy* (5.2.1) *and that* $\lim_{i\to\infty} d_H(M_i, X) = 0$. *Then X is a smooth Riemannian manifold. Furthermore, we have diffeomorphisms $f_i\colon X \to M_i$ such that $f_i^*(g_{M_i})$ converges to the metric tensor of X with respect to the C^∞-topology.*

The proof is similar to and much easier than that of Theorem 4.1.

§6. Pointed and equivariant Hausdorff distances

In this section, we discuss generalizations of Hausdorff convergence to the case of open manifolds and/or manifolds on which groups act.

Definition 6.1. The symbol $\mathcal{MET}_0$ denotes the set of all isometry classes of pointed metric spaces (X, p) such that $B_R(p, X)$ is compact for every R.

Definition 6.2. Let $(X,p), (Y,q) \in \mathcal{MET}_0$ and $\varphi\colon X \to Y$ be a pointed map. We say that φ is an *ϵ-pointed Hausdorff approximation* if $\varphi(B_{1/\epsilon}(p,X)) \subset B_{1/\epsilon}(q,Y)$ and if $\varphi|_{B_{1/\epsilon}(p,X)} : B_{1/\epsilon}(p,X) \to B_{1/\epsilon}(q,Y)$ is an ϵ-Hausdorff approximation.

The pointed Hausdorff distance, $d_{p.H}((X,p),(Y,q))$, stands for the infimum of the numbers ϵ such that there exist ϵ-Hausdorff approximations from X to Y and from Y to X.

[12] This fact is pointed out to the author by Bando.

Example 6.3. Let X_n, o_n, p_n, and q_n be as in Figure 6.4. Then we see easily that

$$\lim_{n\to\infty}(X_n, o_n) = (S^2 - \{\text{point}\}, \text{point})$$
$$\lim_{n\to\infty}(X_n, p_n) = (\mathbf{R}, 0)$$
$$\lim_{n\to\infty}(X_n, q_n) = (T^2 - \{\text{point}\}, \text{point})$$

We remark that in this example the limit space does depend on the choice of base points.

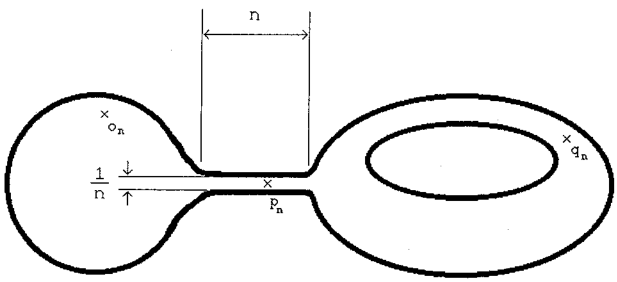

Fig. 6.4

Example 6.5. Let (M, g) be a compact n-dimensional Riemannian manifold and $p \in M$. Then, $\lim_{R\to\infty}((M, Rg), p) = (\mathbf{R}^n, 0)$.

Some more examples are given in section 17. Many results in previous sections can be generalized to the case of pointed Hausdorff convergence. The following is the noncompact version of Theorem 2.2.

Theorem 6.6. *Put*

$$\mathcal{S}_0(n) = \left\{ (M, p) \;\middle|\; \begin{array}{l} M \text{ is a complete Riemannian manifold,} \\ \dim M = n, \;\; \mathrm{Ricci}_M > -(n-1) \end{array} \right\}.$$

Then, $\mathcal{S}_0(n)$ is precompact in $\mathcal{MET}_0$.

The following version of Theorem 5.1 also holds.

Theorem 6.7. *Put*

$$\mathcal{M}_0(n,\epsilon) = \left\{ (M,p) \left| \begin{array}{l} M \text{ is a complete Riemannian manifold.} \\ \mathrm{inj}_M > \epsilon,\ |K_M| \le 1,\ \dim M = n \end{array} \right. \right\}.$$

Let $(M_n, p_n) \in \mathcal{M}_0(n,\epsilon)$, $(X,p) \in \mathcal{MET}_0$. *Assume*

$$\lim_{n\to\infty} d_{p.H}((M_n,p_n),(X,p)) = 0.$$

Then X *is a smooth Riemannian manifold with* $C^{1,\alpha}$*-metric tensor.*

We remark that the noncompact version of Theorem 3.2 does *not* hold. We gave counter examples in 6.3 and 6.5 above.

Next we define equivariant versions of Hausdorff distances.

Definition 6.8. Let G be a group and $\mathcal{MET}(G)$ be the set of all isometry classes of compact metric spaces equipped with an isometric G-action. For $X, Y \in \mathcal{MET}(G)$, a map $\varphi: X \to Y$ is said to be an *ϵ-G-Hausdorff approximation* if φ is an ϵ-Hausdorff approximation and if

$$d(g\varphi(x), \varphi(gx)) < \epsilon$$

holds for each $x \in X$ and $g \in G$.

The *G-Hausdorff distance* between X and Y, $d_{G-H}(X,Y)$, is the infimum of all positive numbers ϵ, such that there exist ϵ-G-Hausdorff approximations from X to Y and from Y to X.

The equivariant versions of Theorems 2.2 and 3.2, hold.

Theorem 6.9. *Let* $\mathcal{S}(n,D,G)$ *be the set of isometry classes of all G-Riemannian manifolds* M *in* $\mathcal{S}(n,D)$. *If* G *is compact, then* $\mathcal{S}(n,D,G)$ *is precompact in* $\mathcal{MET}(G)$.

For each n, D, *and* $v > 0$ *there exists* $\epsilon = \epsilon(n,D,v)$ *such that the following holds. Let* G *be a compact group,* M, N *be n-dimensional compact Riemannian G-manifolds. Suppose that* $M, N \in \mathcal{M}(n,D,v)$ *and that* $d_{G-H}(M,N) = \delta < \epsilon$. *Then there exists a G-diffeomorphism* $f: M \to N$. *Furthermore* $\mathrm{dil}\, f, \mathrm{dil}\, f^{-1} < o(\delta)$.

Remark 6.10. In the case when M and N are (not equivariantly) diffeomorphic, the second part of Theorem 6.8 is proved in Grove-Karcher [GK].

We need another equivariant version of Hausdorff distance in which the groups need not be the same.

Definition 6.11. Let $\mathcal{MET}_{0,eq}$ be the set of all triples (X,p,G) where (X,p) is an element of $\mathcal{MET}_0$ and that G is a closed subgroup of the group of all isometries of X. For $(X,p,G),(Y,q,H) \in \mathcal{MET}_{0,eq}$, a pair of maps $f\colon X \to Y$, $\varphi\colon G \to H$ is said to be an *ϵ-pointed equivariant Hausdorff approximation* if f is an ϵ-pointed Hausdorff approximation and if

$$d(\varphi(g)f(x), f(gx)) < \epsilon$$

holds for each $x \in X$, $g \in G$ satisfying $d(x,p), d(gx,p) < 1/\epsilon$.[13] The *pointed equivariant Hausdorff distance*, $d_{e.H}(\cdot,\cdot)$, is defined as before.

Theorem 6.12. *The set*

$$\{(X,p,G) \in \mathcal{MET}_{0,eq} \mid (X,p) \in \mathcal{MET}_0(n)\}$$

is precompact in $\mathcal{MET}_{0,eq}$.

We can not expect that an analogue of Theorem 3.2 holds, because there exists a continuous family of groups. We have an analogue under the assumption of uniformly discreteness of G. See [F2] Chapter 2 for detail. In a similar way, we can define pointed G-Hausdorff distance and (unpointed) equivariant Hausdorff distance. But we do not need them in this article.

Finally we compare the equivariant Hausdorff convergence and the Hausdorff convergence of the quotient spaces.

Lemma 6.13 ([F2]). *Let X_i be the sequence of compact G-spaces satisfying $\lim_{i,j\to\infty} d_{G-H}(X_i,X_j) = 0$. Then $\lim_{i,j\to\infty} d_H(X_i/G, X_j/G) = 0$.*

Let $(X_i,G_i) \in \mathcal{MET}_{0,eq}$ such that $\lim_{i,j\to\infty} d_{e.H}((X_i,G_i),(X_j,G_j)) = 0$. Then, we have $\lim_{i,j\to\infty} d_{p.H}(X_i/G_i, X_j/G_j) = 0$.

Chapter II. Collapsing Riemannian Manifolds

§7. Pseudo fundamental group

In this chapter, we study the narrow parts of Riemannian manifolds. Put :

$$\mathcal{M}(n,D \mid \Lambda) = \{\, M \mid \dim M = n,\ |K_M| \le \Lambda,\ \text{Diam}\ M < D\,\}$$

---

[13]We do not assume that φ is a homomorphism.

For $p \in M \in \mathcal{M}(n, D \mid \Lambda)$, we put

$$TB_r(p,M) = \{ v \in T_p(M) \mid |v| < r \}$$

Then the exponential map, $\mathrm{Exp}_p: T_p(M) \to M$, is of maximal rank on $TB_{\pi/\Lambda}(p,M)$. Let $(TB_{\pi/\Lambda}(p,M), g_M)$ denote the ball with the metric induced by Exp_p. Since $\mathrm{inj}_{TB_{\pi/\Lambda}(p,M)}(0) \geq \pi/\Lambda$, we can apply the results in Chapter 1 to it. Hence the studies of the small neighborhood of p in M is reduced to the those of the map Exp_p. This map is of maximal rank, but is not in general a covering map, because $TB_{\pi/\Lambda}(p,M)$ is not complete. The covering space is investigated by using a fundamental group. In our case, we use, instead, the pseudo fundamental group, which we define in this section. We begin with the definition of a pseudogroup and its action.[14]

Definition 7.1. $(G, U, \cdot, e, \cdot^{-1})$ is said to be a pseudogroup when

(7.1.1) U is a symmetric subset of $G \times G$.
(7.1.2) $\cdot : U \to G$ is a map.
(7.1.3) $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ if both sides are well defined.
(7.1.4) $e \in G$, $\{e\} \times G \subset U$, $e \cdot g = g \cdot e = g$.
(7.1.5) $\cdot^{-1}: G \to G$ is a map. For $g \in G$, we have $(g, g^{-1}) \in U$ and $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Remark 7.2. When G is a topological space, U is an open set, and maps are continuous, Definition 7.1 coincides with the definition of the local group or the group germ.[15] But, in our main application, G has a discrete topology.

Exercise 7.3. Define the notion of homomorphism between pseudogroups.

Example 7.4. Let M be a Riemannian manifold, $p \in M$ and V is an open connected neighborhood of p. Put

$$I(M,V,p) = \{ g: V \to M \mid g \text{ is an isometric embedding, } g(p) \in V \}$$

For $g_1, g_2, g_3 \in I(M,V)$, we put $g_1 \cdot g_2 = g_3$ if and only if there exists a neighborhood W of p such that $g_2(W) \subset V$ and that $g_1(g_2(x)) = g_3(x)$

[14]There are possibly other versions of the axiom of pseudogroups. They are not in general equivalent to Definition 7.1. For example it is possible to assume that $\cdot^{-1}$ is not everywhere defined but is defined in a neighborhood of the unit. The author does not know how to choose one from them.
[15]See [Po] §23.

for each $x \in W$. By the unique extension property of the isometries, we see that $g_1 \cdot g_2 = g_3$, $g_1 \cdot g_2 = g_4$ implies $g_3 = g_4$. We can prove easily that $I(M, V, p)$ satisfies all the axioms of pseudogroup except (7.1.5).[16]

Definition 7.5. Let $(G, U, \cdots)$ be a pseudogroup and $H \subset G$ such that $g^{-1} \in H$ for each $g \in H$. Then we define a pseudogroup $(H, V, \cdots)$ by taking $V = U \cap (H \times H)$ e.t.c. This pseudogroup is called a subpseudogroup of G.

Definition 7.6. Let (M, p) be a pointed Riemannian manifold and G is a pseudogroup. An *action* of G to (M, p) is a homomorphism from G onto a subpseudogroup of $I(M, B_\epsilon(p, M), p)$.

Definition 7.7. Let $\varphi\colon G \to I(M, B_\epsilon(p, M), p)$ be an action of G to (M, p). We define the equivalence relation $\sim$ on $B_{\epsilon/2}(p, M)$ by

$$x \sim y \Longleftrightarrow \exists g \in G \;\; \varphi(g)x = y.$$

The *quotient space* $(M/G, \bar{p})$ is defined to be the pointed metric space $(B_\epsilon(p, M)/\sim, [p])$.

Now we return to the study of the small neighborhood of p in $M \in \mathcal{M}(n, D \mid \Lambda)$.

Definition 7.8. Let $2\epsilon\Lambda < \pi$. We define the pseudo fundamental group $\pi_1(M, p; \epsilon)$ by

$$\pi_1(M, p; \epsilon) = \{g \in I(TB_{2\epsilon}(p, M), TB_\epsilon(p, M), 0) \mid \mathrm{Exp}_p \circ g = \mathrm{Exp}_p\}.$$

Clearly $\pi_1(M, p; \epsilon)$ is a subpseudogroup of $I(TB_{2\epsilon}(p, M), TB_\epsilon(p, M), 0)$. Hence it acts on $(TB_{2\epsilon}(p, M), 0)$.

Lemma 7.9. *The quotient space $(TB_{2\epsilon}(p, M), 0)/\pi_1(M, p; \epsilon)$ is isometric to $B_\epsilon(p, M)$.*

We give an alternative definition of the pseudo fundamental group.

Definition 7.10. By $L(p, M; \epsilon)$, we denote the the set of all closed loops $\ell\colon [0, 1] \to M$ such that $\ell(0) = \ell(1) = p$ and that the length of ℓ is equal to or smaller than ϵ. We define an equivalence relation $\sim$ on $L(p, M; \epsilon)$ by

$$\ell \sim \ell' \Longleftrightarrow \exists \ell_t \text{ such that } \begin{cases} \ell_0 = \ell, \\ \ell_1 = \ell' \\ \text{the length of } \ell_t \le \epsilon \end{cases}$$

[16]Namely there may be an element without inverse.

Let $\pi_1(p,M;\epsilon)$ be the set of equivalent classes. We define a product $\cdot$ on $\pi_1(p,M;\epsilon)$ by

$$[\ell]\cdot[\ell'] = [\ell''] \iff \exists \ell_t \text{ such that } \begin{cases} \ell_0(s) = \begin{cases} \ell(2s) & \text{if } s \le 1/2 \\ \ell'(2s+1) & \text{if } s \ge 1/2 \end{cases} \\ \ell_1 = \ell'' \\ \text{the length of } \ell_t \le 2\epsilon. \end{cases}$$

Lemma 7.11. *The two definitions of fundamental pseudogroup coincide.*

The proof is similar to the corresponding statement for the fundamental groups and the covering spaces. The following fact is obvious from the first definition (but not from the second one.)

Lemma 7.12. *Let $\epsilon_1 < \epsilon_2 < 2\pi$, $M \in \mathcal{M}(n,\infty \mid \Lambda)$. Then $\pi_1(M,p;\epsilon_1) \subset \pi_1(M,p;\epsilon_2)$. If $g_1, g_2, g_3 \in \pi_1(M,p,\epsilon_1)$ and if $g_1 g_2 = g_3$ in $\pi_1(M,p,\epsilon_2)$ then $g_1 g_2 = g_3$ in $\pi_1(M,p;\epsilon_1)$.*

Here we remark that the pseudogroup is not so natural object and several facts which seems clear at first sight does *not* hold. For example, as is remarked in Buser-Karcher [BK] Remark 3.1.6, the formula

$$((g_1\cdot g_2)\cdot g_3)\cdot g_4 = g_1\cdot(g_2\cdot(g_3\cdot g_4))$$

may not in general hold in pseudogroup even if the both sides are well defined.[17] Therefore we will feel more comfortable if we can embed fundamental pseudogroup to a group. Remark that this was always possible in the case of continuous pseudogroup (namely the case of local group (group germ).)

Definition 7.13. Let G be a pseudogroup. We define a group $\overline{G}$ as follows. For each element g of G, fix a generator w_g. When $g_1\cdot g_2 = g_3$ we put the relation $w_{g_1}\cdot w_{g_2} = w_{g_3}$. Then we obtain a group $\overline{G}$.

The group $\overline{G}$ can be characterized by the following universal property.

Lemma 7.14. *Let G be a pseudogroup, H a group, and $\varphi\colon G \to H$ be a homomorphism. Then there exists a unique homomorphism $\overline{\varphi}\colon \overline{G} \to$*

[17]Because the product $(g_1\cdot g_2)\cdot(g_3\cdot g_4)$ may not be defined.

H such that $\overline{\varphi} \circ \pi = \varphi$. *Here* $\pi: G \to \overline{G}$ *denotes the homomorphism defined by* $\pi(g) = w_g$.

We omit the proof. By Lemma 7.14, we obtain homomorphisms $i: \overline{\pi}_1(M,p;\epsilon) \to \pi_1(M.p)$ and $\pi: \pi_1(M,p;\epsilon) \to \overline{\pi}_1(M,p;\epsilon)$. The examples below show that i need not be injective.

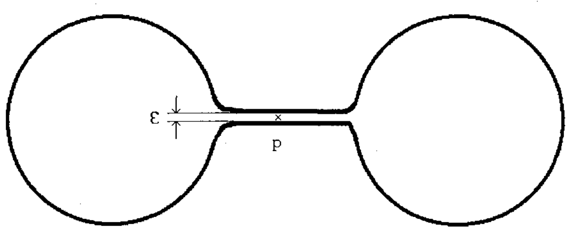

Fig. 7.15

Example 7.15. Let M and p be as in Figure 7.15. Then $\overline{\pi}_1(M,p;\epsilon) = \mathbf{Z}$ and $\pi_1(M,p) = 1$.

Example 7.16. Let $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$.

$$\mathrm{rot}_N = \begin{pmatrix} \cos 2\pi/N & -\sin 2\pi/N & 0 \\ \sin 2\pi/N & \cos 2\pi/N & 0 \\ 0 & 0 & 1 \end{pmatrix} : S^2 \to S^2.$$

Put $\theta_N: S^2 \times \mathbf{R} \to S^2 \times \mathbf{R}; (x,t) \mapsto (\mathrm{rot}_N(x), t + 1/N^2)$, $M_N = S^2 \times \mathbf{R}/\theta_N$, $p = [((1,0,0),0)] \in M_N$. Then, we have

$$\pi_1(M_N, p, k/N) \doteqdot \{\, \theta_N^n \mid n = aN + b, a^2 + b^2 \le k^2 \}$$

It follows that $\overline{\pi}_1(M_N, p, 1/N) \simeq \mathbf{Z} \oplus \mathbf{Z}$. On the other hand, $\pi_1(M_N, p) \simeq \mathbf{Z}$.

Open problem 7.17. Is the map $\pi: \pi_1(M,p;\epsilon) \to \overline{\pi}_1(M,p;\epsilon)$ injective ?

We remark that the map $G \to \overline{G}$ is not necessary injective for pseudogroup G. For example let

$$\widehat{G} = \{a^{n_1} b^{m_1} a^{n_2} \cdots b^{m_k} \mid \Sigma|n_i| + \Sigma|m_i| < 10\},$$

and let $\sim$ be the equivalence relation on $\widehat{G}$ generated by $a^7 \sim b^8$ and $a^8 \sim b^9$. We can define a pseudogroup structure on $G = \widehat{G}/\sim$ in an obvious way. Then, clearly $\overline{G} = 1$, but $a \not\sim 1$.

In the case when ϵ is sufficiently small, we can prove that $\pi_1(M, p; \epsilon)$ is a subpseudogroup of a nilpotent Lie group. Hence, in this case, we have an affirmative answer to Problem 7.17. To be precise we have the following :

Theorem 7.18 (Margulis' Lemma). *There exists a positive numbers ϵ_n and w_n depending only on the dimension n and satisfying the following. Let $M \in \mathcal{M}(n, \infty), p \in M, \epsilon \leq \epsilon_n$. Then there exist a nilpotent Lie group N and a discrete subgroup Γ of $N \propto Aut(N)$, such that*

(7.18.1) $\pi_1(M, p; \epsilon)$ *is a subpseudogroup of* Γ,
(7.18.2) $[\Gamma : \Gamma \cap N] \leq w_n$,
(7.18.3) $\dim N \leq n$.

The proof of this lemma is given in section 13, where we prove a little more. (Namely we determine the topological type of a small neighborhood of p.)

Remark 7.19. Several other versions of Margulis' lemma are known. (See for example [G1], [G2], [G.6], [BK]). Most of them follow from Theorem 7.18 above. But there are some which are not covered by Theorem 7.18. For example [G6] 8.50, [G10] 6.6.B are such ones.

§8. Almost flat manifolds I

In this and the next sections, we shall study the case when the Riemannian manifold itself is narrow. The goal is the following :

Theorem 8.1 (Gromov [G2], Ruh [R2], Buser-Karcher [BK]). *There exists $\epsilon = \epsilon_n$ such that, if a compact Riemannian manifold M satisfies*

$$(\text{Diam } M)^2 \cdot |K_M| \leq \epsilon_n, \tag{8.2}$$

then there exists a nilpotent Lie group N and a discrete subgroup Λ of $N \propto \text{Aut } N$ such that M is diffeomorphic to N/Λ and that $[\Lambda : N \cap \Lambda] < \infty$.

Remark 8.3. The converse is true. (See Example 10.8).

Remark 8.4. An explicit estimate $\epsilon_n \geq \exp(-\exp(\exp n^2))$ is proved in Buser-Karcher [BK]. This estimate is not optimal. It seems quite difficult to obtain a realistic value for ϵ_n .[18]

Remark 8.5. We can prove also that the metric is almost homogeneous. Namely there exists a right invariant metric g on N such that g is Λ invariant and that

$$|\overline{g} - g_M| < o(\epsilon) \tag{8.6.1}$$

$$\left|\nabla^{\overline{g}} - \nabla^{g_M}\right| < o(\epsilon) \tag{8.6.2}$$

where $\overline{g}$ denotes the metric on $M = N/\Lambda$ induced from g. Here and hereafter $o(\epsilon)$ denotes the positive number depending only on ϵ and satisfying $\lim_{\epsilon\to 0} o(\epsilon) = 0$.

