Hausdorff dimension for horseshoes

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Abstract. We shall measure how thick a basic set of a C^1 axiom A diffeomorphism of a surface is by the Hausdorff dimension of its intersection with an unstable manifold. This depends continuously on the diffeomorphism. Generically a C^2 diffeomorphism has attractors whose Hausdorff dimension is not approximated by the dimension of its ergodic measures.

0. *Introduction*

The Smale horseshoe [16] is an invariant set Λ (a 'basic set') of a certain axiom A diffeomorphism of a surface. It behaves like a saddle in that many orbits approach for a while and then leave a neighbourhood of Λ. Bowen and Ruelle [2], [5], showed that, for C^2 diffeomorphisms f satsifying axiom A, the set $W^s(\Lambda)$ of points that do approach Λ has Lebesgue measure zero. However, [3] exhibits a C^1 diffeomorphism where $W^s(\Lambda)$ has positive Lebesgue measure. Here we consider a more delicate way to describe how substantial the basic set Λ is. The unstable manifold $W^u(x)$ of a point x of Λ meets it in a Cantor set. We calculate the Hausdorff dimension of this set and show that it is independent of x and varies continuously with the C^1 diffeomorphism f. We also show that, generically, a C^2 diffeomorphism has attractors whose Hausdorff dimension is *not* approximated by the dimension of its ergodic measures! Finally we consider the bifurcation from Anosov to DA and show that the Hausdorff dimension of the attractor is continuous across the bifurcation.
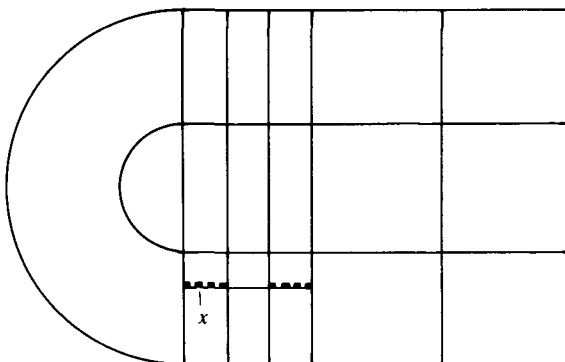

FIGURE 1

1. *Hausdorff dimension in an unstable manifold*

Let $M = M^2$ be a compact surface without boundary and let $f \in \text{Diff}^1(M)$ satisfy axiom A and the No Cycle Property (which are sufficient conditions for Ω-stability

[17]). Let Λ be one of the basic sets for f and let $\Lambda(g)$ be the corresponding basic set for g in some C^1 neighbourhood of f. We denote by $W^u_{loc}(x)$ and $W^u_{loc}(x, g)$ the local unstable manifolds of a point x under the diffeomorphisms f and g.

Consider the set

$$W^u_{loc}(x, g) \cap \Lambda(g).$$

We may regard the Hausdorff dimension of this set as a measure of the 'fatness' of the basic set $\Lambda(g)$ in the unstable direction and hence of its influence on the nearby dynamics. Hausdorff dimension is defined for a metric space X as follows, see [10, chapter 7]. For a non-negative real number p the *Hausdorff p-measure* of X is

$$m_p(X) = \sup_{\varepsilon > 0} \inf \sum_{i=1}^{\infty} [\text{diam}\, (A_i)]^p$$

where the infimum is taken over all covers of X by a countable collection of subsets $A_1, A_2, A_3, \ldots$, each of diameter less than ε. Then the *Hausdorff dimension* $HD(X)$ is defined by

$$HD(X) = \sup\{p; m_p(X) = \infty\} = \inf\{p; m_p(X) = 0\}.$$

We shall also need the *pressure* of a continuous function [14], [18]. Let $C(\Lambda, \mathbb{R})$ denote the continuous real-valued functions on the basic set Λ of f with the sup norm. Then pressure $P: C(\Lambda, \mathbb{R}) \to \mathbb{R}$ is defined by

$$P(\psi) = \limsup_{n \to \infty} n^{-1} \log \sup_E \sum_{x \in E} \exp\left(\sum_{i=0}^{n-1} \psi(f^i x)\right),$$

where the supremum is taken over (n, δ)-separated sets E and δ is an expansive constant for $f|\Lambda$. The variational principle [18] gives an alternative description of pressure as

$$P(\psi) = \sup_\mu \left\{ h_\mu(f) + \int \psi \, d\mu \right\}$$

where the supremum is taken over f-invariant Borel probability measures μ. From this it follows easily, or see [18], that

$$|P(\phi) - P(\psi)| \le \|\phi - \psi\|,$$

P is convex and, for a strictly negative function ψ, the map

$$t \mapsto P(t\psi)$$

is strictly decreasing.