Theorem 8.1 is one of the key stones of the theory of Hausdorff convergence. Unfortunately its proof is long and difficult. The reader who is not so much interested in the proof can skip the rest of this section and the next section.

First remark that the left hand side of (8.2) is invariant under the scale change of the metric. Hence, by rescaling, we may assume $\operatorname{Diam} M = \epsilon, |K_M| \leq 1$. Therefore, using Theorem 5.1 or 5.2, we may assume $\left|\nabla^k R\right| < C(k,n)$. Then by rescaling again we have

$$\operatorname{Diam} M = 1 \tag{8.7.1}$$

$$\begin{gathered} \left|\nabla^k R\right| < C(k,n)\cdot \epsilon^2 \\ |K| < \epsilon^2. \end{gathered} \tag{8.7.2}$$

In this section we shall prove the following :

Lemma 8.8. *Suppose M satisfies* (8.7.1) *and* (8.7.2). *Let $p \in M$. Then, for sufficiently small ϵ, there exists a homomorphism φ_ϵ: $\pi_1(M_\epsilon, p_\epsilon; 100) \to O(n)$, such that*

(8.8.1) *For each closed loop ℓ in $L(M_\epsilon, p_\epsilon; 100)$ we have*

$$d_{O(n)}(h(\ell), \varphi_\epsilon([\ell])) < o(\epsilon),$$

where $h\colon L(M,p;100) \to O(n)$ is the holonomy representation, and $d_{O(n)}(\cdot,\cdot)$ is the biinvariant metric on $O(n)$.

[18]In [R2] Ruh suggests that $\exp(-n^2)$ is a realistic value. The author does not know the reason.

(8.8.2) $\varphi_\epsilon(\pi_1(M,p;100))$ ($\subset O(n)$) *is a finite group independent of* ϵ.

Proof. We take a family M_ϵ satisfying (8.7.1) and (8.7.2) and shall prove that M_ϵ satisfies (8.8.1) and (8.8.2) for sufficiently small ϵ. Put $B_\epsilon = TB_{1/\epsilon}(p_\epsilon, M_\epsilon)$. Let $\pi_\epsilon: B_\epsilon \to M$ be the exponential map. Put $g_\epsilon = \pi_\epsilon^*(g_{M\epsilon})$.

Sublemma 8.9. *On each compact set,* g_ϵ *converges to the flat metric on* $\mathbf{R}^n$ *with respect to the* C^∞*-topology.*

This is an immediate consequence of (8.7.2). Recall

$$\pi_1(M_\epsilon, p_\epsilon; 1/\epsilon) = \left\{ g: TB_{1/\epsilon}(p_\epsilon, M_\epsilon) \to TB_{2/\epsilon}(p_\epsilon, M_\epsilon) \middle| \, \pi_\epsilon \circ g = \pi_\epsilon \right\}.$$

Sublemma 8.10. *There exists a group* G *of isometries of* $\mathbf{R}^n$ *such that a subsequence of* $(B_\epsilon, \pi_1(M_\epsilon, p_\epsilon; 1/\epsilon))$ *converges to* $(\mathbf{R}^n, G)$ *with respect to the equivalent pointed Hausdorff distance.*

This sublemma follows immediately from the pseudogroup version of Theorem 6.12. The group G above is a closed subgroup of $\mathrm{Isom}(\mathbf{R}^n)$ $= \mathbf{R}^n \propto O(n)$, hence is a Lie group.

Sublemma 8.11. G_0*, the connected component of* G*, is nilpotent.*

This is a consequence of Margulis' lemma. But, for the proof of Margulis' lemma (Theorem 7.18), we need Theorem 8.1 ! In fact, we shall prove this sublemma by an induction on dimension. The proof is deferred to section 13.

Sublemma 8.12. *Let* G *be a Lie subgroup of* $\mathrm{Isom}(\mathbf{R}^n)$*. Suppose that* $\mathbf{R}^n/G$ *is compact and that* G_0 *is nilpotent. Then there exists a finite subgroup* H *of* $O(n)$ *such that* $G = \mathbf{R}^n \propto H$.

Proof. If the sublemma is false, then G contains a compact subgroup of positive dimension. The nilpotency of G_0 implies that it contains a maximal compact subgroup T of positive dimension. T is normal in G. Put

$$X = \{x \in R^n | \, \forall g \in T \ \ gx = x\}.$$

It is easy to see that X is nonempty. Since T is a normal subgroup of G, it follows that X is G-invariant. Hence the function $\chi; x \mapsto d(x, X)$ is G-invariant. Therefore, since X/G is compact, we conclude that χ is bounded. But this is impossible because X is convex.[19]

[19]The proof we presented above is similar to the proof of Theorem 16.14.

Now let $(f_\epsilon, \psi_\epsilon): (B_\epsilon, \pi_1(M_\epsilon, p_\epsilon; 1/\epsilon)) \to (\mathbf{R}^n, G)$ be the $O(n)$ Hausdorff approximation. We may assume that f_ϵ is the natural inclusion. Let $\pi: G \to H$ be the natural projection. Put $\widetilde{\varphi}_\epsilon = \pi \circ \psi_\epsilon: \pi_1(M_\epsilon, p_\epsilon; 1/\epsilon) \to H$. Then we have

$$d(\widetilde{\varphi}_\epsilon(\gamma_1)\widetilde{\varphi}_\epsilon(\gamma_2), \widetilde{\varphi}_\epsilon(\gamma_1\gamma_2)) < o(\epsilon),$$

for each $\gamma_1, \gamma_2 \in \pi_1(M_\epsilon, p_\epsilon; 1/\epsilon)$. Therefore, the discreteness of H implies that $\widetilde{\varphi}_\epsilon$ is a homomorphism for sufficiently small ϵ. We obtain $\varphi: \pi_1(M_\epsilon, p_\epsilon; 100) \to O(n)$, by taking the composition of the natural inclusion $\pi_1(M_\epsilon, p_\epsilon; 100) \hookrightarrow \pi_1(M_\epsilon, p_\epsilon; 1/\epsilon)$, the homomorphism $\widetilde{\varphi}_\epsilon$, and the inclusion : $H \hookrightarrow O(n)$. We left the proof of (8.8.1) to the reader as an exercise. The proof of Lemma 8.8 is now complete.

§9. Almost flat manifolds II

In this section, we complete the proof of Theorem 8.1, following Ruh [R2]. We present here a detailed argument of the first half of the paper [R2] and an outline of the second part. First remark that Lemma 7.8 implies that $(TB_{100}(p_\epsilon, M_\epsilon))/\pi_1(M_\epsilon, p_\epsilon; 100)$ is diffeomorphic to M_ϵ. We put

$$\widetilde{M}_\epsilon = TB_{100}(p, M)/\ker \varphi_\epsilon.$$

Then, $\widetilde{M}_\epsilon$ is a finite Galois covering of M_ϵ. The deck transformation group, $\varphi_\epsilon(\pi_1(M_\epsilon, p; 100))$, is independent of ϵ, which we put H.

Lemma 9.1. *There exists a connection $\widetilde{\nabla}^\epsilon$ on $\widetilde{M}_\epsilon$ such that the following holds:*

$$R(\widetilde{\nabla}^\epsilon)(X,Y) = \widetilde{\nabla}^\epsilon_{[X,Y]} - \left[\widetilde{\nabla}^\epsilon_X, \widetilde{\nabla}^\epsilon_Y\right] = 0 \tag{9.1.1}$$

$$X(g_\epsilon(Y,Z)) = g_\epsilon\left(\widetilde{\nabla}^\epsilon_X Y, Z\right) + g_\epsilon\left(X, \widetilde{\nabla}^\epsilon_X Z\right). \tag{9.1.2}$$

$$\left|\widetilde{\nabla}^\epsilon - \widetilde{\nabla}^{(0,\epsilon)}\right|_{C^k} < o(\epsilon \mid k), \tag{9.1.3}$$

where $\nabla^{(0,\epsilon)}$ is the Levi-Civita connection of $(\widetilde{M}_\epsilon, g_\epsilon)$, and $o(\epsilon \mid k)$ are positive numbers depending only on ϵ and k and satisfying $\lim_{\epsilon\to\infty} o(\epsilon \mid k) = 0$, for each k. The C^k-norm is defined by g_ϵ and the geodesic coordinate.

Proof. By the definition of $\widetilde{M}_\epsilon$ and Lemma 8.8 we see that

$$d(h(\ell), 1) < o(\epsilon), \tag{9.2}$$

for each $\ell \in L(\widetilde{M}_\epsilon, p_\epsilon; 100)$. (Here $1 \in O(n)$ is the unit.) In other words, $\widetilde{M}_\epsilon$ is of almost without holonomy. For a Riemannian manifold M, let $\pi : F\widetilde{M}_\epsilon \to \widetilde{M}_\epsilon$ be the bundle of orthonormal frames of $\widetilde{M}_\epsilon$. Namely

$$F\widetilde{M}_\epsilon = \left\{(x,\alpha) \,\middle|\, \alpha : \mathbf{R}^n \to T_x(\widetilde{M}_\epsilon) \text{ is a linear isometry}\right\}$$

Put $F_x\widetilde{M}_\epsilon = \pi^{-1}(x) \subset F\widetilde{M}_\epsilon$. Fix an element $(p_\epsilon, \alpha_0) \in F_{p_\epsilon}(\widetilde{M}_\epsilon)$. We define a section $u: TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon) \to F\left(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)\right)$, as follows. $\alpha_0: \mathbf{R}^n \to T_{p_\epsilon}(\widetilde{M}_\epsilon) = T_0\left(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)\right)$ defines an element $u(0) \in F_0\left(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)\right)$. Then we take u, so that, for each $e \in \mathbf{R}^n$, the section

$$u(e): TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon) \to T(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)); x \longmapsto u(x)(e)$$

satisfies

$$\nabla^{g_\epsilon}_{\frac{\partial}{\partial r}} u(e) = 0,$$

where ∇^{g_ϵ} is the Levi-Civita connection of $(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon), g_{\widetilde{M}_\epsilon})$, and $\frac{\partial}{\partial r}$ is the unit radial vector. (When we identify $T_x\left(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)\right) = Tp_\epsilon(\widetilde{M}_\epsilon)$, we have

$$\frac{\partial}{\partial r} = \frac{x}{|x|} \in TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon) \subset T_{p_\epsilon}(\widetilde{M}_\epsilon) = T_x(TB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon)).\Big)$$

Now, let $\pi_{\epsilon *}: FTB_{1/\epsilon}(p_\epsilon, \widetilde{M}_\epsilon) \to F\widetilde{M}_\epsilon$ be the map induced by the exponential map π_ϵ, which is a local isometry. We take a C^∞-function $\chi: [0, 1/\epsilon) \to [0,1]$ such that

$$\chi(t) \begin{cases} = 1 & \text{for } t \in [0, 1/3\epsilon) \\ = 0 & \text{for } t \in [2/3\epsilon, 1/\epsilon] \end{cases} \tag{9.3.1}$$

$$|\chi'| < 4/\epsilon. \tag{9.3.2}$$

Then, for each $x \in M_\epsilon$, we define a function $h_x: F_xM_\epsilon \to \mathbf{R}$ by

$$h_x(\alpha) = \frac{\sum_{v \in \pi_\epsilon^{-1}(x)} \chi(|v|) \cdot d_{O(n)}(\alpha, \pi_{\epsilon *}(u(v)))}{\sum_{v \in \pi_\epsilon^{-1}(x)} \chi(|v|)}. \tag{9.4}$$

Here we used the isometry $F_xM_\epsilon \simeq O(n)$. Formula (9.2) implies $d(u(v), u(w)) < o(\epsilon)$ for $v, w \in \pi(x)$. Therefore, since $d_{O(n)}(\cdot, \pi_{\epsilon *}(u(v)))$

is convex in a small neighborhood of $\pi_{\epsilon,*}(u(v))$, it follows that h_x assumes its minimum at a unique point, say $\beta(x)$.[20] Thus, we obtain a section $\beta: \widetilde{M}_\epsilon \to F\widetilde{M}_\epsilon$, which gives a connection $\widetilde{\nabla}^\epsilon$ satisfying (9.1.1) and (9.1.2). Next we shall briefly explain the proof of (9.1.3). Let

$$\widetilde{\beta}: TB_{1/\epsilon}(\widetilde{M}_\epsilon, p_\epsilon) \to FTB_{1/\epsilon}(\widetilde{M}_\epsilon, p_\epsilon)$$

be the section induced by β.

Lemma 9.5.
$$\left|\widetilde{\beta} - u\right|_{C^k} < o(\epsilon \mid k)$$

Here $o(\epsilon|k)$ stands for positive numbers depending only on ϵ and k and satisfying $\lim_{\epsilon\to 0} o(\epsilon|k) = 0$. We omit the proof. (See [F4] § 5.) Remark that the connection $\nabla^{(0,\epsilon)}$ corresponds to the section u and that the connection $\widetilde{\nabla}^\epsilon$ corresponds to the section β. Hence, it is easy to see that (9.1.3) follows from Lemma 9.5. The proof of Lemma 9.1 is now complete.

Lemma 9.6. *There exists* ∇^ϵ *such that it satisfies* (9.1.1), (9.1.2) (9.1.3) *and that it is* H*-invariant.*

Proof. Let $\Gamma(F\widetilde{M}_\epsilon)$ be the set of C^∞- sections to the frame bundle. The structure group $O(n)$ acts on $\Gamma(F\widetilde{M}_\epsilon)$ from the right, since $F\widetilde{M}_\epsilon$ is the principal bundle. We can identify the quotient space $\Gamma(FM_\epsilon)/O(n)$ with the set of trivial connections on $\widetilde{M}$. The group H acts on $\Gamma(FM_\epsilon)$ by

$$h(\alpha)(x) = h(\alpha(h^{-1}(x))),$$

where $\alpha \in \Gamma(F\widetilde{M}_\epsilon)$, $x \in \widetilde{M}_\epsilon$, $h \in H$. Let $\beta \in \Gamma(F\widetilde{M}_\epsilon)$ be the element corresponding to $\widetilde{\nabla}^\epsilon$. Since $\nabla^{(0,\epsilon)}$, the Levi-Civita connection, is H-invariant it follows from the Formula (9.1.3) that $\widetilde{\nabla}^\epsilon$ is almost H-invariant. Hence we have a map $\mu: \widetilde{M}_\epsilon \times H \to O(n)$ such that

$$h(\beta(x)) = \beta(hx) \cdot \mu(x,h), \tag{9.7}$$

$$d_{O(n)}(\mu(x,h), \mu(y,h)) < o(\epsilon). \tag{9.8}$$

[20]This is a standard center of mass technique. For detail, see for example [BK] Chapter 8.

Formula (9.7) and [BK] §8 imply that we can approximate $h \mapsto \mu(p_\epsilon, h)$ by a homomorphism $\hat{\mu}$. We define an alternative action ψ of H on $\Gamma(F\widetilde{M}_\epsilon)$ by

$$\psi(h)(\alpha)(x) = h\left(\alpha(h^{-1}(x))\right) \cdot \hat{\mu}(p_\epsilon, h^{-1}),$$

Remark that ψ induces the same action on the set of connections. By (9.8) we have

$$d_{O(n)}(\psi(h)(\beta)(x), \beta(x)) < o(\epsilon), \tag{9.9}$$

For each $x \in \widetilde{M}_\epsilon$ and $h \in H$, we define $\chi_x\colon F_x(M'_\epsilon) \to \mathbf{R}$ by

$$\chi_x(\alpha) = \frac{\sum_{h \in H} d_{O(n)}(\psi(h)(\beta)(x), \alpha)}{\sharp H}$$

Formula (9.9) implies that χ_x assumes its minimum at a unique point $\beta'(x)$. Then, β' is an (H, ψ) invariant section. Hence it defines an H-invariant connection on $F\widetilde{M}_\epsilon$. The proof of Lemma 9.6 is now completed.

Next, we shall deform the connection by Gauge transformation. Let $Gl(\widetilde{M}_\epsilon) = F\widetilde{M}_\epsilon \times_{O(n)} Gl(n, \mathbf{R})$ denotes the associated $Gl(n, \mathbf{R})$ bundle. Using the adjoint representation, we construct the associated bundle $Gl(\widetilde{M}_\epsilon) \times_{\mathrm{Ad}} Gl(n, \mathbf{R})$. Put $\mathcal{G} = \Gamma(Gl(\widetilde{M}_\epsilon) \times_{\mathrm{ad}} Gl(n, \mathbf{R}))$. We have a natural action

$$\mathcal{G} \times \Gamma\left(Gl(\widetilde{M}_\epsilon)\right) \to \Gamma\left(Gl(\widetilde{M}_\epsilon)\right).$$

The set of trivial linear connections on $\widetilde{M}_\epsilon$ can be identified with the quotient space of $\Gamma(Gl(\widetilde{M}_\epsilon))$ by the right action of $Gl(n, \mathbf{R})$. It contains the set of flat $O(n)$ connections as a subset. This inclusion map is identified with the map

$$\Gamma\left(F\widetilde{M}_\epsilon\right)/O(n) \to \Gamma\left(Gl(n, \mathbf{R})\right)/Gl(n, \mathbf{R})$$

induced from the inclusion map : $F\widetilde{M}_\epsilon \to Gl(\widetilde{M}_\epsilon)$. Remark that $\mathcal{G}$ acts on $\Gamma\left(Gl(M'_\epsilon)\right)/Gl(n, \mathbf{R})$.

Lemma 9.10. *For sufficiently small ϵ, there exists an element $g \in \mathcal{G}$ such that the connection $g(\nabla^\epsilon)$ satisfies the following. (∇^ϵ is the connection in Lemma 9.6.)*

(9.10.1) $|\, g - 1\,|_{C^k} < o(\epsilon \mid k)$, *where 1 is the section corresponding to the trivial gauge transformation.*

(9.10.2) *Let*

$$T(g(\nabla^\epsilon))(X,Y) = g(\nabla^\epsilon)_X(Y) - g(\nabla^\epsilon)_Y(X) - [X,Y]$$

be the torsion tensor. Then

$$(g(\nabla^\epsilon))(T(\nabla^\epsilon)) = 0.$$

(9.10.3) $g(\nabla^\epsilon)$ *is H-invariant.*
(9.10.4) g *depends smoothly on* ∇^ϵ.

Sketch of the proof. We regard (9.10.2) as a partial differential equation and solve it by Newton's method. For this purpose we calculate the linearized equation. Let $gl(n;\mathbf{R})$ be the set of all $n \times n$ matrixes. $\Gamma(Gl(\widetilde{M}_\epsilon) \times_{\mathrm{ad}} gl(n;\mathbf{R}))$ is identified to the Lie algebra, Lie($\mathcal{G}$), of $\mathcal{G}$. If $\sigma \in \mathrm{Lie}(\mathcal{G}), \delta \ll 1 + \delta\sigma$ is an element of $\mathcal{G}$. We have

$$\begin{aligned}\frac{d}{d\delta}\left\{(1+\delta\sigma)\left(\nabla^\epsilon\right)_X(Y)\right\}\Big|_{\delta=0} &= \frac{d}{d\delta}\left\{(1+\delta\sigma)(\nabla^\epsilon_X((1-\delta\sigma)(Y)))\right\}\Big|_{\delta=0}\\ &= \sigma(\nabla^\epsilon_X Y) - \nabla^\epsilon_X(\sigma Y)\\ &= \nabla^\epsilon_X(\sigma)(Y).\end{aligned}$$

Therefore

$$\frac{d}{d\delta}\left\{T((1+\delta\sigma)(\nabla^\epsilon))\right\}(X,Y)\Big|_{\delta=0} = \nabla^\epsilon_X(\sigma)(Y) - \nabla^\epsilon_Y(\sigma)(X).$$

Let $\Lambda^k(\widetilde{M}^\epsilon; T\widetilde{M}^\epsilon)$ denote the set of $T\widetilde{M}^\epsilon$-valued k forms on $\widetilde{M}^\epsilon$. Define $d'_k\colon \Lambda^k(\widetilde{M}_\epsilon; T\widetilde{M}_\epsilon) \to \Lambda^{k+1}(\widetilde{M}_\epsilon; T\widetilde{M}_\epsilon)$ by

$$(d'_k\alpha)(X_0,\cdots,X_k) = \sum_{i=0}^{k}(-1)^i(\nabla^\epsilon_{X_i}A)(X_0,\cdots,\widehat{X}_i,\cdots,X_k).$$

Then we have

$$\frac{d}{d\delta}\left\{T((1+\delta\sigma)(\nabla^\epsilon))\right\}(X,Y)\Big|_{\delta=0} = (d'_1\sigma)(X,Y).$$

Therefore the linearized equation for $T(g(\nabla^\epsilon)) = 0$ is

$$d'_1\sigma + T(\nabla^\epsilon) = 0. \tag{9.11}$$

Using the first Bianchi identity,[21] we obtain

$$\begin{aligned} d_2' T(\nabla^\epsilon)(X,Y) &= \mathcal{CYC}\{(\nabla^\epsilon_X T)(Y,Z)\} \\ &= \mathcal{CYC}\{R(X,Y)Z - T(T(X,Y),Z)\}, \end{aligned}$$

where $\mathcal{CYC}$ is the cyclic sum with respect to X, Y and Z. In our case $R = 0$ and $T \ll 1$. Hence we conclude that $d_2' T$ is zero of second order. Therefore there is some hope to solve (9.11) up to second order. We recall Hodge-de Rham-Kodaira decomposition

$$\ker d_k = \operatorname{Im} d_{k-1} \oplus \text{harmonic forms}.$$

Roughly speaking, in our case, the harmonic part corresponds to the parallel part. Therefore we can find σ such that

$$\begin{cases} d_1' \sigma + T(\nabla^\epsilon) = T_1 + T_2 \\ \nabla^\epsilon(T_1) = 0 \\ |T_2| \text{ is zero of second order.} \end{cases}$$

Iterating this construction, we obtain $g \in \mathcal{G}$ such that $(g(\nabla^\epsilon))(T(g(\nabla^\epsilon))) = 0$. Of course there are various difficulties to make the above argument rigorous. The most essential one is the following. Indeed the norm of the torsion tensor goes to zero as ϵ goes to zero. But, then, the manifold $\widetilde{M}_\epsilon$ changes at the same time. Therefore all the constants appeared in the estimates we make during the proof should be independent of ϵ.

We omit the detail of the argument, which is in [R2]. Finally we mention the proof of (9.10.3) and (9.10.4). When we solve the equation (9.11), we put an additional condition[22] so that the solution is unique.[23] This condition is determined by the metric and the connection. Hence the condition and the equation (9.11) are both H-invariant, if so are $\widetilde{M}_\epsilon$ and ∇^ϵ. Therefore, the solution (which is unique) is also H-invariant. As the consequence, the gauge transform g is also H-invariant. The proof of (9.10.4) is similar.

Now we are in the position to complete the proof of Theorem 8.1. Let X_ϵ be the universal covering space of M_ϵ. The connection $g_\epsilon(\nabla)$ induces one, ∇, on X_ϵ. Put

$$\mathcal{N}_\epsilon = \{X \in \Gamma(T(X_\epsilon)) \mid \nabla X \equiv 0\}.$$

[21] See [KN] Chapter III Theorem 5.3 .

[22] See [R2] Mainlemma.

[23] Otherwise there would be no hope to get an estimate on the solution.

For $X, Y \in \mathcal{N}_\epsilon$, we have, by (9.10.2),

$$\begin{aligned}\nabla([X,Y]) &= \nabla(T(X,Y) - (\nabla_X Y - \nabla_Y X)) \\ &= \nabla(T(X,Y)) \\ &= (\nabla T)(X,Y) + T(\nabla X, Y) + T(X, \nabla Y) \\ &= 0.\end{aligned}$$

Hence $(\mathcal{N}_\epsilon, [\cdot,\cdot])$ is a (finite dimensional) Lie algebra. Let N_ϵ be the corresponding Lie group. Fix $p_0 \in X_\epsilon$. Define $I: N_\epsilon \to X_\epsilon$, by

$$I(\exp X) = \phi_1^X(p_0),$$

where $\phi_t^X: X_\epsilon \to X_\epsilon$ is the one parameter group of transformations associated to X, and $\exp: \mathcal{N}_\epsilon \to N_\epsilon$ is the exponential map of Lie group. It is easy to see that I is the diffeomorphism and that $I^*(\nabla)$ is the canonical connection of N. We identify X_ϵ and N_ϵ by I.

Recall the element $\gamma \in \pi_1(M_\epsilon, p_\epsilon; 100)$ is a map : $TB_{100}(p_\epsilon, M\epsilon) \to TB_{200}(p_\epsilon, M_\epsilon)$, which can be extended to a map $\gamma: B_{C_\epsilon}(p_0, N_\epsilon) \to B_{2C_\epsilon}(p_0, N_\epsilon)$, where $\lim_{\epsilon\to 0} C_\epsilon = \infty$. This map γ preserves the connection ∇. Hence it can be extended uniquely to an affine diffeomorphism of (N_ϵ, ∇). Thus the elements of $\pi_1(M_\epsilon, p_\epsilon)$ act on N as affine transformations. On the other hand, the group $\pi_1(M_\epsilon, p_\epsilon)$ acts on N as deck transformations.[24] It is easy to see that $\pi_1(M_\epsilon, p_\epsilon; 100)$ generates $\pi_1(M_\epsilon, p_\epsilon)$ in the group of affine transformations of N_ϵ. Recall that the group of affine transformations of N_ϵ is equal to $N_\epsilon \propto \text{Aut}\, N_\epsilon$. Hence $\pi_1(M_\epsilon, p_\epsilon) \subset N_\epsilon \propto \text{Aut}\, N_\epsilon$. Put $\Lambda_\epsilon = \pi_1(M_\epsilon, p_\epsilon)$ and let Λ'_ϵ be the group generated by $\ker \varphi_\epsilon$.[25] Using Margulis' lemma for Lie group,[26] we can prove that Λ'_ϵ is nilpotent for sufficiently small ϵ. Therefore N_ϵ is a nilpotent Lie group. It is easy to see that $[\Lambda_\epsilon : \Lambda_\epsilon \cap N_\epsilon] < \infty$. The proof of Theorem 8.1 is now complete.

§10. Examples

First we explain the title of this chapter.

Definition 10.1. Let M_i be a sequence of n-dimensional compact Riemannian manifolds and $X \in \mathcal{MET}$. We say that M_i *collapses* to X if $\lim_{i\to\infty} d_H(M_i, X) = 0$ and if the Hausdorff dimension of X is smaller than n.

---

[24]Recall that N_ϵ is identified to the universal covering space of M_ϵ.
[25]Here $\varphi_\epsilon: \pi_1(M_\epsilon, p_\epsilon) \to O(n)$ is as in Lemma 8.8.
[26]See [BK] or [Rag].

In sections 8 and 9, we have studied collapsing Riemannian manifolds to a point. In sections 10,11,12, we shall study collapsing Riemannian manifolds while keeping sectional curvatures and diameters bounded. The first important example was discovered by Berger.

Example 10.2 (Berger sphere). Let $\pi: S^3 \to S^2$ be the Hofp fibration and g_{can} be the standard metric on S^3. For $V \in T(S^3)$, we put

$$g_\epsilon = \begin{cases} \epsilon \cdot g_{can}(V,V) & \text{if } \pi_* V = 0, \\ g_{can}(V,V) & \text{if } V \text{ is perpendicular to the fibre of } \pi. \end{cases}$$

Then we have

$$\sup_{\epsilon \in (0,1]} \left| K_{(S^3, g_\epsilon)} \right| \leq 1.$$

It follows that $(S^3, g_\epsilon) \in \mathcal{M}(3,1)$ for $\epsilon \leq 1$. On the other hand, it is easy to see

$$\lim_{\epsilon \to 0} d_H((S^3, g_\epsilon), (S^2, \overline{g})) = 0.$$

Here $\overline{g}$ is a Riemannian metric on S^2 such that $K_{(S^2, \overline{g})} \equiv 4$.

Remark that $S^2 = S^3/S^1$ in the above example. It is immediate to generalize Example 10.2 to the arbitrary free S^1 action. In fact, the following more general construction is possible.

Proposition-Example 10.3. *Let $(M, g_M) \in \mathcal{M}(n, D)$ and T^k be the k-dimensional torus acting on M by isometry. Assume, for each p,*

$$I_p = \{t \in T^k \mid tp = p\} \neq T^k.$$

Then, there exists a positive number Λ independent of ϵ and a family of metrics g_ϵ on M, such that $g_1 = g_M$ and that

(10.3.1) $$(M, g_\epsilon) \in \mathcal{M}(n, D \mid \Lambda)$$

(10.3.2) $$\lim_{\epsilon \to 0} d_H((M, g_\epsilon), (M/T^k, \overline{g})) = 0,$$

where $\overline{g}$ is the metric on M/T^k induced from g_M.

Construction. Let $I: \mathbf{R} \to T^k$ be the injective homomorphism such that the closure of $I(\mathbf{R})$ is T^k. For $p \in M$ put

$$v(p) = \frac{D}{dt} I(t)(p) \bigg|_{t=0}.$$

Since $I(\mathbf{R})$ is dense in T^k, the assumption $I_p \neq T^k$ implies that v vanishes nowhere. Define g_ϵ on $T_p(M)$ by

$$g_\epsilon(w,w) = \begin{cases} \epsilon \cdot g_M(w,w) & \text{if } w = c \cdot v(p), \\ g_M(w,w) & \text{if } g_M(w, v(p)) = 0. \end{cases}$$

Proof. It is easy to see (10.3.2). We shall calculate the curvature using a normal frame. Let $p \in M$. We choose vector fields $e_1, \ldots, e_{n-k}$ on a neighborhood of p, such that

(10.4.1) $$g_M(e_i, e_j) = \delta_{i,j}, \quad \text{for } i,j = 1, \cdots, n-k$$

(10.4.2) $$e_i \quad (i = 1, \cdots, n-k) \quad \text{are } T^k\text{-invariant.}$$

Put $e_0 = \frac{v}{|v|}$. It is easy to see that we can choose vector fields $e_1, \ldots, e_{n-1}$ on M such that e_i, $i = 0, \ldots, n-1$ are orthonormal frame in a neighborhood of p in (M, g_M). We put

$$[e_i, e_j] = \sum_{k=0}^{n-1} c_{i,j}^k(x)\, e_k.$$

(10.4.2) implies that

(10.5) $$c_{0,i}^j = c_{i,0}^j \equiv 0.$$

Now, the vectors $e_0^\epsilon = e_0/\epsilon$, $e_1^\epsilon = e_1, \ldots, e_{n-1}^\epsilon = e_{n-1}$ are the orthonormal frame of (M, g_ϵ). When we put

$$[e_i^\epsilon, e_j^\epsilon] = \sum_{k=0}^{n-1} c_{i,j}^{(\epsilon)\,k}(x)\, e_k^\epsilon,$$

we have

$$c_{i,j}^{(\epsilon)\,k} = \epsilon^{\sigma(k)-\sigma(i)-\sigma(j)} c_{i,j}^k,$$

where

$$\sigma(i) = \begin{cases} 0 & \text{if } i \neq 0, \\ 1 & \text{if } i = 0. \end{cases}$$

It follows from (10.5) that

$$\left| c_{i,j}^{(\epsilon)\,k} \right| \leq \left| c_{i,j}^k \right|$$

Therefore, since the curvature of (M, g_ϵ) is calculated by the symmetrization of $c_{i,j}^{(\epsilon)\,k}$, it follows that

$$\left|K_{(M,g_\epsilon)}\right| \le \Lambda$$

for some number Λ independent of ϵ. The proof is now complete.

Before proceeding further, we give two examples of infinite families of Riemannian manifolds constructed from torus fibrations.

Example 10.6 (Wallach [Wa1], [Wa2], [WA]). Put

$$T^2 = \left\{ \left(\begin{array}{ccc} e^{2\pi i\theta} & 0 & 0 \\ 0 & e^{2\pi i\varphi} & 0 \\ 0 & 0 & e^{-2\pi i(\theta+\varphi)} \end{array} \right) \middle| \theta, \varphi \in \mathbf{R} \right\} \subset SU(2)$$

Let $S^1_{(n,m)}$ be the subgroup

$$\frac{(\mathbf{R} \cdot (n,m)) \cdot (\mathbf{Z} \oplus \mathbf{Z})}{\mathbf{Z} \oplus \mathbf{Z}},$$

of T^2. Let $M_n = SU(2)/S^1_{(n,n+1)}$. It is easy to see that

$$\lim_{n\to\infty} d_H(M_n, SU(2)/T^2) = 0.$$

On the other hand, locally, M_n comes to be similar and similar to $SU(2)/S^1_{(1,1)}$, when n goes to infinity. Moreover there exists $C_1, C_2 > 0$ such that $C_1 > K_{SU(2)/S^1_{(1,1)}} > C_2$. Therefore,

$$C_1 > K_{M_n} > C_2,$$

for large n. In other words, M_n is a sequence of uniformly positively curved manifolds which collapses to $SU(2)/T^2$. Remark that M_n is a S^1 bundle over $SU(2)/T^2$.

Example 10.7 (Wang-Ziller [WZ]). Let M_i be a Kähler-Einstein manifold with $c_1(M_i) > 0$. Put $c_1(M_i) = q_i\alpha_i$, where α_i is indivisible. Let P be the total space of a principal torus bundle over $B = \Pi M_i$ whose characteristic classes in $H^2(B;\mathbf{Z})$ are linear combination of α_i. Assume that P is simply connected. Then Wang and Ziller proved that P admits an Einstein metric with positive scalar curvature. They also proved that, if we normalize the metric so that the Ricci curvature is equal to the metric tensor, then the absolute value of the sectional curvature of P is estimated by a number depending only on B. We can find an infinite number of P_i satisfying the above conditions. In that case we can prove that P_i converges to B with respect to the Hausdorff distance.

So far, we have constructed examples by making use of the action of the torus (an abelian group.) Next we use a nilpotent group.

Theorem-Example 10.8. *Let N be a nilpotent Lie group, Λ is a discrete subgroup of $N \propto \mathrm{Aut}\, N$ such that N/Λ is compact and that $[\Lambda : \Lambda \cap N] < \infty$. Then, there exists a family of metrics g_ϵ on N/Λ such that*

$$(N/\Lambda, g_\epsilon) \in \mathcal{M}(n, D), \tag{10.8.1}$$

$$\lim_{\epsilon \to 0} d_H((N/\Lambda, g_\epsilon), \mathrm{point}) = 0. \tag{10.8.2}$$

Construction. Let n be the Lie algebra of N. Put

$$\begin{cases} n_0 = n \\ n_1 = [n, n_0] \\ \quad \cdots \\ n_{k+1} = [n, n_k] \\ \quad \cdots \end{cases}$$

The group $H = \Lambda/\Lambda \cap N$ acts on n. It preserves the stratification

$$n_K = \{0\} \subset \cdots \subset n_k \subset \cdots \subset n_0 = n.$$

Choose an H-invariant positive definite quadratic form g on n, and define g_ϵ by :

$$g_\epsilon(v, v) = \epsilon^{2^k} g(v, v)$$

if $v \in n_k$, and if $g(v, n_{k+1}) = 0$. Then g_ϵ is an H-invariant quadratic form on n. Hence, g_ϵ defines a Λ-invariant metric on N. It induces a Riemannian metric g_ϵ on N/Λ.

Proof. The verification of (10.8.2) is easy. To show (10.8.1), we recall the formula

$$|K_{g_\epsilon}(V \wedge W)| = \frac{1}{4} \cdot \frac{|[V, W]|_{g_\epsilon}}{|V \wedge W|_{g_\epsilon}} \tag{10.9}$$

for $V, W \in n = T_e(N) = T_{[e]}(N/\Lambda)$.[27] Hence, it suffices to estimate the right hand side of (10.9). Let $V \in n_k$, $W \in n_\ell, g(V, n_{k+1}) =$

[27]For the proof of (10.9), see for example [BK] Chapter 7.

$g(W, n_{\ell+1}) = 0$, $g(V,W) = 0$, $k \le \ell$. Then we have

$$|V \wedge W|_{g_\epsilon} = |V|_g |W|_g \epsilon^{2^k + 2^\ell} \le |V|_g |W|_g \epsilon^{2^{k+1}}.$$

Therefore, since $[V,W] \subset n_{k+1}$, it follows that

$$|[V,W]|_{g_\epsilon} \le \epsilon^{2^{k+1}} |[V,W]|_g.$$

Hence

$$\frac{|[V,W]|_{g_\epsilon}}{|V \wedge W|_{g_\epsilon}} \le \frac{|[V,W]\ |_g}{|V|_g \mid W|_g} \le C,$$

where C is a number independent of ϵ. It follows that

$$\left| K_{(N/\Lambda, g_\epsilon)} \right| \le C.$$

By rescaling the metric if necessary, we obtain (10.8.1).

A similar construction is possible for solvemanifold.

Theorem-Example 10.10. *Let G be a solvable Lie group and $\Gamma \subset G \propto \mathrm{Aut}\, G$ is a discrete subgroup. Suppose that G/Γ is compact and that $[\Gamma : \Gamma \cap G] < \infty$. Then, there exists a family of metrics g_ϵ on G/Γ such that*

$$(G/\Gamma, g_\epsilon) \in \mathcal{M}(n, D)$$
$$\lim_{\epsilon \to 0} d_H((G/\Gamma, g_\epsilon), [G,G] \setminus G/\Gamma) = 0.$$

Here $[G,G] \setminus G/\Gamma$ is the quotient of the abelian group $G/[G,G]$ by a discrete group. Hence it is a flat (V-)manifold.

We omit the construction.[28] In the above example, there exists a fibration $G/\Gamma \to [G,G] \setminus G/\Gamma \cap [G,G]$, whose fibre $[G,G]/\Gamma \cap [G,G]$ is a nilmanifold. We can generalize Example 10.10 to such fibrations.

Theorem-Example 10.11. *Let $M \in \mathcal{M}(n,D)$, $M' \in \mathcal{M}(n',D)$, $n' \le n$, and $f\colon M \to M'$ be a fibration such that*

(10.12.1) *The fibre of f is diffeomorphic to N/Λ, where N and Λ is as in Example* 10.8,

(10.12.2) *The structure group is contained in*

$$\frac{\mathrm{Cent}\ N}{\mathrm{Cent}\ N \cap \Lambda} \propto \mathrm{Aut}\ \Lambda,$$

[28]See [F4].

where Cent N *is the center of* N.[29]
Then, there exists family of metrics g_ϵ *on* M *such that*

$$\lim_{\epsilon\to 0} d_H((M,g_\epsilon),M')=0 \tag{10.13.1}$$

$$(M,g_\epsilon)\in \mathcal{M}(n,D). \tag{10.13.2}$$

Sketch of the proof. Put $H=\Lambda/\Lambda\cap N\subset \mathrm{Aut}\,N$. Recall that the set of locally homogeneous metrics on N/Λ has a one to one correspondence to the set of all H-invariant quadratic forms on the Lie algebra n of N. Hence, we can construct a smooth family of Riemannian metrics on the fibres of f by defining a metric on the n vector bundle associated to the bundle f. (The condition (10.12.2) certifies that f has an associated n bundle.) We can decompose $T(M)$ to the horizontal and the vertical directions. Then, we can define the Riemannian metric on M by using the metric on M' for the horizontal direction and by using the family of metrics on the fibres for the vertical direction. The estimate of the curvature of the metric is the combination of the methods presented in the proofs of Examples 10.3 and 10.8. The detail of the proof is found in [F4] § 6.

The examples of this section can be analyzed also from the point of view of F-structures. See section 19 for it.

§11. A compactification of $\mathcal{M}(n,D)$

In this section, we shall generalize Theorem 4.1 to $\mathcal{M}(n,D)$.

Theorem 11.1 (Fukaya [F6] Theorem 0.6). *Let* $M_i\in\mathcal{M}(n,D)$ *and* $(X,d)\in\mathcal{MET}$. *Assume* $\lim_{i\to\infty} d_H(M_i,(X,d))=0$. *Then we have the following :*

There exists a C^∞*-manifold* N *with a* $C^{1,\alpha}$*-metric* g_N, *on which there is a smooth and isometric action of the orthogonal group* $O(n)$ *such that*

(11.1.1) (X,d) *is isometric to* $(N,g_N)/O(n)$.
(11.1.2) *For each* $p\in N$, *the isotropy group* $I_p=\{\gamma\in O(n)\mid \gamma p=p\}$ *is an extension of a torus* T^k *by a finite group.*

[29]The skew product is constructed by the following action of Aut Λ to N : an element γ of Aut Λ preserves $\Lambda\cap N$, and the action of γ on $\Lambda\cap N$ can be uniquely extended to an automorphism of N, because of the Marcev's theorem (see [BK] or [Rag]).

Open problem 11.2. Let $(N, g_N)/O(n)$ satisfy (11.1.1) and (11.1.2). Is $(N, g_N)/O(n)$ contained in the closure of $\mathcal{M}(n, \infty)$?

Proof of Theorem 11.1. Let FM_i be the frame bundle of M_i. We define a Riemannian metric on FM_i so that $\pi: FM_i \to M_i$ is a Riemannian submersion, the fibres of π are isometric to the orthogonal group with the standard metric and that the horizontal and the vertical directions with respect to the Levi-Civita connection are perpendicular. Then, $O(n)$ acts on FM_i by isometry such that $FM_i/O(n)$ is isometric to M_i. By O'Neill formula,[30] we have $|K_{FM_i}| \le C$, where C is independent of i. Hence, by Theorem 6.9, we can find a subsequence such that $(FM_i, O(n))$ converges to an element $(N, O(n))$ of $\mathcal{MET}(O(n))$ with respect to the $O(n)$-Hausdorff distance. Then Lemma 6.13 implies that

$$\lim_{i \to \infty} d_H(FM_i, N/O(n)) = 0.$$

Hence $d_H(X, N/O(n)) = 0$. It follows from Theorem 1.5 that X and $N/O(n)$ are isometric.

In the rest of this section we shall prove that N is a C^∞ manifold, the action of $O(n)$ is of C^∞ class and that the metric on N is a Riemannian metric of $C^{1,\alpha}$ class.

First, using Theorems 5.1, 5.2, we obtain M_i^ϵ such that

$$\text{(11.3)} \qquad \begin{cases} d_L(FM_i, FM_i^\epsilon) \le \epsilon \\ \left| \left(\nabla^{FM_i^\epsilon}\right)^k (R(FM_i^\epsilon)) \right| < C(\epsilon, k). \end{cases}$$

We may assume that FM_i^ϵ converges to N^ϵ with respect to the $O(n)$-Hausdorff distance. Then, by Lemma 1.8, we have $d_L(N^\epsilon, N) \le \epsilon$. Hence, by Theorems 4.1 and 6.9, it suffices to show that N^ϵ is a C^∞-Riemannian manifold. Hereafter we shall write M_i, N instead of M_i^ϵ, N^ϵ. Take a Hausdorff approximation $\psi_i: N \to M_i$ and $p_0 \in N$. We shall prove that a neighborhood of p_0 is a C^∞-Riemannian manifold. Put $p_i = \psi_i(p_0)$ and $\overline{p}_i = \pi(p_i) \in M_i$. Then, by Lemma 7.9, we have

$$B_1(\overline{p}_i, M_i) = TB_1(\overline{p}_i, M_i)/\pi_1(\overline{p}_i, M_i; 1).$$

By taking a subsequence if necessary, we may assume that the sequence of the pairs $(TB_1(\overline{p}_i, M_i), \pi_1(\overline{p}_i, M_i; 1))$ converges to a pair (X, G) with

[30] See [O] p.476.

respect to the equivariant Hausdorff distance. Since

$$\begin{cases} \operatorname{inj}_{TB_1(p_i,M_i)}(0) > 1/2 \\ \left|\left(\nabla^{FM_i}\right)^k (R(FM_i))\right| < C(k). \end{cases}$$

Proposition 5.10 implies that X is a C^∞ Riemannian manifold. Remark

$$B_1(p_i, FM_i) \subset F(B_1(\overline{p}_i, M_i)).$$

Therefore

$$B_1(p_0, N) \subset FX/G.$$

Since the action of G on X is effective and isometric, it follows that the action of G on FX is free. We conclude that $B_1(p_0, N)$ is a C^∞-Riemannian manifold.