Now we can state our first theorem.

THEOREM 1. *Let Λ be a basic set for a C^1 axiom A diffeomorphism $f: M^2 \to M^2$ with $(1, 1)$ splitting*

$$T_\Lambda M = E^s \oplus E^u.$$

Define $\phi : W^u(\Lambda) \to \mathbb{R}$ by

$$\phi(x) = -\log \|Df_x | E^u_x\|.$$

Then the Hausdorff dimension of $W^u(x) \cap \Lambda$ is given by the unique δ for which

$$P_{f|\Lambda}(\delta\phi) = 0.$$

δ *is independent of* $x \in \Lambda$, *and* $0 \le \delta \le 1$. *Moreover,* δ *depends continuously on f in the* C^1 *topology on diffeomorphisms.*

Proof. For purposes of calculating Hausdorff dimension it does not matter which Riemannian metric we choose on M since any two are Lipschitz equivalent so let us choose a metric 'adapted' to f, [12]. Now ϕ is a negative function and so $t \mapsto P(t\phi)$ is strictly monotonically decreasing, [18].

$$P(0) = h(f|\Lambda) \ge 0$$

and, for C^2 diffeomorphisms f, we have $P(\phi) \le 0$ by [2]. Hence there is a unique value δ of t for which $P(t\phi) = 0$ and, at least for f of class C^2, it satisfies $0 \le \delta \le 1$.

Given any $t > \delta$ choose $\varepsilon > 0$ such that

$$P(t(\phi + \varepsilon)) < 0.$$

Choose a Markov partition $\mathcal{U}$ for $f|\Lambda$ so fine that ϕ varies by at most ε on the intersection of $W^u(\Lambda)$ with the region of M enclosed by each rectangle. We also need each rectangle to be enclosed by four intervals of stable and unstable manifold and we do not allow rectangles to consist of finitely or infinitely many pieces between each other. (This 'connected' or 'convex' property can be achieved for our two-dimensional case by modifying the construction given in [2, p 81] to divide T_j into up to nine parts for each neighbour T_k as in figure 2(a) instead of four as in figure 2(b). Alternatively see [15].)
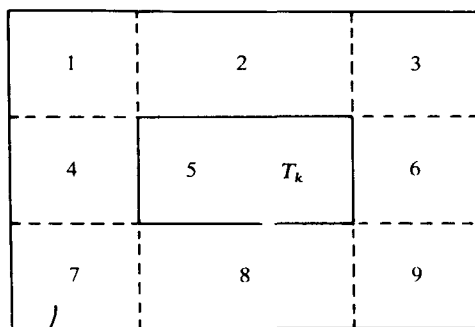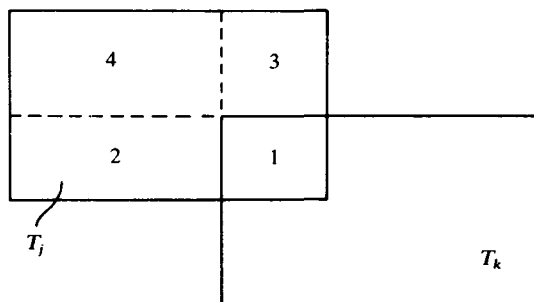

FIGURE 2(a)

FIGURE 2(b)

Consider a compact set

$$W = W^u_{loc}(x) \cap \Lambda.$$

We shall show that the Hausdorff t-measure of W is zero. Since each unstable manifold meets Λ in a countable union of such pieces it cannot then have dimension more than δ.

The compact set W is in a C^1 submanifold of dimension one and the metric on that as a Riemannian manifold in its own right is Lipschitz equivalent to the restriction to W of the metric on M. Thus the Hausdorff dimension of W will be the same for the two metrics and we may choose to work with the former.