Next we shall verify (11.1.2). First we remark that if $\widetilde{p} \in FX$, $p \equiv \widetilde{p} \mod G \in B_1(p_0, N)$, $\bar{p} = \pi(\widetilde{p}) \in X$, then

$$I_p = \{\gamma \in O(n) \mid \gamma p = p\} \simeq \{g \in G \mid g\bar{p} = \bar{p}\} = I'_{\bar{p}}.$$

On the other hand, $I'_{\bar{p}}$ is a compact subgroup of G. Theorem 7.18 implies that the connected component of G is nilpotent. Therefore the connected component of $I'_{\bar{p}}$ is nilpotent. The compactness of $I'_{\bar{p}}$ implies that $I'_{\bar{p}}$ and I_p satisfy (11.1.2). The proof of Theorem 11.1 is now completed.

Finally we remark :

Definition 11.4. A metric space X is said to be a *Riemannian orbifold* (V-manifold) if for each $p \in X$ there exists a neighborhood U of p such that U is isometric to the quotient of a Riemannian manifold by an action of a finite group.

Proposition 11.5. *If we assume* $\dim X = n-1$ *in Theorem* 11.1, *then the group I_p is finite. In other words, X is a Riemannian orbifold.*

Proof. We use the notation in the proof of Theorem 11.1. Since $\dim X = n - 1$, it follows that $\dim G = 1$. Therefore, there exists $\gamma_i \in \pi_1(M_i, p_i; 1)$ such that $\gamma_i \neq 1$ and that $\lim_{i\to\infty} d(\gamma_i(p_i), p_i) = 0$. Let δ be an arbitrary small positive number. We can choose n_i such that

$$0 < \lim_{i\to\infty} d(\gamma_i^{n_i}(p_i), p_i) \leq \delta.$$

By taking a subsequence it necessary, we may assume that $\gamma_i^{n_i}$ converges to an element g_δ of G. Then

$$0 < d(g_\delta(\bar{p}), \bar{p}) \leq \delta. \tag{11.6}$$

Since (11.6) holds for arbitrary small δ, we can find a subgroup Λ of G of positive dimension such that p is not a fixed point of any nontrivial elements of Λ. Therefore, since $\dim G = 1$, it follows that the group $I'_{\bar{p}} = \{g \in G \mid g\bar{p} = \bar{p}\}$ is discrete.

§12. Fibre bundle theorem

In this section we shall combine Theorems 3.2 and 8.1 and prove the following :

Theorem 12.1 (Fukaya [F4], [F5]). *Let $M_i \in \mathcal{M}(n, D)$ and N be compact manifolds. Assume that $\lim_{i\to\infty} d_H(M_i, N) = 0$. Then, for each large i, there exists $f_i : M_i \to N$ such that*

(12.1.1) *f_i is a fibration.*

(12.1.2) *The fibre of f_i is diffeomorphic to N_i/Λ_i, where N_i and Λ_i are as in Example* 10.8.

(12.1.3) *The structure group of f_i is contained in*

$$\frac{\mathrm{Cent}\, N_i}{\mathrm{Cent}\, N_i \cap \Lambda_i} \propto \mathrm{Aut}\, \Lambda_i.$$

(12.1.4) *f_i is an almost Riemannian submersion. In other words, for each $V \in TM_i$ perpendicular to the fibre of f_i, we have*

$$e^{-o(i)} < \frac{|f_{i*}(V)|}{|V|} < e^{o(i)},$$

where $o(i)$ satisfies $\lim_{i\to\infty} o(i) = 0$.

Remark 12.2. We can replace the assumption by : N is a complete Riemannian manifold such that $|K_N| \le C < \infty$, $\mathrm{inj}_N > c > 0$: M_i is a sequence of complete Riemannian manifolds such that $|K_{M_i}| \le 1$. In other words, N is of bounded geometry, and M_i has uniformly bounded curvature.

Sketch of the proof. First we construct f_i. The construction is a modification of one in section 3. We define the map $I : N \to L^2(N)$ by Formula (3.6.1). We modify[31] the definition (3.6.2) of $I_i\colon M_i \to L^2(N)$

[31] This modification is necessary because, in our case, I_i in (3.6.2) is no longer of C^1 class since we have no estimate from below of the injectivity radius of M_i.

to

$$(12.3) \qquad I_i^\epsilon(p_i)(q) = \chi\left(\frac{\int_{x\in B_\epsilon(p_i,M_i)} d(x,\varphi_i(q))\,dx}{\mathrm{Vol}(B_\epsilon(p_i,M_i))}\right).$$

Recall that $\varphi_i\colon N \to M_i$ is a Hausdorff approximation. Roughly speaking, I_i^ϵ is an average of I_i over a small ball. Then, I_i^ϵ is of C^1-class since I_i is almost everywhere smooth and is uniformly Lipschits.[32]

Now $\pi\colon B_\delta(I(N)) \to N$ is defined as in section 3. The proof of Lemma 3.7 works without any change. Hence we obtain a map $f_i\colon M_i \to N$.

To show that f_i is a fibration, it suffices to see that f_i is of maximal rank. Hence it suffices to show Lemma 3.9 in our case. Unfortunately, the argument of the proof of Lemma 3.9 itself is not enough in our situation, because we used Toponogov's Theorem there, which does not necessarily hold for M_i. Hence we have to prove a version of triangle comparison theorem. We omit the detail, which can be found in sections 2 and 3 of [F4].

Next we shall verify (12.1.2) and (12.1.3). For this, first we have to replace f_i by f_i' which has uniform estimate on higher derivatives. As we remarked before the function I_i is not of C^2 class. Hence the map f_i which we have constructed has no estimate on higher derivatives. This difficulty is somewhat similar to one we met at section 5, where the normal coordinate is not enough smooth as we need. Here the maps f_i, which we constructed by using distance function, are not smooth. The method we used in section 5 is to use the harmonic coordinate, namely the solutions of Laplace equations. In our case, we can use eigenfunctions of Laplace operator instead of distance function and can construct an embedding similar to I_i^ϵ. In order to make use of this embedding to our purpose, we have to know C^1-convergence of eigenfunctions. We omit those arguments and refer [F3] and [F5]. The result we obtain is the following :

Lemma 12.4 ([F5] Lemma 1.6). *Let M_i and N be as in Theorem 12.1. Assume*

$$(12.5) \qquad \left|\nabla^k R_{M_i}\right| < C(k),$$

[32]Remark that we can not expect that I_i^ϵ or any such smoothing of I_i has uniform bound on C^2-norm, because the second derivative of I_i has the same size as the volume of the cut locus, which we can not control in our situation.

in addition. Then we have $f_i: M_i \to N$ *such that* f_i *satisfies* (12.1.1) *and that*

$$\left| \frac{\partial^{|\alpha|}(\psi_j^{-1} \circ \pi_i \circ \exp_{x,r})}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n} \right| \le C_\alpha, \tag{12.6}$$

where $\exp_{x,r}: B(r) \to M$ *is the normal coordinate centered at* $x \in M_i$*, and* $\psi_j: \mathbf{R}^n \to N$ *is a coordinate system independent of* i.

Here we remark that we can assume (12.5) without loss of generality. In fact by Theorems 5.1 and 5.2, we can find $M_{i,\epsilon}$ satisfying (12.5) and $d_L(M_i, M_{i,\epsilon}) < \epsilon$. Let N_ϵ be the Hausdorff limit of $M_{i,\epsilon}$. Then, by Lemma 1.8, $d_L(N, N_\epsilon) \le \epsilon$. Hence by Theorem 4.1, N and N_ϵ are diffeomorphic for small ϵ. We can use $M_{i,\epsilon}$ and N_ϵ instead of M_i and N.

Now using Lemma 12.4, we see that the second fundamental forms of the fibres of f_i are uniformly bounded. It follows that their sectional curvatures are uniformly bounded. On the other hand, since f_i are ϵ_i-Hausdorff approximations for some ϵ_i with $\lim_{i\to\infty} \epsilon_i = 0$. It follows that

$$\lim_{i\to\infty} \left| K_{f_i^{-1}(\{\text{point}\})} \right| \cdot \text{Diam}(f_i^{-1}(\{\text{point}\})^2 = 0.$$

This formula and Theorem 8.1 imply (12.1.2).

To prove (12.1.3), we need to recall the proof of Theorem 8.1. In section 9, we constructed, for each M satisfying (8.2), a connection ∇ such that $R(\nabla) = 0, \nabla(T(\nabla)) = 0$. We use the following parametrized version of this construction.

Lemma 12.7 (Fukaya [F5] Theorem 1.1). *Let* $f_i: M_i \to N$ *be as above. Then we have :*

(12.7.1) *For each* $p \in N$ *there exists a flat connection* ∇^p *on* $f_i^{-1}(p)$ *depending smoothly on* p.

(12.7.2) *There exists a nilpotent Lie group* N_i *and a discrete subgroup* Λ *of* $N_i \propto \text{Aut}\, N_i$ *satisfying the conditions of Example* 10.8 *such that* $(f_i^{-1}(p), \nabla^p)$ *is affinely diffeomorphic to* N_i/Λ_i *for each* p.

Sketch of the proof. In section 9, we gave a method to construct a flat connection on a given almost flat manifold. If the connection constructed there depends smoothly on the given almost flat metric, then there is nothing to show. But we used a fixed base point on the manifold in the construction of Lemmas 9.1 and 9.6. Therefore, if the fibration f_i has a section then we have a smooth family of connections satisfying the conclusion of Lemma 9.6. In general case, we can estimate the difference of the two connections determined by the distinct base points and show that the difference is very small with respect to the C^∞-topology. Hence using a rigidity of the structure we can patch the connections and obtain a family satisfying the conclusion of Lemma 9.6. Next using (9.10.4) we can modify this family and obtain the desired family of connections.[33]

Now using Lemma 12.7 we can prove (12.1.3). We can prove that the group of affine diffeomorphisms of N_i/Λ_i is equal to

$$\frac{N_i}{\mathrm{Cent}\, N_i \cap \Lambda_i} \propto \mathrm{Aut}\, \Lambda_i.$$

Hence Lemma 12.7 implies that the structure group can be reduced to this group. On the other hand, the maximal compact subgroup of $\frac{N_i}{\mathrm{Cent}\, N_i \cap \Lambda}$ is $\frac{\mathrm{Cent}\, N_i}{\mathrm{Cent}\, N_i \cap \Lambda}$. Therefore the structure group can be reduced to

$$\frac{\mathrm{Cent}\, N_i}{\mathrm{Cent}\, N_i \cap \Lambda_i} \propto \mathrm{Aut}\, \Lambda_i,$$

as required. We omit the proof of (12.1.4).

Next we study the case when the limit space is not necessary a manifold.

Theorem 12.8 (Fukaya [F6] Theorem 10.1). *Let $M_i \in \mathcal{M}(n, D)$, $X \in \mathcal{MET}$. Assume $\lim_{i\to\infty} d_H(M_i, X) = 0$. Theorem* 11.1 *implies $X = N/O(n)$. Then, for each sufficiently large i, there exists $f_i\colon M_i \to X$ and $\widetilde{f}_i\colon FM_i \to N$ such that*

(12.8.1) $\widetilde{f}_i$ *satisfies* (12.1.1), ...,(12.1.4).

(12.8.2) $\widetilde{f}_i$ *is $O(n)$ map. The flat connections constructed in Theorem* 12.7 *on the fibres on $\widetilde{f}_i$ are $O(n)$-invariant.*

(12.8.3) *The following diagram commutes :*

$$\begin{array}{ccc} FM_i & \xrightarrow{\widetilde{f}_i} & N \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ M_i & \xrightarrow{f_i} & X = N/O(n) \end{array}$$

[33] The detail of the proof is in [F5].

This theorem follows from the following equivariant version of Theorem 12.1. (We apply Theorem 12.1′ to $FM_i = M_i$ and $N = N$.)

Theorem 12.1′. *Let M_i and N be as in Theorem 12.1. Assume that a compact group G acts on M_i and N by isometry, and $\lim_{i\to\infty} d_{G-H}(M_i, N) = 0$. Then we have $f_i: M_i \to N$ which satisfies (12.1), ...,(12.4). Furthermore f_i is a G-map and the affine structures of the fibres are G-invariant.*

Sketch of the proof. We recall the construction of the map in the proof of Theorem 12.1. Our map will be a G-map if so is the embedding $I_i^\epsilon: M_i \to L^2(N)$. We replace I_i^ϵ by

$$I_i'(p_i)(q) = \chi\left(\int_{g\in G} \frac{\int_{x\in B_\epsilon(p_i,M_i)} d(gx, \varphi_i(gq))\,dx}{\mathrm{Vol}(B_\epsilon(p_i,M_i))}\,dg\right),$$

which is a G-map. Then the same construction as Theorem 3.2 works and we obtain a G-fibration. To make the affine structure G-invariant, we use the method of proof of Lemma 9.6. We omit the detail.

§13. Margulis' Lemma

The purpose of this section is to show the following :

Theorem 13.1. *There exists a positive number ϵ_n depending only on n, the dimension, and satisfying the following.*

Suppose that $M \in \mathcal{M}(n,\infty)$, $p \in M$. Then there exists an open neighborhood U of p in M such that

(13.1.1) *U is diffeomorphic to a vector bundle (in the categery of orbifold) over N/Λ, where $\Lambda \subset N \propto \mathrm{Aut}\, N$ is as in Example* 10.8.

(13.1.2) *U contains $B_{\epsilon_n}(p, M)$.*

Remark 13.2. We have dim $N < n$ unless Diam $M < \epsilon_n$.

Remark 13.3. It is easy to see that Theorem 7.18 follows from Theorem 13.1.

Proof. The proof is by contradiction. We assume that there exists $M_\epsilon \in \mathcal{M}(n,\infty), p_\epsilon \in M_\epsilon$ such that $B_\epsilon(p_\epsilon, M_\epsilon)$ is not contained in any neighborhood U satisfying (13.1.1) and (13.1.2). By Theorem 6.6, we may assume that there exists $(X,q) \in \mathcal{MET}_0$ such that

$$\lim_{\epsilon\to 0} d_{p.H.}((M_\epsilon,p_\epsilon),(X,q)) = 0.$$

Then we can find $V_\epsilon \supseteq B_1(p_\epsilon, M_\epsilon)$, $V \supseteq B_1(q, X)$, Y, $f_\epsilon : V_\epsilon \to V$, and $\widetilde{f}_\epsilon : FM_\epsilon \to Y$ such that :[34]

(13.4.1) $V = Y/O(n)$,
(13.4.2) $\widetilde{f}_\epsilon$ is an $O(n)$-map,
(13.4.2) $\widetilde{f}_\epsilon$ satisfies (12.1.1),$\cdots$,(12.1.4).
(13.4.3) The following diagram commutes.

$$\begin{array}{ccc} FV_\epsilon & \xrightarrow{\widetilde{f}_\epsilon} & Y \\ \pi\downarrow & & \pi\downarrow \\ V_\epsilon & \xrightarrow{f_\epsilon} & V \end{array}$$

Since, $V = Y/O(n)$, we can choose a small neighborhood U of q in V, such that

$$U \simeq \partial U \times [0,1] / \sim,$$

where $(x,t) \sim (y,s) \Longleftrightarrow t = s = 1$. Hence, by (13.4), there exists

$$\pi : f_\epsilon^{-1}(\partial U) \to f_\epsilon^{-1}(q)$$

such that

$$f_\epsilon^{-1}(U) \simeq f_\epsilon^{-1}(\partial U) \times [0,1] / \sim,$$

where

$$(x,t) \sim (y,s) \Longleftrightarrow \begin{cases} t = s = 1 \\ \pi(x) = \pi(y). \end{cases}$$

Hence $f_\epsilon^{-1}(q)$ is a deformation retract of $f_\epsilon^{-1}(U)$. Therefore, since $f_\epsilon^{-1}(q) \simeq N/\Lambda$ where N and Λ is as in Example 10.8, it follows easily that $f_\epsilon^{-1}(U)$ satisfies (13.1.1). On the other hand, $f_\epsilon^{-1}(U) \supset B_\epsilon(p_\epsilon, M_\epsilon)$. This is a contradiction.

Finally, we prove Sublemma 8.11. The proof is by induction on dimension. We assume that Sublemma 8.11 (and hence all the results of this chapter) is valid when dimension is smaller than n. Let M_ϵ be a family of n-dimensional Riemannian manifolds satisfying (8.7.1) and (8.7.2). We are to show that the connected component G_0 of the group

$$G = \lim_{\epsilon \to 0} \pi_1(M_\epsilon, p_\epsilon; \epsilon_n)$$

[34](13.4) is a pointed version of Theorem 12.1, and can be proved in a same way.

is nilpotent. This fact is immediate if Theorem 7.18 (or 13.1) holds for M_ϵ. Recall that $\mathrm{Diam}\, M_\epsilon = 1$. Hence, as we remarked in 13.2, we need to apply Theorem 8.1 to a manifold with dimension smaller than n in order to prove Theorem 13.1 for M_ϵ. Therefore, by induction hypothesis, the conclusion of Theorem 13.1 holds for M_ϵ. The proof of Sublemma 8.11 is now complete.

Chapter III. Applications

§14. Finiteness theorems

The following result is first proved without using Hausdorff convergence. The proof we give below is due to Gromov [G7].

Theorem 14.1 (Cheeger [C2], Peters [Pe1]). *For each $n, D, v > 0$, the number of diffeomorphism classes contained in $\mathcal{M}(n,D,v)$ is finite.*

Proof. By Theorem 4.1, there exists $\epsilon = \epsilon(n,D,v)$ such that $M, N \in \mathcal{M}(n,D,v)$ and $d_H(M,N) < \epsilon$ imply that M and N are diffeomorphic. On the other hand, since $\mathcal{M}(n,D,v)$ is totally bounded (Theorem 2.2), there exist a finite number of elements $M_1, \cdots, M_k$ such that, if M is an element of $\mathcal{M}(n,D,v)$, then $d_H(M,M_i) < \epsilon$ for some $i \leq k$. Hence M is diffeomorphic to M_i. Therefore the number of diffeomorphism classes contained in $\mathcal{M}(n,D,v)$ is not greater than k.

For the class $\mathcal{M}(n,D)$, we have the following :

Theorem 14.2 (Fukaya [F6] Theorem 0.15). *For each n, D, there exists a finite set Σ of closed manifold such that the following holds. If $M \in \mathcal{M}(n,D)$, then there exists $N \in \Sigma$ and a map $f\colon FM \to N$ satisfying* (12.1.1), ..., (12.1.4).

The proof is an application of Theorems 12.8 and 2.2 and is given in [F6]. Theorem 14.2 gives an alternative proof of a weaker version of the following :

Theorem 14.3 (Gromov [G4]). *There exists a positive number $C_n(D)$[35] such that the following folds. Let M be an n-dimensional Riemannian manifold satisfying*

(14.4.1) $\qquad K_M \geq -1$

(14.4.2) $\qquad \mathrm{Diam}\, M \leq D$

[35] Gromov gave a huge but explicit number.

then we have

$$\Sigma \operatorname{rank} H_i(M;K) \leq C_n(D),$$

where K is an arbitrary field.

Remark 14.5. It seems to the author that the conclusion of Theorem 4.2 does not hold under the weaker assumption

$$\mathrm{Ricci}_M \geq -(n-1).$$

Theorem 14.3 does not follow from 14.2. To apply Theorem 14.2 we have to assume $K_M \leq 1$, in addition. We have few informations on the limit of a sequence of manifolds satisfying (14.4). If we assume $\mathrm{Vol}\, M \geq v$ in addition to (14.4), an interesting result has recently been obtained by K.Grove, P.Petersen and J.Wu.

Theorem 14.6 (Grove-Petersen-Wu [GPW], [GP1]). *Let M_i be a sequence of Riemannian manifolds satisfying* (14.4.1), (14.4.2), dim M_i > 4, *and* Vol $M_i \geq v > 0$ *for a number v independent of i. Suppose that M_i converges to an element X of $\mathcal{MET}$ with respect to the Hausdorff distance. Then, X is a topological manifold and is homeomorphic to M_i for large i.*

Open problem 14.7. In the situation of Theorem 14.6, does M_i converges to X with respect to the Lipschits distance ?

Example 14.8. Let P^n be a boundary of an $n+1$-dimensional convex polyhedron in $\mathbf{R}^{n+1}$. We can find a sequence of n-dimensional Riemannian submanifolds $M_i \subset \mathbf{R}^{n+1}$ such that

(14.8.1) $$K_{M_i} \geq 0,$$

(14.8.2) $$\lim_{i\to\infty} d_H(M_i, P) = 0,$$

where we give the induced Riemannian metric to M_i and the induced inner metric to P. Then we have a homeomorphism $M_i \to P$, but it seems that P has no natural differentiable structure.

Grove, Petersen and Wu used Theorem 14.6 to show the following :

Theorem 14.9 (Grove-Petersen-Wu [GPW], [GP1]). *For each D, $n > 4$ and $v > 0$, There exist only a finite number of diffeomorphism classes containing n-dimensional Riemannian manifolds satisfying* (14.4) *and* Vol $M \geq v$.

Sketch of the proof of Theorem 14.6. We use the following criteria for a metric space to be a topological manifolds due to Edwards and Quinn.[36] A compact metric space is a topological manifold if :

(14.10.1) X is an ANR.
(14.10.2) X is a homology manifold. In other words, $H_*(X, X - \{p\}) \simeq H_*(\mathbf{R}^n, \mathbf{R}^n - \{0\})$, for each $p \in X$.
(14.10.3) X satisfies disjoint disk property. Namely for each $f_1, f_2 \colon D^2 \to M$ and $\epsilon > 0$, there exists $f_1^\epsilon, f_2^\epsilon \colon D^2 \to M$ such that

$$d(f_1(x), f_1^\epsilon(x)) < \epsilon$$
$$d(f_2(x), f_2^\epsilon(x)) < \epsilon$$
$$f_1^\epsilon(D^2) \cap f_2^\epsilon(D^2) = \phi.$$

The verification of these properties for our limit space, X, is based on the following result in Grove-Petersen [GP1]. We assume

(14.11.1) $$K_M > -1$$
(14.11.2) $$\text{Diam } M \leq D$$
(14.11.3) $$\text{Vol } M \geq v > 0.$$

Lemma 14.12 (Grove-Petersen [GP1] 1.3). *There exists $\alpha(D, v) > 0$ and $r(D, v) > 0$ such that for each $x, y \in M$ with $d(x, y) < r$ we have one of the following :*

(14.12.1) *There exists $v \in T_x(M)$ such that $\angle v\dot{\ell}(0) > \pi + \alpha$, for each minimal geodesic ℓ joining x and y.*
(14.12.2) *There exists $v \in T_y(M)$ such that $\angle v\,\dot{\ell}(0) > \pi + \alpha$, for each minimal geodesic ℓ joining y and x.*

Lemma 14.12 is proved by a volume comparison argument. It implies the following. Put

$$U = \{(x, y) \in M \times M \mid d(x, y) < r\},$$
$$U_0 = \{(x, y) \in U \mid x \neq y\}.$$

Lemma 14.13. *There exists a vector field V on U_0 such that $V(d) < -\beta$ for a positive number β depending only on v and n. Here $d \colon M \times M \to \mathbf{R}$ is the distance.*

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[36] See [Ed] and [Q].