Choose some large n and consider the cover $\mathcal{U}_n$ of W by n-cylinders. These are intervals I so that, for each i with $0 \leq i \leq n$, all y in I have $f^i y$ in the same rectangle. By the Mean Value Theorem there is some particular $y \in I$ for which

$$\text{diam } I = \text{diam } (f^n I) \cdot \|Df^n_y | E^u\|^{-1} = \text{diam } (f^n I) \cdot \exp \sum_{i=0}^{n-1} \phi(f^i y).$$

This point y could be wandering so choose some $z \in I \cap \Lambda$. Then

$$\exp \sum_{i=0}^{n-1} \phi(f^i y) \leq \exp \sum_{i=0}^{n-1} (\phi(f^i z) + \varepsilon).$$

Now $f^n I$ stretches across some rectangle so

$$\text{diam } I \leq m \exp \sum_{i=0}^{n-1} (\phi(f^i z) + \varepsilon),$$

where the constant m is the maximum width of any rectangle.

Let E be the set of these points z chosen one from each interval I in the cover $\mathcal{U}_n$ of W by n-cylinders. The set E is arranged in some order along W and we write

$$E = E_1 \cup E_2$$

where E_1 and E_2 are the odd and even numbered points in this order in E. Now E_1 and E_2 are both $(n+1, \delta)$-separated sets where δ is the minimum distance apart of any two disjoint rectangles. This uses the 'convexity' property of the rectangles. For the cover $\mathcal{U}_n$ of W we find

$$\sum_{I \in \mathcal{U}_n} (\text{diam } I)^t \leq m^t \sum_{z \in E} \exp \sum_{i=0}^{n-1} t(\phi(f^i z) + \varepsilon)$$

$$\leq 2m^t \max_{i=1,2} \sum_{z \in E_i} \exp \sum_{i=0}^{n-1} t(\phi(f^i z) + \varepsilon)$$

$$\leq 2m^t \sup_{E'} \sum_{x \in E'} \exp \sum_{i=0}^{n-1} t(\phi(f^i x) + \varepsilon)$$

$$\leq 2m^t c \sup_{E'} \sum_{x \in E'} \exp \sum_{i=0}^{n} t(\phi(f^i x) + \varepsilon),$$

where c is a constant involving the norm of ϕ and the supremum is taken over $(n+1, \delta)$-separated sets E'. Now this expression has the negative growth rate $P(t(\phi + \varepsilon))$ as n increases so the Hausdorff t-measure of W is zero as claimed and the dimension of W is at most δ.

On the other hand consider $t = \delta$. $P(\delta\phi) = 0$ and $\delta\phi$ has an ergodic equilibrium state μ, say, (which need not be unique when f is C^1). Thus

$$h_\mu + \delta\mu(\phi) = 0.$$

By [11] the subset of W consisting of points that are future generic for μ has Hausdorff dimension h_μ/χ_μ where χ_μ is the positive Lyapunov exponent for μ. But χ_μ is just $-\mu(\phi)$ so this subset of W has Hausdorff dimension

$$-h_\mu/\mu(\phi) = \delta$$

and the dimension of W is precisely δ. This proof is different from [4, lemma 10] because our ϕ need not have any Gibbs state.

It remains to prove that δ depends continuously on f in the C^1 topology. That will then imply that $\delta \leq 1$ for f C^1 since this holds for C^2 diffeomorphisms and they are dense in the C^1 topology. For any g close enough to f in the C^1 topology there is a homeomorphism $h : \Lambda \to \Lambda(g)$ satisfying $hf = gh$. Now h only moves points a small distance and so Dg_{hx} is near Df_{hx}, since g is C^1 near f, and this is near Df_x, since f is C^1. To compare $E^u(g)$ on $\Lambda(g)$ with E^u on Λ consider a family of cones in $T_\Lambda M$ containing E^u, invariant by Df. Extend this to a neighbourhood of Λ still invariant by Df. This family is invariant by Dg and so contains $E^u(g)$. Hence the map

$$\phi_g h : \Lambda \to \mathbb{R} \qquad x \mapsto -\log \|Dg_{hx} | E^u(g)\|$$

is close to $\phi : \Lambda \to \mathbb{R}$. The conjugacy h makes $P_g(t\phi_g)$ vanish for the same value of t as $P_f(t(\phi_g h))$ and this is near the value δ that makes $P(t\phi)$ vanish since $P : C(\Lambda, \mathbb{R}) \to \mathbb{R}$ is continuous. Thus the Hausdorff dimension of $W^u(hx, g) \cap \Lambda(g)$ is near that of $W^u(x, f) \cap \Lambda$ for g C^1 near f. □