Using this vector field V, we can construct a deformation retract $f_t: U \to U$. Namely

$$\begin{cases} f_0(x,y) = (x,y), \\ f_t(x,x) = (x,x), \\ f_1(x,y) \in \{(z,z) \mid z \in M\}. \end{cases}$$

Therefore we have

Lemma 14.14. *Suppose that M satisfies* (14.11.1), (14.11.2), (14.11.3). *Let $f_1, f_2: X \to M$ be continuous maps satisfying $d(f_1(x), f_2(x)) < \epsilon$ for each $x \in X$. Then, f_1 and f_2 are homotopic.*

Lemma 14.15. *There exists a continuous map $\tau: (0,t] \to (0,\infty)$ such that $\tau(0) = 0$ and that the inclusion map : $B_r(p,M) \hookrightarrow B_{\tau(r)}(p,M)$ is 0-homotopic for each $r \in (0,t]$ and $p \in M$.*

Now let M_i and X be as in Theorem 14.6. We can prove that not only M_i but also X satisfies the conclusions of Lemmae 14.14 and 14.15. Proposition 2.10 implies that the Hausdorff dimension of X is not greater than dim M. It follows that X can be embedded into $\mathbf{R}^n$ by a Lipschitz map. Let ϵ be a positive number satisfying

$$\underbrace{\tau(C\tau(C\tau(\cdots\tau(C}_{N \text{ times}}\epsilon)\cdots) < r.$$

Where C is a number determined later. Let W be the ϵ-neighborhood of $I(X)$ in $\mathbf{R}^N$. $W - I(X)$ has a triangulation P such that

$$\text{Diam } \Delta < C \cdot d(\Delta, I(X)),$$

holds for each simplex Δ of P. Let $P^{(k)}$ be the k-skeletons of P. We can find $f^{(0)}: P^{(0)} \to I(X)$ such that $d(f^{(0)}(x), x) = d(x, I(X))$, and that $d(f^{(0)}(x), f^{(0)}(y)) < Cd(x,y)$. Then by using Lemma 14.15, we can extend $f^{(0)}$ to a map $f^{(1)}: P^{(1)} \to I(X)$ satisfying the same condition. Thus, by induction, we can construct a retraction $f: W \to I(X)$. We have proved (14.10.1). The proof of (14.10.2) is based on Lemma 14.15. The author does not know the proof of (14.10.3).

Next we remark that we can not replace Condition (14.4.1) by Ricci $\geq -(n-1)$.

Example 14.16 (Kobayashi-Todorov [KT]). Put $\tau: T^4 \to T^4; [x] \mapsto [-x]$, $X^4 = T^4/\{1,\tau\}$. X^4 is an algebraic variety with sixteen isolated singular points. We obtain the Kummer surface $M \to X$ by blowing up those singular points. Let $\mathbf{C}P_i^1 \subset M$, $i = 1, \cdots, 16$ denote the inverse image of the singular points. By the solution of Calabi conjecture [Ya], we have a Kähler metric g_ϵ on M such that :

$$\text{Ricci}_{(M,g_\epsilon)} \equiv 0, \tag{14.16.1}$$

$$\int_M \omega_{g_\epsilon}^2 = 1, \tag{14.16.2}$$

$$\int_{\mathbf{C}P_i^1} \omega_{g_\epsilon} = \epsilon, \tag{14.16.3}$$

where ω_{g_ϵ} is the Kähler form of (M, g_ϵ). Kobayashi-Todorov proved that

$$\lim_{\epsilon \to 0} d_H((M, g_\epsilon), X) = 0. \tag{14.16.4}$$

Here we give the flat orbifold metric to X. In fact, the convergence in (14.16.4) is stronger than the Hausdorff converges. Namely the Riemannian metric on M_i converges to the flat metric on X in the C^∞-topology outside $\cup \mathbf{C}P_i^1$.

More examples of this kind are discussed in [Kob]. In the case of Kähler-Einstein 4 manifolds, the limit space has at worst the orbifold singularity.

Theorem 14.17 (Nakajima [N], Bando-Kasue-Nakajima [BKN], Anderson [An]). *Let $M_i \in \mathcal{S}(4, D)$ be a sequence of Kähler-Einstein manifolds. Assume that* $\lim_{i\to\infty} d_H(M_i, X) = 0, \text{Vol}\, M_i \geq v > 0$. *Then X is a Kähler Einstein orbifold and the metric converges for C^∞ topology outside the singular point.*

Finally we remark that Kodani [Kod] generalize Theorems 4.1 and 14.1 to manifolds with boundary. Yamaguchi [Y3] and Petersen [Pet] proved some finiteness theorems for manifolds satisfying a condition similar to the conclusion of Lemma 14.15.

§15. Pinching Theorems

We begin with the following simplest case.

Theorem 15.1. *For each $n, D, v > 0$ there exists $\epsilon = \epsilon(n, D, v) > 0$ such that the following holds.*

If M is an n-dimensional Riemannian manifold satisfying

(15.2.1) $$\sigma + \epsilon \geq K_M \geq \sigma - \epsilon$$

(15.2.2) $$\text{Vol } M \geq v$$

(15.2.3) $$\text{Diam } M \leq D,$$

then M admits a Riemannian metric g_0 such that $K_{(M,g_0)} \equiv \sigma$. Here $\sigma = +1, -1$ or 0.

There are various generalizations of Theorem 15.1. Before discussing them we shall give :

Outline of the proof of Theorem 15.1. If the theorem is false, then there exists a sequence M_i such that M_i satisfies (15.2.2),(15.2.3), and

(15.2.1)′ $\sigma + 1/i \geq K_{M_i} \geq \sigma - 1/i$
(15.2.2) M_i does not admit a metric g_o with $K_{(M_i,g_0)} \equiv \sigma$.

In view of Theorem 2.2, we may assume, by taking a subsequence, that M_i converges with respect to the Hausdorff distance to a space M_∞. Theorem 4.1 implies that M_∞ is a Riemannian manifold of $C^{1,\alpha}$-class. The next step is to show :

Lemma 15.3. *M_∞ is a smooth Riemannian manifold and $K_{M_\infty} \equiv \sigma$.*

Using Lemma 15.3, it is easy to complete the proof of Theorem 15.1. In fact, Theorem 3.2 implies that M_∞ and M_i are diffeomorphic for large i. Therefore $K_{M_\infty} \equiv \sigma$ contradicts (15.2.4).

Sketch of the proof of Lemma 15.3. Let $p \in M_\infty$. Choose $p_i \in M_i$ such that $\lim_{i\to\infty} d_{p.H}((M_i,p_i),(M_\infty,p_\infty)) = 0$. Define $I_p: B_r(p,M_\infty) \to B_r(p,M_\infty)$ by $I_p(\exp_p(v)) = \exp_p(-v)$. And define $I_{p_i}: B_r(p_i,M_i) \to B_r(p_i,M_i)$ in a similar way. Using (15.2.1′), we can prove that I_{p_i} becomes closer and closer to be an isometry as i goes to infinity. It follows that I_p is an isometry. Using this fact, we can show that the group of isometries, $I(\widetilde{M}_\infty)$, of the universal covering space $\widetilde{M}_\infty$ of M_∞ acts transitively on $\widetilde{M}_\infty$. Hilbert's fifth problem[37] implies that $I(\widetilde{M}_\infty)$ is a Lie group. Hence, it is easy to show that M_∞ is a homogeneous Riemannian manifold. In particular the metric on M_∞ is smooth. The rest of the proof is easy and is omitted. The detailed argument can be found in [F2] section 10.

We shall give remarks to Theorem 15.1.

[37]See [MZ].

Remark 15.4. First we consider the case $\sigma = -1$.

(1) In this case, Condition (15.2.2) is automatically satisfied for some $v = v(n)$. In fact, Heintze [He] (see also Gromov [G1]) proved that

$$\left.\begin{array}{l} -1 \le K_M < 0 \\ \dim M = n \end{array}\right\} \Longrightarrow \operatorname{Vol} M > v(n).$$

(2) (15.2.3) can be replaced by

$$\operatorname{Vol}(M) \le V \text{ and } M \text{ is complete.}$$

In fact, Gromov [G1] proved

$$\operatorname{Diam} M \le C_n \cdot \operatorname{Vol} M^{P_n}$$

for compact manifold M with $-1 \le K_M < 0, \dim M = n > 3$. In the case when M is not compact but complete, we can use the argument of [F1]. In the case when $\dim M = 3$, we can use the result Thurston [T] volume 2. The detailed arguments are omitted.

(3) We can not remove the condition $\operatorname{Vol} M < V$ when $\dim M > 3$. Counter examples are given in Gromov-Thurston [GT]. When $\dim M = 3$, we conjecture that $K_M < 0$ implies that there exists a metric with constant negative curvature on M.

(4) In the case, $\sigma = -1$, Theorem 15.1 is first obtained in Gromov [G1], [G2].[38]

Remark 15.5. In the case when $\sigma = 0$, Remarks 15.4 (1) and (2) do not hold. In fact almost flat manifold (Example 10.8) shows that we can not remove Condition (15.2.2). The example, $(S^n, N g_{\text{can}}) \times (S^1, N^{-n} g_{\text{can}})$, shows that we can not replace (15.2.3) by $\operatorname{Vol} M < V$. In a sense, Theorem 8.1 is a generalization of the $\sigma = 0$ case of Theorem 15.1.

Remark 15.6. Let $\sigma = 1$.

(1) (15.2.3) holds automatically in this case. In fact, Myers [My] proved that

$$\operatorname{Ricci}_M \ge n - 1 \Longrightarrow \operatorname{Diam} M \le \pi.$$

(2) (15.2.2) is also unnecessary. In fact, when n is even, Klingenberg [Kl] proved that

$$1 \ge K_M \ge \epsilon > 0 \Longrightarrow \operatorname{Vol} M \ge v_n(\epsilon) > 0.$$

[38]The author guess that Gromov had already an idea of Hausdorff convergence when he wrote [G1] and the proof we gave above is essentially the same as one he had at that time.

In the case when n is odd, Cheeger-Gromoll [CGr2] and Klingenberg-Sakai [KS1] proved that

$$\left.\begin{array}{l} 1 \geq K_M \geq 1/4 \\ \pi_1(M) = 1 \end{array}\right\} \Longrightarrow \mathrm{Vol}(M) \geq v_n > 0.$$

Hence, by taking a universal cover and working with equivariant Hausdorff convergence, we can prove Theorem 15.1 in this case without assuming (15.2.2)

For odd dimensional manifolds with uniformly positively curvature, there is no uniform lower bound on their volumes.[39] In fact, Example 10.6 gives a collapsing sequence of such manifolds. However all the known examples of such sequences are constructed by using S^1-fibrations.

Conjecture 15.7. *Let $M_i \in \mathcal{M}(n, D), X \in \mathcal{MET}$. Assume*

(15.8.1) $1 \geq K_{M_i} \geq \epsilon > 0,$ where ϵ is independent of i.
(15.8.2) $\pi_1 M_i = 1$
(15.8.3) $\lim_{i \to \infty} d_H(M_i, X) = 0.$

Then

$$\dim X \geq n - 1.$$

If this is valid, then Proposition 11.5 *implies that X is a Riemannian orbifold. In the case when* $\dim X = n - 1$, *we have a S^1 (singular) fibration. Let $e_i \in H^2(X; \mathbf{R})$ be the Euler class of this fibration. Then*

$$\lim_{i \to \infty} |e_i| \to \infty. \tag{15.9}$$

Remark 15.10. If we assume $\pi_2 M_i = 1$ in addition, then (15.9) can not occur. Hence, in this case, the conjecture is that $X = M_i$ for large i, that is M_i can not collapse.

Remark 15.11. Conjecture 15.7 is closely related to a conjecture by Klingenberg. See [KS2].

In the case when $\sigma = 1$, Theorem 15.1 is a version of the celebrated differential sphere theorem. Our proof of Theorem 15.1 uses indirect argument, hence we can not get an explicit bound for ϵ from our proof.

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[39] even for simply connected manifolds

	Assumption	Conclusion		Assumption	Conclusion
1a	$1 \geq K \geq \frac{1}{4}$ $\pi_1 = 1$	$M \simeq S^n$ or $M \equiv \begin{cases} \mathbb{C}P^* \\ \mathbb{H}P^* \\ C_aP^2 \end{cases}$	1b	$1 \geq K \geq \frac{1}{4} - \epsilon$ $\pi_1 = 1$	$M \simeq S^n$ or $M = \begin{cases} \mathbb{C}P^* \\ \mathbb{H}P^* \\ C_aP^2 \end{cases}$
2a	$\text{Ricci} \geq n-1$ $V \geq \frac{\omega_n}{\#\pi_1}$	$M \equiv S^n/\Gamma$ $\Gamma \subset O(n+1)$	2b	$\text{Ricci} \geq n-1$ $K > -K^2$ $V \geq \frac{\omega_n}{k} - \epsilon$ $\#\pi_1 = k$ $\epsilon = \epsilon(n, k, K)$	$M = S^n/\Gamma$ $\Gamma \subset O(n+1)$
3a	$K \geq 1$ $D = \frac{\pi}{2}$ $\pi_1 = 1$	$M \simeq S^n$ or $M \equiv \begin{cases} \mathbb{C}P^* \\ \mathbb{H}P^* \end{cases}$ or $H_*M = H_*C_aP^*$	3b	$\Lambda^2 \geq K \geq 1$ $D \geq \frac{\pi}{2} - \epsilon$ $V > \epsilon_0 > 0$ $\epsilon = \epsilon(n, \Lambda, \epsilon_0)$	a) $M \simeq S^n$ b) $M \equiv S^n/\Gamma$ $\equiv \mathbb{C}P^{\frac{n}{2}}/\Gamma$ c) $\pi_1 = 1$ $H_*M = \frac{\mathbb{Z}[x]}{(x^k)}$
4a	$K \geq 1$ $D = \frac{\pi}{2}$ $\pi_1 \neq 1$	$\begin{cases} M & \equiv S^n/\Gamma \\ \Gamma & \subset O(n+1) \end{cases}$ or $M \equiv \mathbb{C}P^*/\mathbb{Z}_2$	4b	$\text{Ricci} \geq n-1$ $D \leq \frac{\pi}{2}$ $K \geq -K^2$ $V \geq \frac{1}{2}\omega_n - \epsilon$ $\epsilon = \epsilon(n, K)$	$M \simeq \mathbb{R}P^2$

Table 15.12

Explicit bound is obtained in the original proof in Gromoll [Gr], Shikata [Sh2], Sugimoto-Shiohama [SS] and some other papers.

For positively curved manifolds, there are various versions of sphere theorems. Hausdorff convergence yields ϵ-versions of most of them. They are summarized in Table (15.12).[40]

Notation of the table 15.12. D = Diam M, n = dim M, V = Vol(M), ω_n = Vol(S^n), K = the sectional curvature, Ricci = the Ricci curvature, $M \equiv N \Longleftrightarrow M$ is isometric to N, $M = N \Longleftrightarrow M$ is diffeomorphic to N, $M \simeq N \Longleftrightarrow M$ is homeomorphic to N.

References and remarks to table 15.12.

(1a) Berger [B1] (See also [CE]). This result is a generalization of Rauch [Ra] and Klingenberg [Kl].

(1b) This theorem is in Berger [B2]. At the time when [B2] was published, those results we presented in sections 5 and 6 were not yet well established, then, at that time, though all the essential ideas are in [B2], it was impossible to work out the detail of some parts of the proof, (which is related to the regularity of the limit metric.) The detailed proof can be found in Durumeric [D2].

(2a) Myers [My].

(2b) Katsuda [Ka1] proved

$$\left.\begin{array}{l} \Lambda^2 \ge K \ge -\Lambda^2 \\ \text{Ricci} \ge n-1 \\ V \ge \omega_n - \epsilon(\Lambda) \end{array}\right\} \Longrightarrow \quad M = S^n.$$

Brittain announced the same result. Their arguments are based on Hausdorff convergence. Shiohama [S1] (without using Hausdorff convergence) showed

$$\left.\begin{array}{l} K \ge -\Lambda^2 \\ \text{Ricci} \ge n-1 \\ V > \omega_n - \epsilon(\Lambda) \end{array}\right\} \Longrightarrow \quad M \simeq S^n.$$

Otsu-Shiohama-Yamaguchi [OSY] proved $M = S^n$ under the above assumption. Theorem (2b) is due to Yamaguchi [Y2].

(3a) This result is due to Gromoll-Grove [GG]. The case $D > \pi/2$ is due to Grove-Shiohama [GS].

(3b) This result is due to Durumeric [D1], [D2]. He conjectured that (c) can be replaced by $M = \mathbf{C}P^{n/2}, \mathbf{H}P^{n/4}$, or C_aP^2. In case (c) some restrictions on deg x and k are proved in [D2].

(4a) This result is due to Gromoll-Grove [GG]. Before [GG], Sakai [Sa1] and Sakai-Shiohama [SaS] gave partial results.

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[40] The author is sorry that this table is not the complete list of such results.

(4b) This result is due to Otsu-Shiomaha-Yamaguchi [OSY]. Under the same assumption Grove-Petersen [GP2] proved $M \simeq S^n/\mathbf{Z}_2$.

Remark 15.13. There are also Pinching theorems using first eigenvalues of Laplace operator. For them, see Croke [Cr], Katsuda [Ka1] and Kasue [K1].

Next we shall give a part of the ideas of the proofs of (1b), ..., (4b). The proofs of (1b) and (3b) are similar. We explain one for (1b). The argument is by contradiction and is similar to the proof of Theorem 15.1. Assume that M_i satisfy $\pi_1 = 1$, $\mathrm{Vol}(M_i) > \delta$, $1 \geq K_{M_i} \geq 1/4$. Let M be the limit of M_i for the Hausdorff distance. It suffices to show that $M = \mathbf{C}P^*, \mathbf{H}P^*, C_aP^2$, or $M \simeq S^n$. By assumption it seems natural to expect that $1 \geq K_M \geq 1/4$. If so, then (1a) implies desired conclusion. But it is not so immediate to prove the inequality $1 \geq K_M \geq 1/4$. The difficulty is that M is only of $C^{1,\alpha}$-class and hence the curvature of M is not well defined. Therefore, instead of using (1a) itself, we have to imitate its proof. Roughly speaking, we shall prove that the cut locus of M has the same property as the symmetric spaces. (For example the cut locus of $\mathbf{C}P^n$ is $\mathbf{C}P^{n-1}$.) The detail is in [B2].

We discuss the idea of the proof of (2b) and (4b). They are proved in a similar way. We explain the proof of (2b) in the case $\pi_1 = 1$. The first step, which is essentially due to Katsuda [Ka1], is to show :

$$\left.\begin{array}{l} \mathrm{Ricci} \geq n-1 \\ V > \omega_n - \epsilon(\Lambda) \\ K > -\Lambda^2 \end{array}\right\} \Longrightarrow d_H(M, S^n) < o(\epsilon)$$

The Hausdorff approximation $\varphi: S^n \to M$ is the composition of the inverse of the exponential map : $T_p(S^n) \to S^n$, the linear isometry : $T_p(S^n) \to T_q(M)$, and the exponential map : $T_q(M) \to M$. The assumption on the volumes is used to verify that this map is a Hausdorff approximation. The second step is to show that M is diffeomorphic to S^n under the above assumption. Remark that we can not directly apply Theorem 3.2 in this case, because we do not assume $K_M < \Lambda^2$. Instead we shall use the idea of its proof. In our situation the following embedding is useful. Let

$$S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$$

$$e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0).$$

Define $I \to \mathbf{R}^{n+1}$ by $I(x) = (I_1(x), \cdots, I_{n+1}(x))$, $I_k(x) = \cos d_{S^n}(e_i, x)$. It is easy to see that I is the standard embedding. Using the $o(\epsilon)$-

Hausdorff approximation $\varphi: S^n \to M$, we define $I': M \to \mathbf{R}^{n+1}$ by $I'(x) = (I'_1, \cdots, I'_{n+1})$, $I'_i(x) = \cos d_M(\varphi(e_i), x)$. We modify I' as in (12.3). Then we can construct a diffeomorphism between M and S^n using the above embeddings in a way similar to section 3.

We turn to the other kinds of Pinching theorems.

Theorem 15.14 (Katsuda [Ka2], Min-no - Ruh [MR1], [MR2] gave a related results). *There exists $\epsilon = \epsilon(n, v, D)$ such that $M \in \mathcal{M}(n, D, v)$, $|\nabla R_M| < \epsilon$ implies that M is diffeomorphic to a locally symmetric space.*

Katsuda also proved a pinching theorem which is an ϵ-version of the theorem of Ambrose-Singer [AS].[41] The scheme of the proof of Those results are similar to one of Theorem 15.1. He also asked if there exists a similar pinching theorem for Einstein manifolds.

Next we review the results for almost nonnegatively curved manifolds. Bochner proved that

$$\mathrm{Ricci}_M \geq 0 \Longrightarrow b_1(M) \leq \dim M,$$

where $b_1(M)$ is the first Betti number. The equality holds if and only if M is a flat torus. Gromov and Gallot [Ga] generalized this result to

$$\left.\begin{array}{l} \mathrm{Ricci}_M \geq -\epsilon(n, D) \\ \mathrm{Diam}\ M \leq D \end{array}\right\} \Longrightarrow b_1(M) \leq n.$$

Yamaguchi proved

Theorem 15.15 (Yamaguchi [Y1]). *There exists $\epsilon = \epsilon(n, D)$, such that $M \in \mathcal{M}(n, D)$,* $\mathrm{Ricci}_M > -\epsilon$, *$b_1(M) = k$ implies that M is a fibre bundle over the k-dimensional torus T^k.*

Conjecture 15.16 (Yamaguchi [Y1]). The same conclusion holds without assuming $|K_M| \leq 1$.

Open problem 15.17. Is the fibre in Theorem 15.15 admits a metric with nonnegative curvature ?

Sketch of the proof of Theorem 15.13. Let $\omega_1, \cdots \omega_k$ be the basis of harmonic 1-forms on M. Define $f: M \to T^k$ by

$$f(x) = \left(\int_{x_0}^{x} \omega_1, \cdots, \int_{x_0}^{x} \omega_k\right),$$

[41] Ambrose-Singer theorem gives a characterization of the locally homogeneous space.

where we identify T^k with

$$\frac{\mathbf{R}^k}{\left\{\left(\int_\gamma \omega_1, \cdots, \int_\gamma \omega_k\right) \middle| \gamma \in \pi_1(M)\right\}}.$$

In other words, T^k is the Albanese Torus and f is the Albanese map. Yamaguchi proved that f is a fibration if $\mathrm{Ricci}_M > -\epsilon$.

In the case when $k \geq \dim M - 1$ the conclusion of Theorem 15.15 is that M is diffeomorphic to a torus or a S^1 bundle over a torus. In this case we can determine the limit of those spaces.

Theorem 15.18 (Yamaguchi [Y1]). *Let $X \in \mathcal{MET}$, $M_i \in \mathcal{M}(n, D)$. Assume $b_1(M_i) \geq n-1$, $\mathrm{Ricci}_{M_i} \geq -1/i$, $\lim_{i\to\infty} d_H(M_i, X) = 0$. Then X is a flat torus T^k or its quotient $T^k/\mathbf{Z}_2$.*

When we try to use the theory of Hausdorff convergence to improve Theorem 15.15, we need more informations on the limit of the spaces with almost nonnegative Ricci curvature. In the case when $b_1 = n-1$ this was easier essentially because the space is aspherical in that case.[42] We cannot expect that the limit of almost nonnegatively curved manifolds are manifolds or orbifolds. This does not hold even for positively curved manifolds.[43] In the case when we assume that the manifold is aspherical we have :

Theorem 15.19 (Fukaya-Yamaguchi [FY]). *There exists $\epsilon = \epsilon_n(D)$ such that the following holds. Let M be an n-dimensional Riemannian manifold. Assume*

$$\begin{cases} 1 \geq K_M \geq -\epsilon \\ \pi_k M = 1 \qquad \text{for } k \geq 2 \\ \mathrm{Diam}\ M \leq D. \end{cases}$$

Then, M is diffeomorphic to N/Γ. Where N and Γ is as in Example 10.8.

We close this section with remarking the following natural but quite difficult problem.

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[42]See the next section to find the reason why the aspherical manifolds are easier to study.

[43]See the examples in section 10.

Open problem 15.20. Suppose that M admits a metric with strictly positive curvature. Put

$$\delta_0 = \sup\{\,\delta \mid \exists g \;\; g \;\text{ is a metric on } M \text{ satisfying } 1 \geq K_g \geq \delta\,\}.$$

Does there exists a metric g_0 on M such that $1 \geq K_{g_0} \geq \delta_0$? If exists is it unique ? What kind of good properties does it have ? How about the similar

problems for negatively curved manifolds ?

§16. Aspherical manifolds

An aspherical manifold stands for a manifold M such that $\pi_k M = 1$ for $k > 1$. In other words, a manifold whose universal covering space is contractible. Typical examples of aspherical manifolds are a locally symmetric space $\Gamma \backslash G/K$, a solvmanifold and a nilmanifold $\Gamma \backslash G$. Then, the study of aspherical manifolds can be regarded as a generalization of the theory of discrete subgroups of Lie groups. The notion of the convergence of discrete subgroups or deformation of it is one of the motivations of the definition of the Hausdorff convergence. In the case of aspherical manifolds, the application of the results of Chapters 1 and 2 becomes simpler. A reason is the following :

Theorem 16.1 (Fukaya [F7]). *Let $M_i \in \mathcal{M}(n,D)$, $X \in \mathcal{MET}$. Assume that M_i is aspherical and that $\lim_{i\to\infty} d_H(M_i, X) = 0$. Then X is isometric to a quotient Y/Γ, where Y is a smooth contractible manifold with $C^{1,\alpha}$ Riemannian metric and Γ is a properly discontinuous group of isometries of Y.*

Remark 16.2. If we assume that $\widetilde{M}_i$ is diffeomorphic to $\mathbf{R}^n$ in addition, then Y is also diffeomorphic to an Euclidean space

Remark 16.3. If we assume $K_{M_i} \leq 0$ in addition. Then we also have $K_Y \leq 0$.

Sketch of the proof of Theorem 16.1. Let $\widetilde{M}_i$ be the universal covering space of $\widetilde{M}_i$. Fix $p_i \in \widetilde{M}_i$. The first step is to show that

$$\mathrm{inj}_{M_i}(p_i) \geq \epsilon > 0,$$

where ϵ is independent of i. Then the limit, Z, of $(\widetilde{M}_i, p_i)$ is an n-dimensional Riemannian manifold with $C^{1,\alpha}$-metric. We consider the limit G of the deck transformation group $\pi_1(M_i, \overline{p}_i)$. G acts on Z by isometry. The second step is to show that the isotropy group $I_p = \{g \in$

$G \mid gp = p\}$ is finite for each $p \in Z$. Then $X = Z/G$ is a Riemannian orbifold. Then, using the results of section 12, we can construct the fibration $f: M_i \to X$. Hence the asphericality of M_i implies that the universal covering orbifold Y of X is contractible.

For the arguments of steps 1 and 2, the following results due to Gromov are essential.

Definition 16.4. A metric space Y is said to be geometrically contractible if, for each r, there exists $R(r)$ such that the inclusion map $: B_r(p, Y) \hookrightarrow B_{R(r)}(p, Y)$ is null homotopic for each $p \in Y$.[44]

Lemma 16.5 (Gromov [G9] 4.5 D). *Let Y be a complete and contractible Riemannian manifold, $K \subset Y$ a compact subset and G a group of isometries of Y. Assume $GK = Y$. Then Y is geometrically contractible.*

The assumption of Lemma 16.5 implies that Y is homogeneous modulo a compact factor. Hence the uniformity of the geometry implies the geometrically contractiblity.

Theorem 16.6 (Gromov [G10],[G9] Appendix I E_3). *Let Y be a geometrically contractible complete Riemannian n-manifold, P a polyhedron, $f: Y \to P$ be a continuous map. Assume* $\dim P < n$. *Then*

$$\sup\{\text{Diam } f^{-1}(p) \mid p \in P\} = \infty.$$

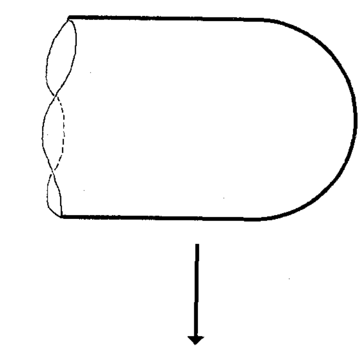

Fig. 16.7

[44]The fact that $R(r)$ is independent of p is essential.

If we take Y, $p \in P = \mathbf{R}_+$, and f as in Figure 16.7, we have

$$\sup\{\mathrm{Diam}\ f^{-1}(p) \mid p \in P\} < \infty.$$

But in this case Y is not geometrically contractible. The proof of Theorem 16.6 uses minimal surface theory in an infinite dimensional Banach space, $L^\infty(Y)$.

Outline of the step 1 *of the proof of Theorem* 16.1. Let (Z,p) be the limit of $(\widetilde{M}_i, p_i)$ with respect to the pointed Hausdorff distance. Assume $\lim_{i\to\infty} \mathrm{inj}_{\widetilde{M}_i}(p_i) = 0$. Then we have dim $Z < n$. On the other hand, we have a map $f_i \colon \widetilde{M}_i \to Z$ such that

$$\lim_{i\to\infty} \mathrm{Diam}\ f^{-1}(p) = 0. \tag{16.8}$$

On the other hand, since $\widetilde{M}_i/\pi_1(M_i) = M_i$ is compact, if follows from Theorem 16.5 that $\widetilde{M}_i$ is geometrically contractible. Hence (16.8) contradicts Theorem 16.6.

Outline of the step 2 *of the proof of Theorem* 16.1. Let G_0 be the connected component of G. Margulis' lemma implies that G_0 is nilpotent. Hence G_0 has a maximal compact subgroup $K \simeq T^k$ contained in the center. By Smith theory[45] K has a fixed point on Z. Using Diam $X/G = D < \infty$, and the fact that K is normal in G, we see that

$$\sup\{\mathrm{Diam}\ Kp \mid p \in Z\} < \infty.$$

Hence by the geometrically contractiblity of Z, we can apply Theorem 16.6 to the map $Z \to Z/K$ and conclude dim $K = 0$. It follows that G_0 contains no compact subgroup. Therefore G_0 acts freely on Z. Consequently Z/G is a Riemannian orbifold as required.

Theorem 16.1 has the following immediate applications.

Corollary 16.9. *For each n, D there exists $v = v(n,D)$ such that the following holds. Let $M \in \mathcal{M}(n,D)$ be an aspherical manifold. Assume that $\pi_1(M)$ is not solvable and does not contain a group isomorphic to $\mathbf{Z}^2$. Then* Vol $M \geq v$.

Corollary 16.10. *For each n, D, there exists a finite set Σ of orbifolds such that, for each aspherical manifold in $\mathcal{M}(n,D)$, there exists*

[45] See [Bo].

an element X *of* Σ *and a fibration* $f: M \to X$ *satisfying* (12.1), ..., (12.4).

The following result is closely related to Theorem 16.1.

Theorem 16.11 (Fukaya-Yamaguchi [FY]). *For each* n, D *there exists a positive number* $\epsilon = \epsilon(n, D)$ *such that* $K_M < \epsilon$ *and* $M \in \mathcal{M}(n, D)$ *imply that the universal covering space of* M *is diffeomorphic to the Euclidean space.*

Remark 16.12. The condition $-1 < K_M$ in Theorem 16.11 cannot be removed. In fact Gromov [G2], Buser-Gromoll [BG][46] constructed a family of metrics g_ϵ on S^3 such $\mathrm{Diam}(M, g_\epsilon) = 1$ and that $K_{M_\epsilon} \leq \epsilon$.

Sketch of the proof of Theorem 16.11. The proof is by contradiction. Suppose that $K_{M_i} < 1/i$, $M_i \in \mathcal{M}(n, D)$ and that the universal covering space of M_i is not diffeomorphic to the Euclidean space. By taking a subsequence, we may assume that M_i converges to a metric space X.

Lemma 16.13. X *is a* $C^{1,\alpha}$*-Riemannian manifold with nonpositive curvature.*[47]

The proof is a modification of the argument of step 2 of the proof of Theorem 16.1. We omit the detail.

Now Lemma 16.13 and an orbifold version of Theorem 12.1, (which can be proved by using Theorem 12.7), imply that there exists a fibration : $M \to X$ whose fibre is an almost flat manifold. Then, both the base, X, and the fibre have the Euclidean space as their universal covering spaces. Hence by a simple argument we can prove that the universal covering space of M is also an Euclidean space.

By studying the fibration in the above argument more closely we can prove the following :

Theorem 16.14 (Fukaya-Yamaguchi [FY]). *Let* M *be as in Theorem* 16.11, *then there exists a fibration* $f: N/\Lambda \to M \to Y/\Gamma$. *Such that the following holds.*

(16.15.1) *The fibre* N/Λ *is as in Example* 10.8.

[46] See also Bavard [Bav].

[47] We have to be careful to say that X is of nonpositive curvature, because the curvature tensor of $C^{1,\alpha}$-Riemannian manifold is not well defined. The definition of nonpositivity in this case is in [FY].

(16.15.2) *Y is a $C^{1,\alpha}$-Riemannian manifold with nonpositive curvature.*
(16.15.3) *The structure group of the fibration is contained in*

$$\frac{\text{Cent } N}{\text{Cent } N \cap \Lambda} \propto L,$$

where L is a subgroup of Aut N *satisfying Condition* 16.16 *below.*

Condition 16.16. There exists a stratification $1 = \Lambda_1 \subset \cdots \Lambda_k = \Lambda \cap N$ such that

(16.16.1) Each Λ_i is L invariant.
(16.16.2) Λ_i/Λ_{i-1} is contained in the center of Λ_k/Λ_{i-1}.
(16.16.3) The image of the homomorphism :

$$L \to \prod_i \text{Aut}(\Lambda_i/\Lambda_{i-1})$$

defined by conjugation is finite.

We omit the proof.

Conjecture 16.17. There exists a positive number ϵ_n depending only on the dimension n such that the following holds. If M is a Riemannian manifold satisfying $-1 \le K_M \le c$, Diam $M \cdot c \le \epsilon_n$, then the universal covering space of M is diffeomorphic to the Euclidean space.

Example 17.12 is related to this conjecture.

§17. Minimal volume

Definition 17.1 (Gromov [G9]). Let M be a closed manifold. We define the *minimal volume*, MinVol M, by

$$\text{MinVol } M = \inf\{ \text{Vol}(M,g) \mid |K_{(M,g)}| \le 1 \}.$$

The *D-minimal volume*, MinVol_D M is defined by

$$\text{MinVol}_D\ M = \inf\{ \text{Vol}(M,g) \mid (M,g) \in \mathcal{M}(n,D) \}.$$

Minimal volume provides an obstruction to collapsing Riemannian manifolds. The study of the minimal volume leads us to the consideration of the phenomena that the manifold splits into many parts, (in other words we have to study pointed Hausdorff convergence). Before discussing it, we give an application of Theorem 16.1 to the proof of a gap theorem for Minvol_D.

Theorem 17.2 (Fukaya [F5]). *For each n and D, there exists a positive number $v(n,D)$ such that if an n-dimensional aspherical manifold M satisfies* $\mathrm{MinVol}_D\, M < v(n,D)$ *then* $\mathrm{MinVol}_D\, M = 0$.

Proof. Suppose the theorem is false. Then we have a sequence of elements M_i of $\mathcal{M}(n,D)$ such that M_i is aspherical, $\lim_{i\to\infty} \mathrm{Vol}(M_i) = 0$, and that $\mathrm{Minvol}_D\, M_i \neq 0$. We may assume that M_i converges to an element X of $\mathcal{MET}$. Theorem 16.1 implies that X is a Riemannian orbifold. Then the orbifold version of Theorem 12.1 implies that there exists a fibration $M_i \to X$ satisfying Conditions (12.1.1),$\cdots$,(12.1.4).Therefore the orbifold version of Example 10.11 implies that there exists a family of metrics g_ϵ on M_i such that $\lim_{\epsilon\to 0}(M_i, g_\epsilon) = X$. It follows that $\mathrm{Minvol}_D\, M_i = 0$. This is a contradiction.

Now we consider the case when the diameter is not necessarily bounded. We have various examples and conjectures but not so many theorems, yet. Those are closely related to the following :

Conjecture 17.3. There exists v_n such that $\mathrm{MinVol}\, M < v_n$ implies $\mathrm{MinVol}\, M = 0$, for an n-dimensional manifold M.

Theorem 17.2 gives an affirmative answer to the same conjecture for MinVol_D in the case when M is aspherical. We can remove this condition, if we can generalize Theorem 10.11 to the case when the base space is not necessary an orbifold. The conjecture for MinVol is more difficult as is illustrated in the following examples.

Example 17.4 (Gromov [G1], see also [E], [Sch]). Let $\epsilon > 0$. We can find a metric g_ϵ on $T^2 - \mathrm{Int}\, B^2$ such that

(17.5.1) $K_{g_\epsilon} \leq 0$.
(17.5.2) $\partial\,(T^2 - B^2)$ is isometric to $(S^1, \epsilon g_0)$, the circle with radius ϵ.
(17.5.3) A neighborhood of $\partial\,(T^2 - B^2)$ is isometric to a direct product $(S^1, \epsilon g_0) \times [0, \delta)$.
(17.5.4) $\mathrm{Vol}(T^2 - B^2, g_\epsilon) < C$, where C is a number independent of ϵ.

We can attach two copies of $(T^2 - B^2, g_\epsilon) \times (S^1, \epsilon g_0)$ by the isometry $\tau\colon \partial(T^2 - B^2) \times S^1 \to (T^2 - B^2) \times S^1; \tau(a,b) = (b,a)$. Then the resulting Riemannian manifold (M^3, g_ϵ) satisfies

(17.6.1) $-1 \leq K_{(M,g_\epsilon)} < 0$,
(17.6.2) $\mathrm{Vol}(M, g_\epsilon) < C\epsilon$, where C is a number independent of ϵ.

(17.6.3) M is not the total space of a nontrivial fibration. When ϵ goes to 0, the diameter of (M, g_ϵ) goes to infinity.[48]

In this example the manifold splits into two manifolds with totally geodesic boundary. More generally we can construct examples which split into manifolds with corners.

Example 17.7 (Fukaya-Januszkiewicz). Take six copies of $(T^2 - B^2, g_\epsilon) \times (T^2 - B^2, g_\epsilon) \times (S^1, \epsilon g_0)$. Call them $M_1, \cdots, M_6$. Attach them according to Figure 17.8. Here ∂M_i and ∂M_{i+1} are identified by $S^1 \times (T^2 - B^2) \times S^1 \to (T^2 - B^2) \times S^1 \times S^1; (a, x, b) \mapsto (x, b, a)$. Then we have a 5 dimensional manifold M with a singular nonpositively curved Riemannian metric g'_ϵ. The singular set of the metric is $T^3 = \cap M_i$. A small neighborhood of this singular set is isometric to the direct product of T^3 and the union of six copies of $\{(x, y) \mid \delta > x \geq 0, \delta > y \geq 0\}$. Then we can modify the metric in this neighborhood so that the metric g_ϵ satisfies

(17.7.1) $-1 \leq K_{(M, g_\epsilon)} \leq 0$,
(17.7.2) $\mathrm{Vol}(M, g_\epsilon) \leq C\epsilon$,
(17.7.3) M has no nontrivial fibration structure.

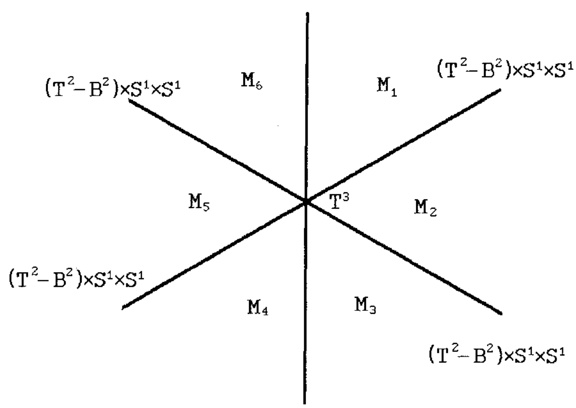

Fig. 17.8

[48] Schroeder [Sch] classified the nonpositively curved metric on this manifold M.

Conjecture 17.9. If a manifold M has nonpositive curvature and small volume, then M splits into manifolds with totally geodesic corners and boundaries, each of which has nontrivial product structure.

The following result is an evidence.

Theorem 17.10 (Buyalo [Bu]). *There exists v_n such that if $-1 \leq K_M \leq 0$ and if $\mathrm{Vol}\, M < v_n$, then $\mathbf{Z}^2 \subset \pi_1 M$.*

Conjecture 17.11. There exists v_n such that if an n-dimensional aspherical Riemannian manifold M satisfies $|K_M| \leq 1, \mathrm{Vol}\, M \leq v_n$, then $\mathbf{Z}^2 \subset \pi_1 M$.

If we assume $\mathrm{Diam}\, M \leq D$ in addition, Corollary 16.9 is the affirmative answer.

By modifying Example 17.4 we obtain the following example providing an evidence to Conjecture 16.17.

Example 17.12. Put $\mathbf{CH}^2 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$. $\mathbf{CH}^2$ has a Kähler metric g with constant negative bisectional curvature.($-1 \leq K_g \leq -1/4$.) Let Γ be a discontinuous group of isometries of $\mathbf{CH}^2$ such that $\mathrm{Vol}\, \mathbf{CH}^2/\Gamma < \infty$, and that $\mathbf{CH}^2/\Gamma$ is noncompact with one end. Then the end is diffeomorphic to N/Λ, where

$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbf{R} \right\}$$

is the Heisenberg group, $\Lambda \subset N \propto \mathrm{Aut}\, N$, $[\Lambda : \Lambda \cap N] < \infty$. As in Example 17.4 we can attach two copies of $\mathbf{CH}^2/\Gamma \times N/\Lambda$ along their boundaries $N/\Lambda \times N/\Lambda$ by the diffeomorphism : $(p, q) \mapsto (q, p)$ and obtain a 7 dimensional manifold M. We can verify that M has a family of Riemannian metrics g_ϵ such that

$$\begin{cases} -1 \leq K_{g_\epsilon} \leq \delta_\epsilon \\ \lim\limits_{\epsilon \to 0} \delta_\epsilon = 0 \\ \lim\limits_{\epsilon \to 0} \mathrm{Diam}(M, g_\epsilon)^2 \cdot \delta_\epsilon = 0. \end{cases}$$

So far we have treated the case when $\mathrm{Minvol}\, M = 0$. More generally we have the following :

Problem 17.13. Let M be a closed Riemannian manifold. Does there exists a metric g on M, such that $|K_{(M,g)}| \leq 1$ and that $\mathrm{Vol}(M,g) = \mathrm{MinVol}\, M$.