From now on let δ and ϕ be denoted by δ_u and ϕ^u. In [3] Bowen constructed a C^1 diffeomorphism with a horseshoe of positive measure. This horseshoe is a product of sets of positive Lebesgue measure in a stable and an unstable manifold and so has $\delta_u = 1$. We now show how unusual this is.

COROLLARY 1. *For a C^1 open dense subset of the C^1 axiom A no-cycle diffeomorphisms of a surface each basic set that is not an attractor has $\delta_u < 1$.*

Proof. The property holds for a C^2 diffeomorphism f by [2, theorem 4.11] which says that

$$P(\phi^u) < 0$$

if the basic set is not an attractor. By the continuity property in theorem 1 we still have $\delta_u < 1$ for a C^1 neighbourhood of f. The union of these neighbourhoods gives the required C^1 open dense set. □

COROLLARY 2. *In the situation of theorem* 1 *the Hausdorff dimension of $W^s(x) \cap \Lambda$ is the unique number δ_s, $0 \leq \delta_s \leq 1$, for which*

$$P(\delta_s \phi^s) = 0$$

where $\phi^s : \Lambda \to \mathbb{R}$ is the negative function given by

$$\phi^s(y) = \log \|Df | E^s_y\|.$$

Proof. Apply theorem 1 to f^{-1}. □

COROLLARY 3. *The Hausdorff dimension δ_u in theorem 1 satisfies*

$$-h_{\text{top}}/m(\phi^u) \le \delta_u \le h_{\text{top}}/(h_{\text{top}} - P(\phi^u)),$$

where h_{top} is the topological entropy of $f|\Lambda$ and $m(\phi^u)$ is the integral of ϕ^u with respect to the measure of maximal entropy.

Proof. The variational inequality for pressure gives

$$0 = P(\delta_u\phi^u) = \sup_\nu (h_\nu + \delta_u\nu(\phi^u))$$
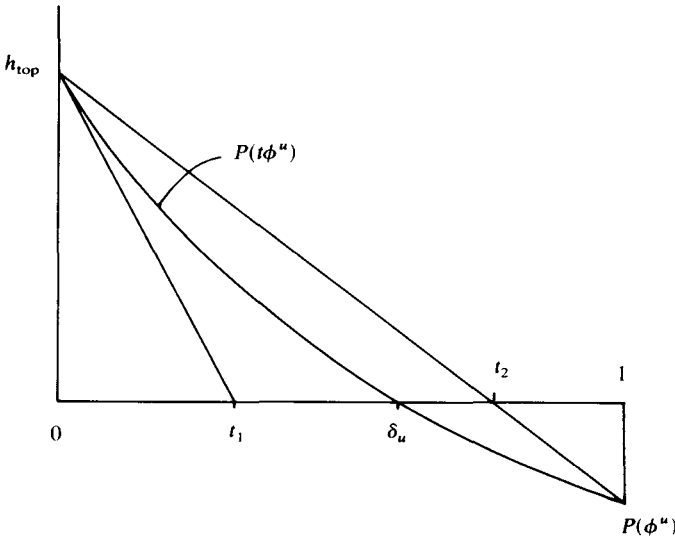$$\ge h_m + \delta_u m(\phi^u) = h_{\text{top}} + \delta_u m(\phi^u)$$

FIGURE 3

from which we have our first inequality. The second comes from the convexity, whereby

$$t_2 = h_{\text{top}}/(h_{\text{top}} - P(\phi^u)) \ge \delta_u. \qquad \square$$

Remark. Notice that $m(\phi^u) < 0$, that m is the unique equilibrium state of the zero function, and that $P(\phi^u) \le 0$. Indeed, Bowen and Ruelle showed that $P(\phi^u)$ has the following significance for C^2 diffeomorphisms [2]. The Lebesgue measure of the set of those points whose orbits remain within ε of Λ from time 0 to n decays like $\exp nP(\phi^u)$. Since the pressure curve

$$t \mapsto P(t\phi^u)$$

is convex there is no precise relation between δ_u and $P(\phi^u)$ although they both indicate the thickness of Λ. In fact the variational inequality shows that the straight lines

$$t \mapsto h_{\text{top}} + tm(\phi^u) \quad \text{and} \quad t \mapsto h_\mu + t\mu(\phi^u),$$

where μ is an equilibrium state for ϕ, lie below the curve $t \mapsto P(t\phi)$. They are actually tangents to the pressure curve at $t = 0$ and 1 provided the curve is differentiable there. The upper and lower bounds for δ_u in corollary 3 are now equal precisely

when
$$m(\phi^u) = \mu(\phi^u)$$
and this implies that $h_\mu = h_{\text{top}}$. In particular this happens when ϕ is constant as in the case of an algebraic Anosov diffeomorphism of T^2.

2. *Hausdorff dimension of invariant measures*

Recently, Young [20] has proposed to define the Hausdorff dimension $\text{HD}(\mu)$ of a probability measure μ as
$$\inf\{\text{HD}(Y); \mu(Y) = 1\}.$$
She proves:

THEOREM (Young). *Let $f : M^2 \to M^2$ be a C^2 diffeomorphism or a C^1 axiom A diffeomorphism. Let μ be an f-invariant ergodic Borel probability measure with Lyapunov exponents $\lambda_1 > 0 > \lambda_2$. Then*
$$\text{HD}(\mu) = h_\mu(1/\lambda_1 - 1/\lambda_2).$$

We shall investigate the relationship of this to the Hausdorff dimension of the basic set on which μ is concentrated. Let Λ be a basic set of a C^2 axiom A diffeomorphism of a surface. Assume that Λ is not just a periodic source or sink in which case it has E^s and E^u one-dimensional. Young's formula gives
$$\text{HD}(\mu) = -h_\mu(1/\mu(\phi^u) + 1/\mu(\phi^s))$$
for an ergodic measure μ on Λ. Now the canonical coordinates are Lipschitz [9] and Λ is locally Lipschitz equivalent to a product of metric spaces
$$(W^u_{\text{loc}}(x) \cap \Lambda) \times (W^s_{\text{loc}}(x) \cap \Lambda).$$
By [1, theorem 2] and [4, lemma 10], the Hausdorff dimension of this product is
$$\delta_u + \delta_s.$$
Pesin has also considered this question in [13].

Let μ_u, μ_s denote respectively the unique equilibrium states of $\delta_u\phi^u$ and $\delta_s\phi^s$. Since
$$P(\delta_u\phi^u) = 0$$
we have
$$h_\mu + \delta_u\mu(\phi^u) \le 0 \quad \text{or} \quad -h_\mu/\mu(\phi^u) \le \delta_u$$
with equality only for $\mu = \mu_u$. Similarly
$$-h_\mu/\mu(\phi^s) \le \delta_s$$
with equality only for $\mu = \mu_s$. Adding the inequalities we obtain
$$\text{HD}(\mu) \le \delta_u + \delta_s = \text{HD}(\Lambda)$$
with equality only if $\mu = \mu_u = \mu_s$. As in [2, corollary 4.15], $\mu_u = \mu_s$ fails for f in a C^2 open dense set. Now
$$-(1/\mu(\phi^u) + 1/\mu(\phi^s))$$
is a positive continuous function of μ and h_μ is an upper semicontinuous function of μ (since $f|\Lambda$ is expansive [6]) so $\text{HD}(\mu)$ depends upper semi-continuously on