Open problem 17.14. Determine the set

$$\mathcal{A}_n = \{\mathrm{MinVol}\, M \mid M \text{ is a closed } n\text{-manifold}\}.$$

The answer to Problem 17.13 is negative in the case $\mathrm{MinVol}\, M = 0$. It seems that the answer is negative also in some of the other cases.[49]

Example 17.15 (Heintze). Let $M_1 = \mathbf{H}^3/\Gamma_1, M_2 = \mathbf{H}^3/\Gamma_2$ be two Riemannian manifolds. Here $\mathbf{H}^3$ is a three dimensional simply connected Riemannian manifold with constant curvature -1. Assume that M_1 and M_2 are noncompact and of finite volume. Assume also that the end of M_i is diffeomorphic to $T^2 \times \mathbf{R}$. The torus at the end of M_i has a conformal structure induced from the metric of M_i. We assume also that the two conformal structures on T^2 coïncide. Let $M = M_1 \cup M_2$ be the closed 3-manifolds attaching M_1 to M_2 along their boundaries. We can define a family of Riemannian metrics g_ϵ on M such that

$$(17.15.1) \quad 0 \geq K_{(M,g_\epsilon)} \geq -1$$

$$(17.15.2) \quad \lim_{\epsilon \to 0} \mathrm{Vol}(M, g_\epsilon) = \mathrm{Vol}\, M_1 + \mathrm{Vol}\, M_2$$

$$(17.15.3) \quad \begin{cases} \lim_{\epsilon \to 0} d_{p.H.}((M, g_\epsilon, p), (M_1, p)) = 0 & \text{if } p \in M_1 \\ \lim_{\epsilon \to 0} d_{p.H.}((M, g_\epsilon, p), (M_2, p)) = 0 & \text{if } p \in M_2. \end{cases}$$

Open problem 17.16.

$$\mathrm{MinVol}\, M = \mathrm{Vol}\, M_1 + \mathrm{Vol}\, M_2?$$

Here the volume of the right hand side is one for the metric with constant negative curvature. The inequality $\leq$ follows from the the above construaction. This problem seems very difficult. In fact we can not even prove the following :

Conjecture 17.17. Let M be a closed manifold. Assume that M admits a metric g_0 such that $K_{g_0} \equiv -1$. Then $\mathrm{MinVol}\, M = \mathrm{Vol}(M, g_0)$.

---

[49]The exact value of the minimal volume is known in few cases, that are the cases when $\mathrm{Minvol} = 0$ or when $\dim M = 2$.

Theorems 17.23 and 26 below implies

$$\text{Vol}(M, g_0) \leq \text{MinVol}\, M < (n-1)^n n!\, \text{Vol}(M, g_0)/R_n, \tag{17.18}$$

where R_n is the minimum of the volumes of all ideal simplices[50] in $\mathbf{H}^n$.[51] For Problem 17.14 we have the following :

Conjecture 17.19. Put

$$\mathcal{V} = \{\text{Vol}(\mathbf{H}^3/\Gamma) \mid \Gamma \text{ is a lattice of } \text{Isom}(\mathbf{H}^3)\}.$$

Then

$$\mathcal{A}_3 = \{a_1 + \cdots + a_k \mid a_i \in \mathcal{V}\}.$$

The reader can easily see that this conjecture is closely related to Problem 17.16 and the Thurston's geometrization conjecture for three manifold.[52] Recall that Thurston [T] proved that $\mathcal{V}$ is isomorphic to $\omega^{\omega^{\cdots}}$ as ordered set. The study of the set $\mathcal{A}_n$, $n > 3$, is much harder and the author has no idea on it. We remark that the results of Cheeger-Gromov [CG5] gives a proof of Conjecture 17.3 for 3 dimensional manifolds.

In [G9], Gromov defined a topological invariant, $\|[M]\|$, the Gromov invariant.[53] This invariant can be regarded as a topological version of minimal volume and provides an obstruction to collapsing Riemannian manifolds.

Definition 17.20. Let X be a topological space and $h \in H_k(M; \mathbf{R})$. We define the *simplicial norm*, $\|h\|$, by

$$\|h\| = \inf \left\{ \Sigma|a_i| \;\middle|\; \begin{array}{l} \Sigma\, a_i\sigma_i \text{ represents } h \\ a_i \in \mathbf{R},\ \sigma_i \text{ are singular simplices} \end{array} \right\}$$

The *Gromov invariant*, $\|[M]\|$, is the simplicial norm of the fundamental class.

Example 17.21. $\|[S^1]\| = 0$. In fact let $\sigma_k\colon [0,1] \to S^1$ be the map with degree k. Then $[S^1] = \frac{1}{k}[\sigma_k]$.

---

[50] that is the simplices all of whose vertices are situated at infinity

[51] $R_2 = \pi, R_3 = \frac{3}{2}\Sigma_{k=1}^{\infty} k^{-2} \sin 2\pi k/3$, and asymptotically $R_n \simeq \sqrt{n}e/n!$. See [M], [HM].

[52] See [Sc].

[53] Gromov called it the *simplicial volume.*

Theorem 17.22 (Gromov [G9] 0.5). *Let M be an oriented Riemannian manifold satisfying* $\mathrm{Ricci}_M \geq -(n-1)$ *then*

$$\mathrm{Vol}\ M > \frac{1}{(n-1)^n \cdot n!} \|[M]\|.$$

Corollary 17.23.

$$\mathrm{MinVol}\ M > \frac{1}{(n-1)^n \cdot n!} \|(M)\|.$$

Theorem 17.24 (Thurston [G9] 0.3). *If* $-1 \leq K_M \leq -\epsilon$, *then*

$$\|[M]\| > C(\epsilon)\,\mathrm{Vol}(M).$$

Corollary 17.25. *If* $-1 \leq K_M \leq -\epsilon$ *then*

$$\mathrm{Minvol}(M) > C_n(\epsilon) \cdot \mathrm{Vol}(M).$$

Gromov [G.10] 6.6 C$'$ implies that $C_n(\epsilon)$ is independent of ϵ.

Theorem 17.26 (Gromov [G9] 0.4). *If* $K_M \equiv -1$ *then*

$$\|[M]\| = R_n\,\mathrm{Vol}(M),$$

where R_n *is as in Formula* (17.17).

Gromov proved also the following weak version of Conjecture 17.13.

Theorem 17.27 (Gromov [G9] 0.5). *There exists* ϵ_n *such that the following holds. Assume that* $M \in \mathcal{M}(n,\infty)$ *and that* $\mathrm{Vol}(B_1(p,M)) < \epsilon_n$ *for each* $p \in M$, *then* $\|[M]\| = 0$.

The proof is a combination of Margulis' Lemma (Theorem 13.1) and the following :

Theorem 17.28 (Gromov [G9] §3). *If* $\pi_1 M$ *contains an index finite nilpotent subgroup then* $\|[M]\| = 0$.[54]

More generally

[54] It suffices to assume the amenability of $\pi_1 M$.

Theorem 17.29 (Gromov [G9] p.41). *Let M be an n-dimensional Riemannian manifold, P a simplicial complex of dimension smaller that n, and $f: M \to P$ a continuous map. Suppose that, for each $p \in P$, there exists an open neighborhood U of $f^{-1}(p)$ such that the image of the homomorphism : $\pi_1(U) \to \pi_1(M)$ contains an index finite nilpotent subgroup.*[55] *Then $\|[M]\| = 0$.*

We omit the proof. Example 7.21 illustrates the situation.

As an application, we can show that, in each dimension there exists at least one manifold for which Problem 17.13 is affirmative.

Theorem 17.30 (Gromov [G9] p.74). *For each n, there exists a complete n-dimensional $C^{1,\alpha}$-Riemannian manifold (M, g) such that the following holds :*

(17.30.1) *g is a limit with respect to the $C^{1,\alpha}$-topology of the metrics g_i on M such that $|K_{(M,g_i)}| \le 1$.*

(17.30.2) *Let g' be a Riemannian metric on M such that $|K_{(M,g')}| \le 1$ and that $d_L((M,g),(M,g')) < \infty$.*[56] *Then we have :*

$$\mathrm{Vol}(M, g) \le \mathrm{Vol}(M, g').$$

Sketch of the proof. The proof is an application of pointed Hausdörff convergence. We need also a noncompact version of Gromov invariant.

Definition 17.31. Let $X \in \mathcal{MET}_0$. Let $S_k(X; \text{size} \to 0)$ be the set of all countable formal sums $\alpha = \Sigma a_i \sigma_i$ satisfying the following :

(17.32.1) σ_i is a singular simplex, $a_i \in \mathbf{R}$.

(17.32.2) For each compact subset K of X, the number of the simplices σ_i which intersects with K is finite. In other words, α is locally finite.

(17.32.3) $\lim_{i \to \infty} \mathrm{Diam}(\sigma_i) = 0$.

(17.32.4) $\Sigma |a_i|$ is finite. We put $\|\alpha\| = \Sigma |a_i|$.

$S_k(X; \text{size} \to 0)$ is a chain complex. Let $H_*(X; \text{size} \to 0)$ be its homology. For $h \in H_k(X; \text{size} \to 0)$ we put

$$\|h\| = \inf\{\|\alpha\| \mid [\alpha] = h\}.$$

---

[55] It suffices to assume the amenability of $\pi_1 M$.

[56] This second condition is automatically satisfied if M is compact. Then, in that case, $\mathrm{Vol}(M, g) = \mathrm{Minvol}\, M$. But, unfortunately we do *not* know if there exists a compact manifold satisfying (17.30.1) and (17.30.2).

Lemma 17.33. *If M is an orientable complete Riemannian manifold satisfying $|K_M| \le 1$ and if* $\mathrm{Vol}(M) < \infty$, *then the fundamental class $[M]$ of M is well defined as an element of $H_n(M; size \to 0)$.*

We omit the proof, which uses a local version of Theorem 17.29.

Now we fix $h > 0$ and put

$$\mathcal{C} = \{M \mid |K_M| \le 1,\ \mathrm{Vol}(M) < \infty, \|[M]\| \ge h\}$$

And set

$$\mu = \inf\{\mathrm{Vol}(M) \mid M \in \mathcal{C}\}.$$

Lemma 17.34. *If $M \in \mathcal{C}$ satisfies* $\mathrm{Vol}(M) = \mu$, *then each connected component of M satisfies* (17.30.2).[57]

The proof is easy.[58] Thus we are only to find (M, g) whose volume assumes the infimum μ. We can find $M_i \in \mathcal{C}$ such that $\lim \mathrm{Vol}(M_i) = \mu$.

Lemma 17.35. *By taking a subsequence if necessary we have $D > 0, N < \infty$, and points $\{p_{i,1}, \cdots, p_{i,N}\}$ on M_i such that the following holds*

(17.35.1) *If $p \in M_i - \cup B_D(p_{i,j}, M_i)$, then* $\mathrm{inj}_{M_i}(p) < \epsilon_n$, *where ϵ_n is as in Theorem* 13.1.

(17.35.2) $\lim_{i\to\infty}(M_i, p_{i,j})$ *converges to an element (X_j, p_j) of $\mathcal{MET}_0$ with respect to the pointed Hausdorff distance.*

(17.35.3) $\lim_{i\to\infty} d(p_{i,j}, p_{i,j'}) = \infty$, *if $j \ne j'$.*[59]

We omit the proof. Roughly speaking, the lemma says that M_i splits into N pieces $X_1, \cdots, X_N$ as i goes to infinity. (See Figure 17.36).

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[57]Remark that we do not assume the connectivity of M in the definition of $\mathcal{C}$.

[58]Remark that the condition $d_L((M, g), (M, g')) < \infty$ implies that $H_*((M, g); \mathrm{size} \to 0) = H_*((M, g'); \mathrm{size} \to 0)$.

[59]This condition is a technical one.

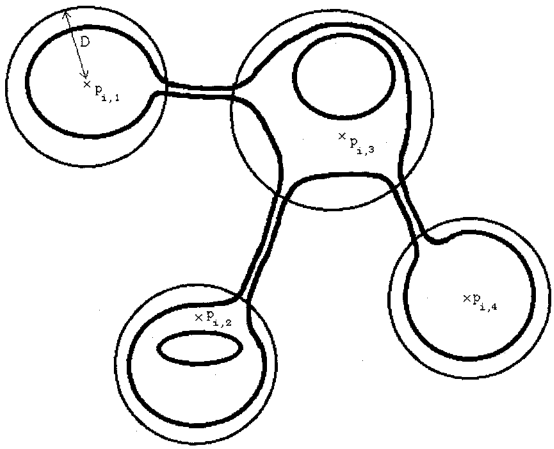

Fig. 17.36

Now put $X = \cup X_j$. By Theorem 6.7, X is a $C^{1,\alpha}$ Riemannian manifold satisfying (17.30.1). Hence it suffices to show

Lemma 17.37.

(17.37.1) $$\mu = \lim \operatorname{Vol} M_i \leq \operatorname{Vol}(X)$$

(17.37.2) $$\|[X]\| \geq h.$$

Sketch of the proof of Lemma 17.37. We discuss only one for (17.37.2). (The proof of (17.37.1) is similar and easier.) By Theorem 3.2 and (17.35.2), we have diffeomorphisms $H_{i,j}: B_{2D}(p_j, X_j) \to B_{2D}(p_{i,j}, M_i)$. Let $\alpha_j = \sum a_{k,j}\sigma_{k,j} \in S_n(X_j; \text{size} \to 0)$ be the cycle representing $[X_j]$ and satisfying $\|\alpha\| < \|[X_j]\| + \delta$, where δ is an arbitrary small positive number. By (17.32.3), we may assume that

$$\sigma_{k,j} \cap B_D(p_j, M_j) \neq \phi \Longrightarrow \sigma_{k,j} \subset B_{2D}(p_j, M_j).$$

For each $\sigma_{k,j}$ contained in $B_{2D}(p_j, X_j)$, we take a singular simplex $\sigma_{k,j,i} = H_{i,j} \circ \sigma_{k,j}$ of M_i. We can find $b_{k,i}$ and $\rho_{k,i}$ such that

$$\sum_{j=1}^{N} \sum_{k=1} a_{j,k}\sigma_{j,k,i} \ + \ \sum_{k=1}^{\infty} b_{i,k}\rho_{k,i}$$

represents the fundamental class $[M_i]$ and that $\rho_{k,i}$ does not intersect with $B_D(p_j, X_j)$. (See Figure 17.38). Then in view of (17.35.1) and

Theorem 17.29,[60] we may assume that

$$\sum |b_{i,k}| < \delta.$$

Therefore we have

$$\|[M]\| \leq \sum \sum |a_{j,k}| + \sum |b_{i,k}| \leq \sum |a_{j,k}| + \delta \leq \|[X]\| + 2\delta.$$

Hence $\|[X]\| \geq h$ as required.

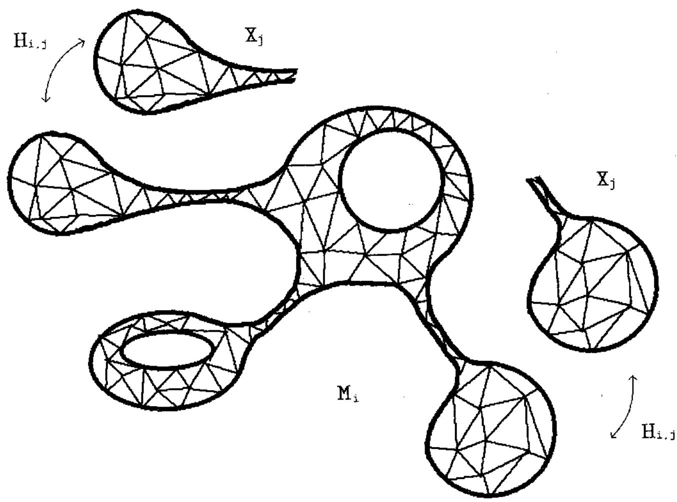

Fig. 17.38

Finally we remark that the invariant defined and studied by O.Kobayashi [Ko] seems closely related to the minimal volume.

§18. Telescope

Hausdorff convergence can be applied to the study of the ends of noncompact complete Riemannian manifolds. In this section, we shall see briefly some of those applications.

Let $((M,g),p)$ be a complete pointed Riemannian manifold. We consider the limit, $\lim_{\epsilon\to 0}((M,\epsilon g),p) = (X,q)$ in $(\mathcal{MET}_0, d_{p.H})$. On the other hand, we define an (inner) metric on $\partial B_D(p,M)$ by

$$d(x,y) = \inf\{\,|\ell| \mid \ell\colon [0,1] \to \partial B_D(p,M), \ell(0) = x, \ell(1) = y\,\},$$

[60] More precisely the local version of it

and study the limit, $\lim_{D\to\infty}(\partial B_D(p, M), d/D) = M(\infty)$, with respect to the Hausdorff distance. Then it is natural to expect that X is isometric to the cone $CM(\infty)$ of $M(\infty)$, and that $M(\infty)$ is closely related to the structure of the ends of M.

Problem 18.1.

(1) Under what condition on M, do $\lim_{\epsilon\to 0}((M, \epsilon g), p)$ and $\lim_{D\to\infty}(\partial B_D(p, M), d/D)$ exist ?

(2) What kind of metric space can $M(\infty) = \lim_{D\to\infty}(\partial B_D(p, M), d/D)$ be ?

(3) Describe the relation between $M(\infty)$ and the structure of the ends of M.

Kasue studied the case of asymptotically flat manifolds.

Theorem 18.2 (Kasue [K2], [K3]). *Let M be a complete n-dimensional Riemannian manifold and $p \in M$. Suppose*

$$(18.3.1) \quad K_M > k \circ r_0, \quad \text{where } r_0(x) = d(p, x), \ \int_0^\infty tk(t)dt < \infty,$$

$$(18.3.2) \quad \kappa_M = \limsup_{t\to\infty} t^2 \cdot \sup\{K_\pi \mid \pi \subset T_x(M), d(x, p) > t\} < \infty.$$

Then $\lim_{\epsilon\to 0}((M, \epsilon g), p) = (X, q)$ and $\lim_{D\to 0}(\partial B_D(p, M), d/D) = M(\infty)$ converge. X is isometric to the cone of $M(\infty)$. Furthermore there exists a sequence of Riemannian manifolds S_D such that $S_D \in \mathcal{M}(n, C \mid \Lambda)$[61] for some Λ and C independent of D and that

$$\lim_{D\to\infty} d_L((\partial B_D(p, M), d/D), S_D) = 0.$$

If we assume $\kappa_M = 0$ and

$$(18.4) \qquad \limsup_{D\to\infty} \frac{\mathrm{Vol}(B_D(p, M))}{D^n} > 0,$$

in addition, then $X - \{q\}$ is a flat Riemannian manifold and $M(\infty)$ is isometric to S^{n-1}/Γ for some finite subgroup Γ of $O(n)$.

Kasue applied this theorem to the studies of the total curvature of complete Riemannian manifolds ([K2] Theorem 3.2), of the Gap phenomena for asymptotically nonpositively curved manifolds ([K2] Theorem 4.1), and of the Busemann function of nonnegatively curved manifolds ([K3] Theorem 4.3). Recently Bando-Kasue-Nakajima [BKN] used

[61] The definition of this symbol is given at the beginning of section 7.

Theorem 18.2 to show that asymptotically flat Ricci flat Kähler manifold satisfying (18.4) is asymptotically Euclidean[62] and applied it to the proof of Theorem 14.17. If we remove the assumption (18.4), $(\partial B_D(p,M), d/D)$ collapses. For example Taub-Nut[63] corresponds to the Berger sphere.[64]

Next we consider the case when M is a locally symmetric space of noncompact type. This case was studied in detail by Gromov-Schroeder [BGS]. We can define a boundary $\partial M = S^{n-1}$ such that $M \cup \partial M = B^n$. For $p \in \partial M$, we put

$$L_p = \left\{ v \in S^{n-1} \;\middle|\; \begin{array}{l} \text{There exists a flat totally geodesic submanifold } F \\ \text{of } M \text{ such that } v \in T_p(\overline{F} \cap \partial M). \end{array} \right\}$$

Remark that dim L_p does depend on p. For example, in the case when $M = \mathbf{H}^2 \times \mathbf{H}^2$, dim $L_p = 2$ if and only if $p = \lim_{t\to\infty}(\ell(t), c)$ or $= \lim_{t\to\infty}(c, \ell(t))$ for some $c \in \mathbf{H}^2$ and some geodesic $\ell\colon [0,\infty) \to \mathbf{H}^2$. Otherwise dim $L_p = 1$. Roughly speaking L_p is determined by the Tits geometry of M. We define a Tits metric, d_T, on ∂M by

$$d_T(x,y) = \inf \left\{ |\ell| \;\middle|\; \begin{array}{l} \ell\colon [0,1] \to \partial M, \ell(0) = x, \ell(1) = y, \\ \forall t \in [0,1] \;\; \dfrac{D}{dt}\ell(t) \in L_{\ell(t)} \end{array} \right\},$$

where $|\ell|$ denotes the length of ℓ with respect to the standard Riemannian metric on S^{n-1}. We have

$$\mathrm{Diam}(S^{n-1}, d_T) < \infty \iff \mathrm{rank}\, M \geq 2.$$

Theorem 18.5 (Gromov-Schroeder [BGS]). *Let M be a symmetric space of noncompact type. Assume* rank $M \geq 2$. *Then,*

$$\lim_{D\to\infty} (\partial B_D(p,M), d/D) = (S^{n-1}, d_T)$$
$$\lim_{\epsilon\to 0}((M, \epsilon g), p) = \text{The cone of } (S^{n-1}, d_T)$$

In [BGS], they defined a Tits metric on the boundaries of nonpositively curved manifolds in more general class. Using it, Ballman-Gromov-Schroeder proved a beautiful generalization of Mostow's rigidity theorem and some other results.

[62]For the definition of these notation see [BKN].
[63]See [Per] p 621 - 623.
[64]Example 10.2

In the case when rank $M = 1$, the limit, $\lim_{D\to\infty}(\partial B_D(p, M), d/D)$, does not exist. Symbolically we can imagine that $\lim_{\epsilon\to 0}((M, \epsilon g)$ is equal to the cone of the sphere, S^{n-1}, equipped with the *discrete topology.*[65] Rigorously speaking $\lim_{\epsilon\to 0}((M, \epsilon g), p)$ does not converge. However $\lim_{D\to\infty}\left(\partial B_D(p, M), \frac{d}{\mathrm{Diam}(\partial B_D(p,M),d)}\right)$ does. To describe its limit we need a notation.