μ and attains its supremum. Thus, if $\mu_u \neq \mu_s$,

$$\sup_\mu \mathrm{HD}(\mu) < \mathrm{HD}\,(\Lambda).$$

As in the proof of theorem 1, for g in some prescribed neighbourhood of f we have

$$\|\phi_g^u \circ h - \phi^u\| < \varepsilon \quad \text{and} \quad \|\phi_g^s \circ h - \phi^s\| < \varepsilon$$

where $h : \Lambda \to \Lambda_g$ is the conjugacy. Then

$$\left| \frac{1}{(h_* \mu)(\phi_g^u)} - \frac{1}{\mu(\phi^u)} \right| \le \frac{\varepsilon}{\mu(\phi_g^u h)\mu(\phi^u)} \le \frac{\varepsilon}{\inf |\phi^u| \cdot (\inf |\phi^u| - \varepsilon)}.$$

Hence $\sup_\mu \mathrm{HD}(\mu)$ depends continuously on the C^2 diffeomorphism f. We can summarise this as:

THEOREM 2. *Let $f : \Lambda \to \Lambda$ be a basic set of a C^2 axiom A diffeomorphism of a surface. Then*

$$\mathrm{HD}(\Lambda) = \delta_u + \delta_s$$

varies continuously with f. So does $\sup_\mu \mathrm{HD}(\mu)$ taken over ergodic measures μ on Λ and the former is strictly greater than the latter except on the nowhere dense set of those diffeomorphisms for which $\mu_s = \mu_u$.   □

Remark. Yorke's suggested formula [7] for the dimension or capacity of an attractor with exponents $\lambda_1 > 0 > \lambda_2$ (with $\lambda_1 + \lambda_2 < 0$) was $1 - (\lambda_1/\lambda_2)$ and when λ_1, λ_2 are calculated for the Bowen–Ruelle measure ν this agrees with Young's formula for

$$\mathrm{HD}\,(\nu) = h_\nu(1/\lambda_1 - 1/\lambda_2)$$

since here $\lambda_1 = h_\nu$ by $P(\phi^u) = 0$. (See [20] for discussion of the relationship between capacity and Hausdorff dimension.) Thus our theorem 2 says that for an open dense set of C^2 axiom A diffeomorphisms of surfaces Yorke's formula gives too low an answer. One would expect to observe this, for example, for an Anosov diffeomorphism of the torus which is a perturbation of a linear one and has ϕ^s constant but ϕ^u not. An invariant subset of an attractor that does have full dimension is

$$\left\{ x \in \Lambda; \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} \to \nu \quad \text{and} \quad \frac{1}{n} \sum_{i=-n+1}^{0} \delta_{f^i x} \to \nu^- \quad \text{as } n \to \infty \right\}$$

where ν^- is the equilibrium state for $\delta_s \phi^s$. Gurevich and Oseledec put a dissipative measure here in [8].

3. *Continuity across a bifurcation*

By theorem 2 an attractor of a C^2 Axiom A diffeomorphism of a surface such as a DA attractor [19] has Hausdorff dimension $1 + \delta_s$ and by theorem 1 this varies continuously with f. So we can ask whether the dimension of the attractor is continuous across the bifurcation from Anosov to DA.

Let f_q, $0 \le q \le 1$ be a path through C^2 diffeomorphisms of the torus T^2 starting at a linear Anosov diffeomorphism f_0 and ending at a DA example f_1 with every f_q agreeing with f_0 outside a fixed neighbourhood V of the origin. Assume that each f_q preserves the stable manifolds of f_0 which are then its own stable manifolds (except for splitting the line though the origin in two).
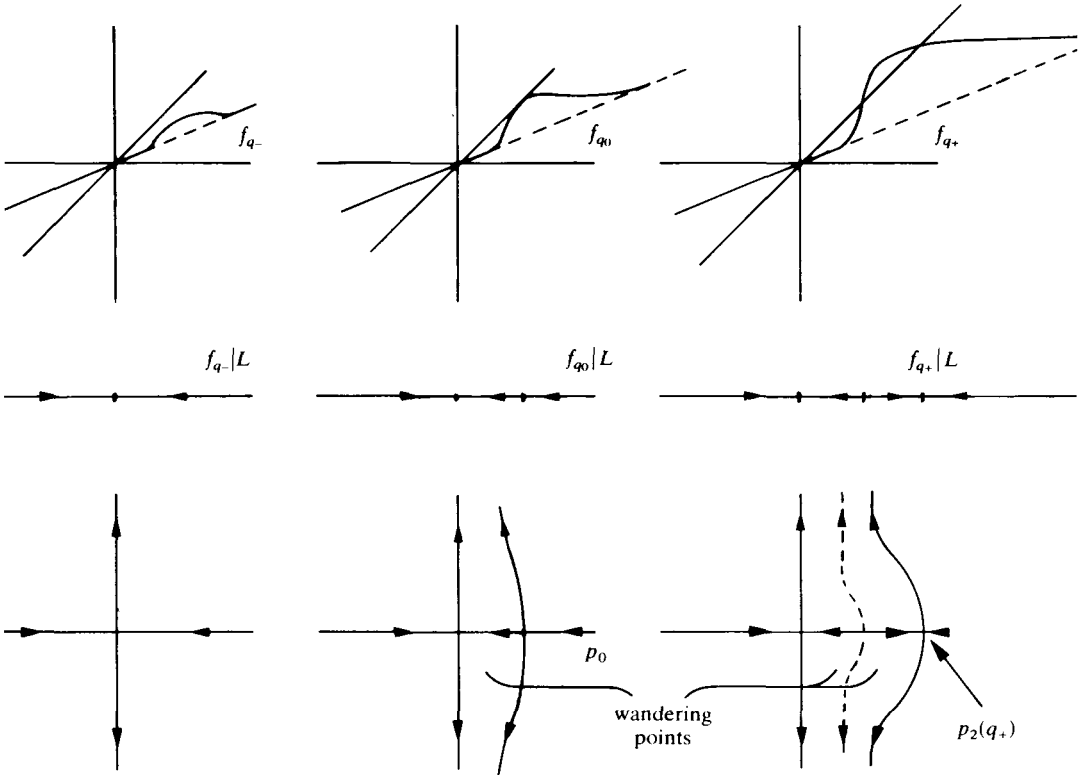
FIGURE 4

Let L denote $W^s_\varepsilon(0, f_0)$ and consider $f_q | L$. Figure 4 shows the graph of $f_q | L$ for three values $q_- < q_0 < q_+$ of q where q_0 is the bifurcation value. It also shows below these graphs the behaviour of f_q on L and on a neighbourhood of the origin. The saddle-node fixed point p of f_{q_0} is fairly near 0 in L.

THEOREM 3. *Under the above hypotheses on f_q, $0 \le q \le 1$, the Hausdorff dimension of $\Omega(f_q)$ is a continuous function of q.*

Proof. We have already proved continuity for $q < q_0$ and for $q > q_0$. So it remains to prove that the dimension tends to 2 as $q \to q_0+$ and that $\Omega(f_{q_0})$ has dimension 2. For the first of these note that the new fixed point $p_2(q_+)$ has its contracting eigenvalue near 1. Specifically,

$$(f_q | L)'(p_2(q)) \to 1$$

from below as $q \to q_0+$. The delta measure at this fixed point is invariant, has zero entropy and integrates ϕ^s_q to give $\phi^s_q(p_2(q))$ which tends to 0 as $q \to q_0+$. Now

$$P_{f_q}(\phi^s_q) = \sup_\mu (h_\mu(f_q) + \mu(\phi^s_q)) \ge \phi^s_q(p_2(q)) \to 0 \qquad \text{as } q \to q_0+.$$

Hence $\delta_s \to 1$ as $q \to q_0+$ and so the dimension of $\Omega(f_q)$, which by theorem 2 is $1 + \delta_s$, tends to 2 as $q \to q_0+$ as required.

For the second statement notice that $\Omega(f_{q_0})$ is the complement of the open half-space to the left of p_0, which consists of points that wander away from p_0. The dimension of $\Omega(f_{q_0})$ in a stable manifold is at least the δ_s associated with a hyperbolic invariant subset Λ of Ω obtained by deleting the orbit of a very small open rectangle containing p_0. This δ_s is arbitrarily close to 1 because the pressure of $\phi_{q_0}^s$ on Λ can be seen to be arbitrarily near 0 by considering the invariant atomic measure on a periodic orbit in Λ that spends most of its orbit near the fixed point p_0 at which $\phi_{q_0}^s$ is 0. Thus $\Omega(f_q)$ has Hausdorff dimension 2 at $q = q_0$ and the dimension is a continuous function of q.   □

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