Definition 18.6. Let M be a connected Riemannian manifold and $L \subset TM$ be a subbundle. Assume that the sections $\Gamma(L)$ of L generate $\Gamma(TM)$ as Lie algebra. We define a metric d_c on M by

$$d_c = \inf\left\{|\ell| \,\middle|\, \ell\colon [0,1] \to M,\ \forall t\ \dot{\ell}(t) \in L\right\}.$$

Our assumption on L implies that $d_c(p, q) < \infty$ for each p and q.

Lemma 18.7. *Let M and L be as in* 18.6. *Define a Riemannian metric g_ϵ on M by*

$$g_\epsilon(V, V) = \begin{cases} 1/\epsilon \cdot g(V, V) & \text{if } V \text{ is perpendicular to } L \\ g(V, V) & \text{if } V \in L. \end{cases}$$

Then we have

$$\lim_{\epsilon\to 0} d_H((M, g_\epsilon), (M, d_c)) = 0.$$

Example 18.8. Let $\pi\colon S^{2n+1} \to \mathbf{C}P^n$, $S^{4n+3} \to \mathbf{H}P^n$, $S^{15} \to C_aP^2$, be Hofp fibrations. Put

$$L = \{V \in TS^N \mid V \text{ is perpendicular to the fibre of } \pi\}.$$

Then L satisfies the condition in 18.6.

Remark 18.9. Gromov called d_c the Carnot metric. This metric had been used in the studies of pseudo convex domain and of Hypoelliptic operator.[66] There it is called control metric. Various properties of d_c is proved in [St].

Remark 18.10. In [Mi], Mitchel calculated the Hausdorff dimension of (M, d_c) in generic cases and proved $\dim(M, d_c) > \dim M$. In the

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[65]The author guess that this fact is one of the motivations of the definition of hyperbolic group in Gromov [G12].
[66]See for example [FL]

case of the Carnot metric constructed in Example 18.8, the Hausdorff dimension is $2n+2$, $4n+6$, 22, respectively.

We return to the case when M is a rank one nonpositively curved symmetric space.

Proposition 18.11.

$$\lim_{D\to\infty}\left(\partial B_D(p,M), \frac{d}{\mathrm{Diam}(\partial B_D(p,M),d)}\right)$$

is isometric to S^{n-1} equipped with Carnot metric associated to the fibration $S^{n-1}\to P$, where P is the compact dual of M.

This fact is implicit in the proof by Mostow of his celebrated Rigidity theorem [Mo]. The recent paper [Pa2] by Pansu is closely related to Proposition 18.11.

Finally we mention a little different way of application presented in Gromov [G11].

Theorem 18.12 (Gromov [G11] pp.113,114). *Let M be a complete n-dimensional Riemannian manifold of nonnegative curvature. Then Assumption* (18.13.1) *implies* (18.13.2).

(18.13.1) *For each $n-1$ dimensional polyhedron P and each continuous map $f: M\to P$, we have*

$$\sup\{\mathrm{Diam}\, f^{-1}(p)\mid p\in P\}=\infty.$$

(18.13.2) *For each $D>0$, we have*

$$\sup\{\mathrm{Vol}(B_D(p,M))\mid p\in M\}=\mathrm{Vol}(B_D(0,\mathbf{R}^n)).$$

Remark 18.14. (18.13.1) is satisfied if M is geometrically contractible or has a compact quotient.[67]

As an example, let us consider the manifolds in Figures 16.7 and 18.15. The manifold in Figure 16.7 does not satisfy (18.13.2) but it does not satisfy (18.13.1) either. On the other hand, the manifold in Figure 18.15 has both properties.

Outline of the proof. Let $p_i\in M$ be a sequence such that (M,p_i) converges to an element (X,q) of $\mathcal{MET}_0$. We decompose X into a direct product $\mathbf{R}^d\times V$.[68]

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[67] See Theorem 16.6.

[68] d may be equal to 0.

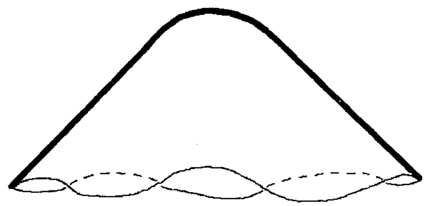

Fig. 18.15

Lemma 18.16. *Assume that, for each sequence q_i of M, the limit of (M, q_i) does not have $\mathbf{R}^{d+1}$ as a direct factor. Then V is compact.*

Sketch of the proof. For simplicity we assume that V is a Riemannian manifold.[69] Then V has nonnegative curvature. Suppose that V is noncompact. Then we can find an isometric embedding $\ell: [0, \infty) \to V$. We can find a sequence $t_i \in [0, \infty)$ such that $\lim t_i = \infty$ and that $(M, \ell(t_i))$ converges to an element (Y, q) of $\mathcal{MET}_0$ with respect to the pointed Hausdorff distance. We can find $q_i \in M$ such that $\lim_{i\to\infty} d_{p.H}((M, q_i), (X, (\ell(t_i), 0)) = 0$.[70] By construction it is easy to see that there exists an isometric embedding $\widetilde{\ell}: \mathbf{R} \to Y$. Then, in the case when Y is a Riemannian manifold, [To2], [CGr1], or [CE] Theorem 8.16 implies that Y can be decomposed into a direct product $Y = Z \times \mathbf{R}$, because Y has nonnegative curvature. In the case when Y is not necessary a manifold, we can prove $Y = \mathbf{R} \times Z$ also by a similar argument. As a consequence we have $\lim_{i\to\infty}(M, q_i) = Z \times \mathbf{R}^{d+1}$. This contradicts the assumption.

We take an integer d satisfying the assumption of Lemma 18.16 and consider all the sequences p_i such that $\lim_{i\to\infty}(M, p_i)$ converges and that the limit has $\mathbf{R}^d$ as the direct factor. Let $\mathbf{R}^d \times V$ be the limit.

Lemma 18.17. *There exists a number D such that* Diam $V < D$. *Here D is independent of the sequences p_i satisfying the above condition.*

Proof. If the lemma is false, then there exists p_i^j such that $\lim_{i\to\infty}(M, p_i^j) = (V_j \times \mathbf{R}^d, (q_j, 0))$, and that $\lim_{j\to\infty} \operatorname{Diam} V_j = \infty$. Then it is easy to find $p_i \in M_i$ such that $\lim(M_i, p_i) = V \times \mathbf{R}^d$ and that V is noncompact. This contradicts Lemma 18.16.

[69]This does not hold in general.
[70]Recall $X = V \times \mathbf{R}^d$.

Lemma 18.18. *There exists C such that*

$$d_{p.H}((B_R(p,M),p),(B_R(0,\mathbf{R}^d),0)) < C$$

for each $p \in M$ and $R > 0$. Here C is independent of p and R.

Sketch of the proof. We shall prove by induction on dim M. We assume that the Lemma is true for the spaces of dimension $< n =$ dim M and is false for M. Then we can find a sequence $p_i \in M$ such that

$$\lim_{i\to\infty} d_{p.H}((B_R(p_i,M),p_i),(B_R(0,\mathbf{R}^d),0)) = \infty. \tag{18.19}$$

We may assume that (M,p_i) converges to an element (X,q) of $\mathcal{MET}_0$. Let us decompose X into a direct product $\mathbf{R}^k \times V$.

If $k = d$ then Diam $V < D$. Therefore $d_{p.H}((B_R(q,X),q),(B_R(0,\mathbf{R}^d),0)) < D$. This contradicts (18.19).

If $k < d$, we can apply the induction hypothesis to V and can prove that $d_{p.H}((B_D(q,V),q),(B_D(0,\mathbf{R}^{d-k}),0)) < \infty$. This also contradicts (18.19).

Using Lemma 18.18 we can construct a d-dimensional polyhedra P and a continuous map $f: M \to P$ such that

$$\sup\{\text{Diam } f^{-1}(p) \mid p \in P\} < \infty.$$

Then by Assumption 18.11.1, we have $d = n$. Therefore we can find a sequence $p_i \in M$ such that (M,p_i) converges to $\mathbf{R}^n$ with respect to the pointed Hausdorff distance. It follows that

$$\lim_{i\to\infty} \mathrm{Vol}(B_D(p_i,M)) = \mathrm{Vol}(B_D(0,\mathbf{R}^n))$$

for each D. (18.13.2) follows immediately.

§19. T- and F- structures

In this section we discuss the approach due to Cheeger-Gromov [CG4], [CG5] e.t.c. to the studies of collapsing Riemannian manifolds.

Definition 19.1. Let M be a C^∞-manifold. A *T-structure* on M stands for a triple $(\{U_i\},\{T^{k_i}\},\{\varphi_i\})$ such that

(19.1.1) $\{U_i\}$ is an open covering of M,
(19.1.2) T^{k_i} is a k_i-dimensional torus,
(19.1.3) $\varphi_i: T^{k_i} \to \mathrm{Diff}(U_i)$ is an effective and smooth action,

(19.1.4) if $U_i \cap U_j \neq \phi$, then $U_i \cap U_j$ is (T^{k_i}, φ_i) and (T^{k_j}, φ_j) invariant,

(19.1.5)

$$\varphi_i(\gamma_i)\varphi_j(\gamma_j)(x) = \varphi_j(\gamma_j)\varphi_i(\gamma_i)(x),$$

for $x \in U_i \cap U_j$, $\gamma_i \in T^{k_i}, \gamma_j \in T^{k_j}$.

F-structure is defined replacing open covering in Definition 19.1 by etale covering. Namely :

Definition 19.2. An *F-structure* on M stands for the collection $(\{U_i\}, \{\widetilde{U}_i\}, \{T^{k_i}\}, \{\varphi_i\}, \{\psi_i\})$ such that

(19.2.1) $\{U_i\}$ is an open covering,

(19.2.2) $\pi_i: \widetilde{U}_i \to U_i$ is a finite Galois covering with Galois (deck transformation) group $= G_i$,

(19.2.3) T^{k_i} is a k_i-dimensional torus,

(19.2.4) $\varphi_i: T^{k_i} \to \mathrm{Diff}(\widetilde{U}_i)$ is an effective and smooth action,

(19.2.5) $\psi_i: G_i \to \mathrm{Aut}\left(T^{k_i}\right)$ is a homomorphism satisfying

$$g_i(\varphi_i(\gamma_i)(x)) = \varphi_i((\psi_i(g_i)(\gamma_i))(x)),$$

for each $g_i \in G_i, \gamma_i \in T^{k_i}, x \in \widetilde{U}_i$,

(19.2.6) if $U_i \cap U_j \neq \phi$, then $\pi^{-1}(U_i \cap U_j)$ is (T^{k_i}, φ_i)-invariant,

(19.2.7) let $V_{i,j}$ be the fibre product in Diagram 19.3. Finite covers $\widetilde{T}^{k_i}$ and $\widetilde{T}^{k_j}$ of T^{k_i} and T^{k_j} act on $V_{i,j}$. Then we have

$$\widetilde{\varphi}_i(\gamma_i)\widetilde{\varphi}_j(\gamma_j)(x) = \widetilde{\varphi}_j(\gamma_j)\widetilde{\varphi}_i(\gamma_i)(x),$$

where $\gamma_i \in \widetilde{T}^{k_i}$, $\gamma_j \in \widetilde{T}^{k_j}$, $x \in V_{i,j}$, and $\widetilde{\varphi}_i: \widetilde{T}^{k_i} \to \mathrm{Diff}(V_{i,j})$, $\widetilde{\varphi}_j: \widetilde{T}^{k_j} \to \mathrm{Diff}(V_{i,j})$, are induced actions.

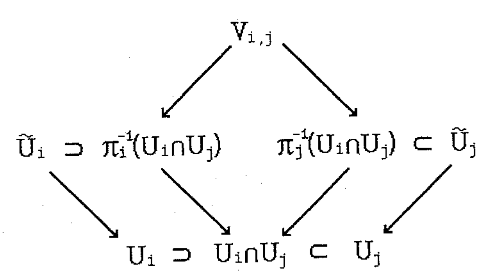

Diagram 19.3

Remark 19.4. Cheeger-Gromov [CG4] used more sophisticated terminology (sheaf of groups and their actions) to define F-structure.

Remark 19.5. In Thanks to (19.2.5) the orbit of the action on U_i through $x \in U_i$ is well defined.

Definition 19.6. A *polarization* of the F-structure is a collection of connected subgroups $H_i \subset T^{k_i}$ such that the following holds.[71]

(19.7.1) H_i is invariant by the adjoint action of $\psi_i(G_i)$.

(19.7.2) The H_i-orbit in U_i through $p \in U_i \cap U_j$, (which is well defined by (19.7.1)), contains the H_j-orbit through p in U_j or is contained in it.

(19.7.3) The dimension of each H_i-orbit is equal to dim H_i. In other words the action of H_i is locally free.

A polarization $H_i \in T^{k_i}$ is a *pure polarization* if

(19.7.4) the H_i-orbit in U_i through $p \in U_i \cap U_j$ coincides to the H_j-orbit in U_j through p. In other words, $\dim H_i = \dim H_j$ for each i and j.

An F-structure is said to be *polarized* if it has a polarization and is said to be *pure polarized* if it has a pure polarization.

An F-structure $(\{U_i\}, \{\widetilde{U}_i\}, \{T^{k_i}\}, \{\varphi_i\}, \{\psi_i\})$ is said to be *pure* if, for each $p \in U_i \cap U_j$ the orbit of T^{k_i} through p coincides to the orbit of T^{k_j} through p.

An F-structure is said to be of *positive dimension*, if all orbits are of positive dimension.

Examples 19.8.

(1) Any effective action of a torus T^k on M defines a T-structure on M. If $I_p = \{g \in T^k \mid g(p) = p\}$ is discrete for each p, then $H = T^k$ is a pure polarization of this T-structure. Next, suppose that I_p is not necessarily discrete but $I_p \neq T^k$ for each p. Let $H \simeq \mathbf{R}$ be a dense subgroup of T^k. Then H defines also a pure polarization of this T-structure.(Compare Example 10.3.)

(2) Let $T^k \to M \to N$ be a fibration. This fibration defines a T-structure on M if the structure group is reduced to $T^k \propto \mathrm{Aut}(T^k)$. This T-structure is pure polarized.

[71] We do *not* assume that H_i is compact.

(3) More generally let M be a manifold, N an orbifold, and $T^k \to M \to N$ be a Zeifert fibration.[72] This fibration defines an F-structure on M if the structure group is reduced to $T^k \propto \mathrm{Aut}(T^k)$. This F-structure is pure polarized.

(4) Let $f: M \to N$ be as in Example 10.11. Put $T^k = \frac{\text{Cent } N}{\text{Cent } N \cap \Lambda}$. Then, using the action of T^k to the fibres, we can define a pure polarized F-structure on M.

(5) Let $f: M \to X$, $\widetilde{f}: FM \to Y$ be as in the conclusion of Theorem 12.7. By (4), FM has a pure polarized F-structure, which is invariant by the action of $O(n)$. Hence it induces a pure F-structure of positive dimension on M. The author does not know whether this F-structure admits a polarization or not.

(6) Let $M^3 = (T^2 - B^2) \times S^1 \cup_{T^2} S^1 \times (T^2 - B^2)$ be as in Example 17.4. We take an open covering $M = U_1 \cup U_2 \cup U_3$ such that $U_1 \simeq U_3 \simeq (T^2 - B^2) \times S^1$, $U_2 \simeq \mathbf{R} \times T^2$. We put $k_1 = k_3 = 1, k_2 = 2$. Then T^{k_i} acts on U_i in an obvious way. These actions give a T-structure on M^3. $H_i = T^{k_i}$ is a polarization of this T-structure. We can prove that this structure is not pure polarized. A similar example can be constructed from Examples 17.7, 17.12.

(7) (Januszkiewicz[73]) Let T^{2n} act on $\mathbf{C}P^{2n}$ by

$$(e^{i\theta_1}, \cdots, e^{i\theta_{2n}})[z_0, \cdots, z_{2n}] = [z_0, e^{i\theta_1} z_1, \cdots, e^{i\theta_{2n}} z_{2n}],$$

where $[z_1, \cdots, z_n]$ is a homogeneous coordinate. This action has $2n+1$ fixed points $p_i = [0, \cdots, \underset{i}{1}, \cdots, 0]$, $i = 0, \cdots, 2n$. Put $B_i = B_\epsilon(p, \mathbf{C}P^{2n})$. Take two copies M_1, M_2 of $\mathbf{C}P^{2n} - \cup B_i$. Define $f: \partial M_1 \to \partial M_2$ by

$$f([z_0, \cdots, z_n]) = \begin{cases} [z_0, \cdots, z_i, \overline{(z_{i+1}/z_i)} \cdot z_i, \cdots, z_{2n}] \\ \qquad \text{if } [z_0, \cdots, z_{2n}] \in \partial B_i, i \neq 2n \\ [\overline{(z_0/z_{2n})} \cdot z_{2n}, z_1, \cdots, z_{2n}] \\ \qquad \text{if } [z_0, \cdots, z_{2n}] \in \partial B_{2n}. \end{cases}$$

We obtain an orientable manifold M by attaching M_1 and M_2 by f. Put $U_1 = M_1$, $U_3 = M_2$, $U_2 =$ a neighborhood of ∂M_1 in M. Then T^{2n} acts on U_1, U_3 and T^{2n+1} acts on U_2. These actions define a T-structure on M. We can prove that this F-structure does not admit a polarization, as follows. By a simple calculation we see that the

[72] See [Or] 5.1.
[73] See [CG4] Example 1.9.

signature of M is equal to 2. Hence by generalized Gauss-Bonnet formula the minimal volume of M is not 0.[74] On the other hand, Theorem 19.10 below shows that if a manifold admits a polarized F-structure then its minimal volume is 0.

Remark 19.9. Example 19.8 (7) shows that signature is not an obstruction for the existence of an F-structure of positive dimension. On the other hand, [CG 4] proved that the Euler number of M vanishes if it has an F-structure of positive dimension.

Theorem 19.10. (Cheeger-Gromov [CG4]). *Let M be a C^∞-manifold. Suppose that M admits an F-structure of positive dimension. Then there exists a family of metrics g_ϵ on M such that*

$$\sup inj_{(M,g_\epsilon)} < \epsilon, \tag{19.10.1}$$

$$\left|K_{(M,g_\epsilon)}\right| \le 1. \tag{19.10.2}$$

If we assume that the F-structure is polarized, then we have

$$\lim_{\epsilon\to 0} \mathrm{Vol}(M, g_\epsilon) = 0. \tag{19.10.3}$$

If we assume that the F-structure is pure polarized, then we have

$$\sup_{\epsilon\in(0,1)} \mathrm{Diam}(M, g_\epsilon) < \infty. \tag{19.10.4}$$

Idea of the proof. Using the proof of Proposition 10.3, we can find a desired family of metrics on each chart U_i. The problem is how to patch them together while keeping Condition (19.10.1). On $U_i \cap U_j$ we have two families of metrics, one is obtained by shrinking each orbit of T^{k_i} and the other is obtained by shrinking each orbit of T^{k_j}. Hence if we patch the metrics directly, then the curvature goes to infinity. To avoid this, we expand the direction normal to both T^{k_i} and T^{k_j} orbits. The detail is found in [CG4].

Theorem 19.11. (Cheeger-Gromov [CG5]). *There exists a positive number ϵ_n depending only on the dimension n satisfying the following. Let M be a complete Riemannian manifold with $|K_M| \le 1$. Then there exists an open set U of M such that*

(19.11.1) *If $p \in M - U$ then $inj_M(p) > \epsilon$,*

[74] See [G9].

(19.11.2) *There exists an F-structure of positive dimension on U.*

Remark 19.12. Theorem 19.11 gives a converse to the first part of Theorem 19.10. Namely if we assume $inj_M < \epsilon_n$, then M has an F-structure of positive dimension.

Open Problem 19.13. Suppose that M admits a family of metrics g_ϵ satisfying (19.10.1), ..., (19.1.3). Does M admit a polarized F-structure of positive dimension ?

Suppose that M admits a family of metrics g_ϵ satisfying (19.1.1), ..., (19.1.4). Does M admit a pure polarized F-structure of positive dimension ?

The question we suggested in Examples 19.8 (5) is closely related to the second problem.

Sketch of the proof of Theorem 19.11. We prove by contradiction. Let (M_ϵ, g_ϵ) be a family such that $|K_{(M_\epsilon,g_\epsilon)}| \le 1$ but M_ϵ does not satisfy the conclusion of Theorem 19.11 for $\epsilon_n = \epsilon$. Take points $p_\epsilon \in M_\epsilon$ such that $inj_{(M_\epsilon,g_\epsilon)}(p_\epsilon) = \delta_\epsilon < \epsilon$. By Theorem 6.6, we may assume that $((M_\epsilon, g_\epsilon/\delta_\epsilon), p_\epsilon)$ converges to a space (X,p) with respect to the pointed Hausdorff distance. Since $inj_{(M_\epsilon,g_\epsilon/\delta_\epsilon)}(p_\epsilon) = 1$ and since $|K_{(M_\epsilon,g_\epsilon/\delta_\epsilon)}| \le \delta_\epsilon \to 0$, it follows that (X,p) is a flat Riemannian manifold and that $inj_X(p) = 1$. Therefore the soul theorem[75] for nonnegatively curved manifolds implies that X is diffeomorphic to a vector bundle over a totally geodesic compact submanifold, the soul, S, of X. Since X is flat, so is S. It follows that a finite covering of X admits an action of a torus. On the other hand, by the proof of Theorem 3.2 we see that a neighborhood of p_ϵ in M_ϵ is diffeomorphic to a neighborhood of p in X. Hence we can find a neighborhood U of p_ϵ, its finite cover $\widetilde{U}_i$ and an action of a torus T^{k_i} on $\widetilde{U}_i$. Thus, it suffices to make these actions compatible in the sense of Definition 19.2. We omit the detail of the argument of this part, which can be found in [CG5]. Essentially this follows from the topological rigidity of compact group action.[76]

In [CG1], [CG2], [CG3], [CG6], Cheeger-Gromov gave interesting applications of Theorem 19.11 to the study of Gauss-Bonnet type theorems for open manifolds with finite volume. Unfortunately this article is already so long that we have no room for mentioning them.

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[75][CGr1], [CE] Chapter 8

[76]See Theorem 6.9.

Note added in proof. Grove-Petersen-Wu said that they proved Theorem 14.9 but not 14.6. 14.6 is now a conjecture.

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